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*Laboratoire Kastler Brossel, CNRS, Ecole normale supérieure & Collège de France*

# The resonant Fermi gas

**Lecture 1** — **Zero-range limit, three-body problem and dynamical symmetry**

**Lecture 2** — **Many-body physics: methods and basic properties**

**Lecture 3** — **2-body and 3-body contacts**

**Benasque 2026**

# Contents

# Contents

***For all slides in one file:***

*https://www.phys.ens.psl.eu/~fwerner/BENASQUE_2026/slides.pdf*

Lecture 1

# Zero-range limit, three-body problem and dynamical symmetry

Refs:


- with Y. Castin: Lect. Notes Phys. **836**, 127 (2012); PRL **97**, 150401 (2006); PRA **74**, 053604 (2006)
- S. Tan, arXiv:cond-mat/0412764
- V. Efimov, Sov. J. Nucl. Phys. **12**, 589 (1971); Nucl. Phys. **A210**, 157 (1973).
- Y. Castin, Comptes Rendus Physique **5**, 407 (2004)

Chapter 1

# Short-range resonant interactions and the universal zero-range limit

## $\boxed{\text{Part 1: Simplified atomic physics}}$

Fermionic atoms in 2 internal states : $\uparrow$ , $\downarrow$
(hyperfine)

$N = N_\uparrow + N_\downarrow$ . $N_\uparrow$ and $N_\downarrow$ are conserved

$\Longrightarrow$ Two species $\begin{cases} N_\uparrow & \text{red fermions} \\ N_\downarrow & \text{blue fermions} \end{cases}$

red and blue are distinguishable .

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$\Longrightarrow$ Two species $\begin{cases} N_\uparrow & \text{red fermions} \\ N_\downarrow & \text{blue fermions} \end{cases}$

red and blue are distinguishable.

e.g. $N_\uparrow = N_\downarrow = 2$:

$$\Psi(\underbrace{\vec{r}_1, \vec{r}_2}_{\text{antisym.}}, \underbrace{\vec{r}_3, \vec{r}_4}_{\text{antisym.}}) = -\Psi(\vec{r}_2, \vec{r}_1, \vec{r}_3, \vec{r}_4) = -\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_4, \vec{r}_3)$$

Interaction: $\underbrace{V_{\uparrow\downarrow}(r)}_{\boxed{V(r)}}$, $\underbrace{V_{\uparrow\uparrow}(r), \; V_{\downarrow\downarrow}(r)}_{\text{negligible effect}}$

Trap: $U(\vec{r})$

$$\boxed{-\frac{\hbar^2}{2m}\sum_{i=1}^{N} \Delta_{\vec{r}_i}\Psi + \sum_{i=1}^{N} U(\vec{r}_i)\Psi + \sum_{\substack{i \leq N_\uparrow \\ j > N_\uparrow}} V(r_{ij})\Psi = E\Psi}$$

$r_{ij} := \|\vec{r}_j - \vec{r}_i\|$

## Part 2: 2-body problem

3D

2 fermions in *different internal states*, $\uparrow$ and $\downarrow$

[ $\Leftrightarrow$ 2 *distinguishable* particles ]

$\psi(\vec{r_1}\,,\vec{r_2})$

no symmetry constraint
when exchanging $\vec{r_1}$ and $\vec{r_2}$

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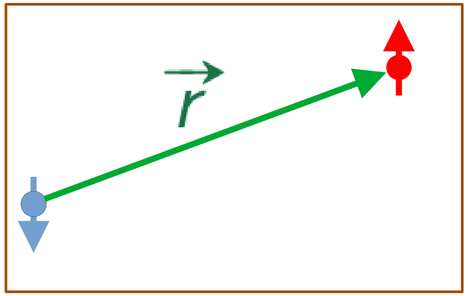


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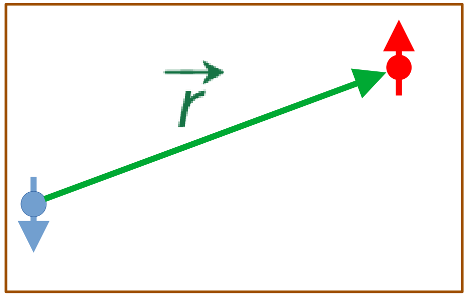


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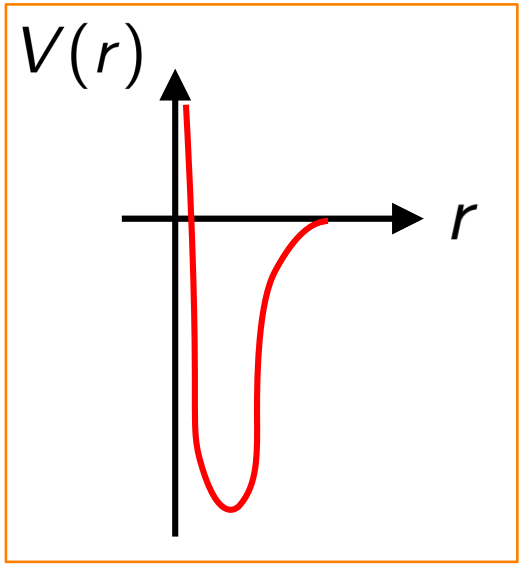
interaction potential $V(r)$
in free space



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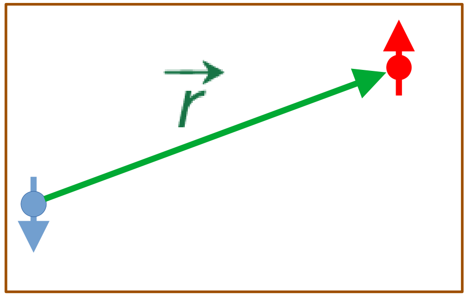


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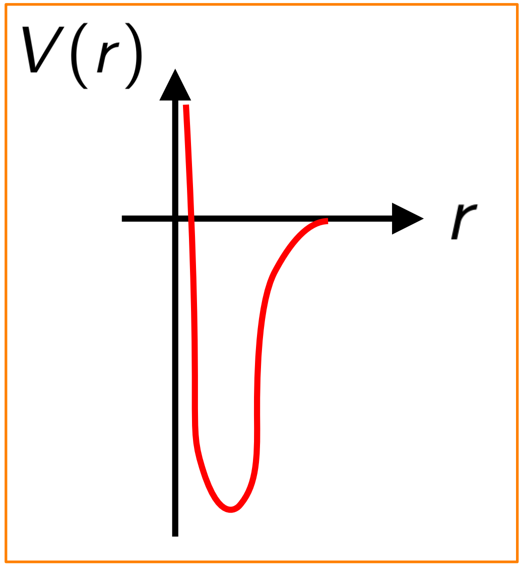


$$-\frac{\hbar^2}{2m}\left(\Delta_{\vec{r}_1} + \Delta_{\vec{r}_2}\right)\psi + V(r)\,\psi = E\,\psi$$

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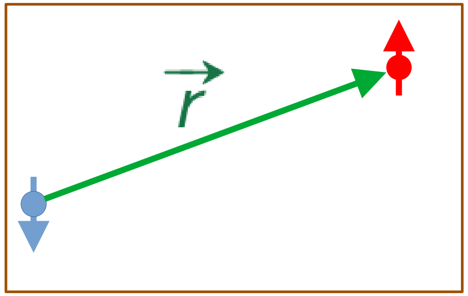


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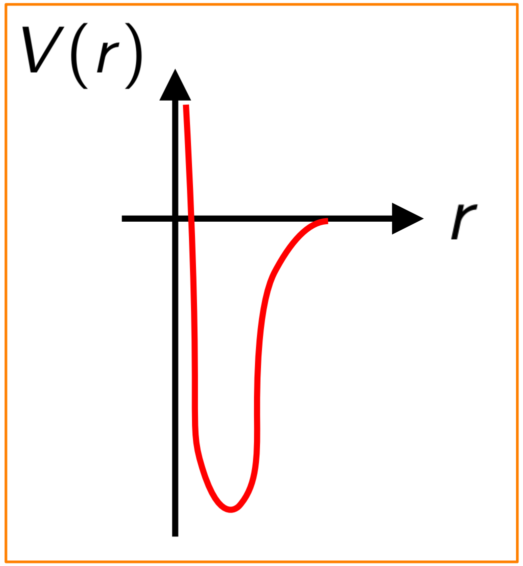


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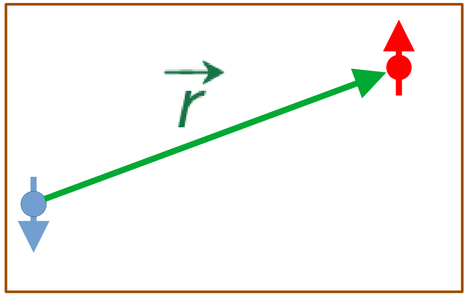


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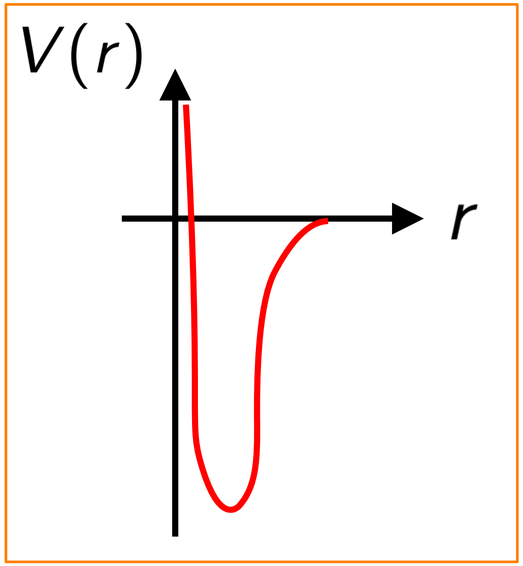


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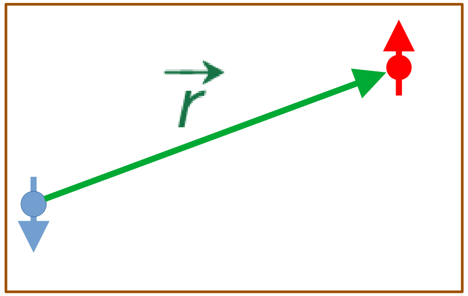


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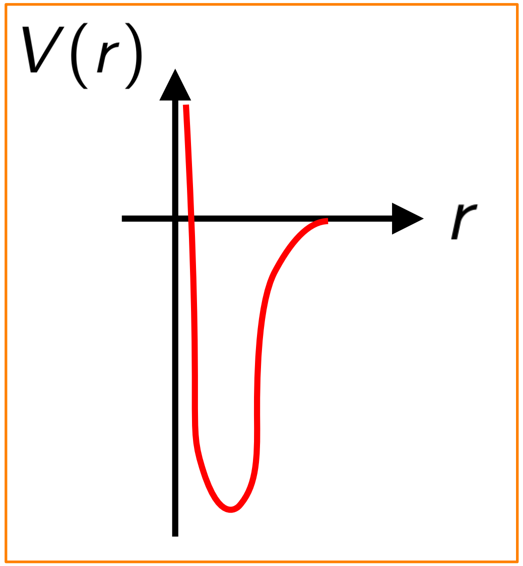


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$$\psi(\vec{r_1}\,,\vec{r_2}) = \psi(\vec{r})\,e^{i\,\vec{K}\cdot\vec{c}}$$

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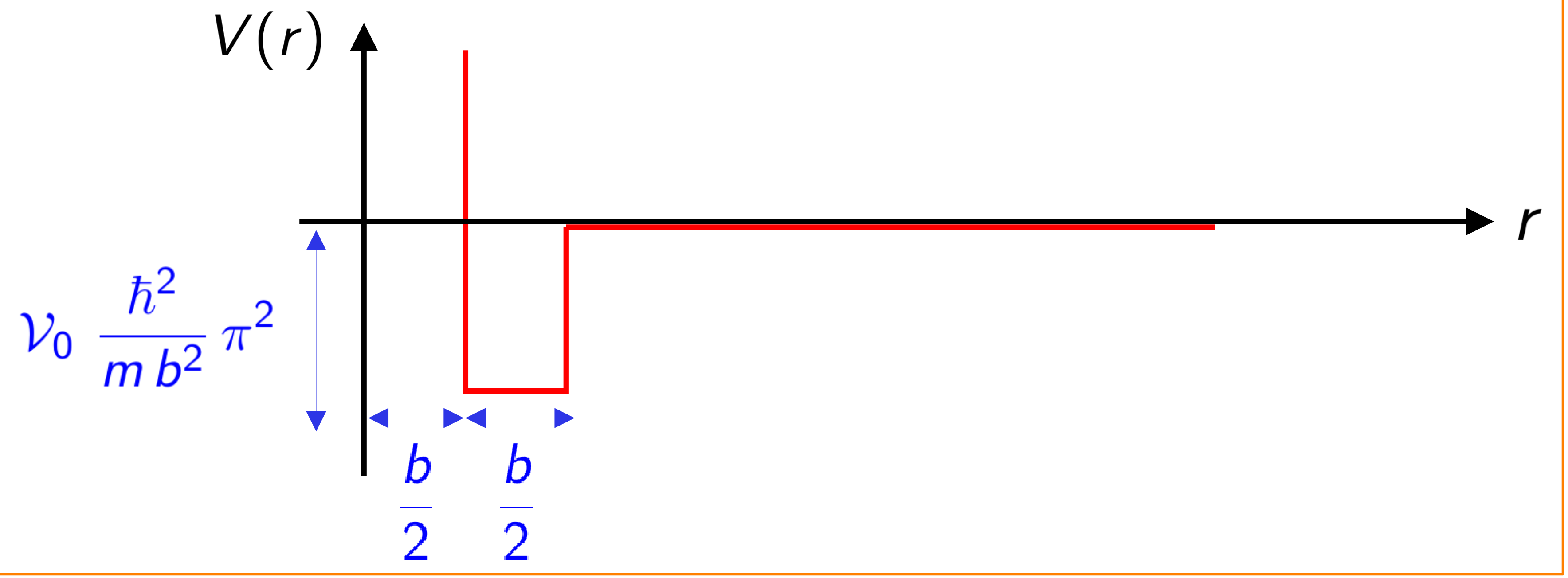

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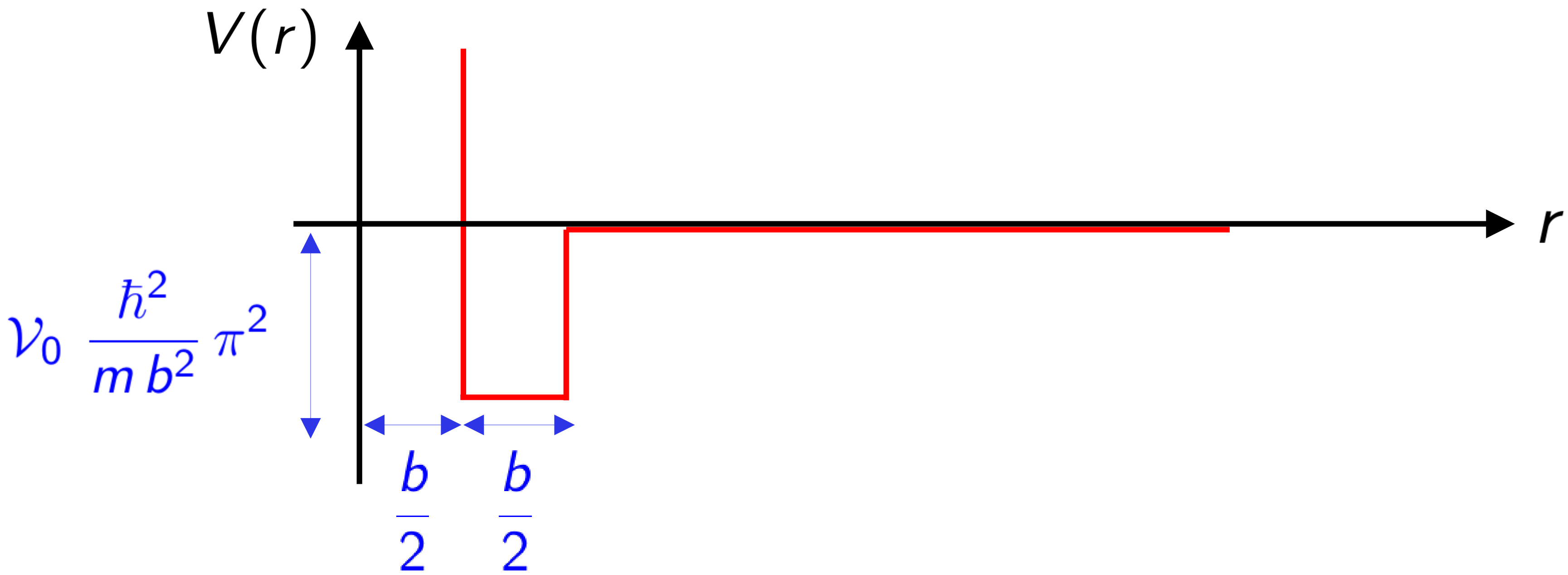


<u>Dimers</u> (= bound states): $E < 0$ $\psi(\vec{r})$ normalizable

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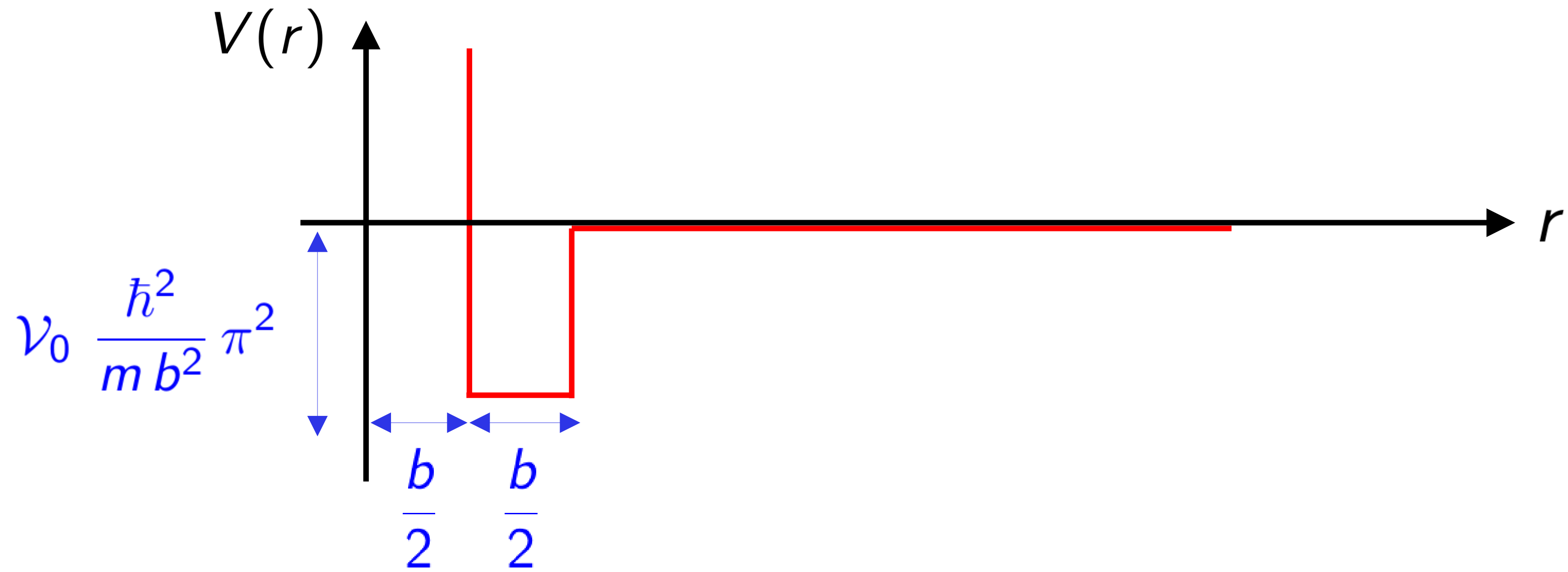


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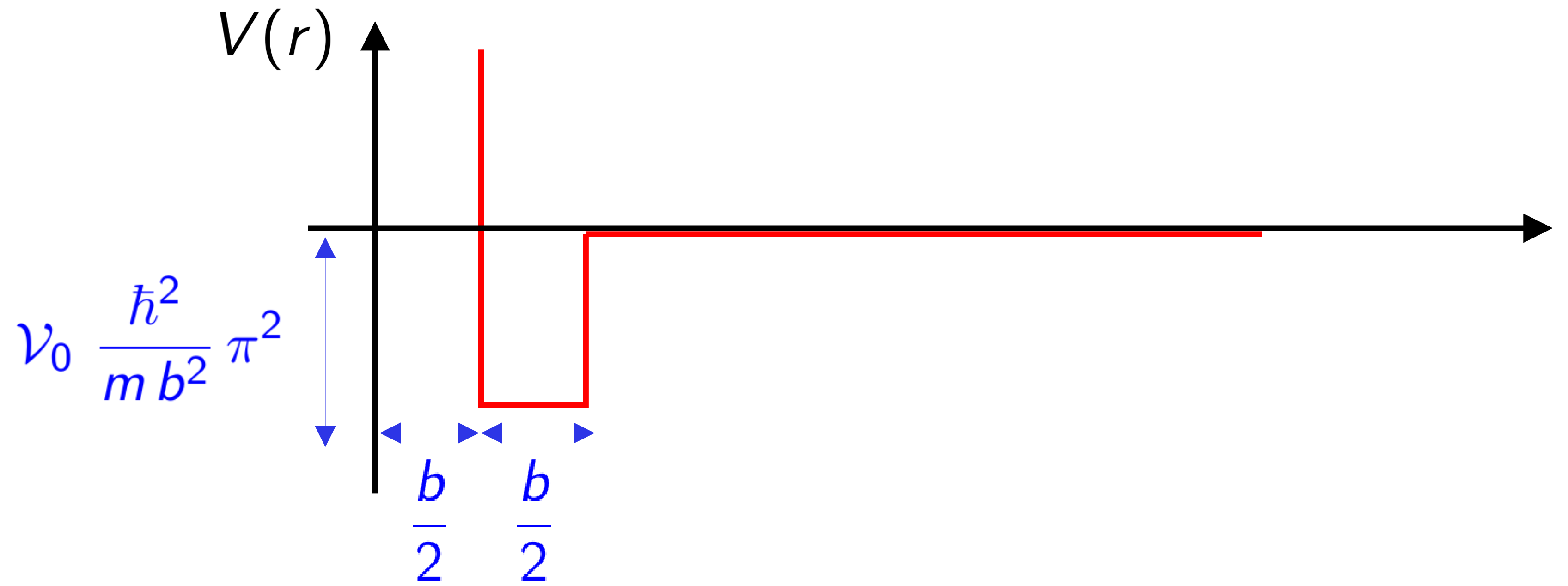


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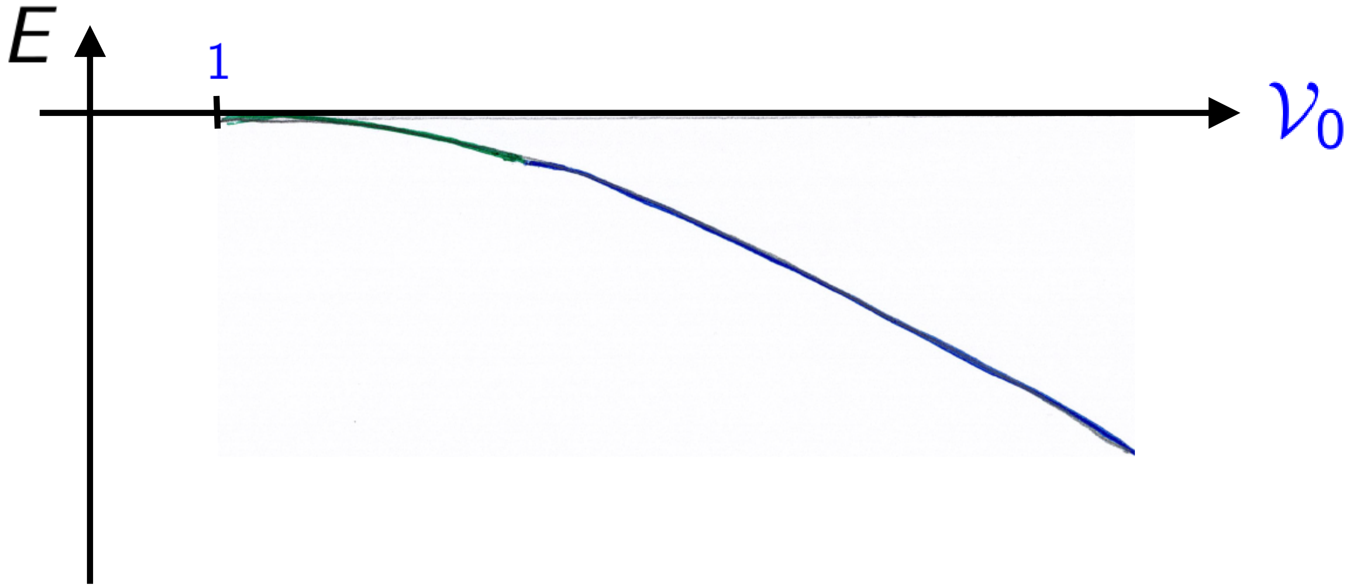

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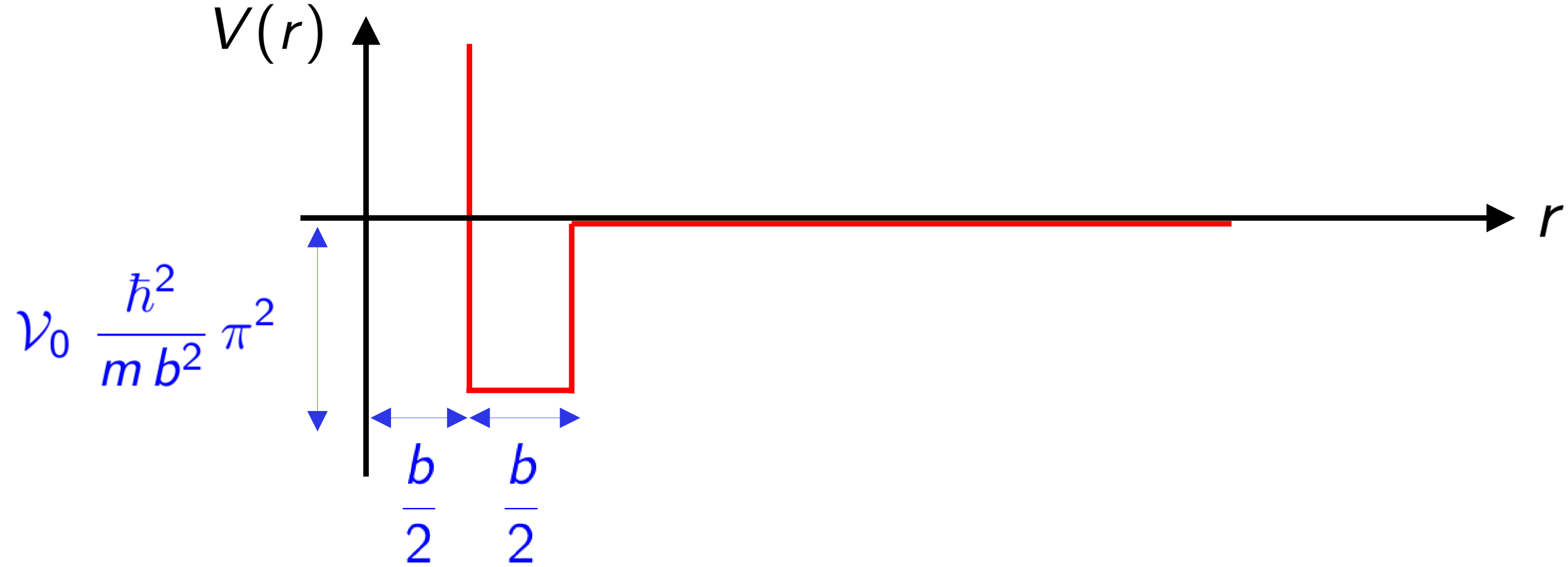


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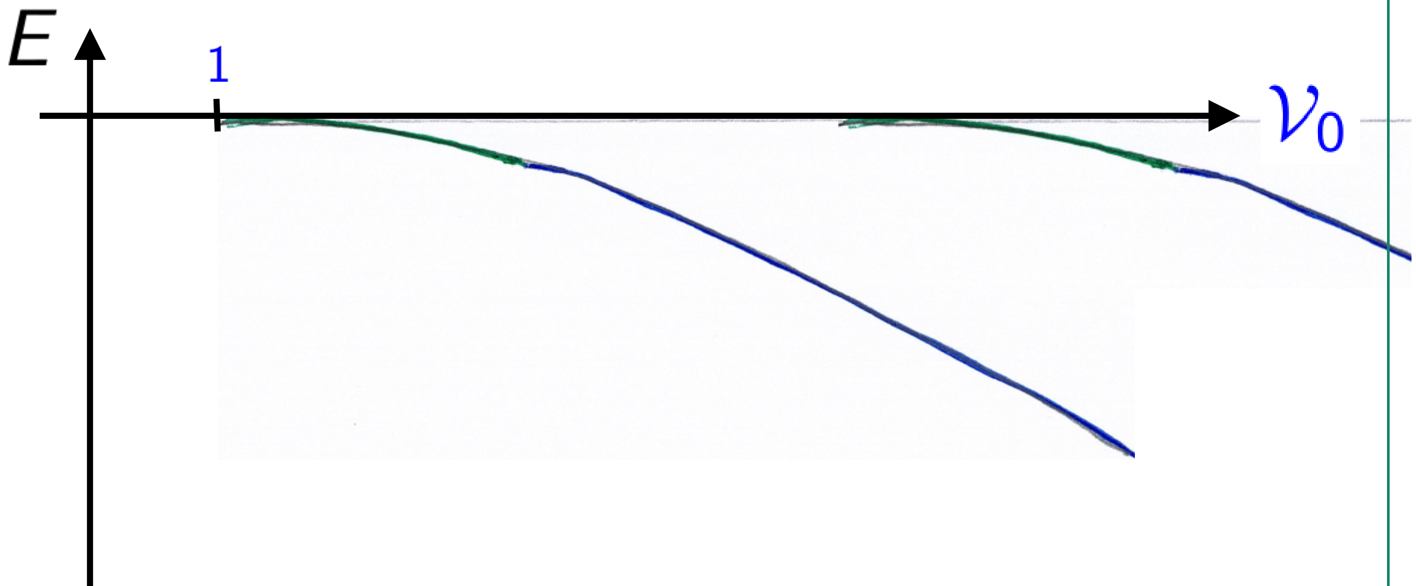

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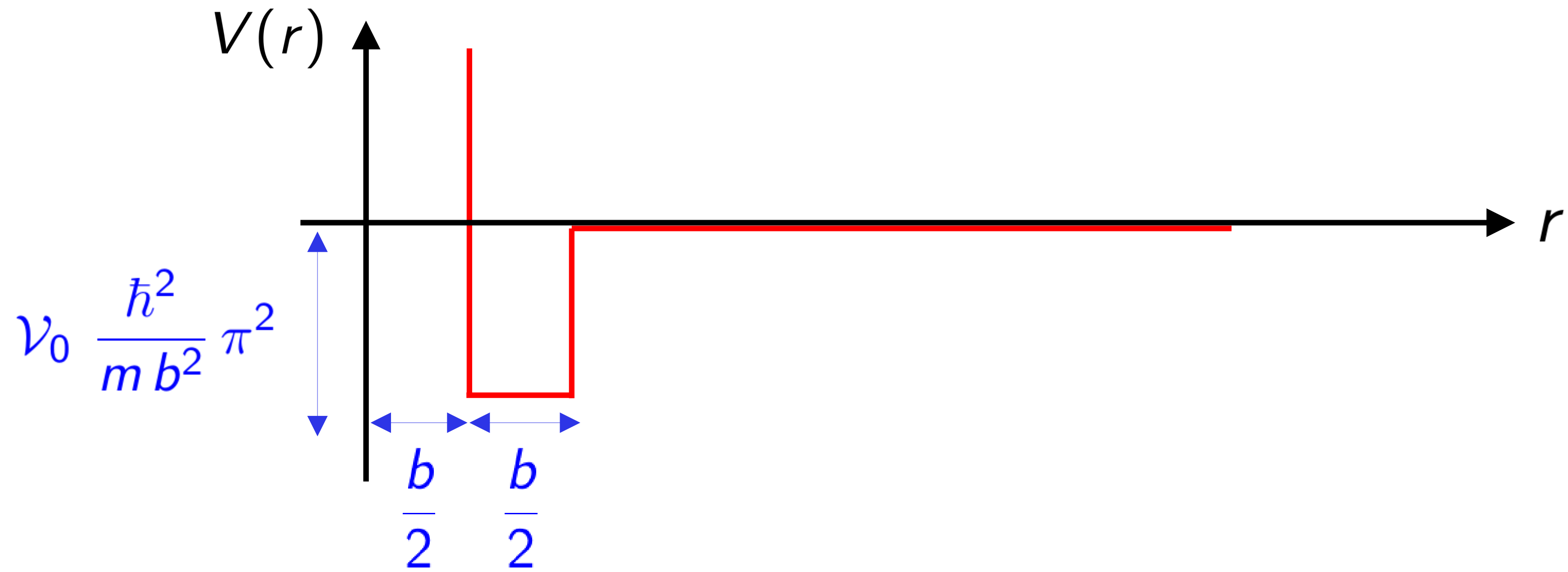


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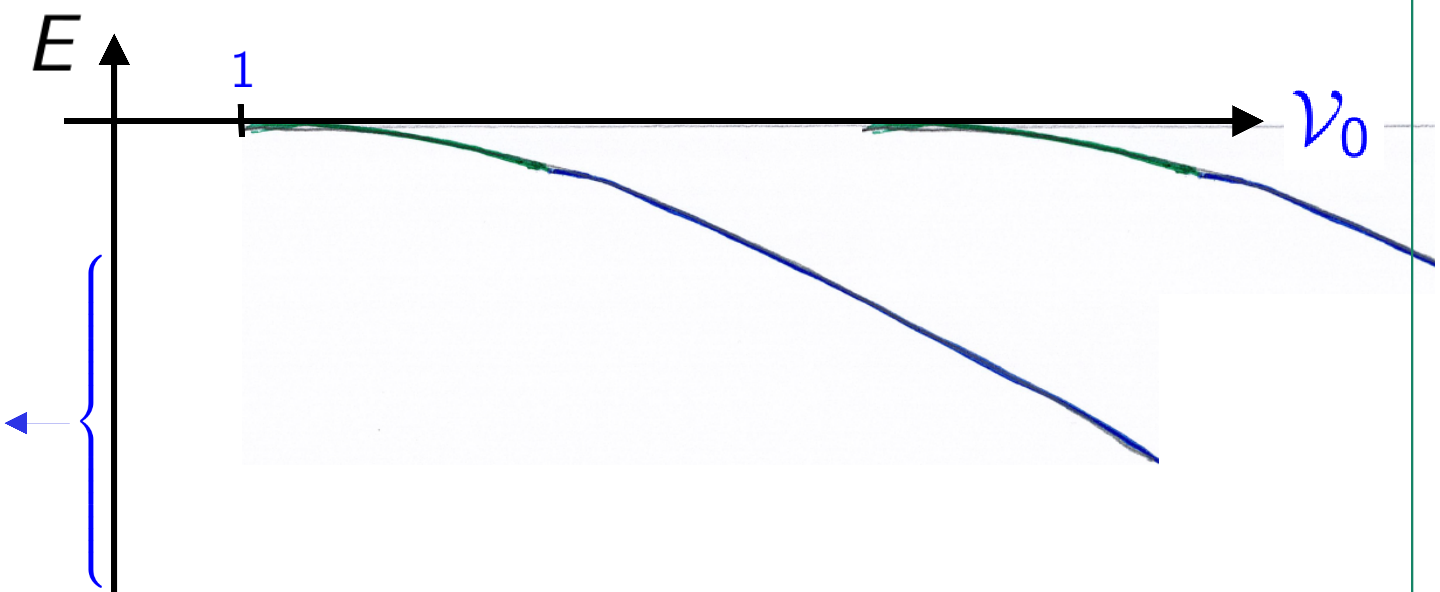


deeply bound: $|E| \sim \frac{\hbar^2}{m\,b^2}$

size $\sim b$

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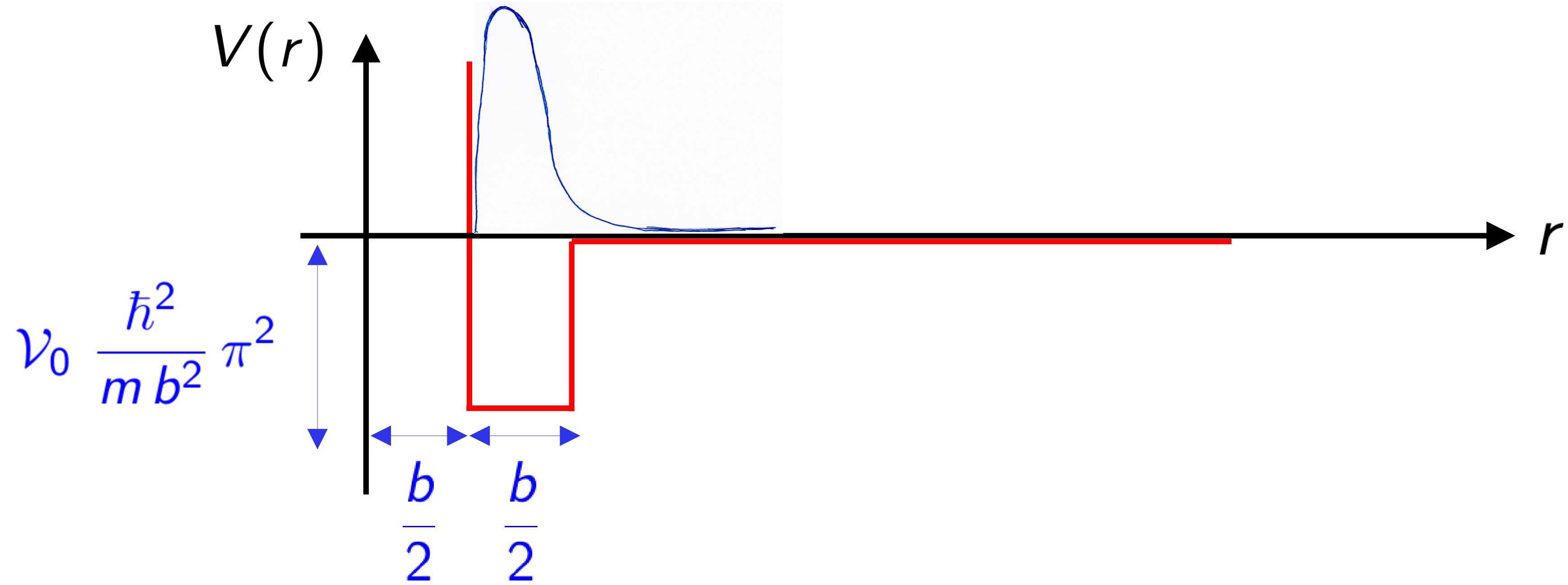


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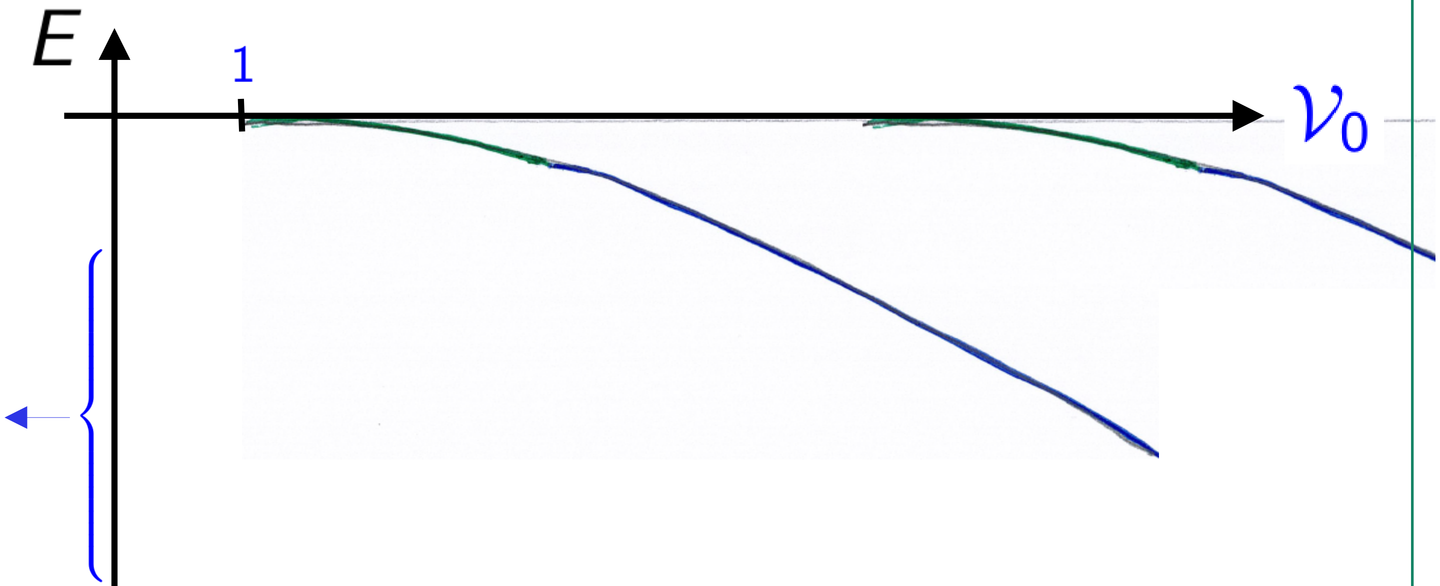


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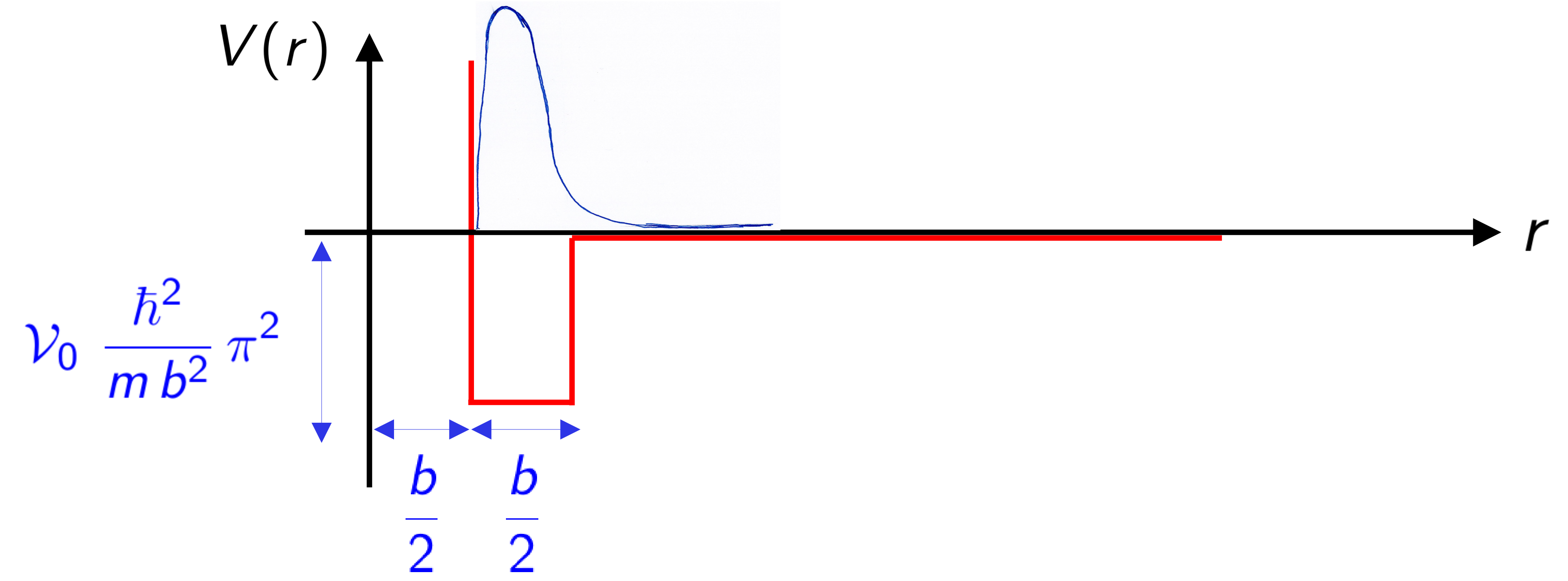


Dimers (= bound states): $E < 0$ $\psi(\vec{r})$ normalizable

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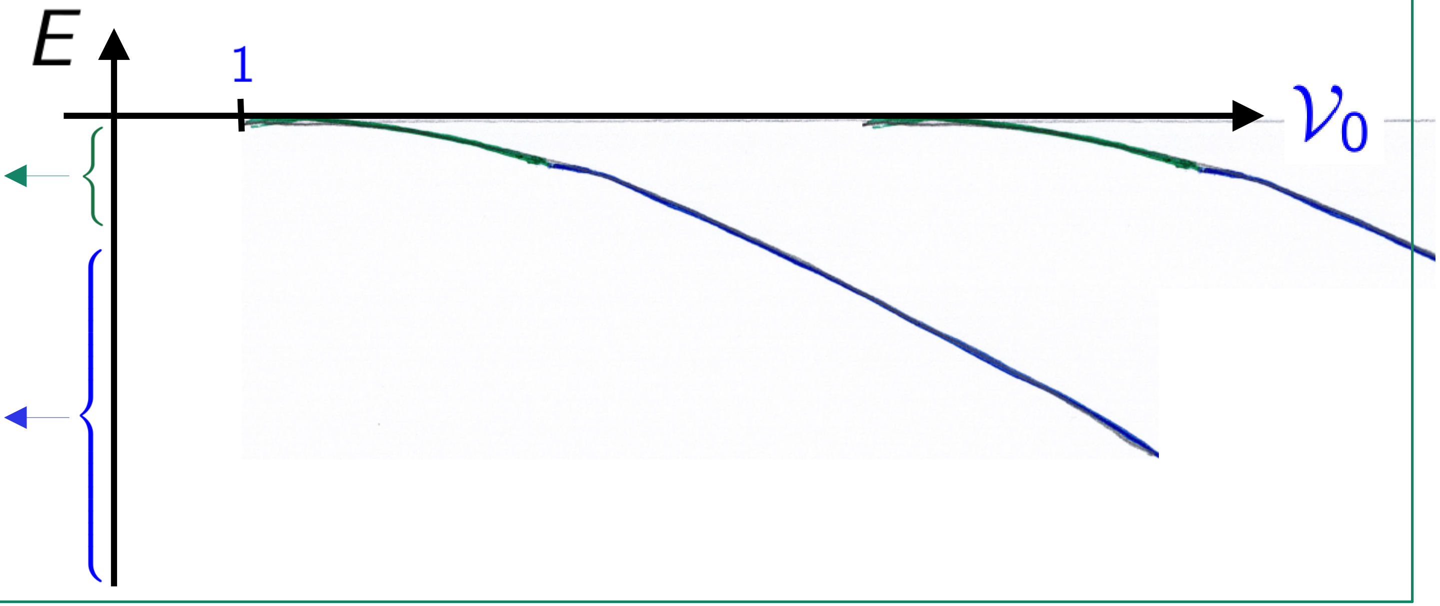

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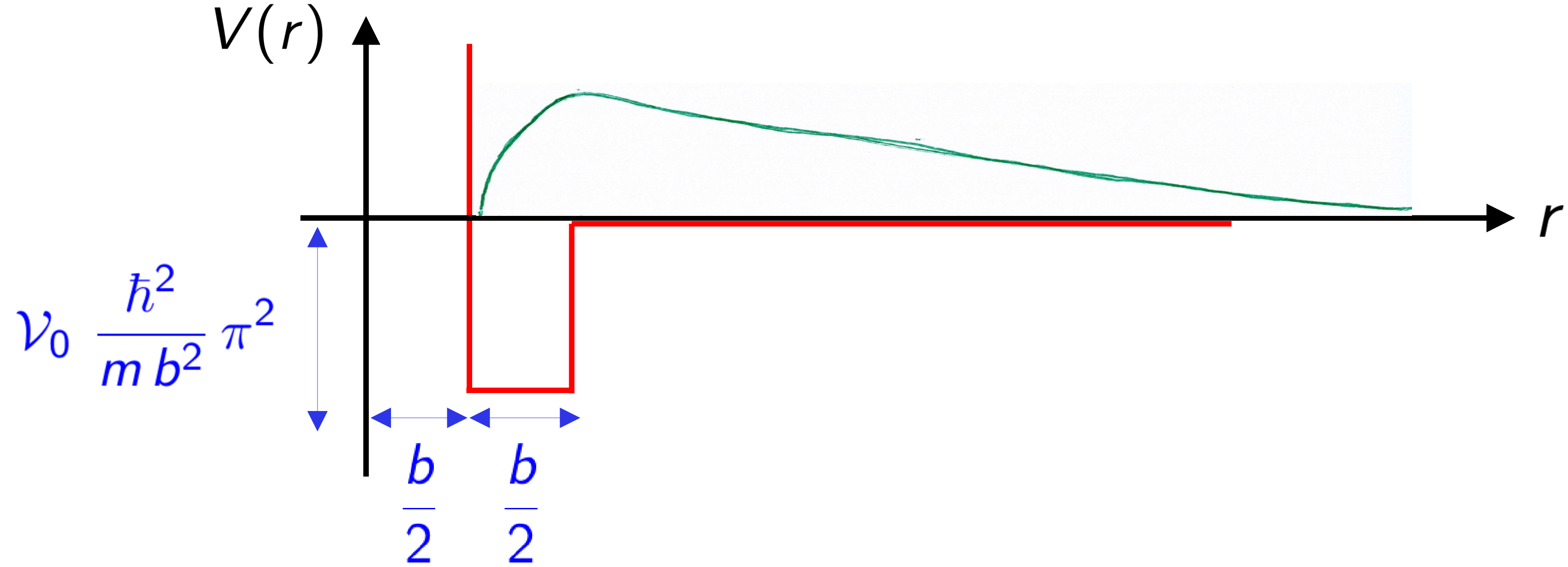


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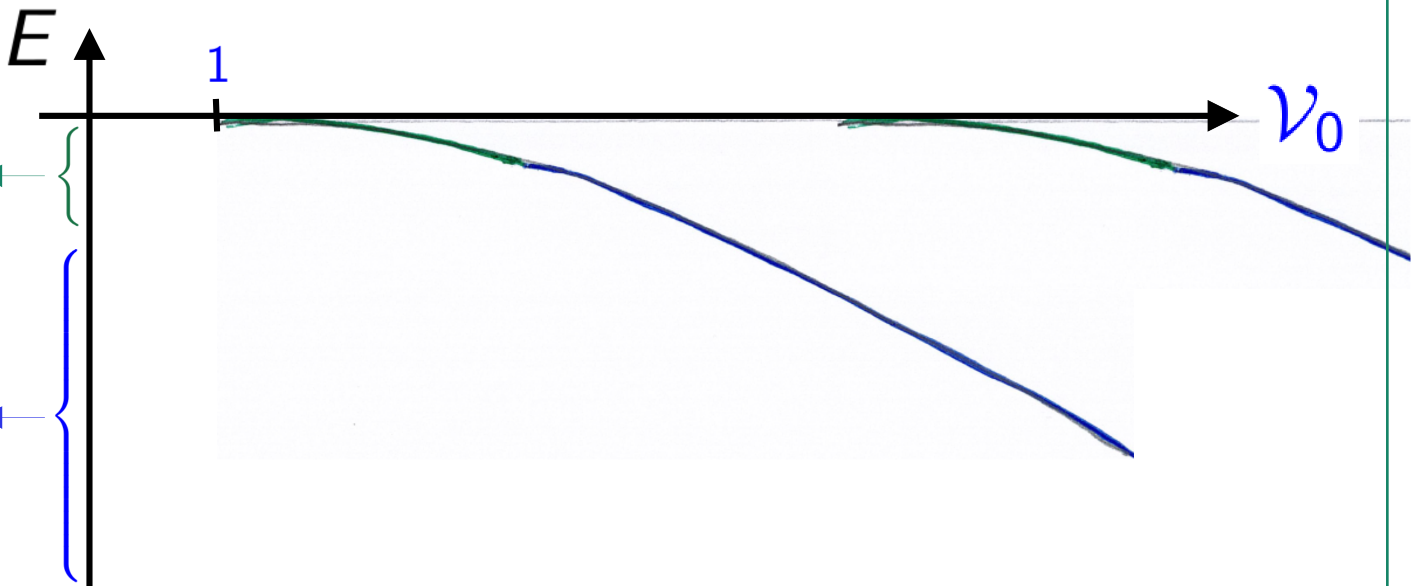

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Scattering states: $E > 0.$ $\qquad E = \frac{\hbar^2 k^2}{m}$

$$\psi(\vec{r}) \underset{r\to\infty}{\simeq} e^{i\,\vec{k}\cdot\vec{r}} + f_k(\hat{r})\,\frac{e^{ikr}}{r} \qquad \left[\ \hat{r} := \frac{\vec{r}}{r}\ \right]$$

$$a := -\lim_{k\to 0} f_k(\hat{r})$$ scattering length

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Universality: When $\mathcal{V}_0 \longrightarrow 1^+$: $\dfrac{a}{b} \longrightarrow +\infty$ and

$$\boxed{E \sim -\frac{\hbar^2}{ma^2}} \qquad \boxed{\psi(r) \longrightarrow \mathcal{N}\,\frac{e^{-r/a}}{r}}$$

for any shape of $V(r)$

(short-ranged)

$$\left[\, f \sim g \ \text{ means } \ \frac{f}{g} \to 1 \,\right]$$

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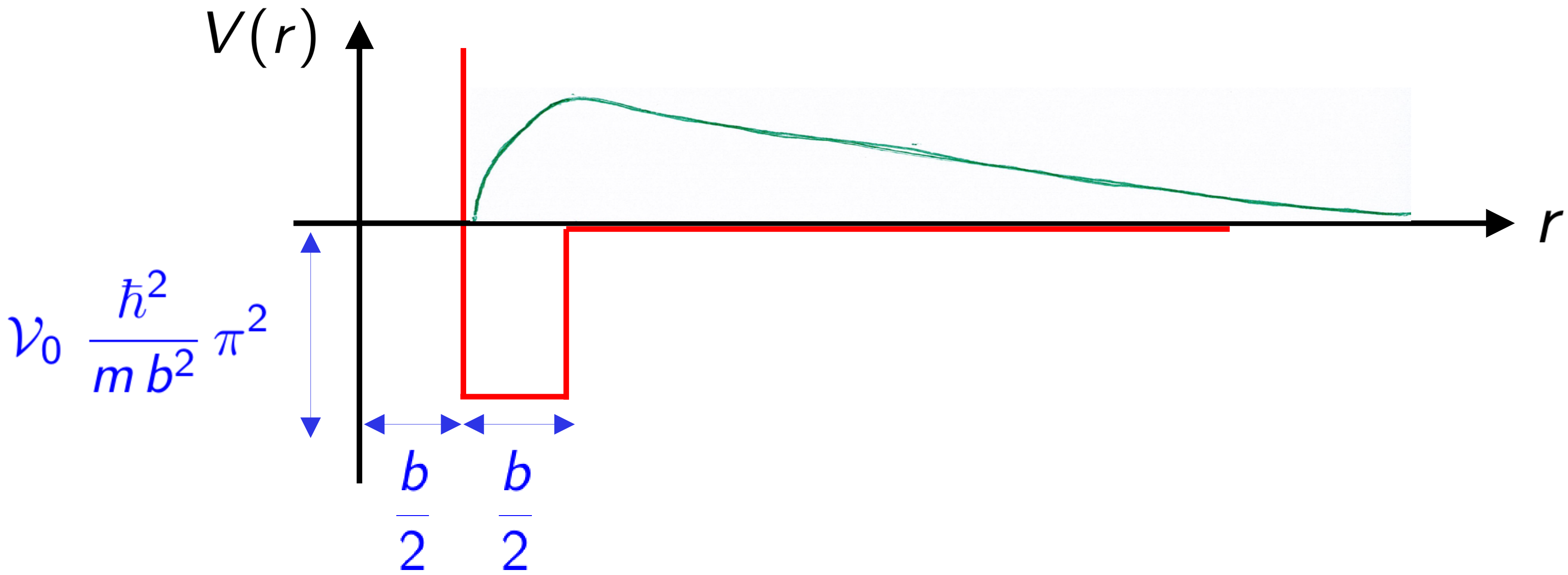


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Zero-range limit: $b \to 0$ , $a$ fixed.

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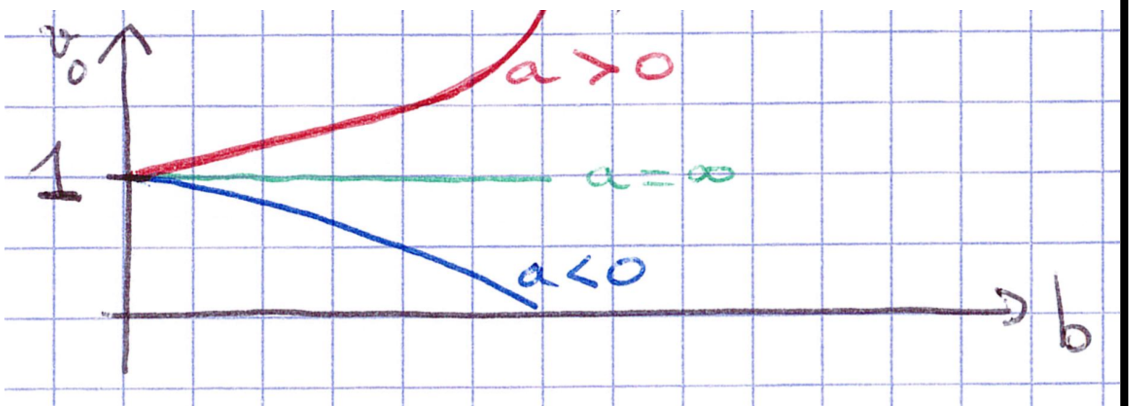

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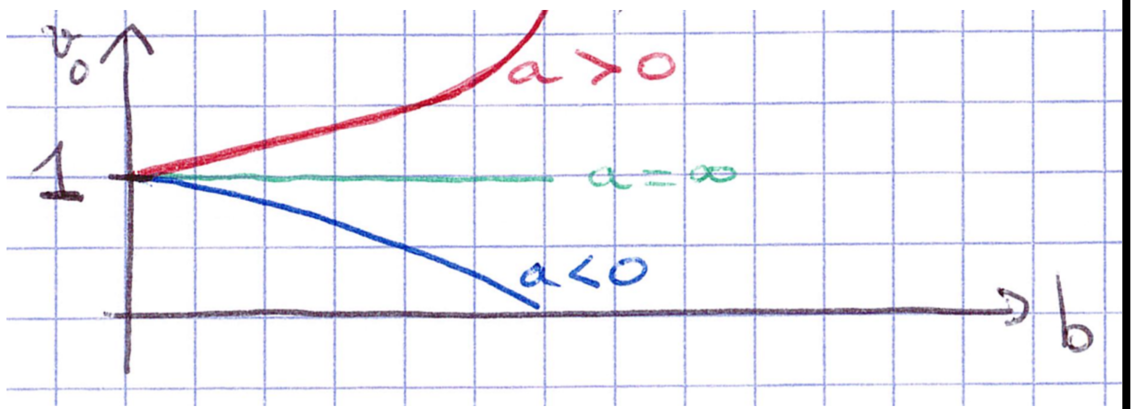

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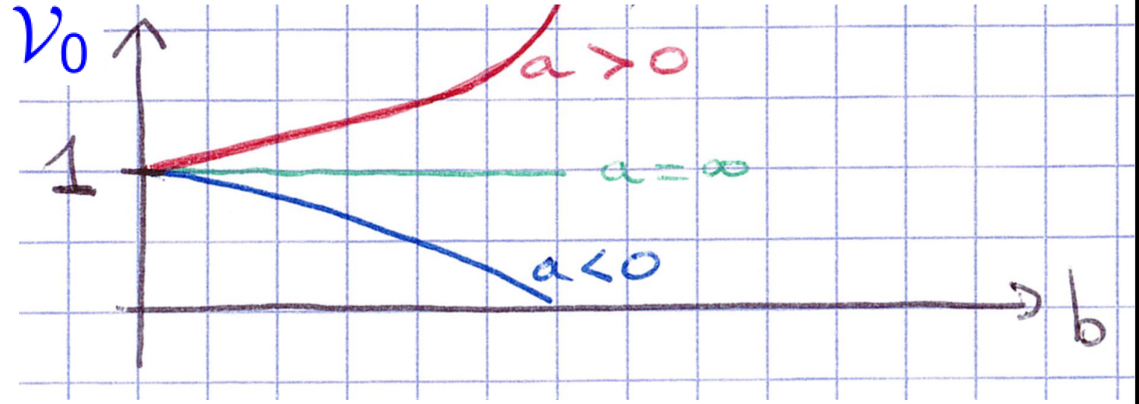

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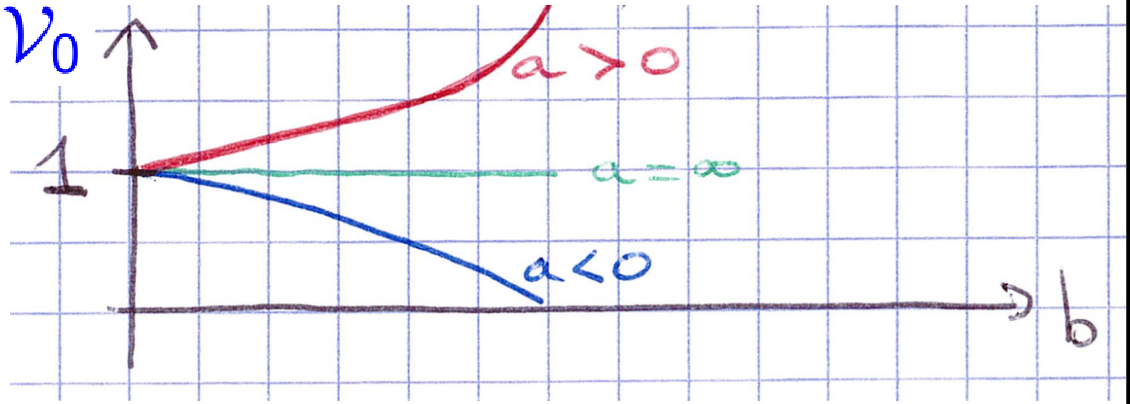


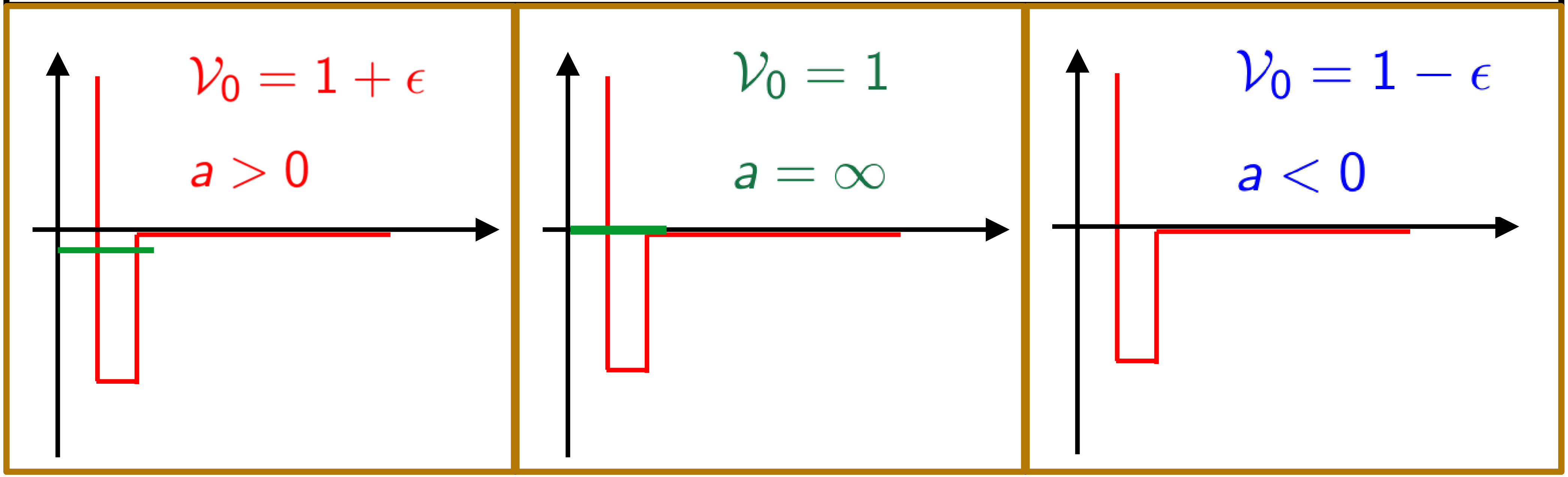

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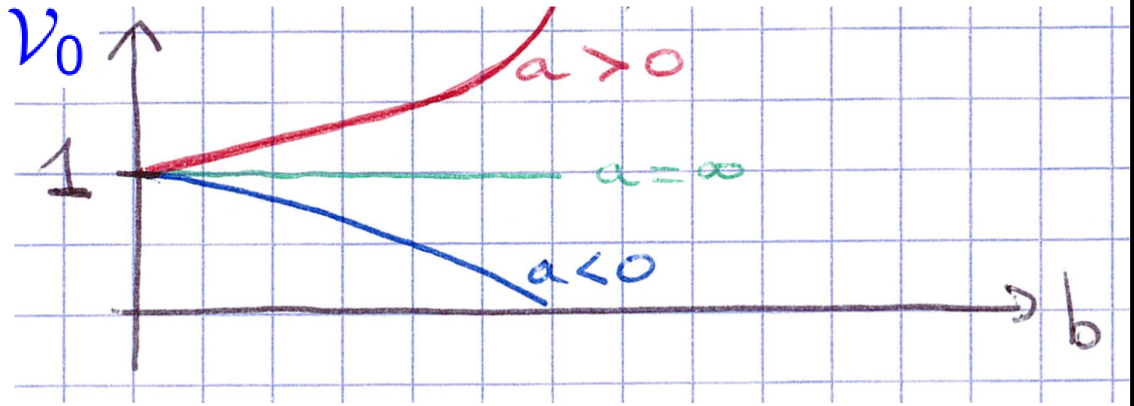


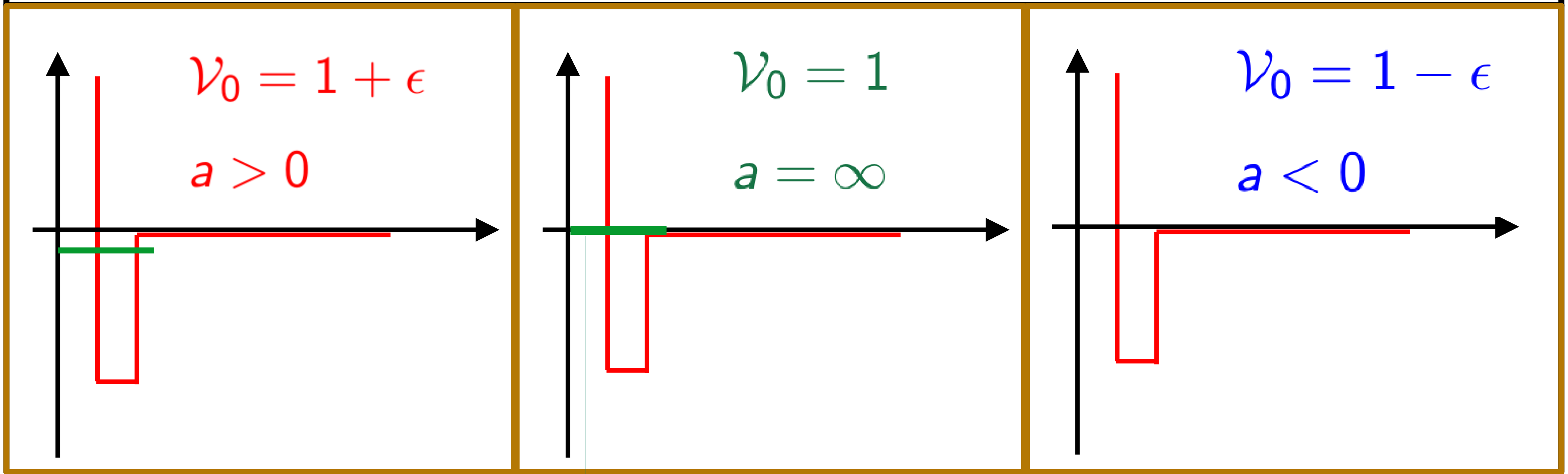


Zero-energy state $\phi_0(r)$

$$-\frac{\hbar^2}{m}\Delta\phi_0 + V(r)\,\phi_0 = 0$$

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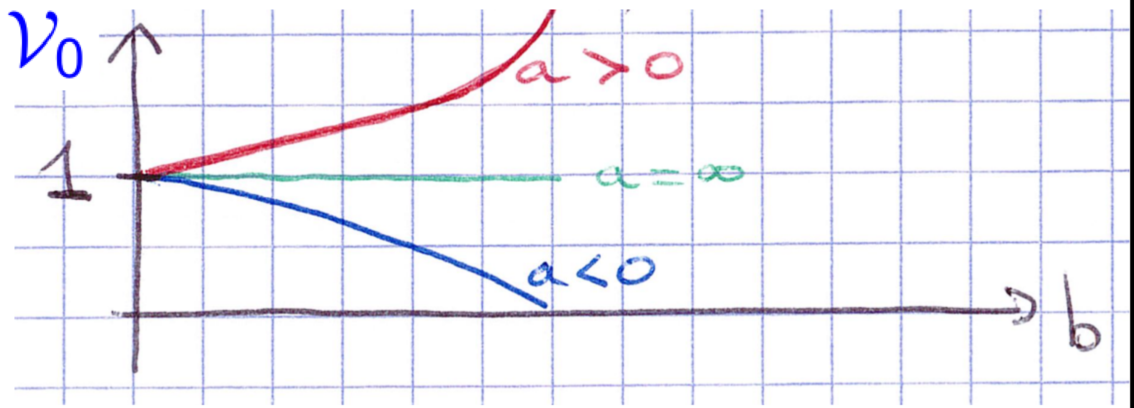


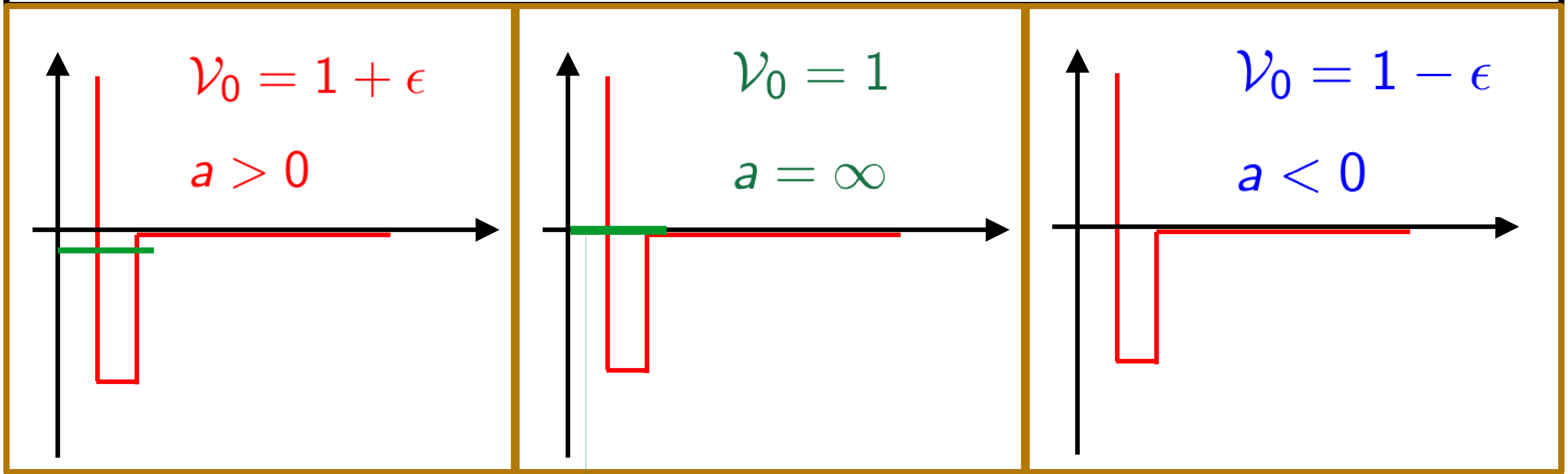


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$a = \infty$: scattering cross-section $\boxed{\sigma_k = \frac{4\pi}{k^2}}$ "unitary limit" $\left(\begin{array}{c}\text{max value allowed}\\ \text{by optical theorem}\end{array}\right)$

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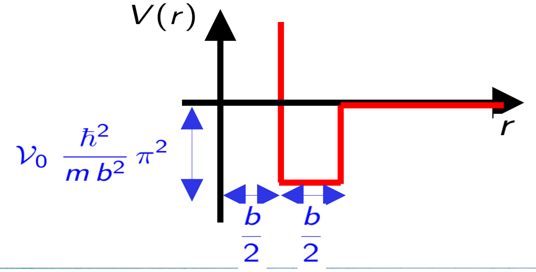

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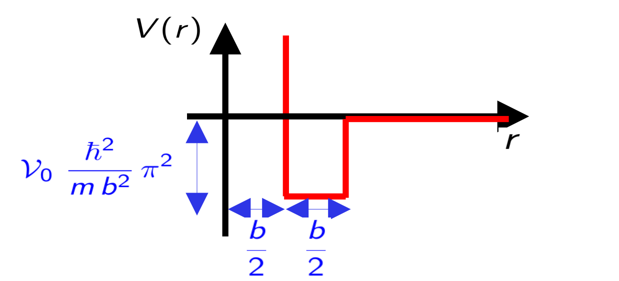


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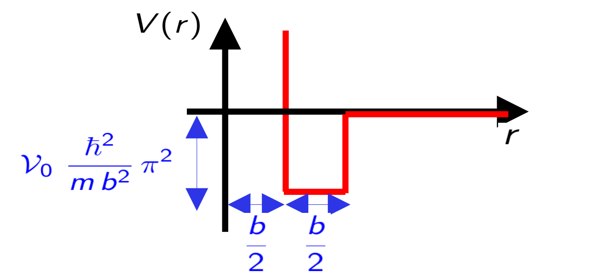


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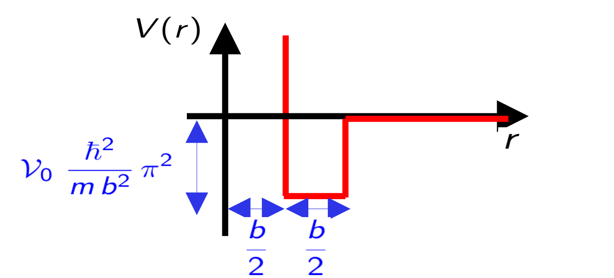


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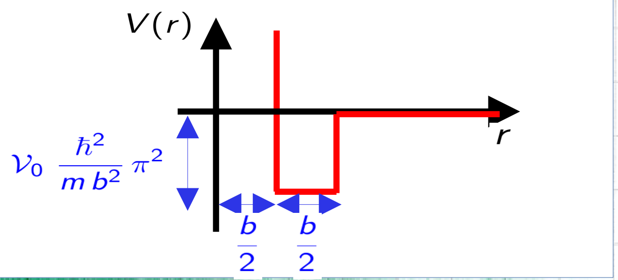


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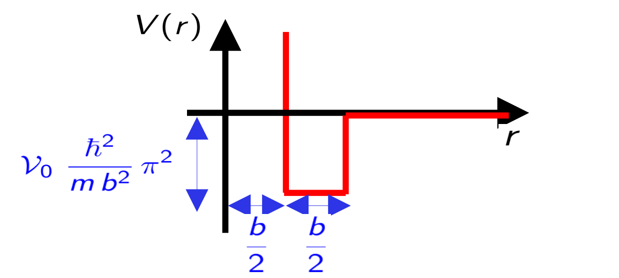


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$N_\uparrow = N_\downarrow$ ( "unpolarized" / "balanced" )

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$\mathcal{V} \to \infty\,,\; N \to \infty$

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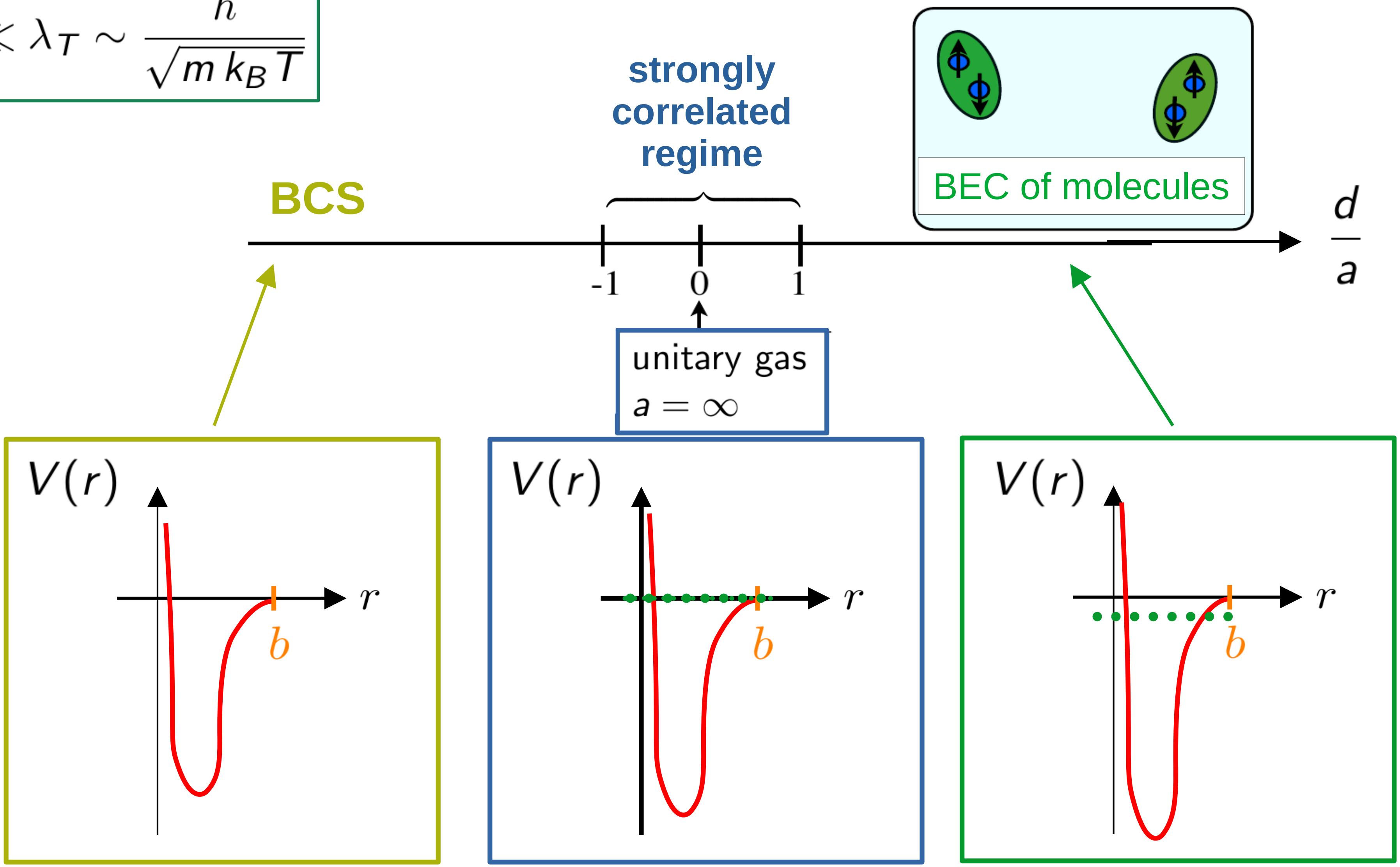

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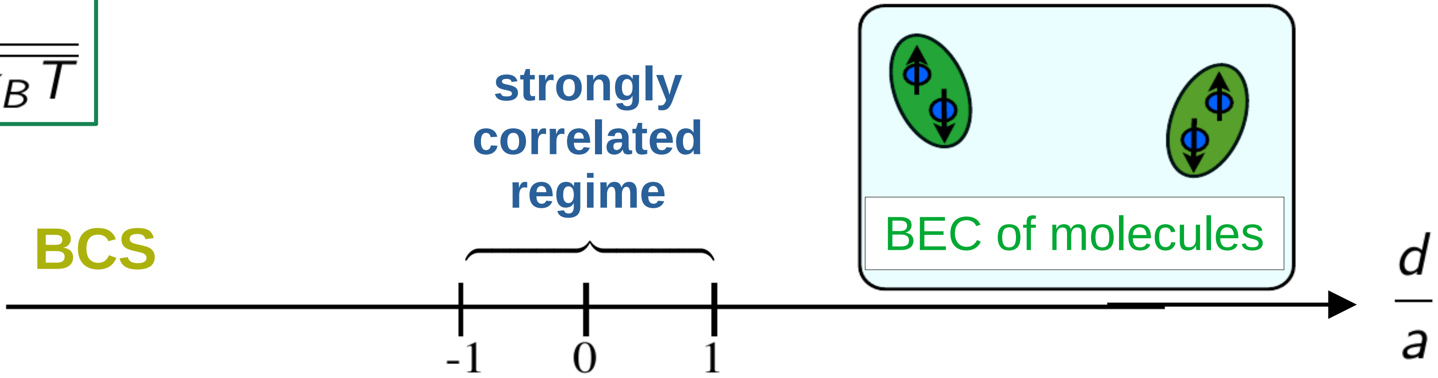

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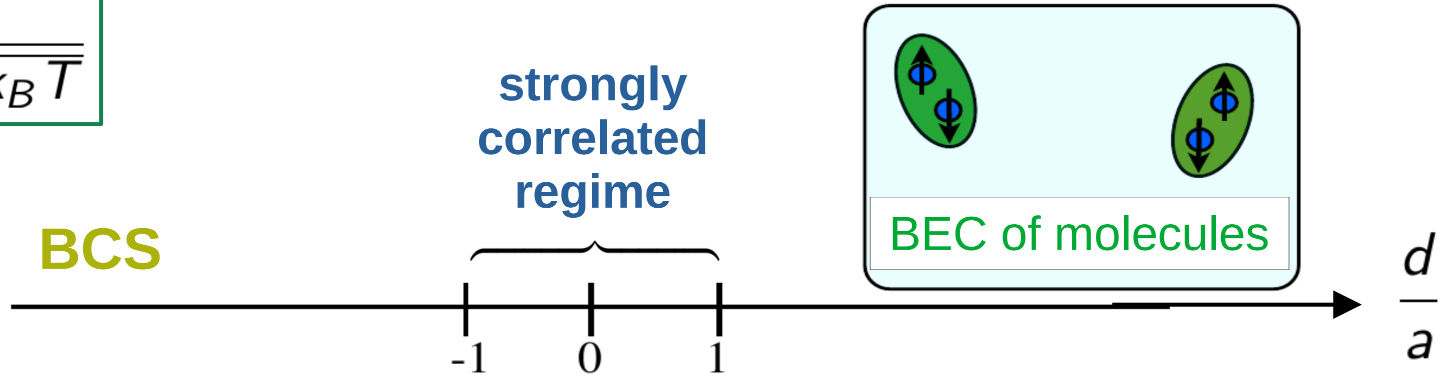


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Ideal Fermi gas

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# Homogeneous gas

$N_\uparrow = N_\downarrow$ ( "unpolarized" / "balanced" )

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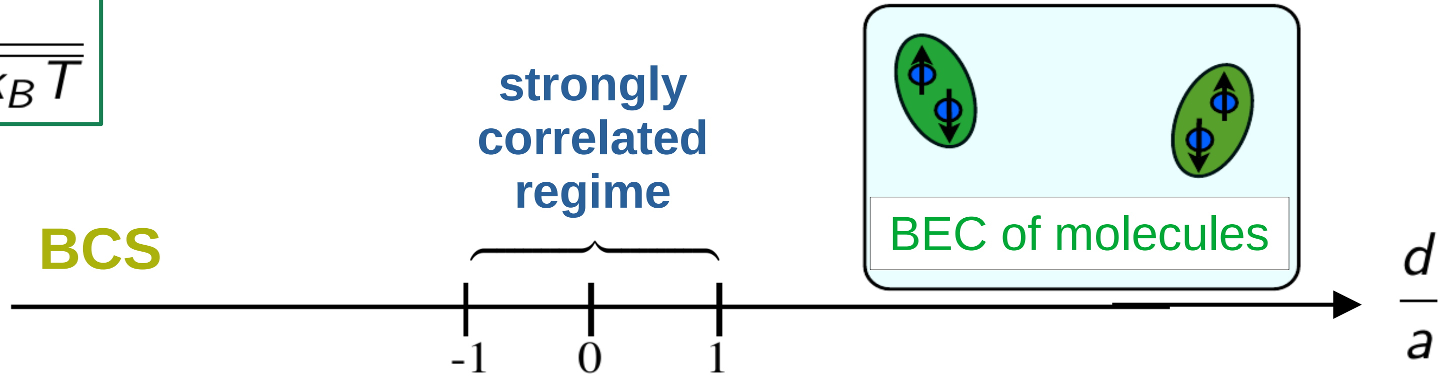


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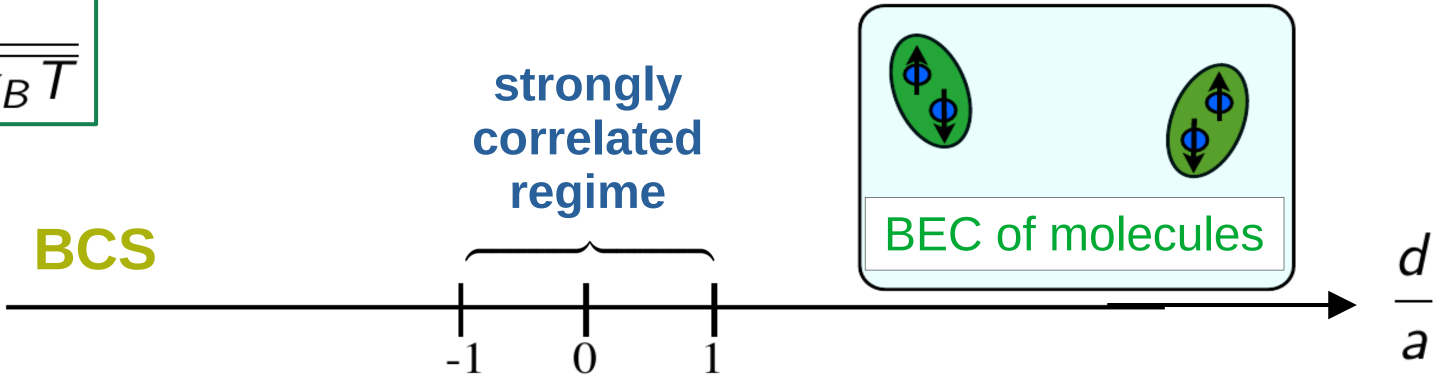


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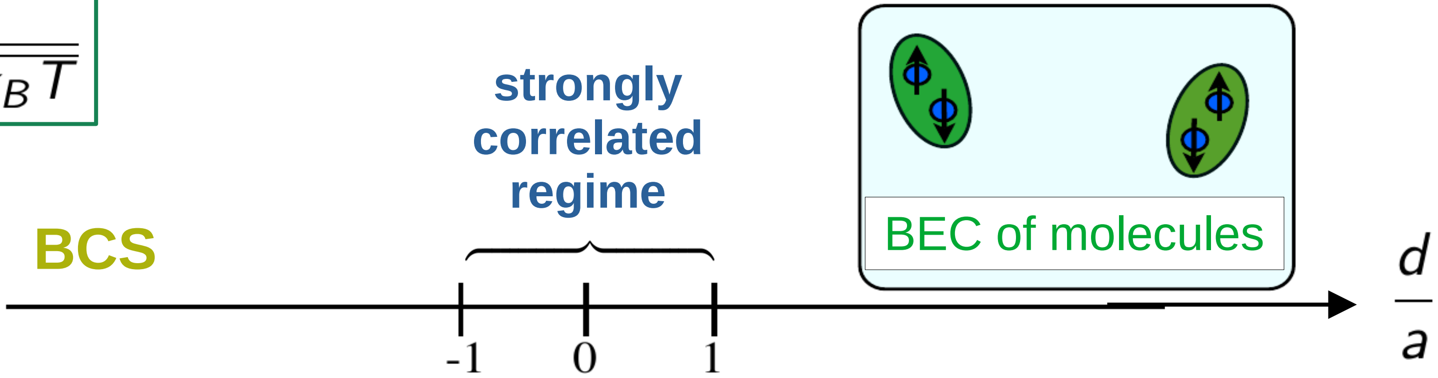


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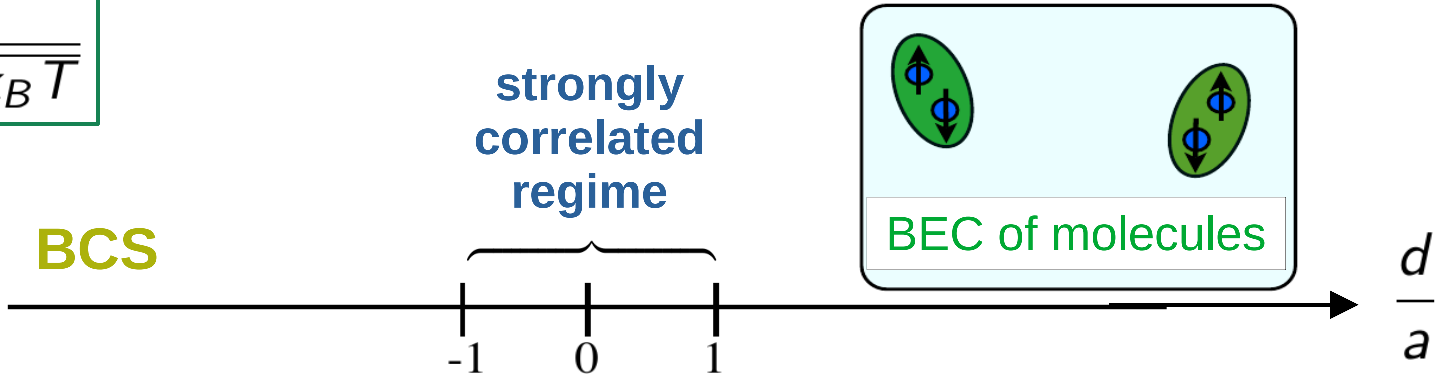


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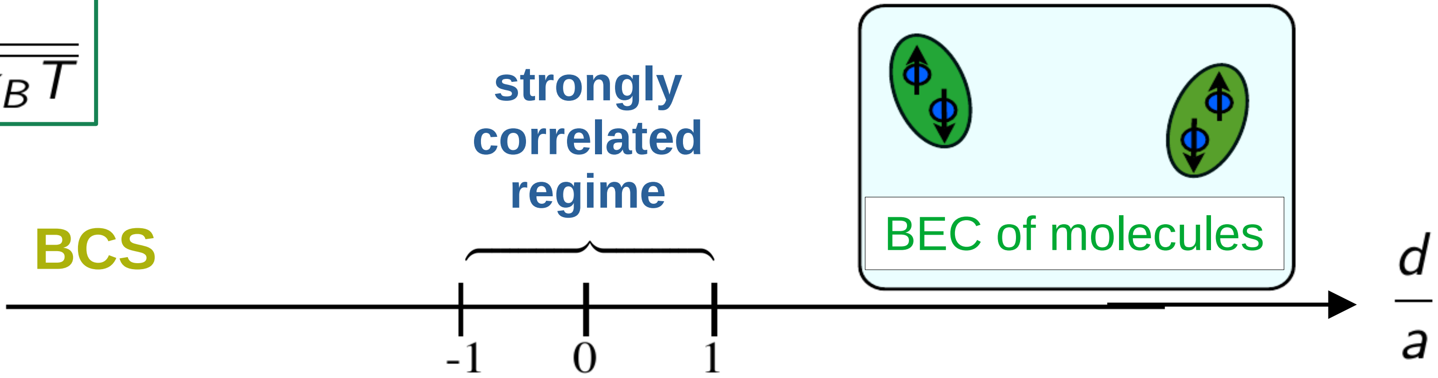


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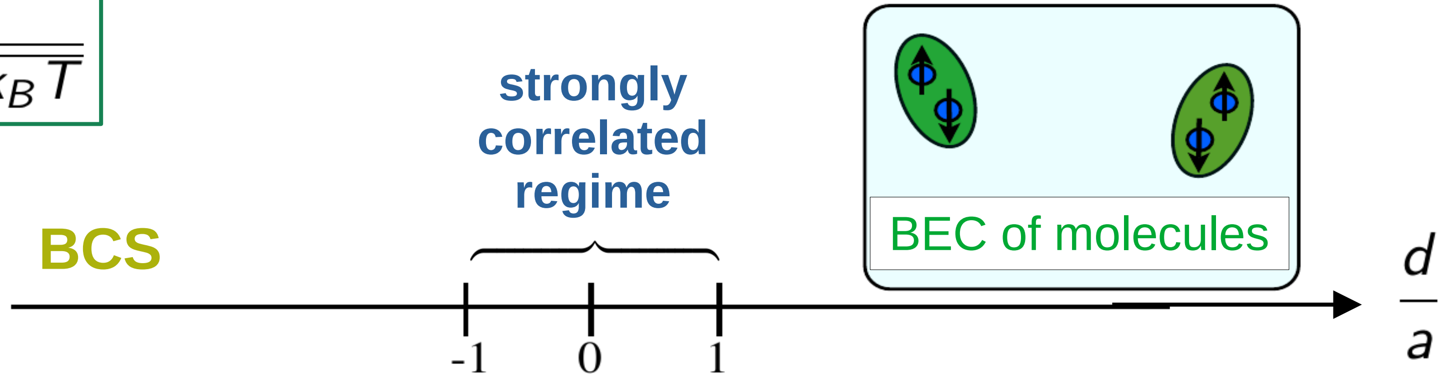


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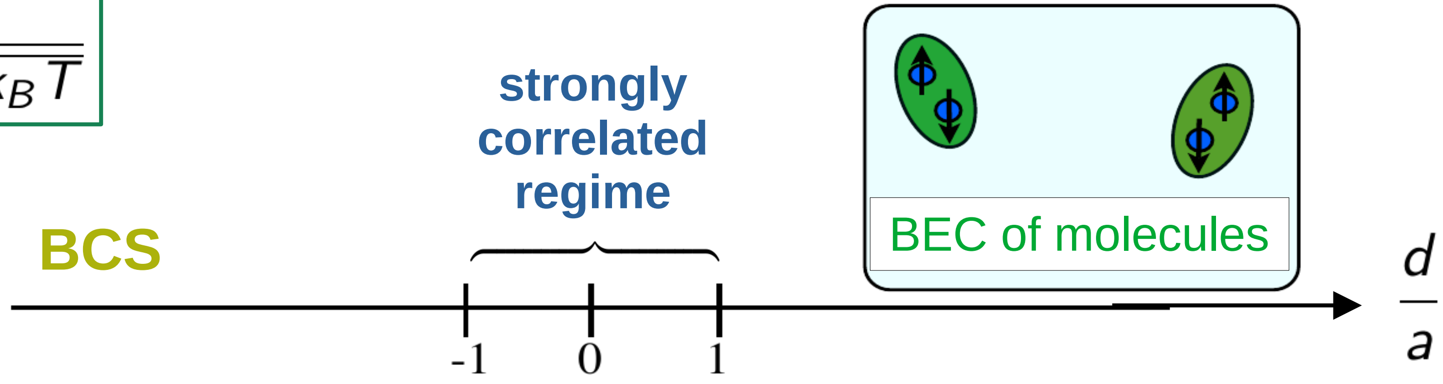


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For typical collisions in the gas: $k \approx k_F \;\Rightarrow\; \sigma_k \approx \dfrac{4\pi}{k_F^2} \approx 1.3\, d^2$

$\boxed{\sigma_k \sim d^2}$ like in a liquid