Spectral Stiff Problems

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We consider a stiff problem that models the vibrations of a body consisting of two materials, one of them very stiff with respect to the other. We study the asymptotic behavior of the eigenvalues and eigenfunctions of the corresponding spectral problem, when the stiffness constant ε of only one of the material tends to 0. We give information about the structure of the eigenfunctions associated with the low and high frequencies (cf.[1-3]). We provide precise estimates for the convergence rates of the eigenelements (cf.[4]). We also outline some questions on the problem which remain unsolved. Further extensions on the problem for different geometries, boundary conditions and operators can be found in [5-6].

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HIGH FREQUENCY VIBRATIONS IN A STIFF PROBLEM*

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The stiff problem here considered models the vibrations of a body consisting of two materials, one of them very stiff with respect to the other. We study the asymptotic behavior of the eigenvalues and eigenfunctions of the corresponding spectral problem, when the stiffness constant of only one of the materials tends to 0. We show that the associated operator has a discrete spectrum "converging", in a certain sense, towards a continuous spectrum in $[0, \infty)$ corresponding to an operator. We also provide information on the structure of the eigenfunctions associated with the high frequencies.

1. Introduction and Statement of the Problem

Let us consider Ω a bounded open domain of $\mathbb{R}^n$, $n \geq 2$, Ω with a smooth boundary divided into two parts Ω_0 and Ω_1 by the interface Σ: $\Omega = \Omega_0 \cup \Omega_1 \cup \Sigma$. We assume that Ω_0 and Ω_1 have a Lipschitz boundary and denote by $\partial_i\Omega$ the part of the boundary of Ω_i contained in $\partial\Omega$, that is to say: $\partial\Omega_i = \partial_i\Omega \cup \Sigma$, $i = 0, 1$. For simplicity, hereafter, we assume that Σ is a part of the plane $\{x_n = 0\}$ and $\Omega_1 \subset \{x_n > 0\}$ and $\Omega_0 \subset \{x_n < 0\}$ (cf. Fig. 1).

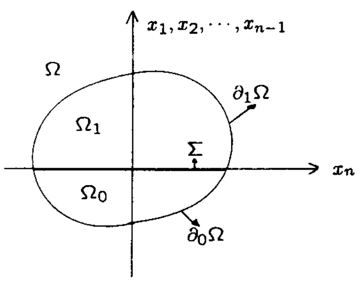

Fig. 1.

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Let us consider the vibration problem:

$$\begin{cases} \dfrac{\partial^2 \mathbf{u}_0^\varepsilon}{\partial t^2} - \Delta u_0^\varepsilon = 0 & \text{in } \Omega_0 \,, \\ \dfrac{\partial^2 \mathbf{u}_1^\varepsilon}{\partial t^2} - \varepsilon \Delta u_1^\varepsilon = 0 & \text{in } \Omega_1 \,, \\ \mathbf{u}_0^\varepsilon = 0 \text{ on } \partial_0 \Omega \,, & \mathbf{u}_1^\varepsilon = 0 \text{ on } \partial_1 \Omega \,, \\ \mathbf{u}_0^\varepsilon = \mathbf{u}_1^\varepsilon \,, & \dfrac{\partial \mathbf{u}_0^\varepsilon}{\partial n} = \varepsilon \dfrac{\partial \mathbf{u}_1^\varepsilon}{\partial n} \text{ on } \Sigma \,, \\ \mathbf{u}^\varepsilon(0,x) = \phi_1(x) \,, & \dfrac{\partial \mathbf{u}^\varepsilon}{\partial t}(0,x) = \phi_2(x) \,. \end{cases} \tag{1.1}$$

The indices 0 and 1 denote the restrictions of the function defined in Ω to the sub-domains Ω_0 and Ω_1 respectively, and ε is a positive small parameter that we shall make to go to 0.

(1.1) models the vibrations of a body occupying the domain Ω, one part of which, Ω_0, is very stiff with respect to the other. For suitable initial data, $\phi_1 \in H_0^1(\Omega)$, $\phi_2 \in L^2(\Omega)$, (1.1) admits a weak formulation:

Find $\mathbf{u}^\varepsilon(t)$ with values in $\mathbf{V}$ such that

$$\left(\frac{d^2 \mathbf{u}^\varepsilon}{dt^2} \,, v\right)_{\mathbf{H}} + a^\varepsilon(\mathbf{u}^\varepsilon \,, v) = 0 \,, \quad \forall\, v \in \mathbf{V} \tag{1.2}$$

$$\mathbf{u}^\varepsilon(0) = \phi_1 \,, \quad \frac{d\mathbf{u}^\varepsilon}{dt}(0) = \phi_2 \,, \tag{1.3}$$

where $\mathbf{V} = H_0^1(\Omega)$, $\mathbf{H} = L^2(\Omega)$.

$$a^\varepsilon(u,v) = a_0(u,v) + \varepsilon a_1(u,v) \,, \quad \forall\, u,v \in \mathbf{V} \,,$$

$$a_i(u,v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx \,, \quad i = 0,1 \,, \quad (u,v)_{\mathbf{H}} = \int_\Omega uv \, dx \,.$$

As a^ε is a bilinear, symmetric, continuous and coercive form on $\mathbf{V}$, (1.2)–(1.3) is a standard vibration problem in the spaces $\mathbf{V}$, $\mathbf{H}$, $\mathbf{V} \subset \mathbf{H}$ with compact imbedding. The spectral problem associated with (1.2)–(1.3) is:

Find λ^ε, $u^\varepsilon \in \mathbf{V}$, $u^\varepsilon \neq 0$ such that:

$$\int_{\Omega_0} \nabla u^\varepsilon \cdot \nabla v \, dx + \varepsilon \int_{\Omega_1} \nabla u^\varepsilon \cdot \nabla v \, dx = \lambda^\varepsilon \int_\Omega u^\varepsilon v \, dx \,, \quad \forall\, v \in \mathbf{V} \,. \tag{1.4}$$

(1.4) is a standard eigenvalue problem. For each fixed $\varepsilon > 0$, let us consider:

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \cdots \xrightarrow{n\to\infty} \infty \,,$$

the sequence of eigenvalues, with the classical convention of repeated eigenvalues. Let $\{u_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions, which are assumed to be an orthonormal basis in $L^2(\Omega)$, i.e.

$$\|u_i^\varepsilon\|_{L^2(\Omega)} = 1 \,. \tag{1.5}$$

In this paper, we deal with the asymptotic behavior of the eigenvalues and the eigenfunctions when $\varepsilon \to 0$. From a physical viewpoint we can postulate that two kinds of eigenfrequencies, corresponding to two different orders of magnitude, can appear: one for the stiffer structure and the other for the less stiff structure.

Mathematically, the coerciveness of the form a^ε and the mini-max principle (see Ref. 5, for instance, for the technique) allow us to obtain an estimate for the eigenvalues. In fact, for each fixed $i = 1, 2, 3, \ldots$, we have

$$C\varepsilon \le \lambda_i^\varepsilon \le C_i \varepsilon\,, \tag{1.6}$$

where C, C_i are constants independent of ε, C independent of i, and, $C_i \to \infty$ when $i \to \infty$.

Estimate (1.6) allows us to assert that there is a spectral concentration phenomenon in the origin: obviously, there are sequences of eigenvalues of order $O(\varepsilon)$ that converge, the so-called *low frequencies*. Moreover, converging sequences of eigenvalues, $\lambda_{i(\varepsilon)}^\varepsilon$, of order $O(1)$ can also exist; these eigenvalues are referred to as *the high frequencies* (in fact, they are in the range of the *medium frequencies*).

In this paper, we study the asymptotic behavior of the high frequencies (cf. Sec. 2) and the corresponding eigenfrequencies (cf. Sec. 3). The asymptotic behavior of the low frequencies has been widely studied with different techniques in Refs. 2, 4, 6 and 9 where it is shown that the low frequencies do not provide a good insight into the vibration problem over all Ω. In Sec. 1.1 we introduce some notations and present, in a suitable way, some results for the low and high frequencies, which will be useful for the proofs throughout the work. In Sec. 4 we give the results for the dimension of the space $n = 1$, as we consider that they may clarify the more general results of Sec. 3.

1.1. *Previous results for the problem*

Results in Refs. 4, 6 and 9 for the asymptotic behavior of the low frequency vibrations allow us to assert that the eigenfunctions associated with low frequencies vanish asymptotically in Ω_0, while they are asymptotically the eigenfunctions of the Dirichlet problem in Ω_1:

$$\begin{cases} -\Delta u_1 = \lambda u_1 & \text{in } \Omega_1\,, \\ \quad u_1 = 0 & \text{on } \partial\Omega_1\,. \end{cases} \tag{1.7}$$

We state here the main convergence result for the low frequencies which will be useful in Sec. 3: see Sec. VII.6 of Ref. 10 for a proof in the framework of the holomorphic perturbation theory, and Refs. 4, 6 and 9 for an asymptotic expansion of the eigenvalues and eigenfunctions.

Lemma 1. *For each i, the values $\lambda_i^\varepsilon/\varepsilon$, converge when $\varepsilon \to 0$ towards the eigenvalue of (1.7) with conservation of the multiplicity (λ_i^ε being the eigenvalues of (1.4)). The* *corresponding eigenfunctions,* u_i^ε, *converge to* u_i *in* $L^2(\Omega)$ *where* u_{i1} *is an eigenfunction associated with the ith eigenvalue of* (1.7), *extended by zero to* Ω_0, *and* $\{u_{i1}\}_{i=1}^\infty$ *is an orthonormal basis of* $L^2(\Omega_1)$.

Different aspects of the high frequencies are considered in Refs. 2 and 6. In both papers, the problem:

$$\begin{cases} -\Delta u_0 = \lambda u_0 & \text{in } \Omega_0\,, \\ u_0 = 0 & \text{on } \partial_0\Omega\,, \\ \dfrac{\partial u_0}{\partial n} = 0 & \text{on } \Sigma \end{cases} \tag{1.8}$$

appears in a natural way (it is enough to take $\varepsilon = 0$ in (1.4)). In relation to this problem, it will prove useful to enunciate the following lemma which provides information about the asymptotic behavior of the high frequencies (see Ref. 2 for the proof of *a*) and Sec. VII.6 of Ref. 10 for that of *b*)):

Lemma 2. (a) *If* $\lambda_{i(\varepsilon)}^\varepsilon \xrightarrow{\varepsilon\to 0} \lambda^0$ *and the corresponding eigenfunctions* $u_{i(\varepsilon)}^\varepsilon$ *converge towards* u^0 *in* $L^2(\Omega)$*-weak as* $\varepsilon \to 0$, *with* $u^0 \neq 0$, *then,* (λ^0, u_0^0) *is an eigenelement of* (1.8), *and* $u_1^0 = 0$. (b) *Each eigenvalue of* (1.8) *is a point of accumulation of eigenvalues of* (1.4).

Note that lemma 2 does not provide much information about the eigenfunctions associated with high frequencies. Other results (cf. Lemma 3 of Sec. 2) are given in Ref. 10 but all are obtained in terms of a very poor convergence of the corresponding spectral families. We will attempt to improve the partial answers given to the problem in the previous papers.

As noted in Ref. 11, $\{u_i^\varepsilon\}_{i=1}^\infty$ is a basis of $L^2(\Omega)$, the eigenfunctions $\{u_{i1}\}_{i=1}^\infty$ of (1.7) form a basis of $L^2(\Omega_1)$, $u_i^\varepsilon \xrightarrow{\varepsilon\to 0} u_i$ in $L^2(\Omega)$, $u_{i0} = 0$, (for each fixed i), and $L^2(\Omega) = L^2(\Omega_0) \oplus L^2(\Omega_1)$. Other eigenfunctions tending, when $\varepsilon \to 0$, to span the orthogonal space, $L^2(\Omega_0)$, should exist. Obviously, we must look for these eigenfunctions among those associated with the high frequencies.

In Sec. 3 we show that only the eigenfunctions associated with $\lambda_{i(\varepsilon)} \approx \lambda^0$, λ^0 being an eigenvalue of (1.8), have asymptotically non-null projections in $L^2(\Omega_0)$.

In the following sections, σ_ε denotes the spectrum of the operator associated with (1.4), and σ_N that associated with (1.8). The explicit calculations for the dimension $n = 1$ (cf. Sec. 4) allow us to intuit the following results for any dimension n of the space:

1. σ_ε *is asymptotically dense in* $[0, \infty)$: in fact, the high frequencies accumulate in $(0, \infty)$ while the low frequencies do so at the point $\{0\}$.

2. *Information can be obtained about the structure of the eigenfunctions associated with the high frequencies*: only those associated with eigenvalues converging (as $\varepsilon \to 0$) towards points of σ_N have a component asymptotically non-null in $L^2(\Omega_0)$. Moreover, all the eigenfunctions have a strongly oscillatory character in $L^2(\Omega_1)$.

Section 2 is devoted to proving the first assertion (cf. Theorem 1): the idea of obtaining spectral properties when the convergence of the solution of some time-dependent problems are known, the Fourier and Laplace transforms, and, techniques of the distributional boundary value of analytic functions theory are used for this proof. Section 3 is devoted to proving assertion 2 above (cf. Theorem 2): the main ideas used for the proofs are some results of the spectral perturbation theory, and the fact that the eigenfunctions of (1.7) are a basis of $L^2(\Omega_1)$ that must be completed, in some way, with the eigenfunctions of (1.4) associated with high frequencies.

2. Spectral Concentration of σ_ε in $[0, \infty)$

Spectral concentration phenomena for self-adjoint operators, $\mathcal{T}^\varepsilon$, on a Hilbert space $\mathbf{H}$, that converge when $\varepsilon \to 0$ in a certain sense to another self-adjoint operator $\mathcal{T}$ on $\mathbf{H}$, have been considered extensively in Refs. 3 and 8. The case in which a continuous spectrum is changed by perturbation into a discrete spectrum was considered in Sec. III.4 of Ref. 8 (see also Example 1.19 of Sec. VII in Ref. 3). In these studies some restrictions on the domains of operators $\mathcal{T}^\varepsilon$, as well as on the convergence of the corresponding spectral families were made. For example, in Ref. 8, $\mathcal{D}(\mathcal{T}^\varepsilon) = \mathcal{D}(\mathcal{T})$, and the strong convergence of the corresponding spectral families holds, as stated in the Rellich theorem. The generalized eigenfunctions are used for the proof in Ref. 8.

We study here a case of spectral concentration where the discrete spectrum of $\mathcal{T}^\varepsilon$ is changed into a continuous one but none of the above conditions are satisfied. All the spectral families associated with operators appearing in this section are considered in the abstract framework of unbounded closed self-adjoint operators on a Hilbert space $\mathbf{H}$, with domain $\mathcal{D}(\mathcal{T}) = \{u : u \in \mathbf{V} \,/\, \mathcal{T}u \in \mathbf{H}\}$ dense on $\mathbf{H}$: the imbedding $\mathbf{V} \subset \mathbf{H}$ is dense and continuous, and $\mathcal{T}$ is an operator associated with a bilinear continuous symmetric and coercive form on $\mathbf{V}$. We state the main result in this section, which confirms the previous assertion about the spectral convergence. We use for its proof a variant of the method of the Fourier transform, with respect to time, of the solutions of the evolution problem (1.2)–(1.3) for suitable initial data and techniques of the distributional boundary value of analytic functions.

Theorem 1. *For any* $\lambda^* > 0$, *there is a sequence* $\lambda^\varepsilon_{i(\varepsilon)}$ *of eigenvalues of* (1.4) *converging to* λ^* *as* $\varepsilon \to 0$.

The proof of the theorem is a consequence of Lemmas 3–5 hereafter. First, let us make some remarks about the statement of the theorem: we observe that when $\lambda^* \in \sigma_N$, the result is a consequence of statement (b) in Lemma 2. For the other values λ^*, we prove that for each neighborhood of λ^*, J, there are eigenvalues $\lambda^\varepsilon_{i(\varepsilon)} \in J$, for small enough ε. The rest of the section is devoted to proving this result, which, in fact, also holds for $\lambda^* \in \sigma_N$. Let us first introduce some notations which will prove useful for the proof. As noted in Ref. 6, the corresponding vibrations associated with $\lambda^\varepsilon_{i(\varepsilon)} \approx \lambda^*$ have a short wavelength, and we dilate the space variable x by introducing a new variable y.

Let us consider problem (1.4) written in the y variable, $y = x/\sqrt{\varepsilon}$:

$$\frac{1}{\varepsilon}\int_{\Omega_0/\sqrt{\varepsilon}} \nabla_y u^\varepsilon \cdot \nabla_y v\, dy + \int_{\Omega_1/\sqrt{\varepsilon}} \nabla_y u^\varepsilon \cdot \nabla_y v\, dy = \lambda^\varepsilon \int_{\Omega/\sqrt{\varepsilon}} u^\varepsilon v\, dy\,, \quad \forall\, v \in H_0^1\left(\frac{\Omega}{\sqrt{\varepsilon}}\right). \tag{2.1}$$

Throughout this section Ω_ε denote $\Omega_\varepsilon = \{y \in \mathbb{R}^n / \sqrt{\varepsilon} y \in \Omega\}$, and we assume that the elements of $H_0^1(\Omega_\varepsilon)$ and $H_0^1(\mathbb{R}^{n+})$ are extended to $\mathbb{R}^n$ with the value 0.

Let $\mathcal{A}^\varepsilon$ be the positive self-adjoint anticompact operator on $L^2(\Omega_\varepsilon)$ associated with the form on the left-hand side of (2.1). $\mathcal{A}^\varepsilon$ has a discrete spectrum, $\sigma_\varepsilon = \{\lambda_i^\varepsilon\}_{i=1}^\infty$, and let $\{e_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions, $e_i^\varepsilon \in H_0^1(\Omega_\varepsilon)$, which are assumed to be an orthonormal basis in $L^2(\Omega_\varepsilon)$. Let $\mathcal{B}^\varepsilon$ be $\mathcal{B}^\varepsilon = \mathcal{A}^\varepsilon + I$, ($I$ the unitary operator) whose spectrum is $\{\mu_i^\varepsilon\}_{i=1}^\infty$, shifted from σ_ε: $\mu_i^\varepsilon = \lambda_i^\varepsilon + 1$.

Let $\mathcal{A}$ be the operator on $L^2(\mathbb{R}^{n+})$ associated with the Laplacian operator in the upper half-space $\mathbb{R}^{n+}$ having a Dirichlet condition on the plane $\mathbb{R}^{n-1}$. As is known, $\mathcal{A}$ is a non-negative self-adjoint operator in $L^2(\mathbb{R}^{n+})$ but is not an anticompact operator as the imbedding $H_0^1(\mathbb{R}^{n+}) \subset L^2(\mathbb{R}^{n+})$ is no longer compact. The fact that $\mathcal{A}$ has a continuous spectrum $\sigma(\mathcal{A}) = [0, \infty)$ is proved by extending the functions of $L^2(\mathbb{R}^{n+})$ to $x_n < 0$ as $u(x_1, x_2, \ldots, x_n) = -u(x_1, x_2, \ldots, -x_n)$ and by using the Fourier transform on the space $\{u \in L^2(\mathbb{R}^n) / u(x_1, x_2, \ldots, x_n) = -u(x_1, x_2, \ldots, -x_n)\}$ (cf. for example Chap. 2 of Ref. 13 for the technique). Let $\mathcal{B}$ denote the operator $\mathcal{B} = \mathcal{A} + I$. Obviously, $\mathcal{B}$ is the operator associated with the bilinear symmetric continuous and coercive form on $H_0^1(\mathbb{R}^{n+})$:

$$\int_{\mathbb{R}^{n+}} \nabla_y u \cdot \nabla_y v\, dy + \int_{\mathbb{R}^{n+}} uv\, dy\,, \quad \forall\, u, v \in H_0^1(\mathbb{R}^{n+})\,,$$

and $\sigma(\mathcal{B}) = [1, \infty)$.

Let us consider the evolution problem analogous to (1.2)–(1.3), in the spaces $L^2(\Omega_\varepsilon)$ and $H_0^1(\Omega_\varepsilon)$, for some initial data:

$$\begin{cases} \dfrac{d^2\mathbf{u}^\varepsilon}{dt^2} + \mathcal{B}^\varepsilon \mathbf{u}^\varepsilon = 0\,, \\ \mathbf{u}^\varepsilon(0) = 0\,, \ \dfrac{d\mathbf{u}^\varepsilon}{dt}(0) = f_\varepsilon\,. \end{cases} \tag{2.2}$$

Let the limit problem be

$$\begin{cases} \dfrac{d^2\mathbf{u}^*}{dt^2} + \mathcal{B}\mathbf{u}^* = 0\,, \\ \mathbf{u}^*(0) = 0\,, \ \dfrac{d\mathbf{u}^*}{dt}(0) = f_+\,, \end{cases} \tag{2.3}$$

in the spaces $L^2(\mathbb{R}^{n+})$ and $H_0^1(\mathbb{R}^{n+})$. The indices ε and $+$ denote the restriction of the function f in $L^2(\mathbb{R}^n)$ to Ω_ε and $\mathbb{R}^{n+}$, respectively. At the moment, f can be any function of $L^2(\mathbb{R}^n)$ for which some restrictions have to be made in Lemma 4. We have the following results for the solutions of (2.2) and (2.3) (cf. Ref. 6 and Sec. VII.6 of Ref. 10 for the proof):

Lemma 3. *Let $\mathbf{u}^\varepsilon(t)$ be the solution of (2.2) ($\mathbf{u}^*(t)$ that of (2.3)), with values in $H_0^1(\Omega_\varepsilon)$ ($H_0^1(\mathbf{R}^{n+})$ respectively), that we assume to be extended by 0 to $\mathbb{R}^n - \Omega_\varepsilon$ ($\mathbb{R}^{n-}$, respectively). Then,*

$$\begin{cases} \mathbf{u}^\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \mathbf{u}^* \ \text{in } L^\infty(-\infty,\infty,H^1(\mathbb{R}^n)) \text{ weak-}^*, \\ \dfrac{d\mathbf{u}^\varepsilon}{dt} \overset{\varepsilon\to 0}{\longrightarrow} \dfrac{d\mathbf{u}^*}{dt} \ \text{in } L^\infty(-\infty,\infty,L^2(\mathbb{R}^n)) \text{ weak-}^*, \end{cases} \tag{2.4}$$

as well as the convergence for the corresponding spectral families:

$$\langle \mathcal{E}^\varepsilon(\lambda)w_\varepsilon, v_\varepsilon\rangle_{L^2(\Omega_\varepsilon)} \overset{\varepsilon\to 0}{\longrightarrow} \langle \mathcal{E}(\lambda)w_+, v_+\rangle_{L^2(\mathbb{R}^{n+})}, \text{ in } L^\infty(-\infty,\infty) \text{ weak-}^*, \tag{2.5}$$

$\mathcal{E}^\varepsilon(\lambda)$, $\mathcal{E}(\lambda)$ being the spectral families associated with $\mathcal{B}^\varepsilon$ and $\mathcal{B}$, respectively.

As is well known, the spectral family $\mathcal{E}^\varepsilon(\lambda)$ is a piecewise constant function with values in $\mathcal{L}(L^2(\Omega_\varepsilon))$, the discontinuities being the eigenvalues of $\mathcal{B}^\varepsilon$: $\mu_i^\varepsilon = \lambda_i^\varepsilon + 1$. This is not the case of the spectral family $\mathcal{E}(\lambda)$ which is a continuous and increasing one in $[1,\infty)$.

In the framework of Lemma 3, when the limit spectral family has a point of discontinuity, the Fourier transform techniques (cf. for example, Sec. V.XII of Ref. 10) usually allow us to characterize this point as a point of accumulation of the discontinuities of $\mathcal{E}^\varepsilon(\lambda)$. As this is not the case, we try to get a similar result for the points of $\sigma(\mathcal{B})$ using both Fourier and Laplace transforms. Because of the relation between the Laplace transform of the solutions of (2.2) and (2.3) and the corresponding resolvent operators, the Fourier transforms, considered as the distributional boundary value of the Laplace transforms on the imaginary axis $\mathrm{Re}(p) = 0$, provide some local information about the behavior of the spectrum of $\mathcal{B}^\varepsilon$ in a neighborhood of a singular point of the resolvent of the limit problem, that is to say, in a neighborhood of the spectral points of $\mathcal{B}$.

According to the semigroup theory, we can write the Laplace transform of $\mathbf{u}^\varepsilon$ and $\mathbf{u}^*$ in terms of the resolvent operators:

$$\mathcal{L}[\mathbf{u}^\varepsilon](p) = (\mathcal{B}^\varepsilon + p^2)^{-1} f_\varepsilon, \text{ for } \mathrm{Re}(p) > 0 \tag{2.6}$$

and

$$\mathcal{L}[\mathbf{u}^*](p) = (\mathcal{B} + p^2)^{-1} f_+, \text{ for } \mathrm{Re}(p) > 0. \tag{2.7}$$

Obviously, $\mathcal{L}[\mathbf{u}^\varepsilon](p) \in H_0^1(\Omega_\varepsilon)$ and $\mathcal{L}[\mathbf{u}^*](p) \in H_0^1(\mathbb{R}^{n+})$.

Let us introduce the scalar functions g^ε and g defined as:

$$g^\varepsilon(t) = \begin{cases} \langle \mathbf{u}^\varepsilon(t), f\rangle_{L^2(\mathbb{R}^n)} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \tag{2.8}$$

$$g(t) = \begin{cases} \langle \mathbf{u}^*(t), f\rangle_{L^2(\mathbb{R}^n)} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \tag{2.9}$$

where we observe that the scalar products defining g^ε and g are in fact scalar products in $L^2(\Omega_\varepsilon)$ and $L^2(\mathbb{R}^{n+})$, respectively. It is evident that $g^\varepsilon, g \in \mathcal{S}'_+$, $\mathcal{S}'_+$ being the space of the tempered distributions with support contained in $[0,\infty)$.

From (2.4), we deduce $g^\varepsilon \xrightarrow{\varepsilon\to 0} g$ in $\mathcal{S}'(\mathbb{R})$; therefore, the convergence of the corresponding Fourier transforms also holds:

$$\mathcal{F}[g^\varepsilon] \xrightarrow{\varepsilon\to 0} \mathcal{F}[g] \text{ in } \mathcal{S}'(\mathbb{R}) . \tag{2.10}$$

We define the boundary value on $\mathcal{S}'$ of $\mathcal{L}[g^\varepsilon](\alpha+i\beta)$ ($\mathcal{L}[g](\alpha+i\beta)$ respectively) on the imaginary axis $\alpha=0$ as $\mathcal{F}[g^\varepsilon](\beta)$ ($\mathcal{F}[g](\beta)$ respectively) in the way stated in Sec. II.2 of Ref. 1:

$$\langle \mathcal{F}[g^\varepsilon](\beta), \varphi(\beta)\rangle = \lim_{\alpha\to 0^+} \langle \mathcal{L}[g^\varepsilon](\alpha+i\beta), \varphi(\beta)\rangle , \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}) , \tag{2.11}$$

$$\langle \mathcal{F}[g](\beta), \varphi(\beta)\rangle = \lim_{\alpha\to 0^+} \langle \mathcal{L}[g](\alpha+i\beta), \varphi(\beta)\rangle , \quad \forall\, \varphi \in \mathcal{S}(\mathbb{R}) . \tag{2.12}$$

We try to obtain spectral information from (2.11)–(2.12). Dealing with the Fourier transform of g^ε we have the following result.

Lemma 4. *Let us consider any open bounded interval, $J \subset (1,\infty)$. Then,*

$$\mathcal{F}[g^\varepsilon](\beta) = -\sum_{j=1}^{\infty} \frac{\langle f, e_j^\varepsilon\rangle^2_{L^2(\mathbb{R}^n)}}{\beta+\sqrt{\mu_j^\varepsilon}} \left(i\pi\delta\left(\beta-\sqrt{\mu_j^\varepsilon}\right) + \mathcal{P}\left(\frac{1}{\beta-\sqrt{\mu_j^\varepsilon}}\right)\right) \text{ in } \mathcal{D}'(J) ,$$

where $\delta(\beta-\nu)$ and $\mathcal{P}(\frac{1}{\beta-\nu})$ are the translated of the Dirac and Principal value distributions to the point ν.

Proof. Since the solution of (2.2) is $\mathbf{u}^\varepsilon(t) = 1/\sqrt{\mathcal{B}^\varepsilon}\sin(\sqrt{\mathcal{B}^\varepsilon}t)f_\varepsilon$, we can write:

$$\mathbf{u}^\varepsilon(t) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\mu_j^\varepsilon}} \langle f, e_j^\varepsilon\rangle_{L^2(\mathbb{R}^n)} \sin(\sqrt{\mu_i^\varepsilon}t) e_j^\varepsilon ,$$

where e_j^ε are the eigenfunctions of (2.1). Thus, the Laplace transform of g^ε is:

$$\mathcal{L}[g^\varepsilon](p) = \mathcal{L}[g^\varepsilon](\alpha+i\beta) = \sum_{j=1}^{\infty} \frac{\langle f, e_j^\varepsilon\rangle^2_{L^2(\mathbb{R}^n)}}{\beta+\sqrt{\mu_j^\varepsilon}-\alpha i} \frac{-1}{\beta-\sqrt{\mu_j^\varepsilon}-\alpha i} ,$$

p being $p=\alpha+i\beta$, with $\alpha>0$.

Let $\tau=\tau(\beta)$ be any smooth function, $\tau\in\mathcal{D}(J)$. Let us pass to the limit in the relation $\langle \mathcal{L}[\mathbf{u}^\varepsilon](p), \tau(\beta)\rangle_{\mathcal{D}'(J)\times\mathcal{D}(J)}$ when $\alpha\to 0^+$. The classical Sokhotsky formula,

$$\lim_{\alpha\to 0^+} \frac{1}{\beta-i\alpha} = i\pi\delta(\beta) + \mathcal{P}\left(\frac{1}{\beta}\right) \text{ in } \mathcal{D}'(\mathbb{R}) , \tag{2.13}$$

and convergence (2.11) allow us to obtain:

$$\langle \mathcal{F}[g^\varepsilon](\beta)\,,\tau(\beta)\,\rangle_{\mathcal{D}'(J)\times\mathcal{D}(J)}$$
$$= \left\langle -\sum_{j=1}^{\infty} \frac{\langle f\,,e_j^\varepsilon\rangle^2_{L^2(\mathbb{R}^n)}}{\beta+\sqrt{\mu_j^\varepsilon}} \left(i\pi\delta\left(\beta-\sqrt{\mu_j^\varepsilon}\right) + \mathcal{P}\left(\frac{1}{\beta-\sqrt{\mu_j^\varepsilon}}\right)\right)\,,\tau(\beta)\right\rangle_{\mathcal{D}'(J)\times\mathcal{D}(J)},$$

where we must observe that the number of terms of the summation in which singular distributions appear is finite. Therefore, the lemma is proved. □

Dealing with the Fourier transform of g we have the following result.

Lemma 5. *Let λ^* and δ be any positive real number such that $\sqrt{\lambda^*+1}-\delta>1$. Let J_δ be the interval $J_\delta=(\sqrt{\lambda^*+1}-\delta,\sqrt{\lambda^*+1}+\delta)\subset(1,\infty)$. Let λ_1,λ_2 be two fixed real positive numbers such that $[\sqrt{\lambda_1},\sqrt{\lambda_2}]\subset J_\delta$. Let us consider the initial datum f of (2.2) and (2.3) as a particular $f\in L^2(\mathbb{R}^n)$ such that: $\|\mathcal{E}(\lambda)f_+\|_{L^2(\mathbb{R}^{n+})}$ is a constant for $\lambda\geq\lambda_2$ and the integral*

$$\int_{\lambda_1}^{\lambda_2} d\|\mathcal{E}(\lambda)f_+\|^2_{L^2(\mathbb{R}^{n+})}>0\,. \tag{2.14}$$

Then, for any $\tau\in D(J_\delta)$, we have:

$$\langle \mathcal{F}[g](\beta)\,,\tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)}$$
$$=\int_1^{\lambda_2} \frac{-1}{\sqrt{\lambda}+\beta}\left\langle\left(i\pi\delta(\beta-\sqrt{\lambda})+\mathcal{P}\left(\frac{1}{\beta-\sqrt{\lambda}}\right)\right),\tau(\beta)\right\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)}$$
$$\times\, d\|\mathcal{E}(\lambda)f_+\|^2_{L^2(\mathbb{R}^{n+})}\,.$$

Proof. As the spectrum of $\mathcal{B}$ fills the interval $[1,\infty)$, the corresponding spectral family takes the value 0 for $\lambda<1$ and it is not a constant in any open interval contained in $(1,\infty)$. Therefore, for λ_1,λ_2 as the lemma states, there is a function $\tilde f\in L^2(\mathbb{R}^{n+})$ such that $\mathcal{E}(\lambda_1)\tilde f\neq\mathcal{E}(\lambda_2)(\tilde f)$. Hence, $\|\mathcal{E}(\lambda)\tilde f\|_{L^2(\mathbb{R}^{n+})}$ is an increasing function in $[\lambda_1,\lambda_2]\subset(1,\infty)$, and (2.14) is true for $f=\tilde f$. In the case when $\mathcal{E}(\lambda)\tilde f$ is not constant for $\lambda>\lambda_2$, we take $f_+=\mathcal{E}(\lambda_2)\tilde f$ which satisfies all the properties stated in the lemma.

Let us consider (2.7) for $p=\alpha+i\beta$, $\alpha>0$:

$$\mathcal{L}[\mathbf{u}^*](\alpha+i\beta)=(\mathcal{B}+(\alpha+i\beta)^2)^{-1}f_+\,,\ \text{for } \alpha>0\,. \tag{2.15}$$

By writing the resolvent operator in terms of the spectral family, and taking into account the definition (2.9) of g, (2.15) reads:

$$\mathcal{L}[g](\alpha+i\beta)=\langle(\mathcal{B}+(\alpha+i\beta)^2)^{-1}f_+\,,f\rangle_{L^2(\mathbb{R}^n)}=\int_{-\infty}^{+\infty}\frac{1}{\lambda+p^2}\,d\|\mathcal{E}(\lambda)f_+\|^2_{L^2(\mathbb{R}^{n+})}\,.$$

For simplicity, we denote by $e(\lambda)$ the function $e(\lambda) = \|\mathcal{E}(\lambda)f_+\|^2_{L^2(\mathbb{R}^{n+})}$. $e(\lambda)$ is a continuous nondecreasing bounded function in $[1, \lambda_2] \subset J_\delta$ which takes the value 0 when $\lambda < 1$ and the constant value $e(\lambda_2)$ when $\lambda \geq \lambda_2$.

For any $\tau \in \mathcal{D}(J_\delta)$, and for fixed $\alpha > 0$, there is no problem in writing:

$$\langle\langle(\mathcal{B}+p^2))^{-1}f_+ \,, f\rangle^2_{L^2(\mathbb{R}^n)} \,, \tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)} = \int_{\sqrt{\lambda^*+1}-\delta}^{\sqrt{\lambda^*+1}+\delta} \tau(\beta)d\beta \int_1^{\lambda_2} \frac{1}{\lambda+p^2}\, de(\lambda)\,.$$

The classic Fubini theorem justifies the interchange of the integrals:

$$\langle\mathcal{L}[g](p) \,, \tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)} = \int_1^{\lambda_2} de(\lambda) \int_{\sqrt{\lambda^*+1}-\delta}^{\sqrt{\lambda^*+1}+\delta} \frac{\tau(\beta)}{(\sqrt{\lambda}+ip)(\sqrt{\lambda}-ip)}\, d\beta\,. \tag{2.16}$$

Note that when $\alpha = \mathrm{Re}(p) \to 0^+$, in formula (2.16) the only possible singular function appearing in the integral is $1/(\sqrt{\lambda}+i\alpha-\beta)$, as $\sqrt{\lambda}$ ranges from $\sqrt{\lambda_1}$ to $\sqrt{\lambda_2}$, and $\sqrt{\lambda_1}, \sqrt{\lambda_2} \in J_\delta$. Let us prove that we can apply the dominated convergence theorem in order to introduce the limit when $\alpha = \mathrm{Re}(p) \to 0^+$ inside the integral in λ. That is to say, we show that the integral

$$\left| \int_{\sqrt{\lambda^*+1}-\delta}^{\sqrt{\lambda^*+1}+\delta} \frac{\tau(\beta)}{\lambda+p^2}\, d\beta \right| \leq CR(\lambda)\,, \tag{2.17}$$

where C is a constant independent of α and $R(\lambda)$ is an integrable function, in the interval $[1, \lambda_2]$, for the measure $de(\lambda)$.

Let us consider the functions:

$$r_{\lambda\alpha}(\beta) = \frac{1}{\alpha + i(\beta - \sqrt{\lambda})} \quad \text{and} \quad s_{\lambda\alpha}(\beta) = \frac{\tau(\beta)}{\alpha - i(\beta + \sqrt{\lambda})}\,.$$

Obviously,

$$\int_{J_\delta} \frac{\tau(\beta)}{\lambda+p^2}\, d\beta = \int_{\sqrt{\lambda^*+1}-\delta}^{\sqrt{\lambda^*+1}+\delta} \frac{\tau(\beta)}{\lambda+p^2}\, d\beta = \int_{-\infty}^{+\infty} r_{\lambda\alpha}(\beta)\overline{s_{\lambda\alpha}(\beta)}\, d\beta\,.$$

Because of the Parseval identity, and the explicit calculations of the inverse Fourier transform of $r_{\lambda\alpha}(\beta)$, we obtain:

$$\begin{aligned}\int_{J_\delta} \frac{\tau(\beta)}{\lambda+p^2}\, d\beta &= 2\pi \int_{-\infty}^{+\infty} \mathcal{F}^{-1}[r_{\lambda\alpha}](t) \cdot \overline{\mathcal{F}^{-1}[s_{\lambda\alpha}](t)}\, dt \\ &= 2\pi \int_{-\infty}^{+\infty} e^{-\alpha t} H(t) e^{i\sqrt{\lambda}t} \cdot \overline{\mathcal{F}^{-1}[s_{\lambda\alpha}](t)}\, dt\,,\end{aligned}$$

where $H(t)$ is the Heaviside function. On account of $\alpha > 0$, and $\beta > 1$ in $\mathrm{Supp}(\tau)$, the following inequality is evident:

$$\left| \int_{J_\delta} \frac{\tau(\beta)}{\lambda+p^2} d\beta \right| \leq 2\pi \int_0^{\infty} e^{-\alpha t} \int_{-\infty}^{+\infty} \left| \frac{\tau(\beta)}{\alpha - i(\beta+\sqrt{\lambda})} \right| d\beta\, dt \leq \frac{C}{\sqrt{\lambda}}\,,$$

with C a constant independent of α and β, and we conclude that (2.17) holds.

Now, taking limits in (2.16), as $\alpha = \mathrm{Re}(p) \to 0^+$, we deduce:

$$\lim_{\alpha\to 0^+} \langle \mathcal{L}[g](\alpha + i\beta)\,,\tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)}$$
$$= \int_1^{\lambda_2} de(\lambda) \lim_{\alpha\to 0^+} \int_{J_\delta} \frac{\tau(\beta)}{(\sqrt{\lambda}-\beta+i\alpha)(\sqrt{\lambda}+\beta-i\alpha)}\,d\beta\,.$$

Then, from (2.12) and (2.13) we obtain:

$$\langle \mathcal{F}[g](\beta)\,,\tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)}$$
$$= \int_1^{\lambda_2} \frac{-1}{\sqrt{\lambda}+\beta}\Big\langle (i\pi\delta(\beta-\sqrt{\lambda}) + \mathcal{P}\Big(\frac{1}{\beta-\sqrt{\lambda}}\Big)\Big)\,,\tau(\beta)\Big\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)} de(\lambda)\,,$$

and the lemma is proved. □

Proof of Theorem 1. We show that for any $\lambda^* > 0$, and for any J_δ as stated in Lemma 5, there exists $j(\varepsilon)$ such that $\sqrt{\mu^\varepsilon_{j(\varepsilon)}} \in J_\delta$, for each small enough ε ($\varepsilon \le \varepsilon_0(\delta)$).

Taking the imaginary part in (2.10) we can write, for any $\tau \in \mathcal{D}(J_\delta)$:

$$\mathrm{Im}[\langle \mathcal{F}[g^\varepsilon](\beta)\,,\tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)}] \xrightarrow{\varepsilon\to 0} \mathrm{Im}[\langle \mathcal{F}[g](\beta), \tau(\beta)\rangle_{\mathcal{D}'(J_\delta)\times\mathcal{D}(J_\delta)}]\,. \tag{2.18}$$

From Lemmas 4 and 5, (2.18) reads:

$$\lim_{\varepsilon\to 0}\sum_{j=1}^{\infty} \langle f\,,e^\varepsilon_j\rangle^2_{L^2(\mathbb{R}^n)}\pi\frac{\tau(\sqrt{\mu^\varepsilon_j})}{2\sqrt{\mu^\varepsilon_j}} = \int_1^{\lambda_2} \frac{\pi}{2\sqrt{\lambda}}\tau(\sqrt{\lambda})\,de(\lambda)\,.$$

It is enough to take the function $\tau \in \mathcal{D}(J_\delta)$, $\tau \ge 0$, $\tau(\beta) > 0$ in the interval $[\sqrt{\lambda_1}, \sqrt{\lambda_2}] \subset (1, \lambda_2)$, to get

$$\int_1^{\lambda_2} \frac{\pi}{2\sqrt{\lambda}}\tau(\sqrt{\lambda})\,de(\lambda) \neq 0\,.$$

Then, we deduce that for small ε there are $\mu^\varepsilon_{j(\varepsilon)}$ such that $\tau\Big(\sqrt{\mu^\varepsilon_{j(\varepsilon)}}\Big) \neq 0$, so that $\sqrt{\mu^\varepsilon_{j(\varepsilon)}} \in J_\delta$, and Theorem 1 is proved.

Remark 1. We point out that the technique used for the proof of Theorem 1 can be generalized to a more general case: more general domains depending on ε, for operators $\mathcal{A}^\varepsilon$, or the case in which $\mathcal{A}$ has a mixed spectrum: isolated eigenvalues and a continuous spectrum σ_c, as the main condition required for the proof is that for $\lambda^* \in [a, b]$ there be suitably chosen numbers λ_1, λ_2 and functions f such that $[\lambda_1, \lambda_2] \subset (a, b)$ and

$$\int_{\lambda_1}^{\lambda_2} d\|\mathcal{E}(\lambda)f\|^2 > 0\,;$$

here $[a, b]$ is any interval of the continuous spectrum set σ_c.

The Fourier–Laplace technique in this section has been applied in Ref. 5 for the case of a limit eigenvalue problem with a discrete spectrum, dealing with a boundary homogenization problem, where the limit of the solutions of the corresponding hyperbolic problems is only characterized through the Laplace transform. □

3. On the Structure of the Eigenfunctions Associated with High Frequencies

Convergence results for high frequencies are given in Ref. 10 in terms of the convergence of the solution of time-dependent problems for some suitable initial data, as well as in terms of convergences of the spectral families. Nevertheless, no information about the structure of the corresponding eigenfunctions seems to be obtained from these results. Two different limit hyperbolic problems are obtained in Ref. 10: the first one provides some information about the vibrations in Ω_0 and none about those in Ω_1; the other one, written in the local variable y (cf. Sec. 2), gives information about the vibrations in $\Omega_1/\sqrt{\varepsilon}$, and the corresponding vibrations vanishing in Ω_0. A fact common to both problems is that the imbedding of the spaces $\mathbf{V} \subset \mathbf{H}$ where limit problems are posed is noncompact. Nevertheless, the operator associated with the first problem has a pure point spectrum, $\sigma_N \cup \{0\}$, while that associated with the second one has a continuous spectrum: the whole real positive axis $[0, \infty)$. As $\sigma_N \subset (0, \infty)$, the eigenfunctions associated with eigenvalues λ^ε near the points of σ_N should have a different behavior from those associated with the rest of the eigenvalues (asymptotically near a point of $(0, \infty) - \sigma_N$). To differentiate this behavior is the aim of Propositions 1 and 2, while to prove their strongly oscillatory character in Ω_1 is that of Proposition 3. We gather all these results in the following theorem.

Theorem 2. *Let λ be any positive real number. Let I_{δ^ε} denote the interval $[\lambda - \delta^\varepsilon, \lambda + \delta^\varepsilon]$ having eigenvalues $\lambda^\varepsilon_{i(\varepsilon)}$ of (1.4), and δ^ε converging to 0 as $\varepsilon \to 0$. Then, the following assertions hold:*

1. *$\lambda \in \sigma_N$ if and only if there are $\{\delta^\varepsilon\}_\varepsilon$ and $\{u^\varepsilon\}_\varepsilon$, each u^ε belonging to the eigenspace associated with all the eigenvalues in I_{δ^ε}, of norm 1 in $L^2(\Omega)$, and, such that $\|u^\varepsilon\|_{L^2(\Omega_0)} > a > 0$, for some constant a independent of ε.*
2. *For any $\lambda \in \sigma_N$ and any eigenfunction U_0 associated with λ, the sequence u^ε in statement 1 can be chosen such that:*

$$u^\varepsilon = \alpha^\varepsilon U_0 + \sum_{j=p(\varepsilon)}^{p'(\varepsilon)} \alpha_j^\varepsilon u_{j1} + o_\varepsilon(1) \text{ in } L^2(\Omega)\,, \tag{3.1}$$

   with α^ε a constant, $\alpha^\varepsilon = O_s(1)$.
3. *For any $\lambda \notin \sigma_N$, any δ^ε and any sequence $\{u^\varepsilon\}$, u^ε of norm 1 in $L^2(\Omega)$ and u^ε in the eigenspace associated with all the eigenvalues in I_{δ^ε}, we have:*

$$u^\varepsilon = \sum_{j=p(\varepsilon)}^{p'(\varepsilon)} \alpha_j^\varepsilon u_{j1} + o_\varepsilon(1) \text{ in } L^2(\Omega). \tag{3.2}$$

α_j^ε, in statements 2 and 3, are the Fourier coefficients of the expansion of u_0^ε in Fourier series of the eigenfunctions, $\{u_{j1}\}_{j=1}^\infty$ of (1.7), and, $p(\varepsilon), p'(\varepsilon)$ are two functions converging to ∞ as $\varepsilon \to 0$.

The proof of Theorem 2 is a consequence of Propositions 1, 2 and 3 below (cf. also Remarks 3 and 4). It will prove useful for their proofs to state here the following result (see Ref. 12 for its proof).

Lemma 6. *Let $\mathcal{A} : \mathbf{H} \longrightarrow \mathbf{H}$ be a linear self-adjoint positive and compact operator on a Hilbert space $\mathbf{H}$. Let $u \in \mathbf{H}$, with $\|u\|_{\mathbf{H}} = 1$ and $\lambda, r > 0$ such that $\|\mathcal{A}u - \lambda u\|_{\mathbf{H}} < r$. Then, there exists an eigenvalue λ_i of $\mathcal{A}$ satisfying $|\lambda - \lambda_i| < r$. Moreover, for any $r^* > r$ there is $u^* \in \mathbf{H}$ with $\|u^*\|_{\mathbf{H}} = 1$ such that*

$$\|u - u^*\|_{\mathbf{H}} < \frac{2r}{r^*},$$

u^* belonging to the eigenspace associated with all the eigenvalues of the operator $\mathcal{A}$ lying on the segment $[\lambda - r^*, \lambda + r^*]$.

Proposition 1. *Let $\lambda^0 \in \sigma_N$, and U_0 be an associated eigenfunction such that $\|\nabla U_0\|_{L^2(\Omega_0)} = 1$. Then, there is a sequence $\{d^\varepsilon\}_\varepsilon$, $d^\varepsilon \xrightarrow{\varepsilon \to 0} 0$, such that $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ has eigenvalues of (1.4): $\lambda_{i(\varepsilon)}^\varepsilon, \lambda_{i(\varepsilon)+1}^\varepsilon, \dots, \lambda_{i(\varepsilon)+K(\varepsilon)}^\varepsilon$. Moreover, there is $\tilde{u}_{i(\varepsilon)}^\varepsilon \in H_0^1(\Omega)$, with $\|\tilde{u}_{i(\varepsilon)}^\varepsilon\|_{L^2(\Omega)} = 1$, $\tilde{u}_{i(\varepsilon)}^\varepsilon$ belonging to the eigenspace associated with all the eigenvalues $\lambda_{i(\varepsilon)}^\varepsilon$ in $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$, such that*

$$\|\tilde{u}_{i(\varepsilon)}^\varepsilon\|_{L^2(\Omega_0)} \geq a,$$

for some constant $a > 0$, and

$$\tilde{u}_{i(\varepsilon)}^\varepsilon = \alpha^\varepsilon U_0 + o_\varepsilon(1) \text{ in } L^2(\Omega_0), \tag{3.3}$$

where α^ε is:

$$\alpha^\varepsilon = \sqrt{\|\nabla \tilde{u}_{i(\varepsilon)}^\varepsilon\|_{L^2(\Omega_0)}^2 + \varepsilon \|\nabla \tilde{u}_{i(\varepsilon)}^\varepsilon\|_{L^2(\Omega_1)}^2}, \quad \alpha^\varepsilon = O_s(1).$$

Proof. Let $\mathbf{V}^\varepsilon$ be the space $H_0^1(\Omega)$ with the norm defined by

$$\|u\|_{\mathbf{V}^\varepsilon}^2 = \|\nabla u_0\|_{L^2(\Omega_0)}^2 + \varepsilon \|\nabla u_1\|_{L^2(\Omega_1)}^2, \quad \forall\, u \in H_0^1(\Omega).$$

Let A^ε be the operator defined by:

$$\langle A^\varepsilon u, v \rangle_{\mathbf{V}^\varepsilon} = \int_\Omega uv\, dx, \quad \forall\, u, v \in \mathbf{V}^\varepsilon.$$

It is evident that A^ε is a positive, compact and symmetric operator on $\mathbf{V}^\varepsilon$ whose eigenvalues are $1/\lambda^\varepsilon$, λ^ε being the eigenvalues of (1.4).

Let $\varphi_\alpha^\varepsilon$ be the smooth function which takes the value 1 in $x_n \le \varepsilon^\alpha$, and 0 outside the half-space $x_n \ge 2\varepsilon^\alpha$:

$$\varphi_\alpha^\varepsilon(x) = \varphi\Big(\frac{x_n}{\varepsilon^\alpha}\Big),$$

where α is a positive number such that $\frac{1}{2} < \alpha < 1$, $\varphi \in C^\infty(\mathbb{R})$, $\varphi(r) = 1$ if $r \le 1$ and $\varphi(r) = 0$ if $r \ge 2$. Obviously, all the derivatives of $\varphi_\alpha^\varepsilon$ are 0 except the derivative $\partial\varphi_\alpha^\varepsilon/\partial x_n$ which is different from zero in $x_n \in [\varepsilon^\alpha, 2\varepsilon^\alpha]$ and it is bounded by a C_0/ε^α, with C_0 a constant independent of ε. In fact, all the constants C and C_i, $i = 1, \dots, 4$, appearing hereafter in this proof are independent of ε.

Let U_0 be as the proposition states. We extend U_0 to Ω_1: let U be the function in $H_0^1(\Omega)$, a fixed extension of U_0 (for example, U can be a harmonic function in Ω_1).

We consider $w^\varepsilon = U\varphi_\alpha^\varepsilon$ and $\tilde{w}^\varepsilon = \frac{w^\varepsilon}{\|w^\varepsilon\|_{\mathbf{V}^\varepsilon}}$. Obviously $w^\varepsilon, \tilde{w}^\varepsilon \in \mathbf{V}^\varepsilon$ and $w_0^\varepsilon = U_0$ in Ω_0. The convergence of $\tilde{w}_0^\varepsilon$ towards U_0 in $H^1(\Omega_0)$, as $\varepsilon \to 0$, is a consequence of the convergence:

$$\|w^\varepsilon\|_{\mathbf{V}^\varepsilon} \xrightarrow{\varepsilon\to 0} 1, \tag{3.4}$$

which holds from the fact $\|\nabla U_0\|_{L^2(\Omega_0)} = 1$ and

$$\varepsilon\|\nabla w^\varepsilon\|^2_{L^2(\Omega_1)} \xrightarrow{\varepsilon\to 0} 0. \tag{3.5}$$

In order to prove (3.5) we take into account the definition of $\varphi_\alpha^\varepsilon$, the estimates for this function and its derivatives, and the relation:

$$\|u\|^2_{L^2(\{x_n<2\varepsilon^\alpha\}\cap\Omega_1)} \le C_1\varepsilon^\alpha\|\nabla u_1\|^2_{L^2(\Omega_1)}, \quad \forall\, u \in H_0^1(\Omega) \tag{3.6}$$

(cf. p. 134 of Ref. 7, for example). Then, we easily obtain the estimate:

$$\varepsilon\|\nabla w_1^\varepsilon\|^2_{L^2(\Omega_1)} \le C_2\varepsilon^{1-\alpha}, \tag{3.7}$$

so, the convergence (3.5) is also true. Taking into account that (λ^0, U_0) is an eigenelement of (1.8), $w^\varepsilon \in \mathbf{V}^\varepsilon$, and the definition of A^ε, we have:

$$\langle A^\varepsilon w^\varepsilon - \frac{1}{\lambda^0}w^\varepsilon, v\rangle_{\mathbf{V}^\varepsilon} = \int_{\{x_n<2\varepsilon^\alpha\}\cap\Omega_1} U\varphi_\alpha^\varepsilon v\,dx - \frac{\varepsilon}{\lambda^0}\int_{\Omega_1}\nabla(U\varphi_\alpha^\varepsilon)\cdot\nabla v\,dx,$$
$$\forall\, v \in H_0^1(\Omega).$$

Then, from the Swarchtz inequality, (3.5)–(3.7), we deduce:

$$\Big|\Big\langle A^\varepsilon\tilde{w}^\varepsilon - \frac{1}{\lambda^0}\tilde{w}^\varepsilon, v\Big\rangle_{\mathbf{V}^\varepsilon}\Big| \le C\varepsilon^p\|v\|_{\mathbf{V}^\varepsilon}, \quad \forall\, v \in H_0^1(\Omega),$$

where $p = \min(\alpha - 1/2, (1-\alpha)/2) > 0$. So, for each small enough ε, we have:

$$\|A^\varepsilon\tilde{w}^\varepsilon - \frac{1}{\lambda^0}\tilde{w}^\varepsilon\|_{\mathbf{V}^\varepsilon} \le C\varepsilon^p.$$

We apply Lemma 6 with $\mathcal{A} = A^\varepsilon$, $\mathbf{H} = \mathbf{V}^\varepsilon$, $u = \tilde{w}^\varepsilon$, $r = C\varepsilon^p$ and $r^* = 2\varepsilon^{p/2}$: we deduce that there are eigenvalues, $\{\lambda^{-1}_{i(\varepsilon)+j}\}_{j=1}^{K(\varepsilon)}$, of A^ε in $I_\varepsilon = [\frac{1}{\lambda^0} - 2\varepsilon^{p/2}, \frac{1}{\lambda^0} + 2\varepsilon^{p/2}]$. Moreover, for each ε, there is $\tilde{U}^\varepsilon_{i(\varepsilon)} \in H^1_0(\Omega)$, with $\|\tilde{U}^\varepsilon_{i(\varepsilon)}\|_{\mathbf{V}^\varepsilon} = 1$, $\tilde{U}^\varepsilon_{i(\varepsilon)}$ belonging to the eigenspace associated with all of the eigenvalues in I_ε, such that

$$\|\tilde{U}^\varepsilon_{i(\varepsilon)} - \tilde{w}^\varepsilon\|_{\mathbf{V}^\varepsilon} \le C\varepsilon^{p/2}\,. \tag{3.8}$$

As $\tilde{w}^\varepsilon \xrightarrow{\varepsilon\to 0} U_0$ in $H^1(\Omega_0)$, (3.8) leads us to assert that $\tilde{U}^\varepsilon_{i(\varepsilon)} \xrightarrow{\varepsilon\to 0} U_0$ in $H^1(\Omega_0)$. In order to normalize $\tilde{U}^\varepsilon_{i(\varepsilon)}$ in $L^2(\Omega)$ we prove:

$$C_3 \le \|\tilde{U}^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega)} \le C_4\,. \tag{3.9}$$

The first estimate in (3.9) is evident because $\|\tilde{U}^\varepsilon_{i(\varepsilon)}\|^2_{L^2(\Omega_0)} \xrightarrow{\varepsilon\to 0} 1/\lambda_0$. As $\tilde{U}^\varepsilon_{i(\varepsilon)}$ belongs to the eigenspace associated with the eigenvalues in I_ε, we can write

$$\tilde{U}^\varepsilon_{i(\varepsilon)} = \sum_{j=1}^{K(\varepsilon)} \beta^\varepsilon_j u^\varepsilon_{i(\varepsilon)+j}\,,$$

for some constants β^ε_j. Then, on account of $\|u^\varepsilon_{i(\varepsilon)+j}\|^2_{L^2(\Omega)} = \frac{1}{\lambda^\varepsilon_{i(\varepsilon)+j}}\|u^\varepsilon_{i(\varepsilon)+j}\|^2_{\mathbf{V}^\varepsilon}$, and the orthogonality, in $\mathbf{V}^\varepsilon$ and $L^2(\Omega)$, of the eigenfunctions of (1.4), we obtain

$$\|\tilde{U}^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega)} \le C_4\|\tilde{U}^\varepsilon_{i(\varepsilon)}\|_{\mathbf{V}^\varepsilon} = C_4\,.$$

Therefore, the right-hand side estimate in (3.9) is also true.

Then, we have proved that statements in proposition hold for $d^\varepsilon = O(\varepsilon^{p/2})$, $\tilde{u}^\varepsilon_{i(\varepsilon)} = \frac{\tilde{U}^\varepsilon_{i(\varepsilon)}}{\|\tilde{U}^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega)}}$ and $\alpha^\varepsilon = \frac{1}{\|\tilde{U}^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega)}} = \|\tilde{u}^\varepsilon_{i(\varepsilon)}\|_{\mathbf{V}^\varepsilon}$.

Proposition 2. *Let us consider* $\lambda^* > 0$, *let* $\{\delta^\varepsilon\}_\varepsilon$ *be any sequence such that* $\delta^\varepsilon \to 0$ *as* $\varepsilon \to 0$, *let* $\{\lambda^\varepsilon_{i(\varepsilon)}, \lambda^\varepsilon_{i(\varepsilon)+1}, \dots, \lambda^\varepsilon_{i(\varepsilon)+\tilde{K}(\varepsilon)}\}$ *be all the eigenvalues of* (1.4) *in* $[\lambda^* - \delta^\varepsilon, \lambda^* + \delta^\varepsilon]$, *and* u^ε *any function in the eigenspace* $[u^\varepsilon_{i(\varepsilon)}, u^\varepsilon_{i(\varepsilon)+1}, \dots, u^\varepsilon_{i(\varepsilon)+\tilde{K}(\varepsilon)}]$ *with* $\|u^\varepsilon\|_{L^2(\Omega)} = 1$.

(i) *If there is some subsequence* $\{u^{\varepsilon_k}\}_k$, $\|u^{\varepsilon_k}\|_{L^2(\Omega_0)} > a > 0$, *for some constant* a, *then* (λ^*, U^*) *is an eigenelement of* (1.8), *where* $U^* \in H^1(\Omega_0)$ *is the limit of* u^{ε_k} *in* $L^2(\Omega_0)$, $\varepsilon_k \to 0$.

(ii) *If* $\lambda^* \notin \sigma_N$, *then*

$$\|u^\varepsilon\|_{L^2(\Omega_0)} \xrightarrow{\varepsilon\to 0} 0\,.$$

Proof. Let $\lambda^* > 0$, $\{\delta^\varepsilon\}_\varepsilon$, $\tilde{K}(\varepsilon)$, u^ε be as stated in the proposition. We prove assertion (ii) by contradiction.

Let us assume that λ^* is not an eigenvalue of (1.8), and the sequence $\|u^\varepsilon\|_{L^2(\Omega_0)}$ does not converge to zero as $\varepsilon \to 0$. On account that $u^\varepsilon = \sum_{j=1}^{\tilde{K}(\varepsilon)} \beta^\varepsilon_j u^\varepsilon_{i(\varepsilon)+j}$, for certain constants β^ε_j, the relation

$$\|u^\varepsilon\|_{\mathbf{V}^\varepsilon} \le C_5\|u^\varepsilon\|_{L^2(\Omega)} \tag{3.10}$$

is proved in the same way as (3.9), with minor modifications, and it allows us to assert that there is a subsequence $\{u^{\varepsilon_k}\}_k$, converging towards u_0^* in $H^1(\Omega_0)$-weak, as $\varepsilon_k \to 0$, with $u_0^* \neq 0$ and, such that $u^{\varepsilon_k} \overset{k\to\infty}{\longrightarrow} u_1^*$ in $L^2(\Omega_1)$-weak and $\sqrt{\varepsilon_k} u^{\varepsilon_k} \overset{k\to\infty}{\longrightarrow} v^*$ in $H^1(\Omega_1)$-weak. So that $v^* = 0$.

Considering (1.4), for each eigenelement $(\lambda^\varepsilon_{i(\varepsilon)+j}, u^\varepsilon_{i(\varepsilon)+j})$, $j = 1, 2, \ldots, \tilde{K}(\varepsilon)$, and multiplying it by β_j^ε we easily obtain, for $\varepsilon = \varepsilon_k$, the relation:

$$\int_{\Omega_0} \nabla u^\varepsilon \cdot \nabla v \, dx + \varepsilon \int_{\Omega_1} \nabla u^\varepsilon \cdot \nabla v \, dx = \lambda^* \int_\Omega u^\varepsilon v \, dx$$

$$+ \sum_{j=1}^{\tilde{K}(\varepsilon)} (\lambda^\varepsilon_{i(\varepsilon)+j} - \lambda^*) \beta_j^\varepsilon \int_\Omega u^\varepsilon_{i(\varepsilon)+j} v \, dx \,, \quad \forall\, v \in H_0^1(\Omega) \,. \tag{3.11}$$

Then, on account of $|\lambda^{\varepsilon_k}_{i(\varepsilon_k)+j} - \lambda^*| \leq \delta^{\varepsilon_k}$, we take limits in (3.11) when $\varepsilon_k \to 0$ to obtain the relation

$$\int_{\Omega_0} \nabla u^* \cdot \nabla v \, dx = \int_\Omega u^* v \, dx \,, \quad \forall\, v \in H_0^1(\Omega) \,,$$

satisfied by u^*. So, $u_1^* = 0$ and u_0^* is the eigenfunction of (1.8) associated with λ^*. Therefore, result (ii) holds. It is evident that the demonstration of assertion (i) is contained in the previous proof. □

Corollary 1. *Let λ be a positive real number, $\lambda \notin \sigma_N$, and let $\{\lambda^\varepsilon_{i(\varepsilon)}\}_\varepsilon$ be any sequence of eigenvalues of* (1.4) *converging to λ as $\varepsilon \to 0$. Then, the corresponding eigenfunctions, $u^\varepsilon_{i(\varepsilon)}$, satisfy:*

$$\|u^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega_0)} \overset{\varepsilon\to 0}{\longrightarrow} 0 \,.$$

Remark 2. In the proof of Proposition 1 we have obtained that $d^\varepsilon = O(\varepsilon^{p/2})$, for a particular real p, $0 < p < \frac{1}{2}$. If there is only one eigenvalue, $\lambda^\varepsilon_{i(\varepsilon)}$ in $[\lambda^0 - d^\varepsilon, \lambda^0 + d^\varepsilon]$ with multiplicity 1, then, function $\tilde{u}^\varepsilon_{i(\varepsilon)}$ in (3.3) is the corresponding eigenfunction. Moreover, in this case, Corollary 1 would be a complementary result to characterize eigenfunctions associated with eigenvalues $\lambda^\varepsilon \approx \lambda^0$: if $\lambda^0 \in \sigma_N$, then $u^\varepsilon_{i(\varepsilon)} = \alpha^\varepsilon U_0 + o_\varepsilon(1)$ in $L^2(\Omega_0)$; if $\lambda^0 \notin \sigma_N$, then $u^\varepsilon_{i(\varepsilon)} = o_\varepsilon(1)$ in $L^2(\Omega_0)$ (cf. formula (4.6) in Sec. 4). That is to say, the projections in $L^2(\Omega_0)$ of the eigenfunctions are asymptotically different from zero if and only if the corresponding eigenvalues converge towards an eigenvalue of (1.8). Otherwise, Proposition 2 is the complementary result of Proposition 1. □

Proposition 3. *Let λ be a positive real number, and let $\lambda^\varepsilon_{i(\varepsilon)}$ be the sequence of eigenvalues of* (1.4) *converging to λ, as $\varepsilon \to 0$. Then, for the corresponding eigenfunctions $u^\varepsilon_{i(\varepsilon)}$, there are functions $U^\varepsilon, V^\varepsilon$, with $U^\varepsilon \in L^2(\Omega_0)$, $U^\varepsilon|_{\Omega_1} = 0$ and $V^\varepsilon \in L^2(\Omega_1)$, $V^\varepsilon|_{\Omega_0} = 0$, and such that*

$$u^\varepsilon_{i(\varepsilon)} = U^\varepsilon + V^\varepsilon \,, \tag{3.12}$$

and

$$V^\varepsilon = \sum_{j=p(\varepsilon)}^{p'(\varepsilon)} \alpha_j^\varepsilon u_{j1} + o_\varepsilon(1) \text{ in } L^2(\Omega_1), \text{ or } \|V^\varepsilon\|_{L^2(\Omega_1)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0\,, \tag{3.13}$$

where α_j^ε *are the Fourier coefficients of the expansion of* $u_{i(\varepsilon)0}^\varepsilon$ *in Fourier series of the eigenfunctions* $\{u_{j1}\}_{j=1}^\infty$ *of* (1.7), *and,* $p(\varepsilon), p'(\varepsilon)$ *are two functions which converge to* ∞ *as* $\varepsilon \to 0$.

Proof. Let p be any fixed integer, $p \geq 1$, and $\{\lambda_j^\varepsilon\}_{j=1}^p$ the first p eigenvalues of (1.4) and $\{u_j^\varepsilon\}_{j=1}^p$ the corresponding eigenfunctions that are assumed to have norm 1 in $L^2(\Omega)$.

Lemma 1 and the decomposition $L^2(\Omega) = L^2(\Omega_0) \oplus L^2(\Omega_1)$ allow us to write:

$$u_j^\varepsilon = u_{j1} + r_j^\varepsilon + s_j^\varepsilon$$

and

$$u_{i(\varepsilon)}^\varepsilon = U^\varepsilon + V^\varepsilon$$

with $V^\varepsilon, u_{j1}, r_j^\varepsilon \in L^2(\Omega_1)$, $U^\varepsilon, s_j^\varepsilon \in L^2(\Omega_0)$, and $r_j^\varepsilon, s_j^\varepsilon = o_\varepsilon(1)$ in $L^2(\Omega)$. Besides, V^ε can be expanded in Fourier series of the eigenfunctions of (1.7)

$$V^\varepsilon = \sum_{j=1}^\infty \alpha_j^\varepsilon u_{j1}\,, \text{ with } \alpha_j^\varepsilon = \frac{\langle V^\varepsilon, u_{j1}\rangle_{L^2(\Omega_1)}}{\|u_{j1}\|^2_{L^2(\Omega_1)}}\,.$$

Because of (1.6), λ_j^ε must be different from $\lambda_{i(\varepsilon)}^\varepsilon$, for $i = 1, 2, \ldots, p$, and, therefore, the corresponding eigenfunctions u_j^ε and $u_{i(\varepsilon)}^\varepsilon$ are orthogonal in $L^2(\Omega)$. Let us consider:

$$0 = \Big\langle u_{i(\varepsilon)}^\varepsilon, \sum_{j=1}^p \alpha_j^\varepsilon u_j^\varepsilon \Big\rangle_{L^2(\Omega)} = \Big\langle U^\varepsilon + \sum_{j=1}^\infty \alpha_j^\varepsilon u_{j1}\,, \sum_{j=1}^p \alpha_j^\varepsilon u_{j1} + \sum_{j=1}^p \alpha_j^\varepsilon r_j^\varepsilon + \sum_{j=1}^p \alpha_j^\varepsilon s_j^\varepsilon \Big\rangle_{L^2(\Omega)}\,.$$

Using the orthogonality of U^ε and u_{j1}, U^ε and r_j^ε, and V^ε and s_j^ε, we have

$$\sum_{j=1}^p |\alpha_j^\varepsilon|^2 = -\Big\langle U^\varepsilon, \sum_{j=1}^p \alpha_j^\varepsilon s_j^\varepsilon \Big\rangle_{L^2(\Omega_0)} - \Big\langle V^\varepsilon, \sum_{j=1}^p \alpha_j^\varepsilon r_j^\varepsilon \Big\rangle_{L^2(\Omega_1)}\,.$$

On account that $\|U^\varepsilon\|_{L^2(\Omega_0)} \leq 1$ and $\|V^\varepsilon\|_{L^2(\Omega_1)} = \sum_{j=1}^\infty |\alpha_j^\varepsilon|^2 \leq 1$, the α_j^ε are bounded by a constant independent of ε and the convergence

$$\sum_{j=1}^p |\alpha_j^\varepsilon|^2 \stackrel{\varepsilon\to 0}{\longrightarrow} 0$$

holds for each fixed p. Then, using a classic argument of diagonalization we can assert that there is a sequence $p(\varepsilon) \stackrel{\varepsilon\to 0}{\longrightarrow} \infty$ such that

$$\Big\| V^\varepsilon - \sum_{j=p(\varepsilon)}^\infty \alpha_j^\varepsilon u_{j1} \Big\|_{L^2(\Omega_1)} \stackrel{\varepsilon\to 0}{\longrightarrow} 0\,.$$

Therefore, the proposition is proved. □

Remark 3. We observe that the proof of formulas (3.12)–(3.13) also holds in the case where the eigenfunctions $u^\varepsilon_{i(\varepsilon)}$ are replaced by the functions $\tilde{u}^\varepsilon_{i(\varepsilon)}$ or u^ε involved in Propositions 1 and 2, with minor modifications. □

Remark 4. We observe that in the case considered in Remark 2, (3.12) can be written:

$$u^\varepsilon_{i(\varepsilon)} = \alpha^\varepsilon U_0 + \sum_{j=p(\varepsilon)}^{p'(\varepsilon)} \alpha^\varepsilon_j u_{j1} + o_\varepsilon(1) \text{ in } L^2(\Omega)\,, \tag{3.14}$$

where U_0 is the eigenfunction associated with the eigenvalue λ of problem (1.8) in Ω_0. This would also be the case if $\|V^\varepsilon\|_{L^2(\Omega_1)} \xrightarrow{\varepsilon\to 0} 0$, as $\|u^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega_0)} > a > 0$.

In the case where $\|u^\varepsilon_{i(\varepsilon)}\|_{L^2(\Omega_0)}$ does not converge to 0, then (3.14) holds for any sequence ε_k such that $\|u^{\varepsilon_k}_{i(\varepsilon_k)}\|_{L^2(\Omega_0)} \geq a > 0$, for some eigenfunction U_0 associated with λ.

On the other hand, on account of Corollary 1, when $\lambda \notin \sigma_N$:

$$u^\varepsilon_{i(\varepsilon)} = \sum_{j=p(\varepsilon)}^{p'(\varepsilon)} \alpha^\varepsilon_j u_{j1} + o_\varepsilon(1) \text{ in } L^2(\Omega)\,. \tag{3.15}$$

□

Note that the terms appearing in the Fourier expansion (cf. (3.13)–(3.15)) correspond to eigenfunctions of (1.7) associated with large eigenvalues of this problem. Therefore, each eigenfunction is likely to be a strongly oscillating function.

4. Results for the Dimension n=1

For the case of the dimension $n = 1$, if we take into account that all the eigenvalues of (1.4) have multiplicity 1 and that some explicit computations can be performed, results in Sec. 3 can be improved. Here, we simply write the main formulas in order to illustrate the results in Sec. 3.

Let us consider the eigenvalue problem (1.4) in relation to the vibrations of a string placed in $(-1,1)$, the stiffer part Ω_0 in $(-1,0)$, and, the less stiff part Ω_1 in $(0,1)$,

$$\begin{cases} -\dfrac{d^2u^\varepsilon}{dx^2} = \lambda^\varepsilon u^\varepsilon\,, & x \in (-1,0)\,,\\ -\varepsilon\dfrac{d^2u^\varepsilon}{dx^2} = \lambda^\varepsilon u^\varepsilon\,, & x \in (0,1)\,,\\ u^\varepsilon(-1) = 0\,, & u^\varepsilon(1) = 0\,,\\ u^\varepsilon(0^-) = u^\varepsilon(0^+)\,, & \dfrac{du^\varepsilon}{dx}(0^-) = \varepsilon\dfrac{du^\varepsilon}{dx}(0^+)\,. \end{cases} \tag{4.1}$$

With the exception of the values λ^ε such that $\cos\sqrt{\lambda} = 0$ or $\sin\sqrt{\lambda} = 0$ (cf. Remark 5), simple calculations show us that the eigenvalues λ^ε of (4.1) are the λ roots of the equation:

$$\sqrt{\varepsilon}\tan\sqrt{\lambda} + \tan\sqrt{\frac{\lambda}{\varepsilon}} = 0\,, \tag{4.2}$$

and the corresponding eigenfunctions are:

$$u^\varepsilon(x) = \begin{cases} \alpha_\varepsilon\left(\cos\sqrt{\lambda^\varepsilon}x + \frac{1}{\tan\sqrt{\lambda^\varepsilon}}\sin\sqrt{\lambda^\varepsilon}x\right), & x \in (-1,0) \\ \alpha_\varepsilon\left(\cos\sqrt{\frac{\lambda^\varepsilon}{\varepsilon}}x - \frac{1}{\tan\sqrt{\frac{\lambda^\varepsilon}{\varepsilon}}}\sin\sqrt{\frac{\lambda^\varepsilon}{\varepsilon}}x\right), & x \in (0,1) \end{cases} \tag{4.3}$$

where α_ε is a constant such that $\int_{-1}^{1} u^\varepsilon(x)^2\,dx = 1$. In fact,

$$\alpha_\varepsilon^2 = \frac{2\varepsilon\sin^2\sqrt{\lambda^\varepsilon}}{\varepsilon + \varepsilon\sin^2\sqrt{\lambda^\varepsilon} - \cos^2\sqrt{\lambda^\varepsilon}}\,. \tag{4.4}$$

Obviously, for $\lambda^\varepsilon = O(1)$, the eigenfunctions (4.3) do not have a regular asymptotic expansion in $\Omega_1 = (0,1)$; we observe that they are strongly oscillating functions. Problem (1.8) is posed now in $(-1,0)$ ((1.7) in $(0,1)$, respectively). Its eigenvalues are $(\frac{(2k-1)\pi}{2})^2, k = 1,2,3,\ldots$, and the corresponding eigenfunctions $\cos(\frac{(2k-1)\pi}{2}x)$ ($(k\pi)^2$, $\sin(k\pi x)$, respectively).

Theorem 1 in Sec. 2 allows us to assert that for each $\lambda > 0$ there is a sequence $\lambda^\varepsilon_{i(\varepsilon)}$ of eigenvalues of (4.1) converging towards λ when $\varepsilon \to 0$. Moreover, in this case, on account of (4.2), one sequence $\lambda^\varepsilon_{i(\varepsilon)}$ can be taken in such a way that for ε small enough,

$$|\lambda^\varepsilon_{i(\varepsilon)} - \lambda| \le C\varepsilon\,, \tag{4.5}$$

where C is a constant independent of ε.

Results in Theorem 2 can now be stated in the following way:

For any sequence $\lambda^\varepsilon_{i(\varepsilon)} \xrightarrow{\varepsilon\to 0} \lambda$, with $\lambda > 0$, the corresponding eigenfunctions (4.3) can be approached in $L^2(-1,1)$ by

$$u^\varepsilon_{i(\varepsilon)}(x) \approx \begin{cases} \tilde{\alpha}^\varepsilon\cos\left(\frac{(2k-1)\pi}{2}x\right), & x \in (-1,0) \\ \displaystyle\sum_{j=j(\varepsilon)}^{\infty} \alpha^\varepsilon_j\sin(j\pi x)\,, & x \in (0,1) \end{cases}, \tag{4.6}$$

where $\tilde{\alpha}^\varepsilon = \alpha_\varepsilon$ if $\lambda = ((2k-1)\pi/2)^2$ for some k and $\tilde{\alpha}^\varepsilon = 0$ otherwise, α^ε_j are the coefficients of the expansion of $u^\varepsilon_{i(\varepsilon)}$ in Fourier series of the eigenfunctions of (1.7) in $L^2(0,1)$:

$$\alpha^\varepsilon_j = \frac{2j\pi\alpha_\varepsilon}{j^2\pi^2 - \frac{\lambda^\varepsilon_{i(\varepsilon)}}{\varepsilon}}\,,$$

and $j(\varepsilon)$ is a function converging to ∞ as $\varepsilon \to 0$.

Equation (4.2) allows us to assert that $|\alpha_\varepsilon|$ is bounded independently of ε, and, in the case when the sequence of the eigenvalues satisfy (4.5) for $\lambda = ((2k-1)\pi/2)^2$, then, we can take $\tilde{\alpha}^\varepsilon = 1$ in formula (4.6).

Remark 5. We observe that the possible eigenvalues $\lambda = \lambda^\varepsilon$ such that $\cos\sqrt{\lambda} = 0$ (and $\cos\sqrt{\frac{\lambda}{\varepsilon}} = 0$) are not included in Eq. (4.2). Each one of these values, $\lambda = ((2k-1)\pi/2)^2$, is an eigenvalue of (4.1) only for certain values of ε: those of the sequence $\varepsilon_n = ((2k-1)/(2n+1))^2$. In this case, the corresponding eigenfunctions are: $u^{\varepsilon_n}(x) = \beta_{\varepsilon_n}\cos(\frac{(2k-1)\pi}{2}x), x \in (-1,0)$ (i.e. the eigenfunction of (1.8) in $(-1,0)$), and $u^{\varepsilon_n}(x) = \beta_{\varepsilon_n}\cos(\frac{(2k-1)\pi}{2\sqrt{\varepsilon_n}}x), x \in (0,1)$). Now $\beta_{\varepsilon_n} = O_s(1)$. This result merely reaffirms Eq. (4.6). Similar results, with minor modifications, are obtained in the case $\sin\sqrt{\lambda} = 0$ (and $\sin\sqrt{\frac{\lambda}{\varepsilon}} = 0$), excluded from formula (4.3): $u^{\varepsilon_n}(x) = \gamma_{\varepsilon_n}\sqrt{\varepsilon_n}\sin(k\pi x), x \in (-1,0)$, and $u^{\varepsilon_n}(x) = \gamma_{\varepsilon_n}\sin(\frac{k\pi}{\sqrt{\varepsilon_n}}x), x \in (0,1)$; $\gamma_{\varepsilon_n} = O_s(1)$. □

We note that it is a basic fact for n to take the value 1: on the one hand, the eigenvalues are characterized through the roots of Eq. (4.2) and, on the other hand, we have the explicit formula (4.3) for the eigenfunctions, which involves the corresponding eigenvalues. (4.6) confirms the strongly oscillatory character of the eigenfunctions, when $\varepsilon \to 0$ in $(0,1)$, already noted in (4.3). In fact, all the results in this section can be obtained by means of simple computations, without using the technique outlined in the previous sections.

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Asymptotically sharp uniform estimates in a scalar spectral stiff problem

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Abstract

Estimates of convergence rates for rescaled eigenvalues of the stiff Neumann problem for the Laplacian are obtained. The bounds are expressed in terms of the stiffness ratio and properties of the limit spectrum both for low and middle frequency ranges. ***To cite this article: M. Lobo et al., C. R. Mecanique 331 (2003).***

Résumé

Estimations asymptotiques uniformes pour le spectre d'un problème scalaire raide. On obtienent des estimations de la vitesse de convergence des valeurs propres, convenablement mises a l'échelle, d'un problème de Neumann raide pour l'operateur de Laplace. Des bornes correspondantes a ces estimations sont exprimées en termes du rapport des raideurs et des propriétés du spectre limite dans les rangs des fréquences basses et moyennes. ***Pour citer cet article : M. Lobo et al., C. R. Mecanique 331 (2003).***

Keywords: Acoustics; Waves; Vibrations; Stiff problem; Spectral analysis; Low frequencies; Middle frequencies

Mots-clés : Acoustique ; Ondes ; Vibrations ; Problème raide ; Analyse spectrale ; Basses fréquences ; Moyennes fréquences

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Soient $\Omega^{\pm} \subset \mathbb{R}^n$ domaines bornées avec des frontières lipschitziennes $\partial\Omega^{\pm}$ and $\Omega^+ \cap \Omega^- = \emptyset$. Soit h le rapport de raideur des deux parties $\Omega^{\pm}$. On introduit les ensembles $(n-1)$-dimensionels $\Upsilon = \partial\Omega^+ \cap \partial\Omega^-$, $\Sigma^{\pm} = \partial\Omega^{\pm} \setminus \overline{\Upsilon}$. On considère la junction Ω des deux corps $\Omega^{\pm}$, oú Ω^+ est la partie plus raide du corps, et on

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suppose que la zone de contact Υ est à mesure positive $\text{meas}_{n-1}\Upsilon$. Les ensembles Υ et $\Sigma^{\pm}$ sont aussi entourés par des frontières lipschitziennes $(n-2)$-dimensionelles. On impose une condition de Neumann sur la surface $\partial\Omega$. On peut signaler les references [1–4] pour la condition de Dirichlet sur $\partial\Omega$ et pour le cas où Σ^- est l'ensemble vide, ainsi que [5] pour des operateurs differentiels et conditions aux limites plus generales.

Nous donc considèrons le problème spectral (2) pour l'operateur de Laplace Δ. La formulation faible du problème (2) nous ramène a celui de trouver $\Lambda(h) \in \mathbb{R}$ et une solution no triviale $u(h,\cdot) \in H^1(\Omega)$ du problème (3). Comme il est bien connu, pour chaque h fixé, les valeurs propres du problème (3), forment la suite (4) avec la convention des valeurs propres repetées. Nous étudions le comportement asymptotique des éléments propres $(\Lambda(h), u(h,\cdot))$ de (3), lorsque $h \to 0$.

Une petite modification des aproximations obtenues dans [6] et [2] (cf. §5.7–5.10 dans [7]) permet d'établir le résultat de convergence (5), où la limite (6) représente le spectre du problème (7). La formulation faible de (7) est donée par (8). Nous remarquons que la condition aux limites $(7)_2$, qui est non locale de type Steklov, peut être obtenue en utilisant des developpements asymptotiques comme dans [5].

Par ailleurs, des approximations asymptotiques, developpées dans [2], pour les fréquences moyennes, présentent les proprietés suivantes : 1) Pour chaque $\lambda > 0$ il y a une suite de valeurs propres $\Lambda_{N(h)}(h)$ dans (4) qui converge vers λ pour $h \to 0$. 2) Si $\beta_k > 0$ est une valeur propre du problème spectral de Neumann (9), il existe une valeur propre $\Lambda_{N(h)}(h)$ du problème (2) vérifiant (10). On doit signaler que, en accord avec (5), dans l'un et l'autre cas, l'indice de la valeur propre $N(h) \to +\infty$ pour $h \to +0$.

L'objet principal de nôtre note est celui de déterminer la vitesse de convergence dans la relation (5), et mettre en evidence la dependence en k (l'indice de la valeur propre β_k de (9)) des bornes dans les estimations asymptotiques de type (10).

Nous avons utilisé le procedure de reduction directe et inverse, décrite dans [8,9], pour obtenir les estimations des Propopositions 1 et 2 et des Téorèmes 1 et 2 dans la Section 1. Ces estimations permettent d'obtenir des bornes pour la vitesse de convergence en améliorant les résultats (5) et (10). Nous remarquons que les constantes $\mathbf{h}_p$ et $\mathbf{c}_p$, qui apparaîtrent dans les Propositions et les Téorèmes 1 et 2, ne dependent du paramètre h, ni de l'indice $k = 1, 2, \ldots$ de la valeur propre. Les bornes dépendent de la constante de raideur h et du numéro k de la valeur propre des problèmes limites et sont plus précises dans le cas de dimension 1 (cf. Section 3) où l'on connait les valeurs propres des problèmes aux limites (voir (24) et (25)) ainsi que la distance entre elles. En fait, pour $n = 1$, les valeurs propres λ_k et β_k dans (5) et (10) sont données par (25), et pour $hk^3 = \mathrm{O}(1)$ l'on écrit (5) comme $|h^{-1}\Lambda_k(h) - \lambda_k| = \mathrm{O}(hk^4)$, tandis que pour $hk^4 = \mathrm{O}(1)$ la borne $C_k h^{1/4}$ dans (10) est d'ordre $\mathrm{O}(k^3 h^{1/4})$.

1. The stiff spectral problem and the limit problems

Let $\Omega^{\pm} \subset \mathbb{R}^n$ be bounded domains with Lipschitz boundaries $\partial\Omega^{\pm}$ and $\Omega^+ \cap \Omega^- = \emptyset$. Let h denote the ratio of stiffness of the two parts $\Omega^{\pm}$. We introduce the $(n-1)$-dimensional sets

$$\Upsilon = \partial\Omega^+ \cap \partial\Omega^-, \qquad \Sigma^{\pm} = \partial\Omega^{\pm} \setminus \overline{\Upsilon} \tag{1}$$

Considering the junction Ω of two bodies $\Omega^{\pm}$, Ω^+ the stiffer part of the body, we assume that the contact zone Υ has a positive measure $\text{meas}_{n-1}\Upsilon$. The sets (1) are surrounded by Lipschitz $(n-2)$-dimensional surfaces. We impose a Neumann condition on the boundary $\partial\Omega$. We refer to [1,2] for a Dirichlet condition on $\partial\Omega$, to [3,4] for the Dirichlet condition and the case $\partial\Omega \cap \partial\Omega^+ = \emptyset$, and to [5] for more general boundary conditions and differential operators. We consider the following spectral problem involving the Laplace operator Δ,

$$\begin{aligned}
&-h\Delta u^-(h,x) = \Lambda(h)u^-(h,x), \quad x \in \Omega^- \\
&-\Delta u^+(h,x) = \Lambda(h)u^+(h,x), \quad x \in \Omega^+ \\
&\partial_n u^{\pm}(h,x) := n^{\pm}(x)^{\mathrm{T}}\nabla u^{\pm}(h,x) = 0, \quad x \in \Sigma^{\pm} \\
&u^-(h,x) = u^+(h,x), \quad h\partial_n u^-(h,x) = \partial_n u^+(h,x), \quad x \in \Upsilon
\end{aligned} \tag{2}$$

The weak formulation of problem (2) is: to find $\Lambda(h) \in \mathbb{R}$ and a nontrivial function $u(h,\cdot) \in H^1(\Omega)$ such that

$$h(\nabla u^-, \nabla v^-)_{\Omega^-} + (\nabla u^+, \nabla v^+)_{\Omega^+} = \Lambda(h)(u,v)_\Omega \quad \forall v = \{v^-, v^+\} \in H^1(\Omega) \tag{3}$$

As is known, for fixed h, the eigenvalues of problem (3) form the sequence

$$0 = \Lambda_1(h) < \Lambda_2(h) \leqslant \cdots \leqslant \Lambda_j(h) \leqslant \cdots \to +\infty \tag{4}$$

where we adopt the convection of repeated eigenvalues. The problem that we address in the paper consists in studying the asymptotic behavior of the spectral pairs $(\Lambda(h), u(h,\cdot))$ of problem (3), as $h \to 0$.

A slight modification of the approaches in [6] and [2] (cf. §5.7–5.10 in [7]) establishes the convergence

$$h^{-1}\Lambda_k(h) \to \lambda_k \quad \text{as } h \to +0 \tag{5}$$

with conservation of the multiplicity, where

$$0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots \to +\infty \tag{6}$$

stands for the spectrum of the resulting problem

$$\begin{aligned} &-\Delta U(x) = \lambda U(x), \quad x \in \Omega^-; \qquad \partial_n U(x) = 0, \quad x \in \Sigma^- \\ &\lambda U(x) = (\mathrm{meas}_n \Omega^+)^{-1} \int_{\Upsilon} \partial_n U(x)\,\mathrm{d}s, \quad x \in \Upsilon \end{aligned} \tag{7}$$

The boundary condition $(7)_2$, which is nonlocal and of Steklov's type, is derived by rewriting the computations in [5] with minor modifications. The weak formulation of (7) reads: to find a number $\lambda \in \mathbb{R}$ and a nontrivial function $U^0 \in \mathcal{H}_0$ such that

$$(\nabla U^0, \nabla V)_{\Omega^-} = \lambda \left\{ (U^0, V)_{\Omega^-} + \frac{(\mathrm{meas}_n \Omega^+)}{(\mathrm{meas}_{n-1}\Upsilon)^2} \int_{\Upsilon} U^0(x)\,\mathrm{d}s \int_{\Upsilon} V(x)\,\mathrm{d}s \right\} \quad \forall V \in \mathcal{H}_0 \tag{8}$$

where $\mathcal{H}_0 = \{V \in H^1(\Omega^-) \mid V \text{ is a constant on } \Upsilon\}$.

At the same time, the approximations developed in [2] uphold the following facts. First, for each $\lambda > 0$ there exists a sequence of eigenvalues $\Lambda_{N(h)}(h)$ in (4) converging towards λ as $h \to 0$. Second, for any eigenvalue $\beta_k > 0$ of the Neumann spectral problem

$$-\Delta W(x) = \beta W(x), \quad x \in \Omega^+; \qquad \partial_n W(x) = 0, \quad x \in \partial\Omega^+ \tag{9}$$

there exists an eigenvalue $\Lambda_{N(h)}(h)$ of problem (2) such that

$$\left|\Lambda_{N(h)}(h) - \beta_k\right| \leqslant C_k h^{1/4} \tag{10}$$

We note that, in view of convergence (5), in both cases, the eigenvalue number $N(h)$ grows indefinitely as $h \to +0$.

The principal goal of our paper is to detect the convergence rate in (5) and to clarify the dependence on k of bounds in the asymptotic accuracy estimates of type (10). We use the so-called direct and inverse reductions, as outlined in [8,9], to obtain these estimates. We state the general results in Section 2. In the one-dimensional case, a detailed analysis of the results is performed in Section 3. In this case, i.e., $n = 1$, the eigenvalues λ_k and β_k of the resulting problems (5) and (10) are given by (25) and the estimates obtained are even more precise than in the case where $n > 1$. As a matter of fact, when $n = 1$, among the results we have that, under the restriction $hk^3 = \mathrm{O}(1)$, the discrepancy in (5) reads $|h^{-1}\Lambda_k(h) - \lambda_k| = \mathrm{O}(hk^4)$, while the restriction $hk^4 = \mathrm{O}(1)$ provides the bound $C_k h^{1/4} = \mathrm{O}(k^3 h^{1/4})$ in (10).

We emphasize that in what follows all constants $\mathbf{h}_p$ and $\mathbf{c}_p$ do not depend on both either the parameter h or the eigenvalue number $k = 1, 2, \ldots$.

2. Estimates for convergence rates

The following assertion results from the *inverse reduction procedure* when an approximation to a solution of the original problem (2) is constructed from a solution of the resulting problem (7) and the classical lemma on "almost eigenvalues" is applied. In the sequel $\varkappa_k$ denotes the multiplicity of the eigenvalue λ_k in (6),

$$\lambda_{k-1} < \lambda_k = \cdots = \lambda_{k+\varkappa_k-1} < \lambda_{k+\varkappa_k} \tag{11}$$

Proposition 1. *There exist constants $\mathbf{h}_1 > 0$ and $\mathbf{c}_1 > 0$ such that, for any eigenvalue λ_k of problem* (8) *with multiplicity $\varkappa_k$ and for any integer $l \in (0, \varkappa_k]$, the condition*

$$h \leqslant \mathbf{h}_1 l^{-1}(1+\lambda_k)^{-1} \tag{12}$$

provides at least l eigenvalues $\Lambda_j(h), \ldots, \Lambda_{j+l-1}(h)$ of problem (3) *satisfying the estimate*

$$\left|\Lambda_p(h) - h\lambda_k\right| \leqslant \mathbf{c}_1 l h^2 (1+\lambda_k)^2 \tag{13}$$

To compensate for obvious lacks of information on spectrum (4) given by Proposition 1, we employ the *direct reduction procedure* which, in contrast to the inverse reduction, provides an approximation solution to the resulting problem (7) based on a solution to the original problem (2). These two reductions lead to completed results on relations between spectra (4) and (6) (see [8,9] and [5] for a detailed discussion).

For the eigenvalue λ_k in (11), we introduce the relative distance d_k from $(1+\lambda_k)^{-1}$ to the nearest point $(1+\lambda_j)^{-1} \neq (1+\lambda_k)^{-1}$, that is

$$d_k = \min\left\{\frac{1+\lambda_k}{1+\lambda_{k-1}} - 1, 1 - \frac{1+\lambda_k}{1+\lambda_{k+\varkappa_k}}\right\} \tag{14}$$

Theorem 1. *There exist constants $\mathbf{h}_2 > 0$ and $\mathbf{c}_2 > 0$ such that, for any $\varkappa_k$-multiple eigenvalue λ_k (cf.* (11)) *of the resulting problem* (8), *the condition*

$$h \leqslant \mathbf{h}_2 \varkappa_k^{-1}\left(1+d_k^{-1}\right)^{-1}(1+\lambda_k)^{-1} \tag{15}$$

ensures that the inclusion

$$\left(1+h^{-1}\Lambda_p(h)\right)^{-1} \in \left[\frac{1}{2}\left(\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k+\varkappa_k}}\right), \frac{1}{2}\left(\frac{1}{1+\lambda_k} + \frac{1}{1+\lambda_{k-1}}\right)\right] \tag{16}$$

occurs only for the eigenvalues $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k-1}(h)$ of the original problem (3). *These eigenvalues satisfy the estimate*

$$\left|\Lambda_p(h) - h\lambda_k\right| \leqslant \mathbf{c}_2 h^2 (1+\lambda_k)^2 \tag{17}$$

Moreover, under the condition

$$h \leqslant \mathbf{h}_2 \varkappa_k^{-1}\left(1+d_k^{-1}\right)^{-1}\left\{1+\frac{1}{2}(\lambda_k + \lambda_{k+\varkappa_k})\right\}^{-1} \tag{18}$$

the segment

$$\left[\frac{h}{2}(\lambda_k + \lambda_{k-1}), \frac{h}{2}(\lambda_k + \lambda_{k+\varkappa_k})\right] \tag{19}$$

contains only the above-mentioned eigenvalues $\Lambda_k(h), \ldots, \Lambda_{k+\varkappa_k-1}(h)$ as well.

Note that Theorem 1 reveals convergence (5) with a fixed k and, in addition, formula (17) delineates the convergence rate. In particular, any segment

$$[\mu_-, \mu_+] \subset (\lambda_{k-1}, \lambda_k) \tag{20}$$

becomes free of the points $h^{-1}\Lambda_p(h)$ for a sufficiently small $h > 0$ and the next assertion, originating in the direct reduction as well, provides a bound for such h.

Proposition 2. *The segment* $[h\mu_-, h\mu_+]$ *related to* (20) *does not contain an eigenvalue of problem* (3) *as long as*

$$h \leqslant \mathbf{h}_3\big(1+\rho^{-1}\big)^{-1}(1+\lambda_k)^{-2} \tag{21}$$

where $\rho = \min\{\lambda_k - \mu_+, \mu_- - \lambda_{k-1}\}$ *and* $\mathbf{h}_3$ *is a certain constant which does not depend either on the endpoints* $\mu_\pm$ *and the eigenvalue* λ_k, *nor on the parameter* h.

For the medium frequencies, only the inverse reduction can be applied, which leads to the following assertion.

Theorem 2. *There exist constants* $\mathbf{h}_4 > 0$ *and* $\mathbf{c}_4 > 0$ *such that, for any eigenvalue* β_k *of problem* (9) *of multiplicity* $\varkappa_k$ *and for any integer* $l \in (0, \varkappa_k]$, *the condition*

$$h \leqslant \mathbf{h}_4 l^{-4}(1+\beta_k)^{-2} \tag{22}$$

provides at least l *eigenvalues* $\Lambda_j(h), \ldots, \Lambda_{j+l-1}(h)$ *of problem* (3) *verifying estimate* (10) *with the constant*

$$C_k = \mathbf{c}_4 l(1+\beta_k)^{3/2} \tag{23}$$

3. Analysis in the one-dimensional case

In order to comment on and clarify our results for many-dimensional domains, we consider the simplest one-dimensional problem (2), where

$$\Omega^- = (-T, 0), \quad \Omega^+ = (0, T), \quad \Sigma^\pm = \{\pm T\}, \quad \Upsilon = \{0\}$$
$$\Delta u = u'', \quad u' := \partial_x u, \quad \partial_{n^\pm} u = \pm u'$$

In this case, we have the explicit solutions of the resulting spectral problems (7) and (9),

$$\begin{aligned} &-U''(x) = \lambda U(x), \quad x \in (-T, 0); \quad U'(-T) = 0, \quad \lambda U(0) = T^{-1}U'(0) \\ &-W''(x) = \beta W(x), \quad x \in (0, T); \quad U'(-T) = 0, \quad U'(0) = 0 \end{aligned} \tag{24}$$

Specifically, the spectra of problems $(24)_1$ and $(24)_2$ are respectively composed of the simple eigenvalues

$$\lambda_k = T^{-2} z_k^2 \quad \text{and} \quad \beta_k = T^{-2}\pi^2 k^2 \tag{25}$$

where $k = 1, 2, \ldots$ and z_k are nonnegative roots of the transcendental equation $z = -\operatorname{tg} z$. It is not difficult to find out that $z_1 = 0$ and, for $k = 2, 3, \ldots$ as $k \to \infty$,

$$z_k = \pi\left(k - \frac{3}{2}\right) + \left[\pi\left(k - \frac{3}{2}\right)\right]^{-1} - \frac{4}{3}\left[\pi\left(k - \frac{3}{2}\right)\right]^{-3} + O\left(\left[\pi\left(k - \frac{3}{2}\right)\right]^{-5}\right) \tag{26}$$

These computations allow us to redefine the assertions and asymptotic formulae presented in the previous section while adapting them to ordinary differential equations. Indeed, by modifying constants $\mathbf{h}_p$ and $\mathbf{c}_p$ in the bounds from restrictions (12), (15), (18) and (22) and estimates (13), (17) and (10) together with (23), on account of (25) and (26), we can replace $1 + \lambda_k$, $1 + d_k^{-1}$, and $1 + \beta_k$ by k^2, k, and k^2, respectively (note that $\Lambda_1(h) = \lambda_1 = \beta_1 = 0$ and, therefore, we do not need to discuss these eigenvalues). Moreover, $l = \varkappa_k = \kappa_k = 1$ because the eigenvalues are simple.

The upper bounds in (13) and (17) now become equal to $\mathbf{c}_{5,6}h^2k^4$. At the same time, restriction (15) with the bound $\mathbf{h}_6 k^{-3}$ is much harder than restriction (12) with the bound $\mathbf{h}_5 k^{-2}$. To explain this disagreement, we compare results presented in Proposition 1 and Theorem 1. By (26), we have

$$0 < \mathbf{c}_0 hk \leqslant |h\lambda_k - h\lambda_{k\pm1}| \leqslant \mathbf{C}_0 hk \tag{27}$$

Since restriction (12) provides $\mathbf{C}_0 hk \leqslant \mathbf{C}_0 \mathbf{h}_5 k^{-1}$, the interval $(h\lambda_k - \mathbf{c}_5 h^2k^4, h\lambda_k - \mathbf{c}_5 h^2k^4)$ of width $2\mathbf{c}_5 h^2k^4 \leqslant 2\mathbf{c}_5\mathbf{h}_5^2$, indicated in (13), can contain the eigenvalues $h\lambda_{k\pm1}$ and $\Lambda_{k\pm1}(h)$ in addition to $h\lambda_k$ and $\Lambda_k(h)$. On the other hand, under the harder restriction (15), the width of the interval generated by (17) satisfies $2\mathbf{c}_6 h^2k^4 \leqslant 2\mathbf{c}_6\mathbf{h}_6^2 k^{-2}$ and therefore can be made smaller than the bound $\mathbf{C}_0 hk \leqslant \mathbf{C}_0 \mathbf{h}_6 k^{-2}$ in (27) by a proper choice of $\mathbf{h}_6$. In other words, Proposition 1 cannot specify how many eigenvalues of the problem (3) fulfill estimate (13) and hence it describes only a *collective asymptotics* of eigenvalues. In contrast, Theorem 1 detects exactly one eigenvalue $\Lambda_k(h)$ which verifies estimate (17) and inclusion (16) so that the *individual asymptotics* of the eigenvalue is met and, for fixed k, the convergence $h^{-1}\Lambda_k(h) \to \lambda_k$ with the rate $O(hk^4)$ is thus confirmed. Furthermore, restriction (18), obtaining the similar bound $\mathbf{h}_8 k^{-3}$ as in (15), ensures absence of "extraneous" eigenvalues $\Lambda_j(h)$ with $j \neq k$ in segment (19).

Rewriting (12) under the form $h(1+\lambda_k) \leqslant \mathbf{h}_1$, we see that Proposition 1 covers the whole low-frequency range $[0, \mathbf{C}_1)$ of spectrum (4) but Theorem 1 only its part $[0, h^{1/3}\mathbf{C}_2)$ (recall that the necessary restriction (15) contains the value $1 + d_k^{-1}$ which, owing to the left inequality in (27), is of order $(1+\lambda_k)^{1/2}$ for the one-dimensional problem). Neither Propositions 1 and 2, nor Theorem 1 work inside the middle-frequency range $[\mathbf{C}_1, h^{-1}\mathbf{C}_3)$ of the spectrum. By virtue of (25), bound (23) in estimate (10) turns into $\mathbf{c}_7 h^{1/4} k^3$. Detecting collective asymptotics of eigenvalues (cf. [1,2,5]), Theorem 2 is only related to the part $[\mathbf{C}_1, h^{-1/2}\mathbf{C}_4)$ of the middle-frequency range of spectrum (4), since the bound in (21) takes the form $\mathbf{h}_7 k^{-4} = \mathbf{C}_4^2 \beta_k^{-2}$ with $\mathbf{C}_4 = T^{-2}\pi^2 \mathbf{h}_7^{1/2}$.

For the sake of brevity, we do not demonstrate here asymptotic forms for eigenvectors which can also be naturally divided into *individual* and *collective* (the latter are known as *quasimodes*). It is self-understood that individual asymptotics of eigenvectors are available provided the corresponding eigenvalues admit the individual asymptotics as well, i.e., only on a narrow part of the low-frequency range (cf. [5] for cases $n = 2, 3$). We also refer to [2] for more precise results on structures of eigenfunctions associated with the middle-frequencies of another scalar stiff problem.

Finally, let us note that the Weyl asymptotics for eigenvalues predicts that $\lambda_k = O(k)$ for $\Omega \subset \mathbb{R}^2$ and $\lambda_k = O(k^{2/3})$ for $\Omega \subset \mathbb{R}^3$. Such asymptotic behavior of eigenvalues reduces the growth in k of the bounds discussed above. However, we do not know proofs of such kind of results for nonlocal boundary conditions of Steklov's type. Moreover, conclusions derived from Theorem 1 are crucially based on the lower bound in (27) but, in general, estimates of this kind are not available in many-dimensional domains. It should be mentioned that if two different eigenvalues happen to be close each to the other, the value $(1 + d_k^{-1})^{-1}$ becomes small and condition (15) excessively restrictive. Hence, in such cases it is worth using the collective asymptotic forms too.

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ALTAS FRECUENCIAS EN UN PROBLEMA "STIFF" RELATIVO A LAS VIBRACIONES DE UNA CUERDA

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Resumen. El estudio de las vibraciones en un problema "Stiff" surge de una manera natural en problemas que modelizan las vibraciones de un sistema formado por dos materiales con distintas características. El problema considerado aquí modeliza las vibraciones de un cuerpo que consta de dos materiales uno "muy rígido" con respecto al otro. Se dan los resultados obtenidos sobre el comportamiento asintótico de las altas frecuencias para el problema de las vibraciones de una cuerda cuando la rigidez de uno de los materiales tiende a cero. El estudio comportamiento asintótico de las bajas frecuencias puede encontrarse en [1], [2], [4] y [5].

1 Planteamiento del problema.

Se considera Ω un dominio acotado de $\mathbf{R}^n$, $n \geq 1$, Ω de frontera regular y dividido por Σ en dos partes Ω_0 y Ω_1 cada de ellas de frontera Lipschitziana: $\Omega = \Omega_0 \cup \Omega_1 \cup \Sigma$. Se denota por $\partial_i\Omega$ la parte de la frontera de Ω_i contenida en $\partial\Omega$, es decir: $\partial\Omega_i = \partial_i\Omega \cup \Sigma$, $i = 0, 1$.

Se plantea un problema standard de vibraciones en los espacios $\mathbf{V} = H_0^1(\Omega)$ y $\mathbf{H} = L^2(\Omega)$:

Encontrar $\mathbf{u}^\varepsilon(t)$ con valores en $\mathbf{V}$ tal que

$$(1)\qquad (\frac{d^2\mathbf{u}^\varepsilon}{dt^2}, v)_{\mathbf{H}} + a^\varepsilon(\mathbf{u}^\varepsilon(t), v) = 0\,, \forall v \in \mathbf{V}$$

$$(2)\qquad \mathbf{u}^\varepsilon(0) = \phi_1 \quad , \quad \frac{d\mathbf{u}^\varepsilon}{dt}(0) = \phi_2\,,$$

dónde

$$a^\varepsilon(u, v) = a_0(u, v) + \varepsilon a_1(u, v)\,, \forall u, v \in \mathbf{V}$$

$$a_i(u,v) = \int_{\Omega_i} \nabla u . \nabla v \, dx \, , i = 0, 1 \, ,$$

$$(u,v)_{\mathbf{H}} = \int_{\Omega} uv \, dx \, ,$$

los datos iniciales $\phi_1 \in \mathbf{V}$, $\phi_2 \in \mathbf{H}$, y ε es un pequeño parámetro positivo, que se hará tender a 0.

El problema (1)-(2) modeliza las vibraciones de un cuerpo situado en el dominio Ω, una de cuyas partes Ω_0 es muy rígida con respecto a la otra. El término $a^\varepsilon(u,v)$ contiene información sobre las caracteísticas elásticas de los materiales.

Como se sabe, existe una única solución de (1)-(2) en $L^\infty(-\infty,\infty,\mathbf{V})$ y admite un desarrollo en serie de Fourier:

$$\mathbf{u}^\varepsilon(t) = \sum_{i=1}^{\infty} (\phi_{1i}^\varepsilon \cos \sqrt{\lambda_i^\varepsilon} t + \frac{\phi_{2i}^\varepsilon}{\sqrt{\lambda_i^\varepsilon}} \sin \sqrt{\lambda_i^\varepsilon} t) u_i^\varepsilon .$$

Los coeficientes ϕ_{ij}^ε dependen de los datos iniciales:

$$\phi_j = \sum_{i=1}^{\infty} \phi_{ji}^\varepsilon u_i^\varepsilon \, , \quad \phi_{ji}^\varepsilon = \frac{(u_i^\varepsilon, \phi_j)_{\mathbf{H}}}{\|u_i^\varepsilon\|_{\mathbf{H}}} \, , \, j = 1, 2 \, ,$$

y, λ_i^ε, u_i^ε, son los valores propios y funciones propias del problema:

$$(3) \qquad \begin{cases} -\Delta u_0^\varepsilon = \lambda^\varepsilon u_0^\varepsilon & \text{en } \Omega_0, \\ -\varepsilon \Delta u_1^\varepsilon = \lambda^\varepsilon u_1^\varepsilon & \text{en } \Omega_1, \\ u_0^\varepsilon = 0 \text{ en } \partial_0 \Omega \quad , & u_1^\varepsilon = 0 \text{ en } \partial_1 \Omega, \\ u_0^\varepsilon = u_1^\varepsilon \quad , & \frac{\partial u_0^\varepsilon}{\partial n} = \varepsilon \frac{\partial u_1^\varepsilon}{\partial n} \text{ en } \Sigma \, . \end{cases}$$

En (3), y en lo sucesivo sigue, los subíndices 0 y 1 indican las restricciones de una función $u \in L^2(\Omega)$ a los dominios Ω_0 y Ω_1 respectivamente.

Así, el estudio del problema de vibraciones (1)-(2) nos lleva al estudio de los elementos propios de (3). Vamos a estudiar el comportamiento asintótico de estos valores propios y funciones propias cuando $\varepsilon \to 0$.

El problema (3) admite una formulación variacional equivalente:

Encontrar λ^ε, $u^\varepsilon \in \mathbf{V}$, $u^\varepsilon \neq 0$ tales que:

$$(4) \qquad \int_{\Omega_0} \nabla u^\varepsilon . \nabla v \, dx + \varepsilon \int_{\Omega_1} \nabla u^\varepsilon . \nabla v \, dx = \lambda^\varepsilon \int_{\Omega} u^\varepsilon v \, dx, \, \forall v \in \mathbf{V} \, ,$$

(4) es un problema de valores propios para el operador asociado a la forma bilineal, simétrica, continua y coerciva a^ε sobre $\mathbf{V}$, admitiendo por tanto un espectro discreto. Sea para cada $\varepsilon > 0$ fijo:

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \cdots \xrightarrow{n\to\infty} \infty\,,$$

la sucesión de valores propios, dónde cada valor propio se cuenta tantas tantas veces como su multiplicidad. Sean $\{u_i^\varepsilon\}_{i=1}^\infty$ las correspondientes funciones propias que forman una base ortonormal en $L^2(\Omega)$, es decir,

$$\|u_i^\varepsilon\|_{L^2(\Omega)} = 1.$$

Desde un punto de vista Físico cabe esperar que haya dos tipos de valores o frecuencias propias de dos órdenes de magnitud distintos: Uno relativo a la estructura menos rígida y otro a la más rígida. Desde un punto de vista matemático, la coercividad de la forma a^ε y el principio del mini-max nos permiten obtener la siguiente estimación para los valores propios (ver [3] para este tipo de técnicas):

Para cada $i = 1, 2, 3, \cdots$ fijo, se tiene

$$C\varepsilon \leq \lambda_i^\varepsilon \leq C_i\varepsilon \tag{5}$$

dónde C, C_i son constantes independientes de ε, C independiente de i.

La relación (5) nos permite afirmar que hay sucesiones de valores propios de orden $O(\varepsilon)$ que convergen; nos referiremos a estos valores propios como a *las bajas frecuencias.* Por otro lado, puesto que $C_i \to \infty$ cuando $i \to \infty$, también puede haber sucesiones de valores propios de orden $O(1)$ que converjan; hablaremos de ellos como de *las altas frecuencias.*

El problema del comportamiento asintótico de las bajas frecuencias ha sido abordado en [1], [2], [4] y [5] utilizando distintas técnicas. Su estudio puede ser llevado a cabo en en el contexto de la *teoría de las perturbaciones holomorfas* y nada sobre lo ya dicho en estos trabajos se puede añadir: Las funciones propias asociadas a frecuencias propias pequeñas son asintóticamente nulas en Ω_0, mientras que en Ω_1 son las funciones propias de un problema de valores propios para el operador de Laplace, planteado en Ω_1, con condiciones de Dirichlet sobre $\partial\Omega_1$.

Es decir, las vibraciones asociadas a las bajas frecuencias son "poco significativas" en la parte más rígida Ω_0. Sin embargo, para determinados estímulos es evidente que se producen vibraciones significativas en Ω_0 y por tanto las altas frecuencias juegan un papel primordial en el estudio del comportamiento asintótico de las vibraciones de todo Ω.

En [1] y [4] se consideran distintos aspectos de las altas frecuencias. En el estudio de éstas aparece de una manera natural (basta con tomar $\varepsilon = 0$ en (3)) el problema de valores propios:

$$
(6) \qquad \begin{cases} -\Delta u_0 &= \lambda u_0 \text{ en } \Omega_0, \\ u_0 &= 0 \text{ en } \partial_0\Omega, \\ \frac{\partial u_0}{\partial n} &= 0 \text{ en } \Sigma. \end{cases}
$$

De los resultados obtenidos, relativos a las altas frecuencias, se pueden extraer, entre otras, las siguientes conclusiones (ver [1] y [4] para mayor detalle):

- Determinadas funciones propias u^ε asociadas a valores propios $\lambda^\varepsilon = O(1)$ se aproximan en Ω_0 por funciones propias de (6) mientras que no se tiene información sobre ellas y, por tanto, sobre las correspondientes vibraciones en Ω_1.
- Para determinados datos iniciales ($\phi_1 = 0$) la solución de (1)-(2) es asintóticamente nula en Ω_0, mientras que en Ω_1, "localmente", oscila como si el medio no fuera acotado.

En la siguiente sección damos resultados más concretos sobre la estructura de las funciones propias asociadas a las altas frecuencias para $n = 1$.

2 Resultados en dimensión $n = 1$.

Se considera el problema de valores propios (3) relativo a las vibraciones de una cuerda situada en $(-1, 1)$, la parte más rígida Ω_0 en $(-1, 0)$ y la menos rígida Ω_1 en $(0, 1)$,

$$
(7) \qquad \begin{cases} -\frac{d^2u^\varepsilon}{dx^2} = \lambda^\varepsilon u^\varepsilon, & x \in (-1,0), \\ -\varepsilon\frac{d^2u^\varepsilon}{dx^2} = \lambda^\varepsilon u^\varepsilon, & x \in (0,1), \\ u^\varepsilon(-1) = 0 \quad , & u^\varepsilon(1) = 0, \\ u^\varepsilon(0^-) = u^\varepsilon(0^+) \quad , & \frac{du^\varepsilon}{dx}(0^-) = \varepsilon\frac{du^\varepsilon}{dx}(0^+). \end{cases}
$$

Exceptuando los λ^ε tales que $\cos\sqrt{\lambda} \neq 0$ (ver observación 1), un simple cálculo nos muestra que los valores propios λ^ε de (7) son los λ raices de la ecuación:

$$(8) \qquad \sqrt{\varepsilon}\tan\sqrt{\lambda} + \tan\sqrt{\frac{\lambda}{\varepsilon}} = 0\,.$$

Las correspondientes funciones propias son:

$$(9) \quad u^\varepsilon(x) = \begin{cases} C_\varepsilon(\sqrt{\varepsilon}\tan\sqrt{\lambda^\varepsilon}\cos\sqrt{\lambda^\varepsilon}x + \sqrt{\varepsilon}\sin\sqrt{\lambda^\varepsilon}x)\,, & x \in (-1,0) \\ C_\varepsilon(-\tan\sqrt{\frac{\lambda^\varepsilon}{\varepsilon}}\cos\sqrt{\frac{\lambda^\varepsilon}{\varepsilon}}x + \sin\sqrt{\frac{\lambda^\varepsilon}{\varepsilon}}x)\,, & x \in (0,1) \end{cases}$$

dónde C_ε es una constante que depende de la normalización que se tome para u^ε. Nosotros suponemos que es tal que:

$$\int_{-1}^{1} u^\varepsilon(x)^2\,dx = 1\,.$$

El problema (6) planteado ahora en $(-1,0)$ es:

$$(10) \qquad \begin{cases} -\frac{d^2u_0}{dx^2} &= \lambda u_0\,,\ x \in (-1,0) \\ u_0(-1) &= 0\,, \\ u_0'(0) &= 0. \end{cases}$$

Los valores propios de (10) son $(\frac{(2k+1)\pi}{2})^2\,, k = 0,1,2,\cdots$, y las correspondientes funciones propias $\cos(\frac{(2k+1)\pi}{2}x)$. Si se considera $\lambda^\varepsilon \approx (\frac{(2k+1)\pi}{2})^2$ para algún k, está claro que la correspondiente función propia u^ε se puede aproximar por $\alpha_\varepsilon\cos(\frac{(2k+1)\pi}{2}x)$ en $(-1,0)$ (α_ε una constante), pero u^ε no admite un desarrollo regular en $(0,1)$. Los siguientes resultados nos dan información sobre estas funciones propias en $(0,1)$, así como sobre las otras funciones propias asocidas a valores propios no necesariamente "próximos" al los de (10).

Teorema 1 *Para cada $\lambda > 0$ existe una sucesión $\lambda^\varepsilon_{i(\varepsilon)}$ de valores propios de (7) que converge a λ cuando $\varepsilon \to 0$.*

Teorema 2 *Si $\lambda^\varepsilon_{i(\varepsilon)} \xrightarrow{\varepsilon\to 0} \lambda$ y $\lambda > 0$ no es valor propio de (10), entonces*

$$\|u^\varepsilon_{i(\varepsilon)}\|_{L^2(-1,0)} \xrightarrow{\varepsilon\to 0} 0$$

Notamos que en las demostraciones de los teoremas 1 y 2 es importante el hecho de que la dimensión del espacio sea $n = 1$: por un lado los valores propios se caracterizan a través de las raices de de la ecuación (8) y, por otro lado, se tiene la fórmula explícita de las funciones propias (9) en función de estos valores propios. Solo damos una idea esquemática de la demostración de los teoremas 1 y 2.

La demostración del teorema 1 en el caso de que λ sea un valor propio del problema (10) se basa en el tipo argumentos que pueden encontrarse en [3] y [4]: se utiliza la típica idea de obtener información sobre la convergencia de los valores propios cuando se conoce la convergencia de las soluciones de los problemas de evolución asociados para ciertos datos iniciales.

En el caso de que $\lambda \neq (\frac{(2k+1)\pi}{2})^2$ para cualquier k, se demuestra facilmente que cualquier entorno $(\lambda - \delta, \lambda + \delta)$, que no contenga valores propios de (10), contiene raices de la ecuación (8) para ε suficientemnte pequeño.

La demostración del teorema 2 se basa en el simple cálculo de las integrales

$$\int_{-1}^{0} u^\varepsilon(x)^2 dx \ \text{ y } \int_0^1 u^\varepsilon(x)^2 dx \,,$$

y en la comprobación de que el orden de magnitud de la primera es inferior al de la segunda. Este resultado no es cierto en el caso de que el $\lambda = (\frac{(2k+1)\pi}{2})^2$ para algún k.

Los teoremas 1 y 2, y resultados más sofisticados de la teoría de perturbaciones espectrales (ver [6], por ejemplo) nos permiten obtener el siguiente resultado, que nos da idea del carácter fuertemente oscilante de las funciones propias u^ε en $(0,1)$.

Teorema 3 *Sea* $\lambda^\varepsilon_{i(\varepsilon)} \xrightarrow{\varepsilon \to 0} \lambda$. *Entonces las correspondientes funciones propias* u^ε *asociadas a* $\lambda^\varepsilon_{i(\varepsilon)}$ *son de la forma:*

$$u^\varepsilon = \alpha_\varepsilon \cos\left(\frac{(2k+1)\pi}{2} x\right)|_{(-1,0)} + \sum_{j=j(\varepsilon)}^{\infty} \alpha_k^\varepsilon \sin(j\pi x)|_{(0,1)} + o_\varepsilon(1)\,,$$

dónde $o_\varepsilon(1) \xrightarrow{\varepsilon \to 0} 0$ *en* $L^2(-1,1)$ *y* $j(\varepsilon) \xrightarrow{\varepsilon \to 0} \infty$. *Además,*

$$\alpha^\varepsilon = 0, \ \textit{si}\ \forall k\, \lambda \neq \left(\frac{(2k+1)\pi}{2}\right)^2,$$

y

$$\alpha^\varepsilon \neq 0\,,\ si\ \lambda = (\frac{(2k+1)\pi}{2})^2.$$

Observación 1 Se observa que en la ecuación (8) no están incluidos los posibles valores propios $\lambda = \lambda^\varepsilon$ tales que $\cos\sqrt{\lambda} = 0$ (y $\cos\sqrt{\frac{\lambda}{\varepsilon}} = 0$). Cada uno de éstos $\lambda = ((\frac{(2k+1)\pi}{2})^2$ sólo es valor propio de (7) para unos determinados valores de ε: los de la sucesión $\varepsilon_n = (\frac{2k+1}{2n+1})^2$. Las correspondientes funciones propias son en este caso: $u^\varepsilon(x) = \cos(\frac{(2k+1)\pi}{2}x)\,, x \in (-1,0)$ (i.e. la función propia de (10) en $(-1,0)$), y, $u^\varepsilon(x) = \cos(\frac{(2k+1)\pi}{2\sqrt{\varepsilon}}x)\,, x \in (0,1)$. Este resultado solo reafirma lo ya dicho en los teoremas 1-3.

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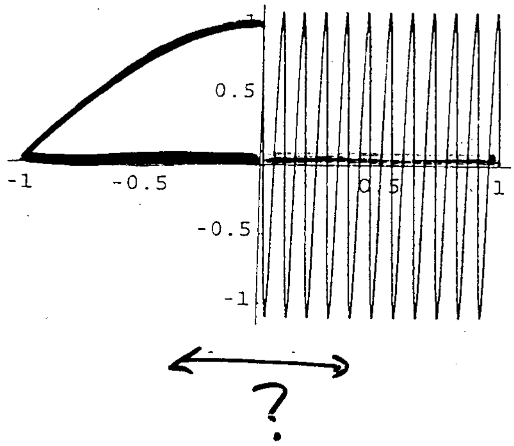

Further questions:

- For $n = 1$ and $\lambda^{\varepsilon} \approx \lambda$, $\exists$ approaches of u^{ε} in $H^1(\Omega)$?
- For $n \geq 2$, are there eigenfunctions concentrating asymptotically their support or their energy along lines or curves?
- Normalization of the eigenfunctions?

Extension of the results for:

- Different boundary conditions
- Different geometries for the domains Ω_i
- Linear elasticity