

# **The resonant Fermi gas**

## **Lecture 1**

**Zero-range limit,  
three-body problem  
and dynamical symmetry**

## **Lecture 2**

**Many-body physics:  
methods and basic properties**

## **Lecture 3**

**2-body and 3-body contacts**

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Part 2: Two-body problem

Part 3:  $N$ -body problem

- Zero-range limit and ZRM
- Homogeneous gas

### **Chapter 2: Solution of the unitary three-body problem**

### **Chapter 3: Symmetry properties of the unitary $N$ -body problem**

Part 1: Separability of the hyperradius

Excursion:  $s$  for  $N \rightarrow \infty$

Part 2: Short-distance scaling law

Part 3: The Castin mode

## Lecture 2: Many-body physics: methods and basic properties

### **Chapter 0: Lattice model**

### **Chapter 1: Many-body methods**

Part 1: Virial expansion

Part 2: Simple variational wavefunctions

BCS ansatz, Chevy ansatz

Part 3: Quantum Monte Carlo

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- Diagrams for the lattice model
- Diagrams for the zero-range model, diagrammatic Monte Carlo
- Appendices

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Part 1: Unpolarized unitary gas

Equation of state, long-range order, pairing gap, spectral function, second sound

Part 2: Fermi polaron

Part 3: Polarized gas

## Lecture 3: 2-body & 3-body contacts ( $C_2$ & $C_3$ )

- $C_2$
- Number of nearby pairs and triplets
- 3-body loss rate
- $C_3$  in the non-degenerate limit
- $C_3$  for the degenerate unitary gas : Heidelberg experiment

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***For all slides in one file:***

*[https://www.phys.ens.psl.eu/~fwerner/BENASQUE\\_2026/slides.pdf](https://www.phys.ens.psl.eu/~fwerner/BENASQUE_2026/slides.pdf)*

## Lecture 1

# Zero-range limit, three-body problem and dynamical symmetry

### Refs:

- with Y. Castin: Lect. Notes Phys. **836**, 127 (2012); PRL **97**, 150401 (2006); PRA **74**, 053604 (2006)
- S. Tan, arXiv:cond-mat/0412764
- V. Efimov, Sov. J. Nucl. Phys. **12**, 589 (1971); Nucl. Phys. **A210**, 157 (1973).
- Y. Castin, Comptes Rendus Physique **5**, 407 (2004)

# Short-range resonant interactions and the universal zero-range limit

Part 1: Simplified atomic physics

Fermionic atoms in 2 internal states:  $\uparrow$ ,  $\downarrow$   
(hyperfine)

$$N = N_{\uparrow} + N_{\downarrow}.$$

$N_{\uparrow}$  and  $N_{\downarrow}$  are conserved

$\Rightarrow$  Two species

$\left\{ \begin{array}{l} N_{\uparrow} \text{ red fermions} \\ N_{\downarrow} \text{ blue fermions} \end{array} \right.$

red and blue are distinguishable.

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e.g.  $N_{\uparrow} = N_{\downarrow} = 2$ :

$$\Psi(\underbrace{\vec{r}_1, \vec{r}_2}_{\text{antisym.}}, \underbrace{\vec{r}_3, \vec{r}_4}_{\text{antisym.}}) = -\Psi(\vec{r}_2, \vec{r}_1, \vec{r}_3, \vec{r}_4) = -\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_4, \vec{r}_3)$$

Interaction:  $\underbrace{V_{\uparrow\downarrow}(r)}_{V(r)}$ ,  $\underbrace{V_{\uparrow\uparrow}(r)}_{\text{negligible effect}}$ ,  $\underbrace{V_{\downarrow\downarrow}(r)}_{\text{negligible effect}}$

Trap:  
 $U(\vec{r})$

$$-\frac{\hbar^2}{2m} \sum_{i=1}^N \Delta_{\vec{r}_i} \Psi + \sum_{i=1}^N U(\vec{r}_i) \Psi + \sum_{\substack{i \leq N_{\uparrow} \\ j > N_{\uparrow}}} V(r_{ij}) \Psi = E \Psi$$

$$r_{ij} := \|\vec{r}_j - \vec{r}_i\|$$

## Part 2: 2-body problem

3D

2 fermions in *different internal states*,  $\uparrow$  and  $\downarrow$

[  $\Leftrightarrow$  2 *distinguishable* particles ]

$$\psi(\vec{r}_1, \vec{r}_2)$$

no symmetry constraint  
when exchanging  $\vec{r}_1$  and  $\vec{r}_2$

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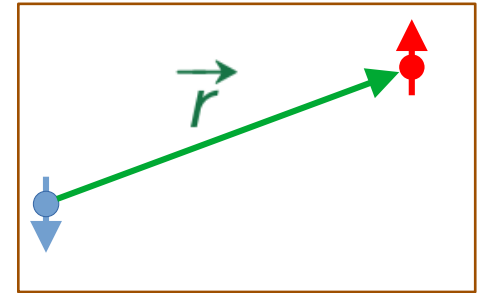
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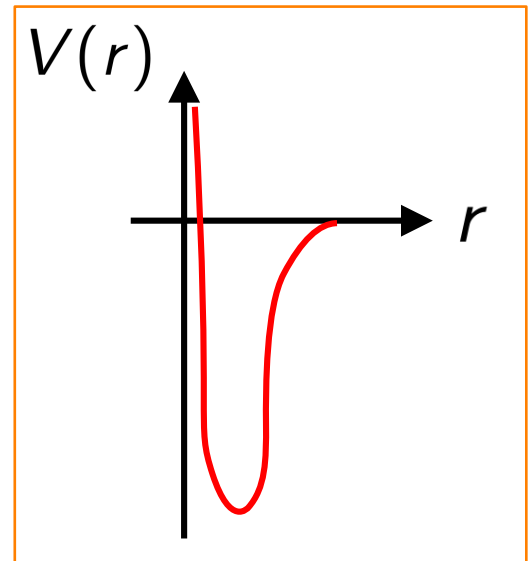
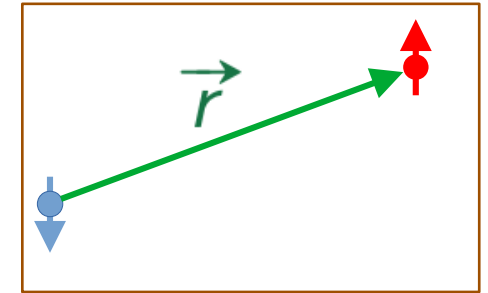
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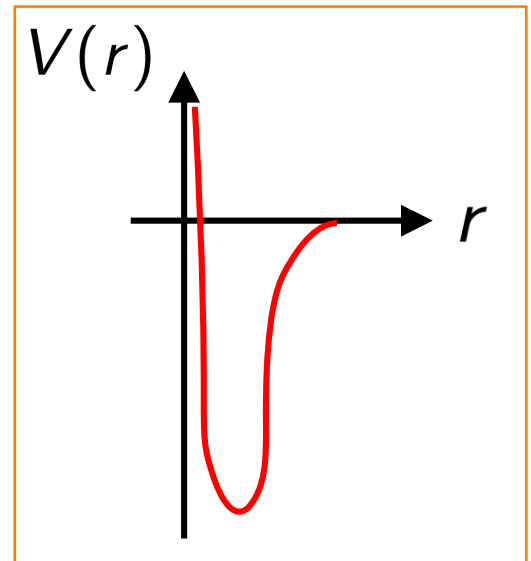
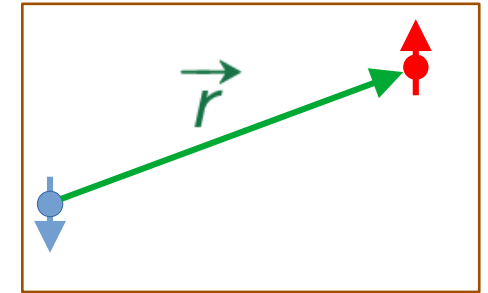
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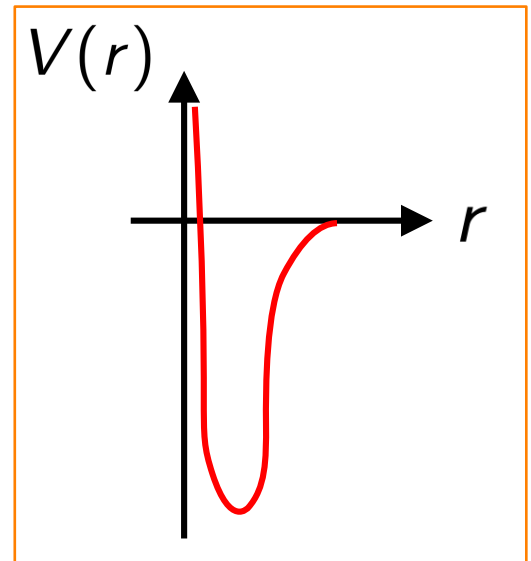
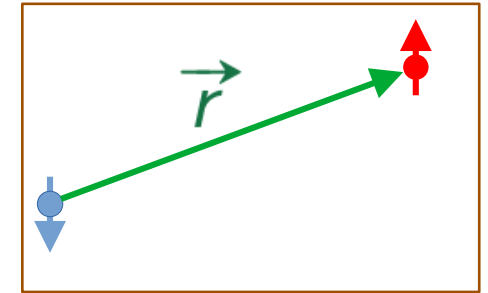
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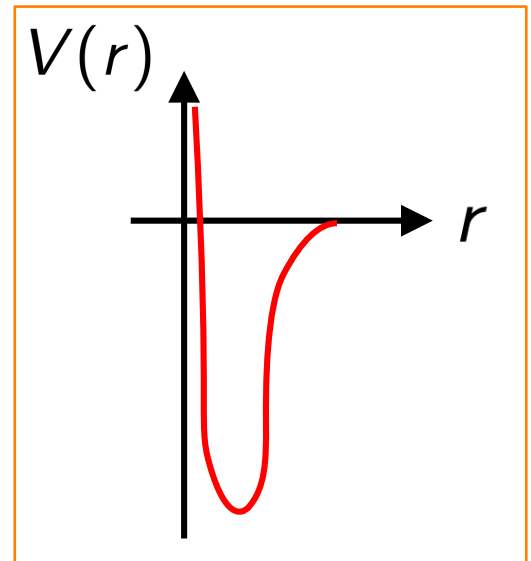
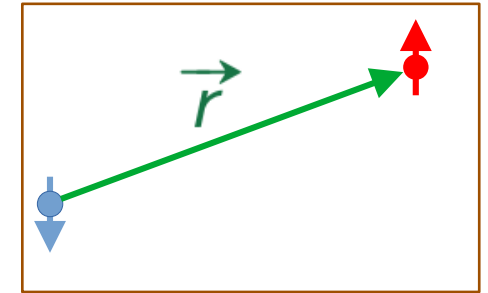
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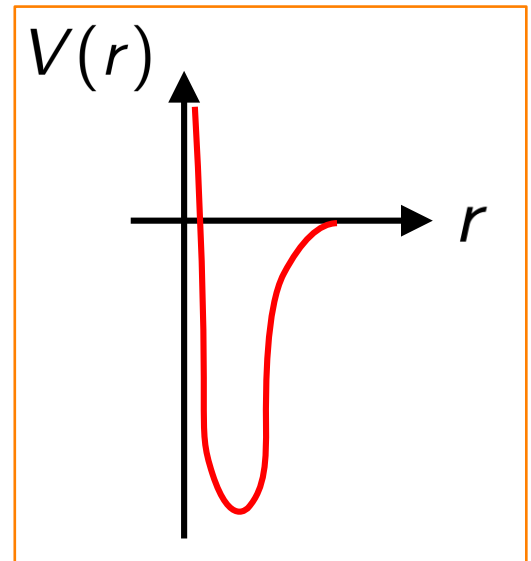
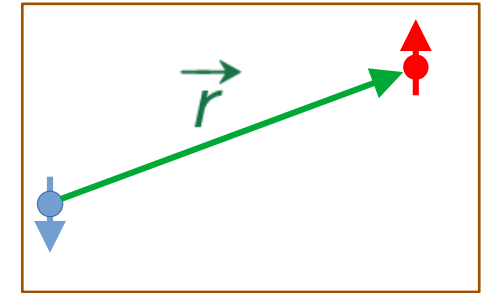
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$$\vec{c} = \frac{\vec{r}_1 + \vec{r}_2}{2}$$

$$\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}) e^{i\vec{K}\cdot\vec{c}}$$

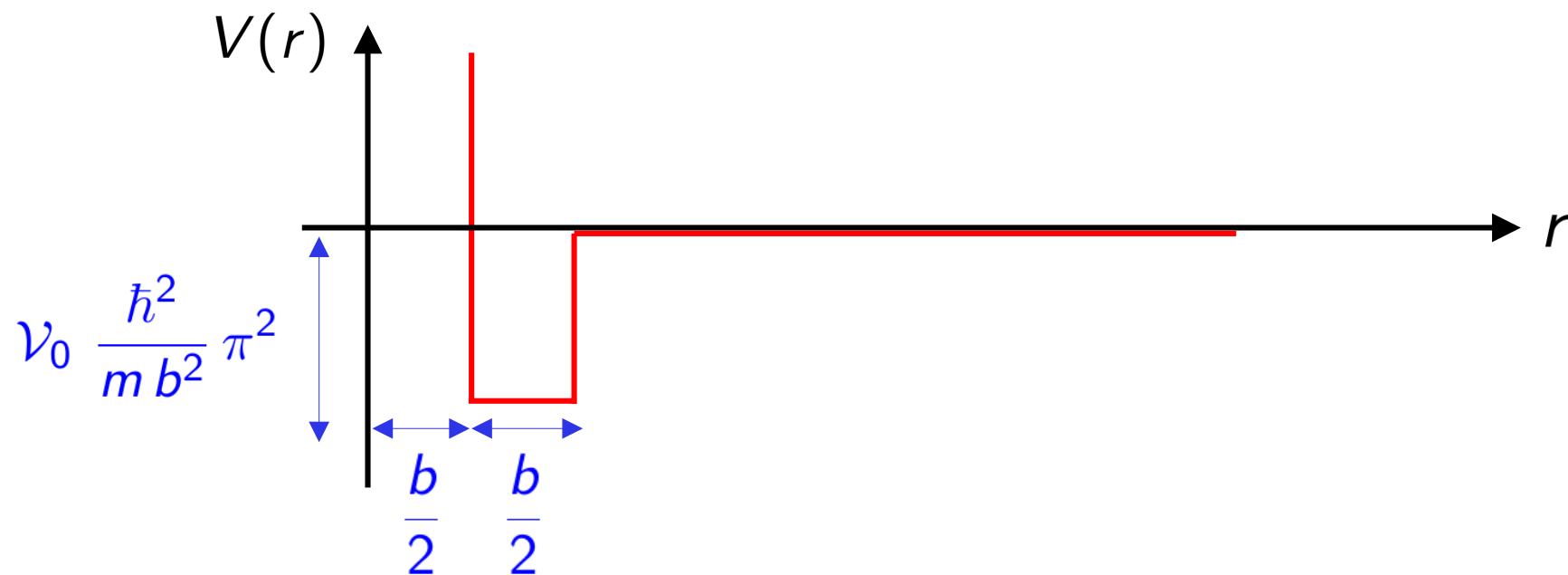
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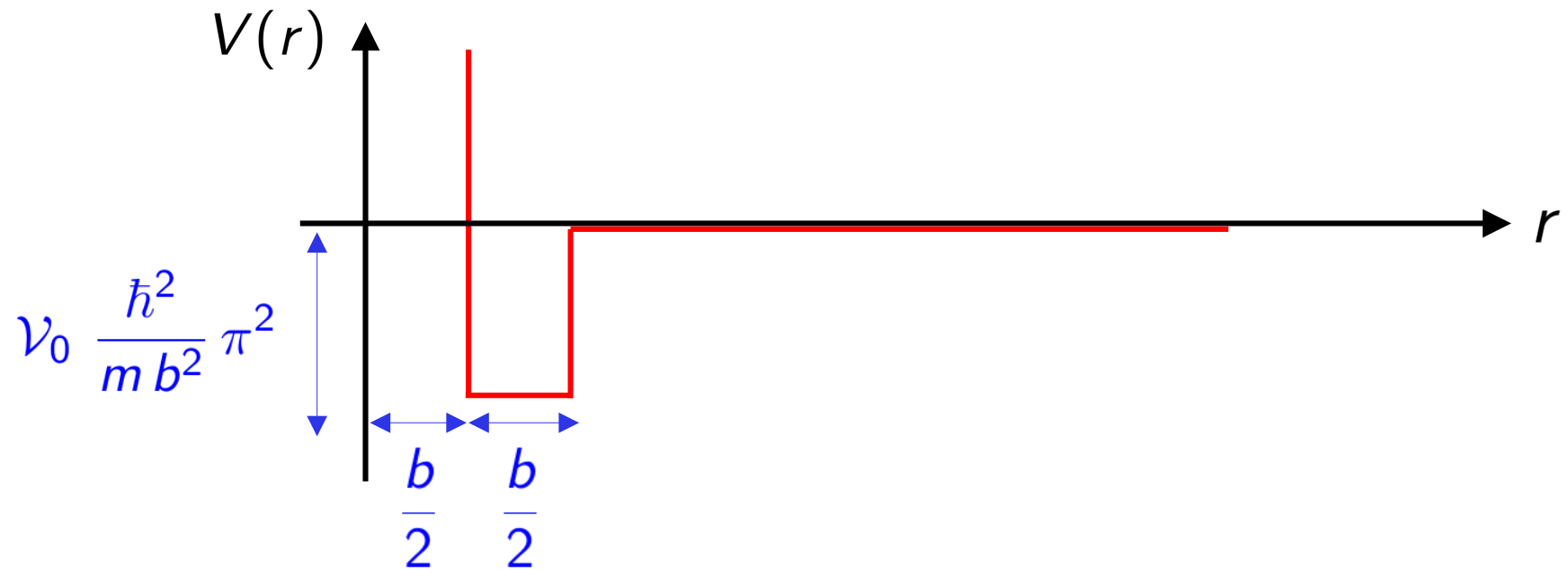
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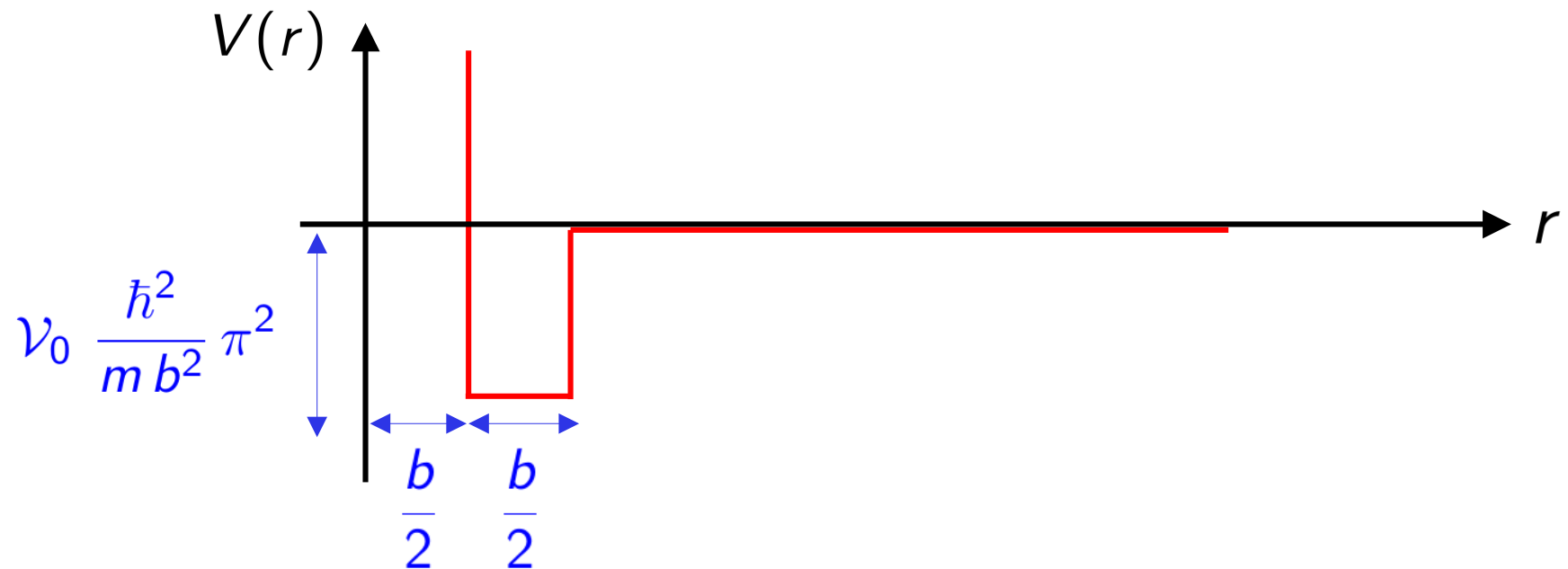


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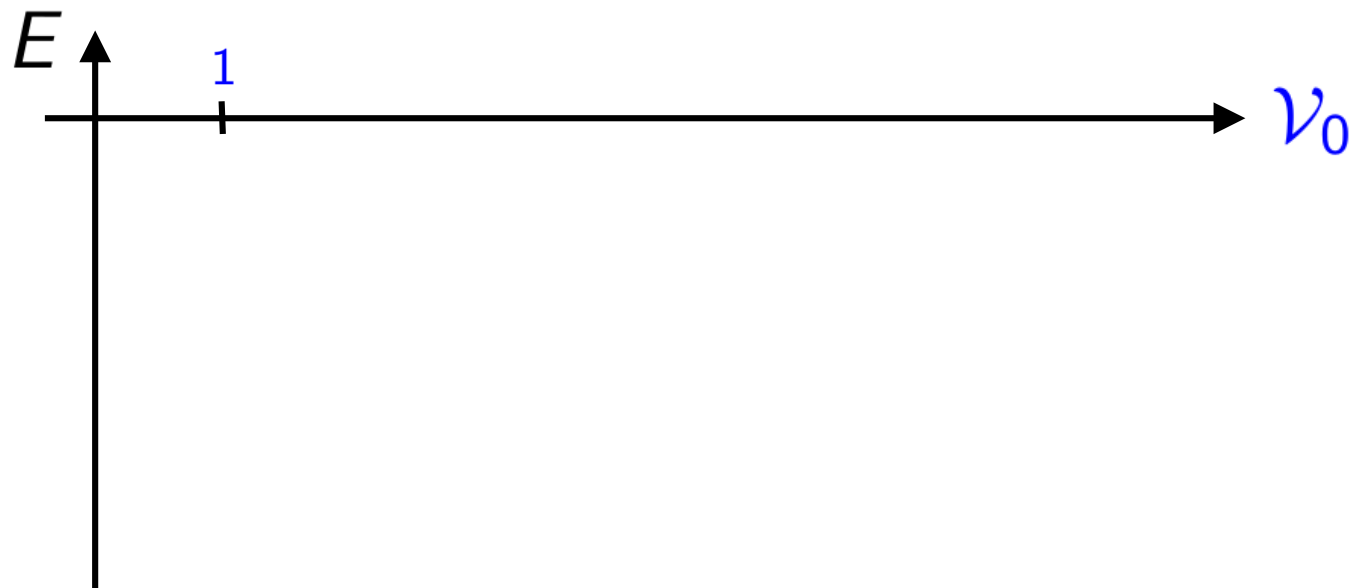


Dimers (= bound states):  $E < 0$   $\psi(\vec{r})$  normalizable

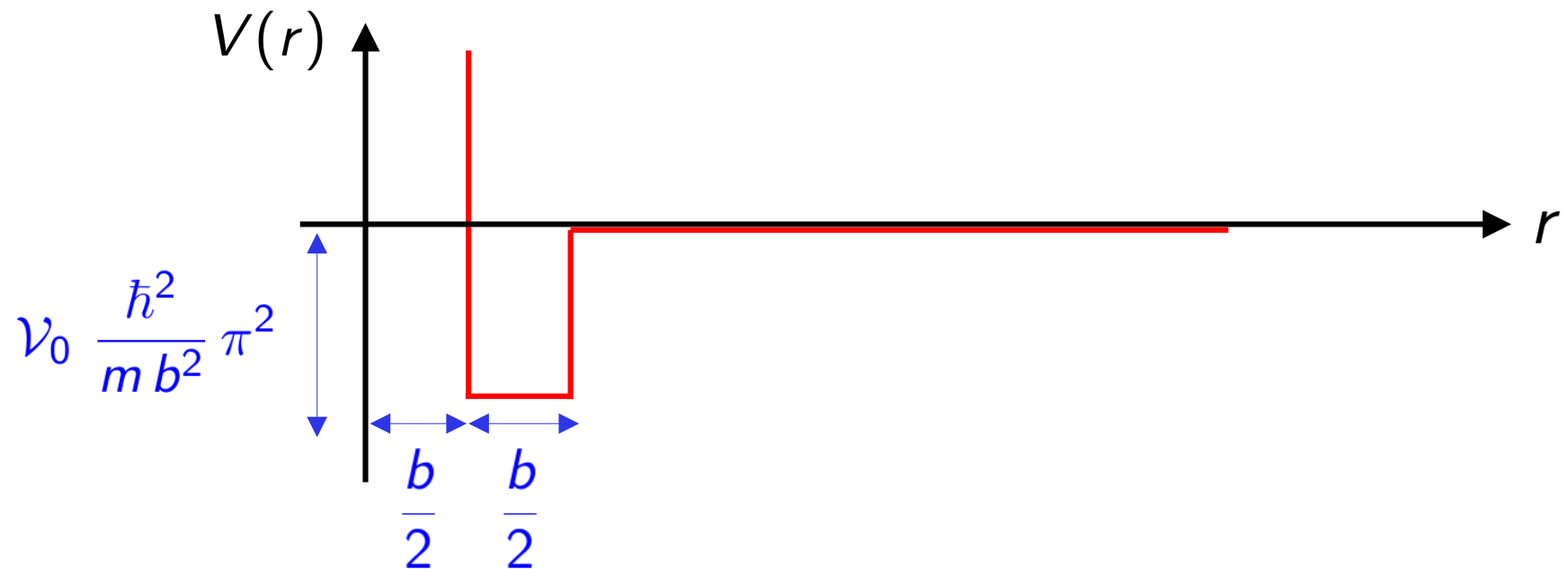
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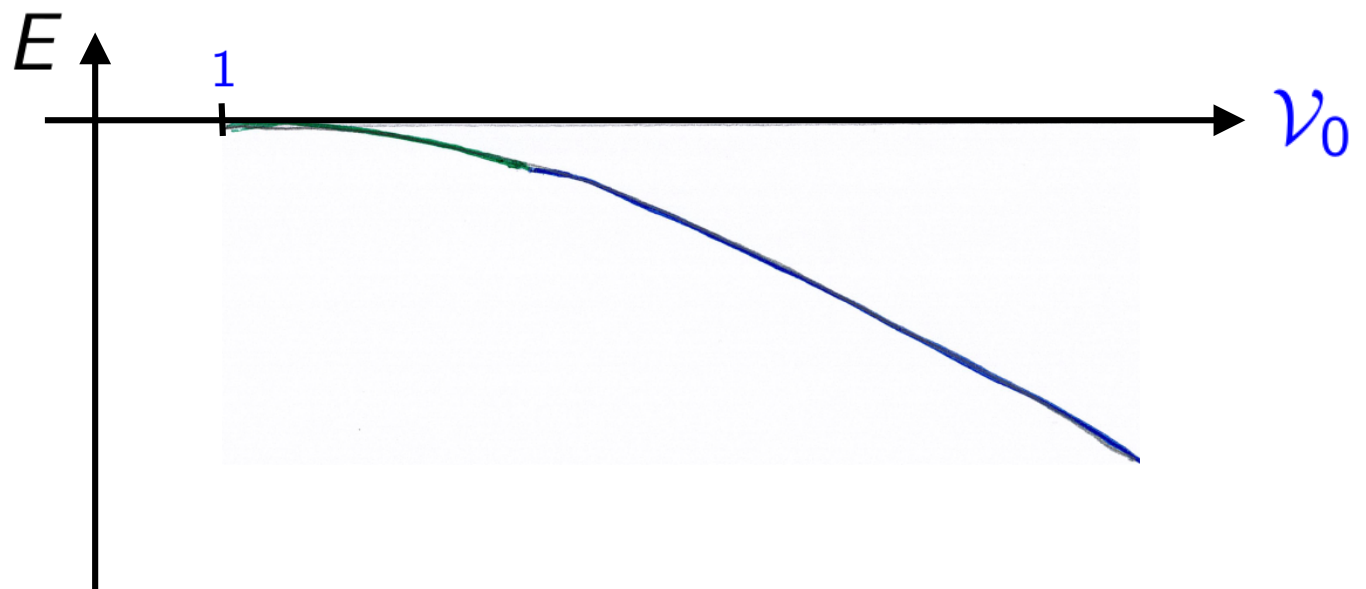
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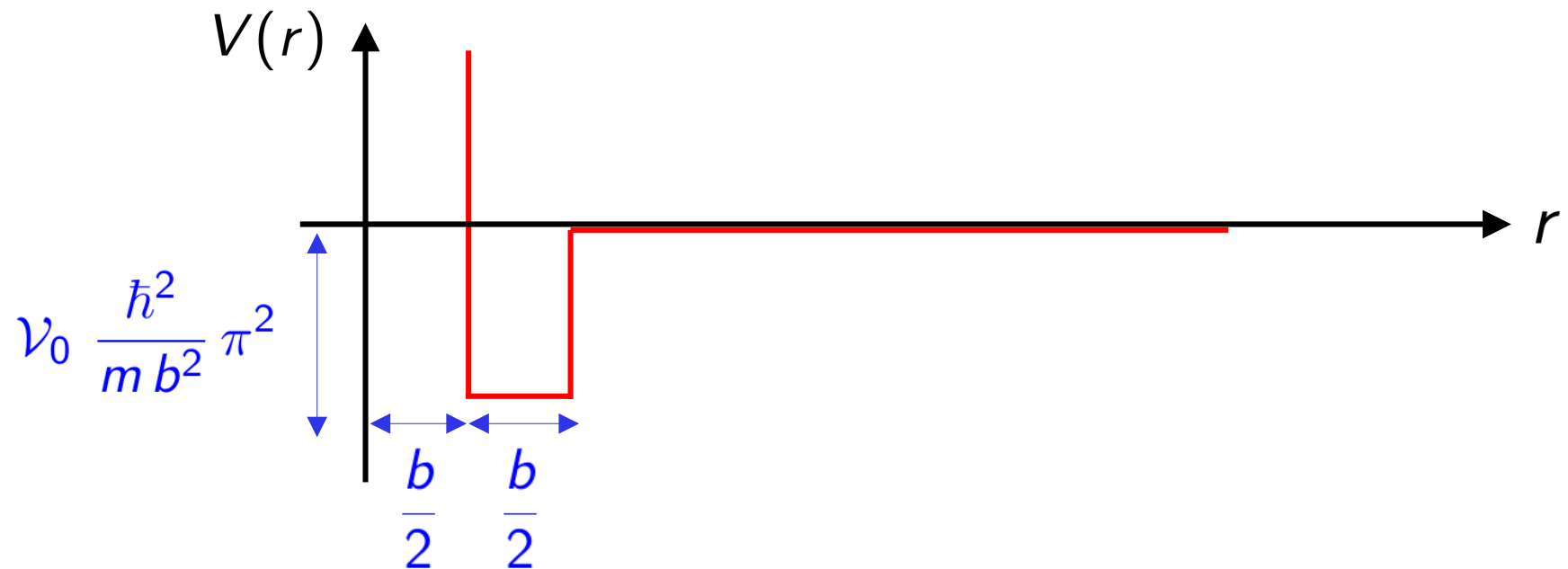
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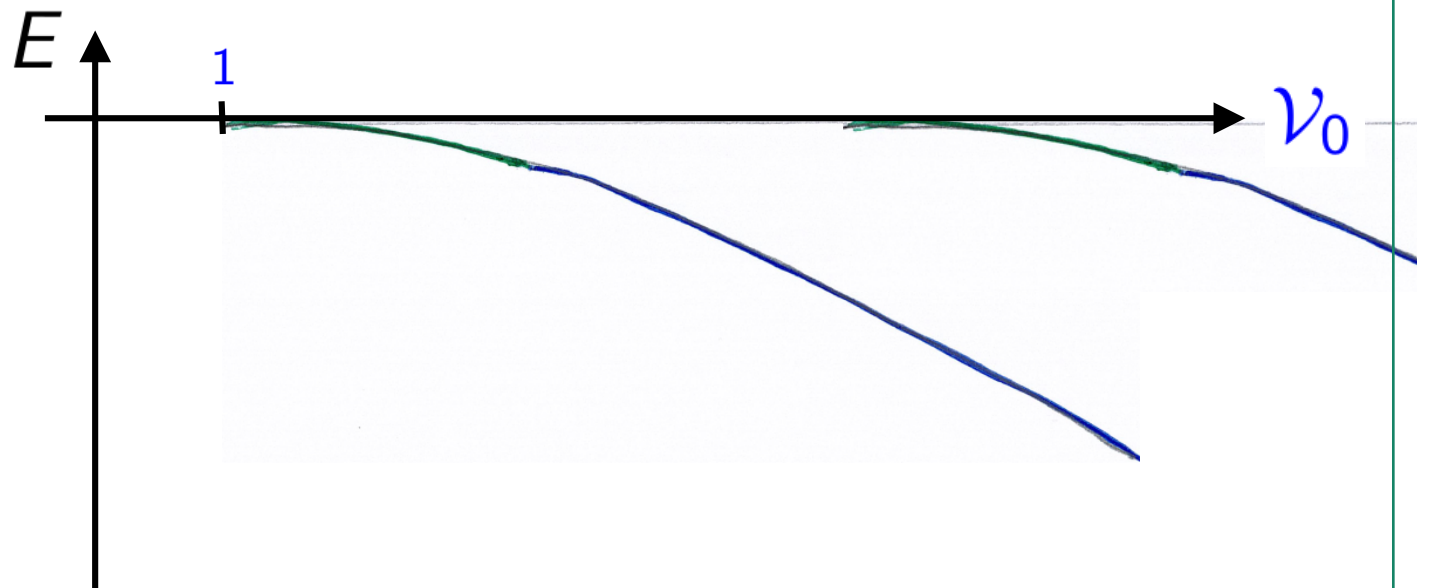
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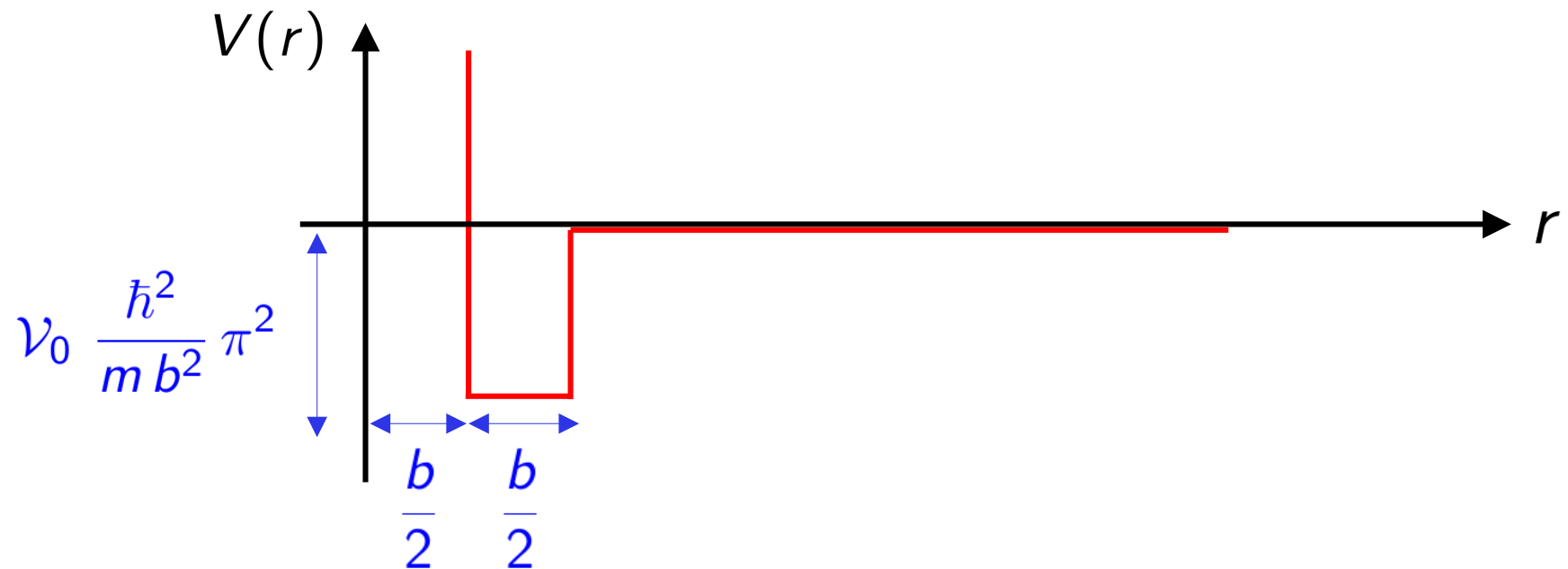
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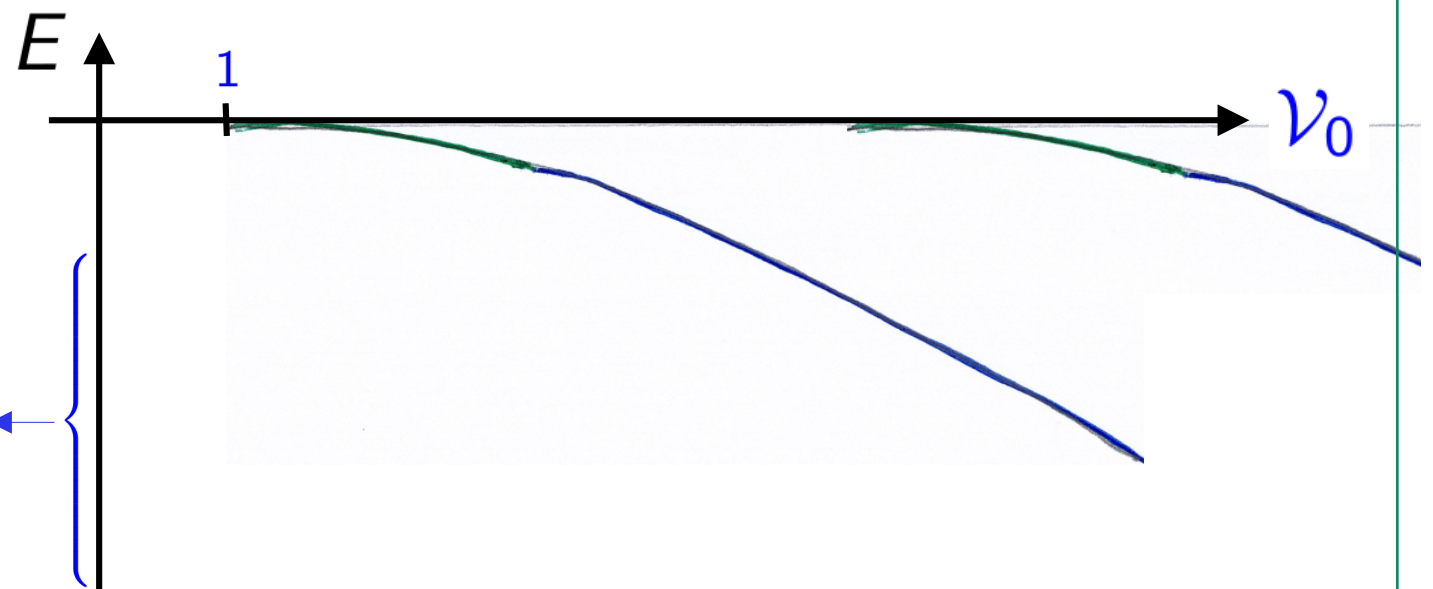
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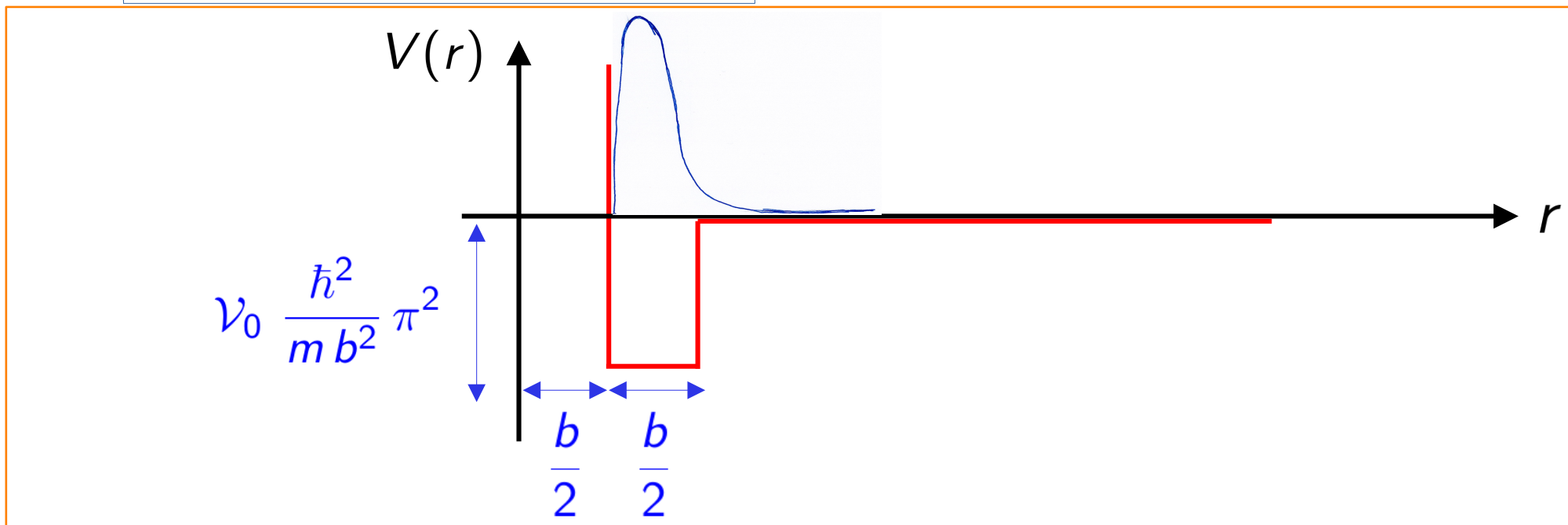


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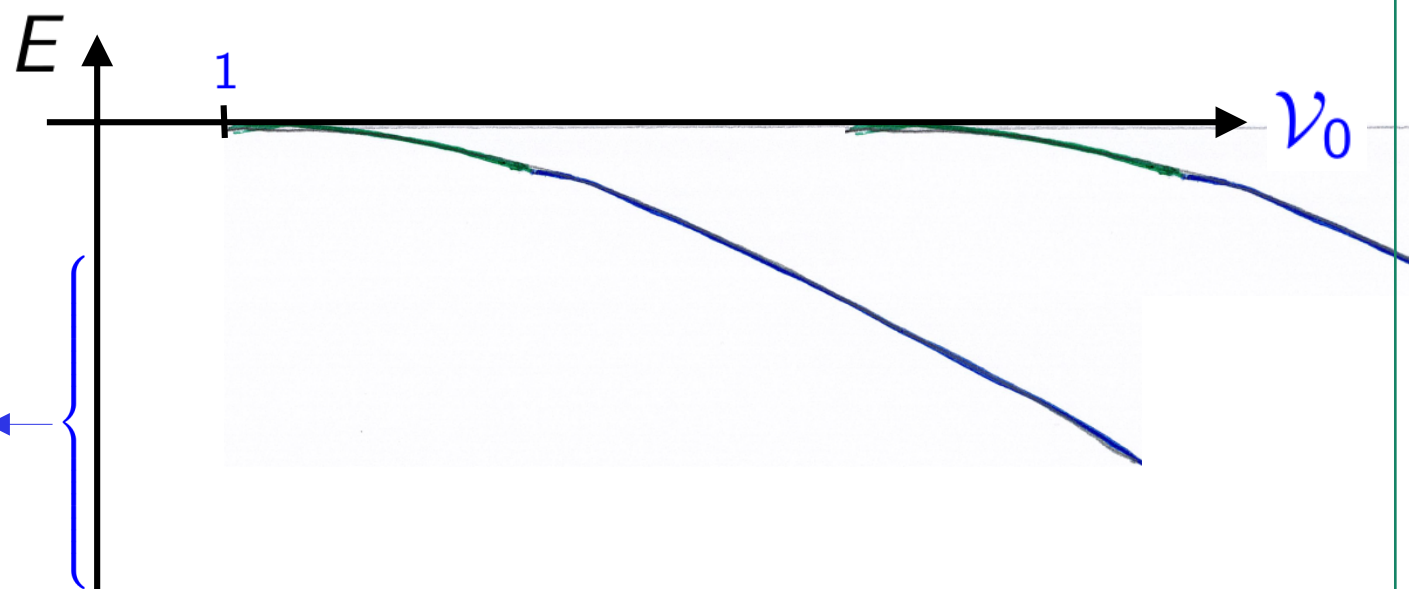


deeply bound:  $|E| \sim \frac{\hbar^2}{m b^2}$   
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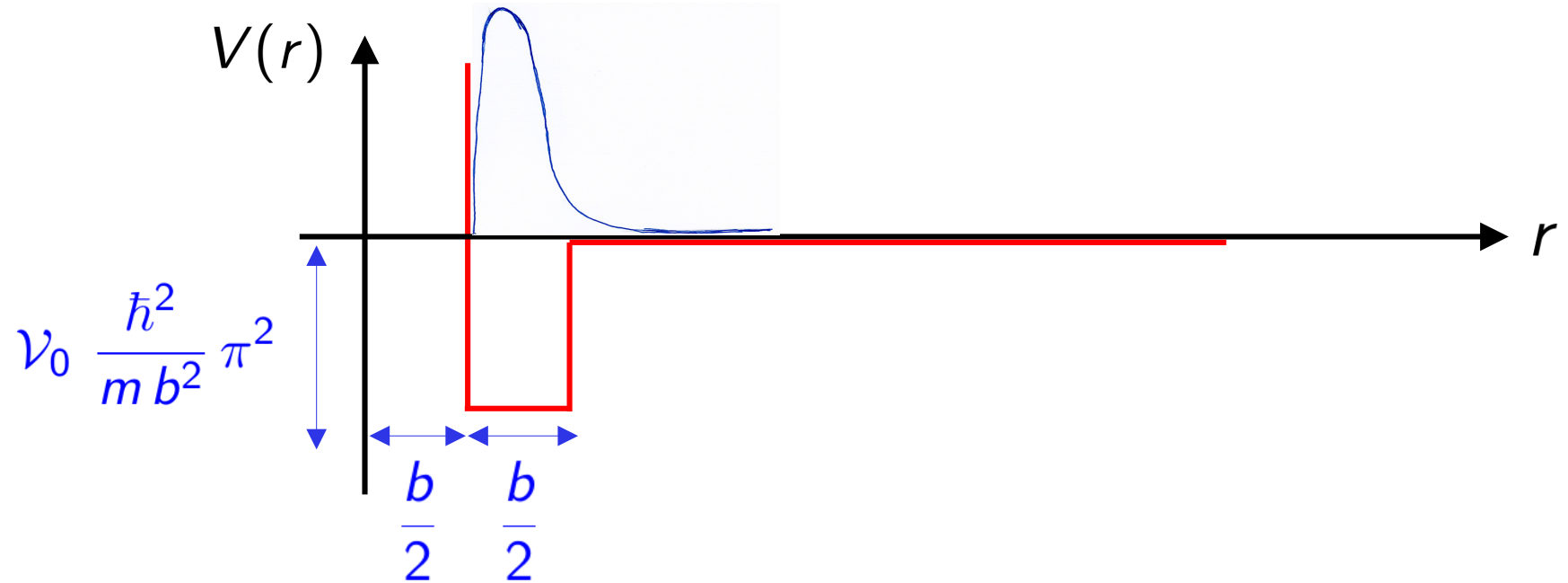


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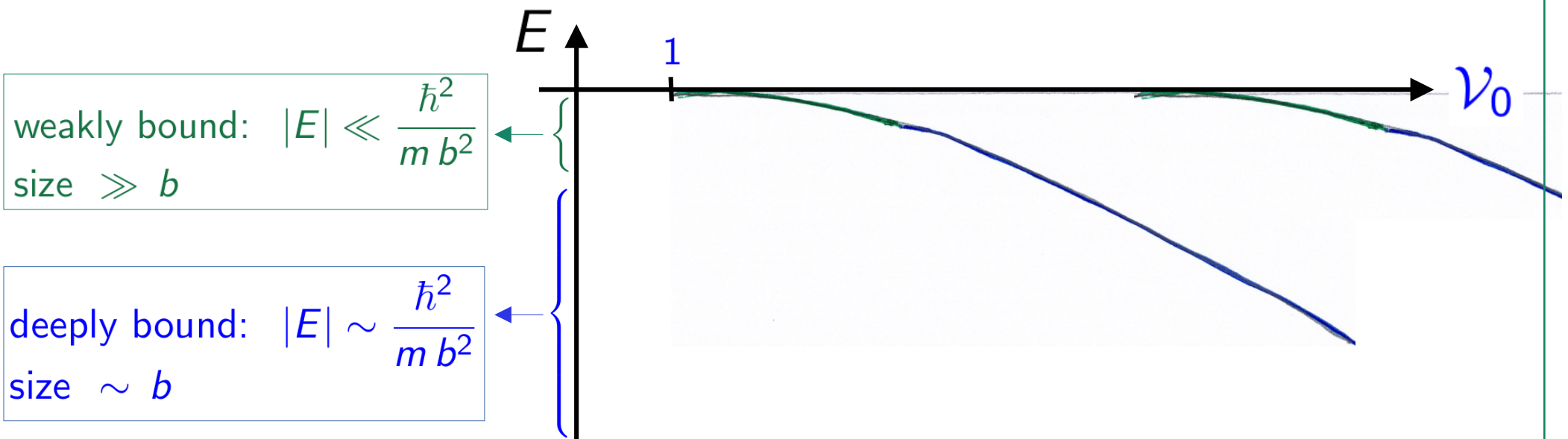


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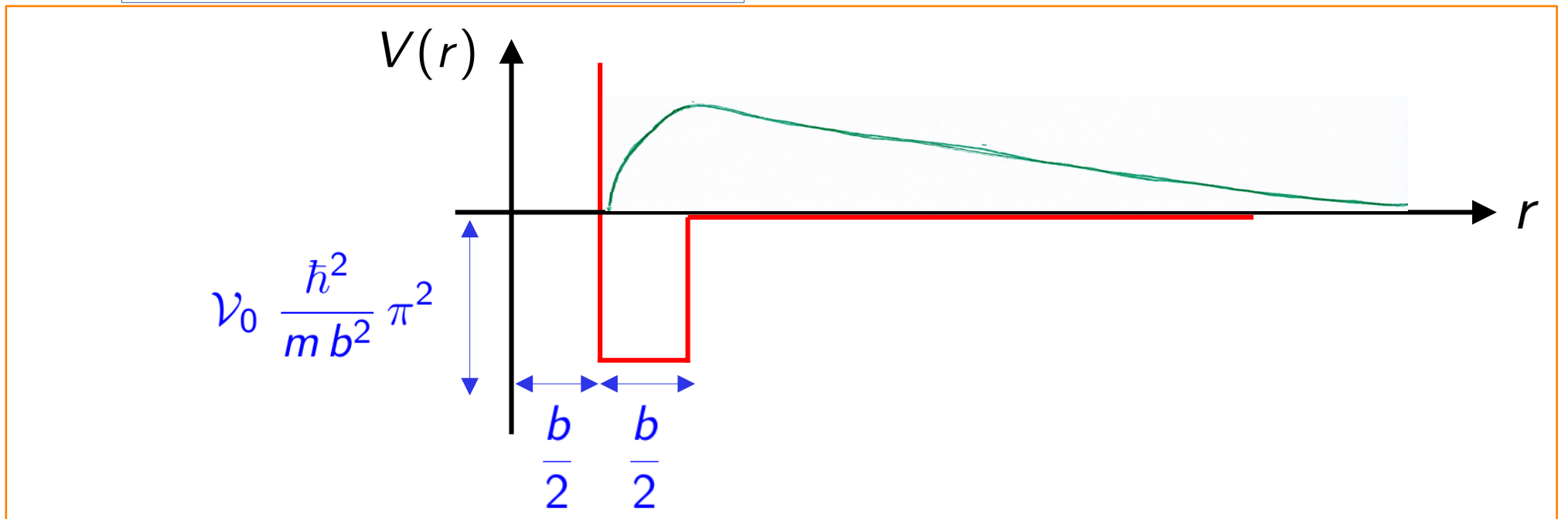
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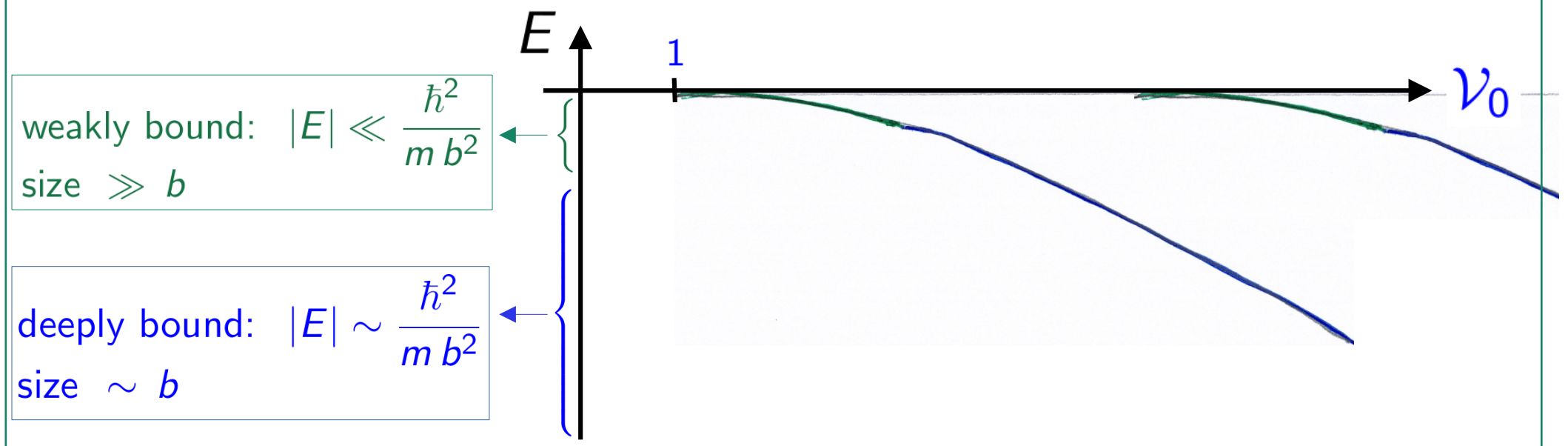
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Scattering states:  $E > 0.$   $E = \frac{\hbar^2 k^2}{m}$

$$\psi(\vec{r}) \underset{r \rightarrow \infty}{\simeq} e^{i \vec{k} \cdot \vec{r}} + f_k(\hat{r}) \frac{e^{ikr}}{r} \quad \left[ \hat{r} := \frac{\vec{r}}{r} \right]$$

$a := - \lim_{k \rightarrow 0} f_k(\hat{r})$  scattering length

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Universality: When  $\mathcal{V}_0 \rightarrow 1^+$ :  $\frac{a}{b} \rightarrow +\infty$  and

$$E \sim - \frac{\hbar^2}{ma^2}$$

$$\psi(r) \rightarrow \mathcal{N} \frac{e^{-r/a}}{r}$$

for any shape of  $V(r)$

(short-ranged)

$$\left[ f \sim g \text{ means } \frac{f}{g} \rightarrow 1 \right]$$

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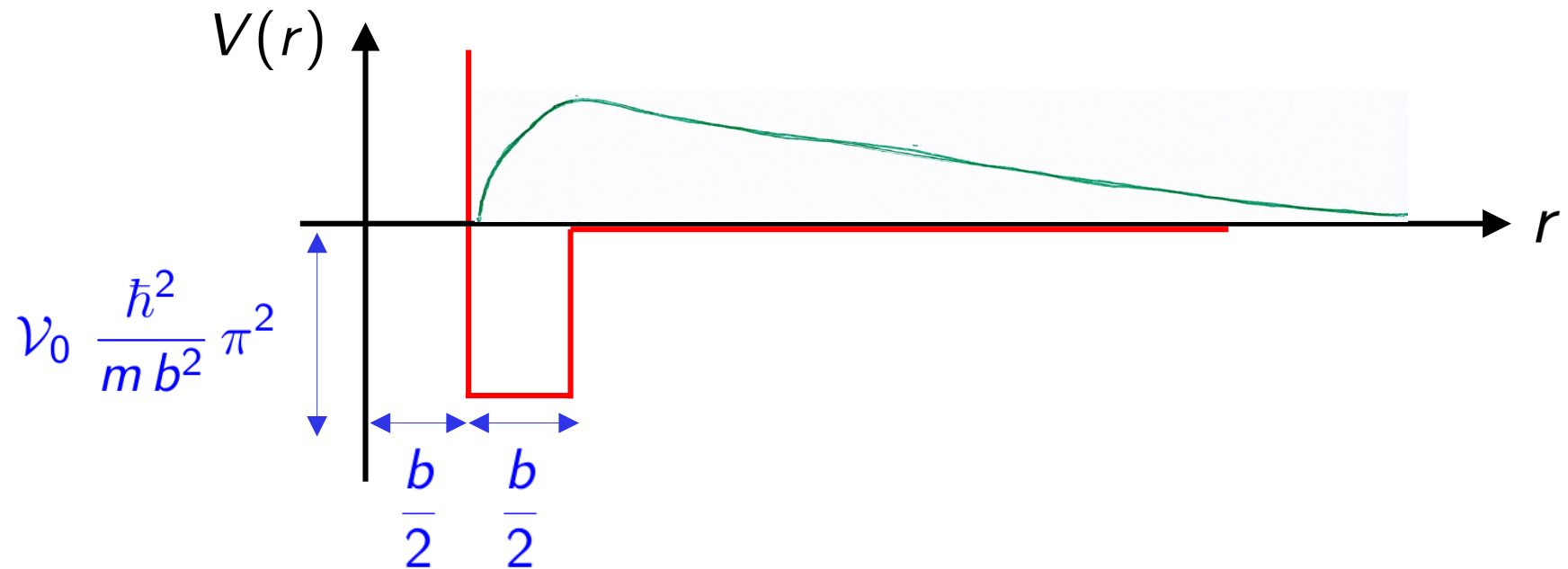
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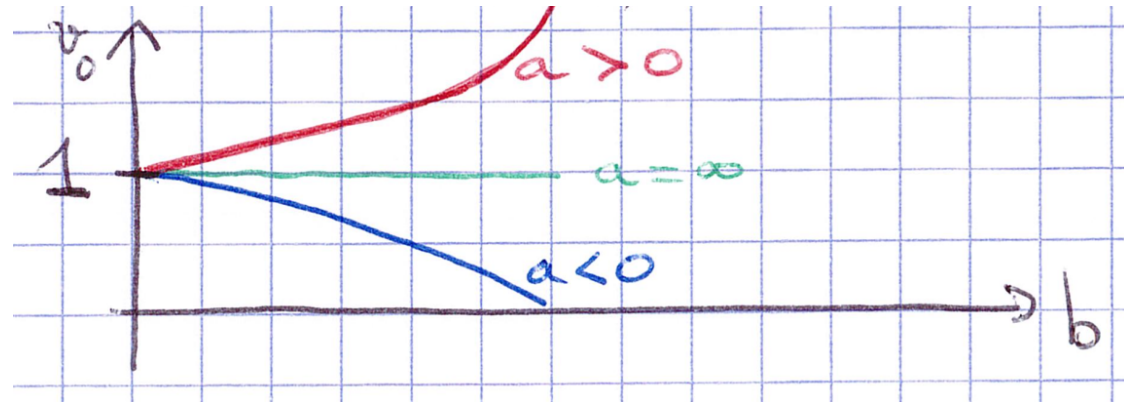
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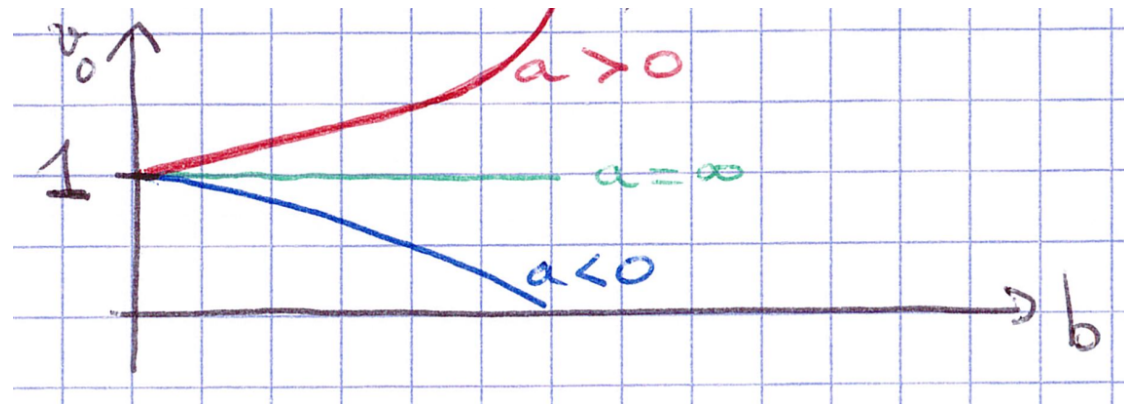
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$$\mathcal{V}_0 \rightarrow 1$$



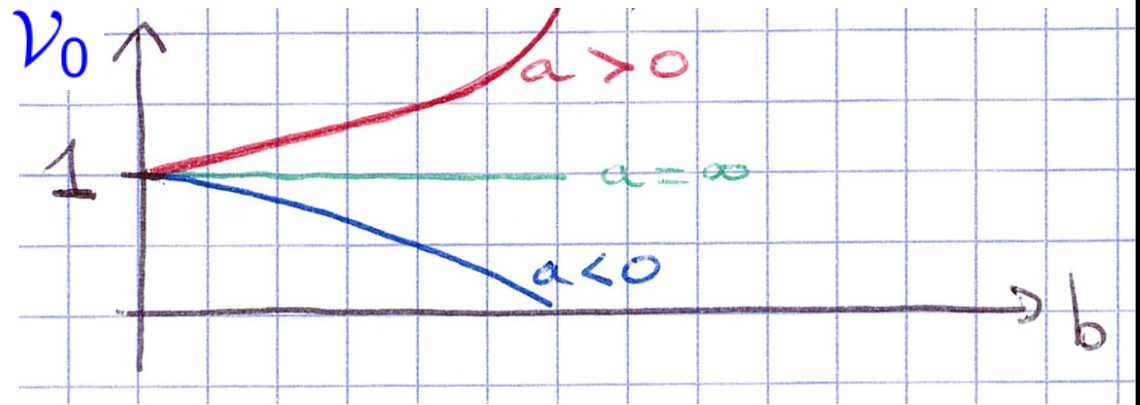
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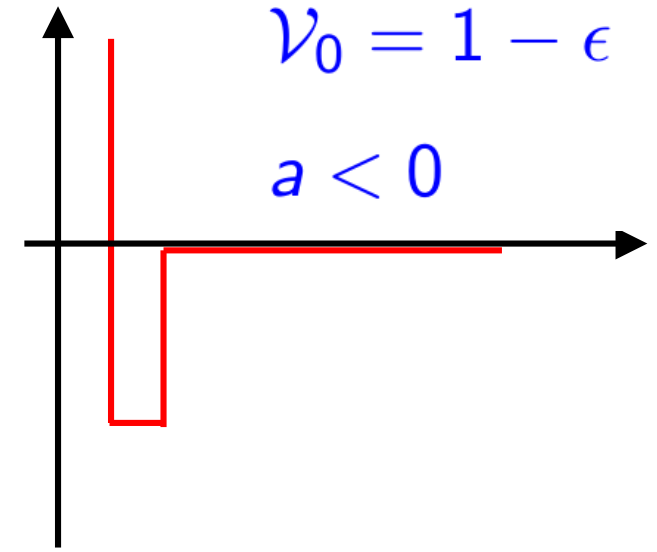
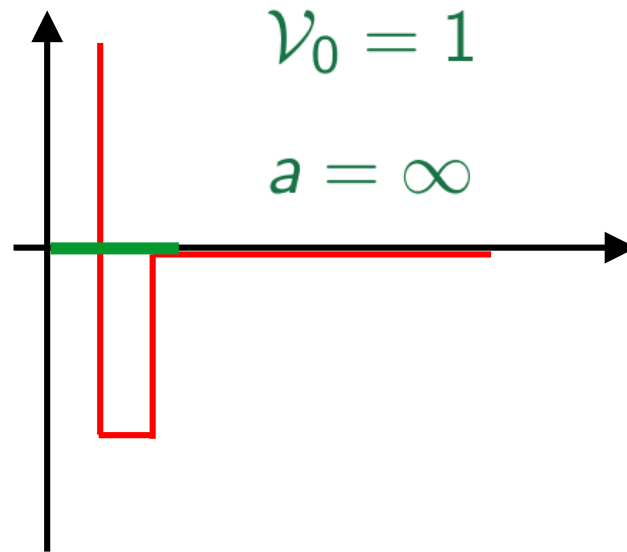
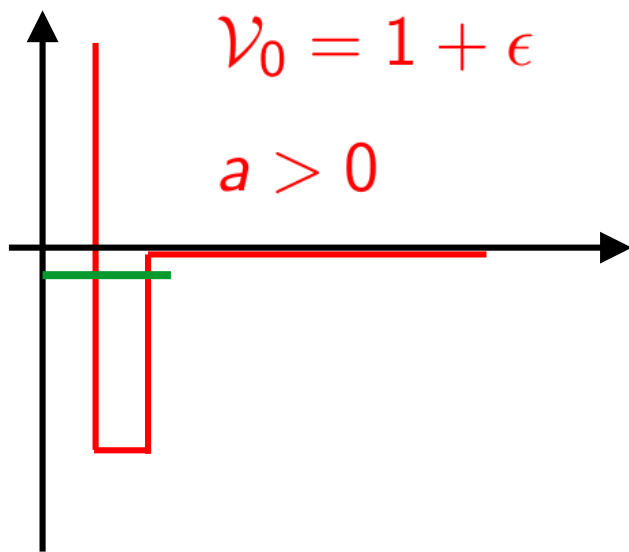
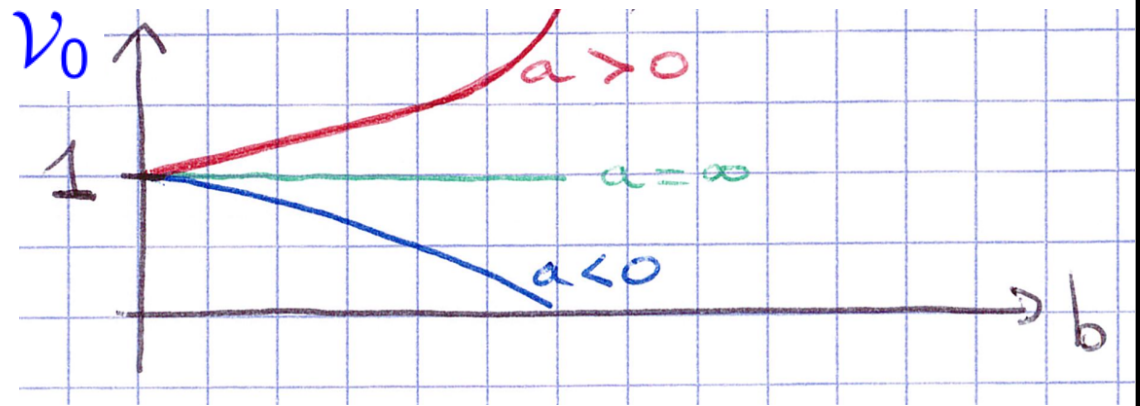
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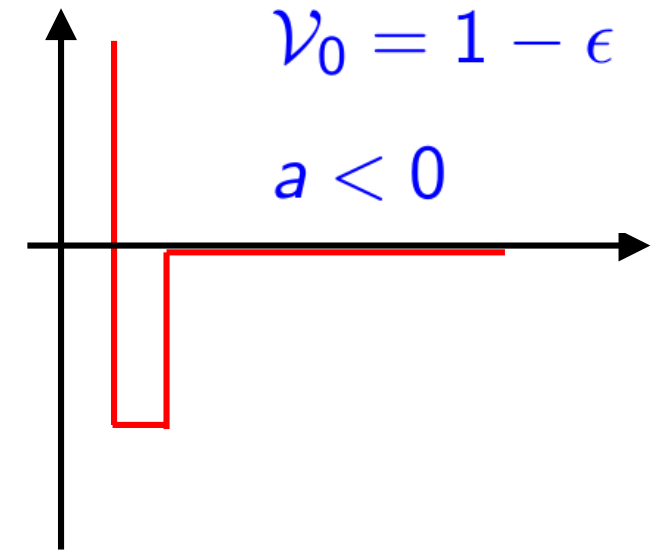
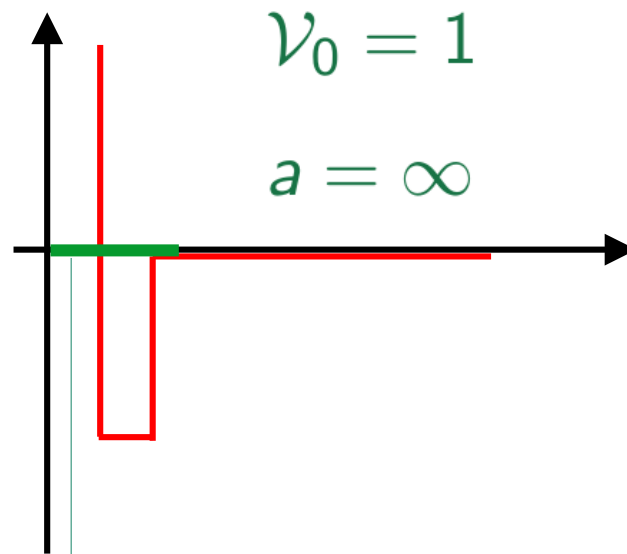
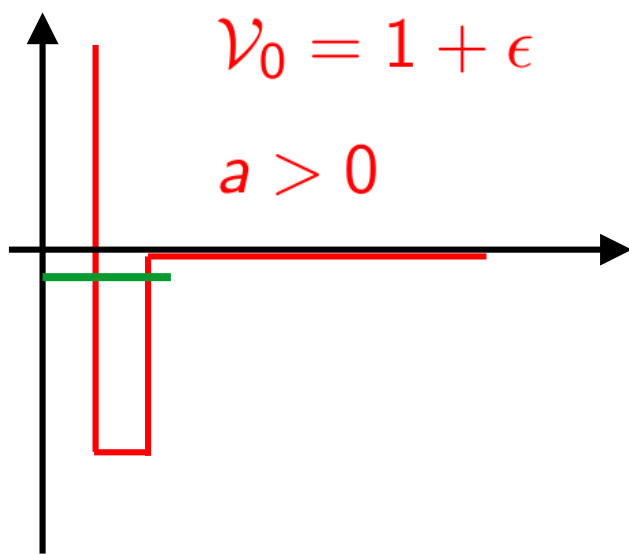
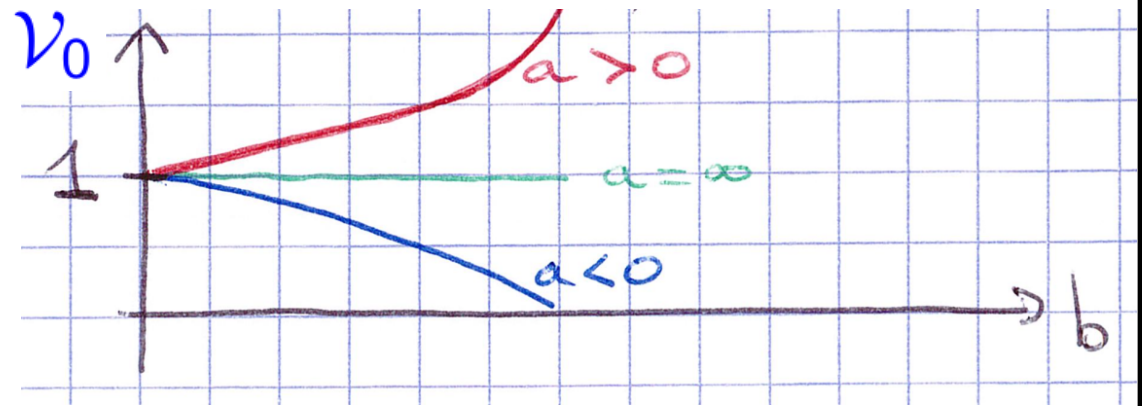
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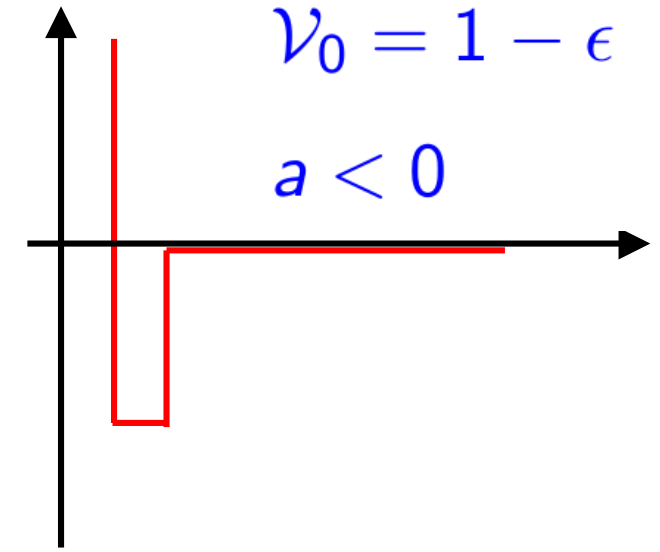
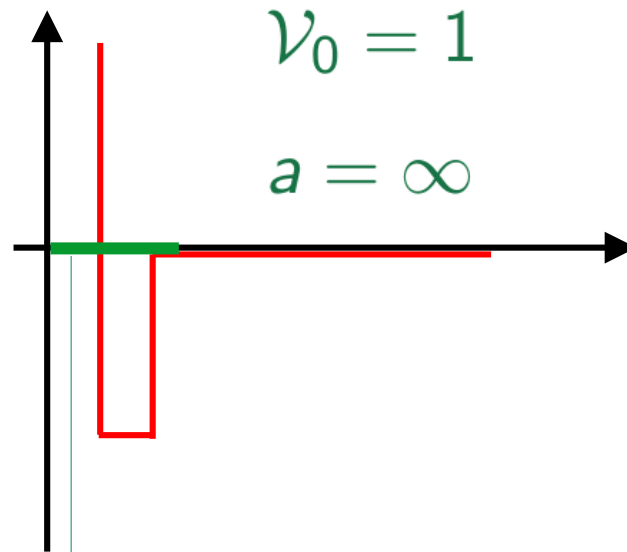
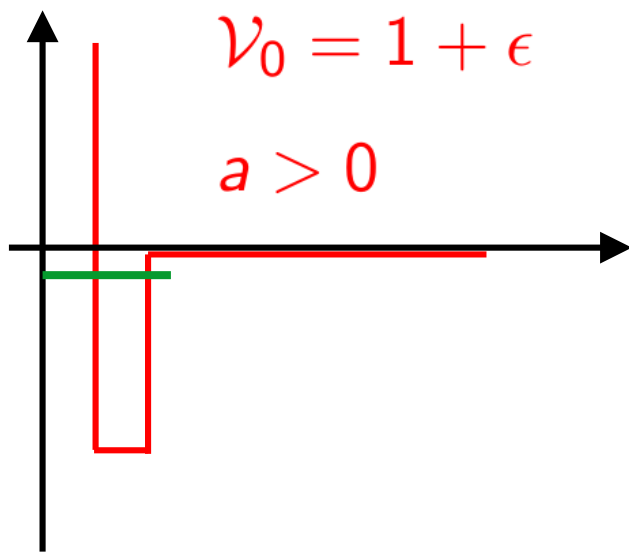
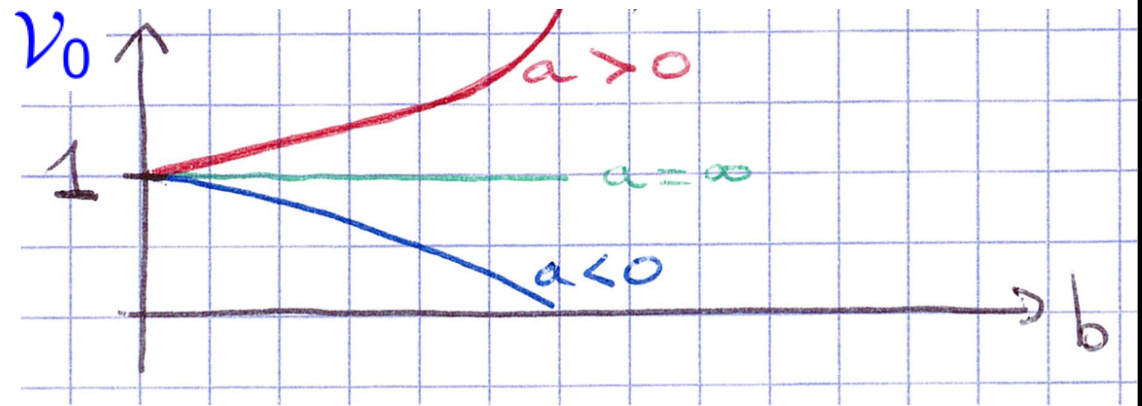


Zero-energy state  $\phi_0(r)$

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$a = \infty$ : scattering cross-section

$$\sigma_k = \frac{4\pi}{k^2}$$

“unitary limit”

(max value allowed  
by optical theorem)

# Part 3 : N-body problem

Part 3 : N-body problem

**Zero-range limit & ZRM**

## Part 3 : N-body problem

Take  $N=4$ ,  $N_{\uparrow} = N_{\downarrow} = 2$   
(just to alleviate notations)

---

## Zero-range limit & ZRM

$$\Psi(\underbrace{\vec{r}_1, \vec{r}_2}_{\text{antisym.}}, \underbrace{\vec{r}_3, \vec{r}_4}_{\text{antisym.}})$$

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antisym.      antisym.

Finite-Range Model:  
(FRM)

$$-\frac{\hbar^2}{2m} \sum_{i=1}^4 \Delta_{\vec{r}_i} \Psi + \sum_{i=1}^4 U(\vec{r}_i) \Psi + \sum_{\substack{i \leq 2 \\ j \geq 3}} V(r_{ij}) \Psi = E \Psi$$

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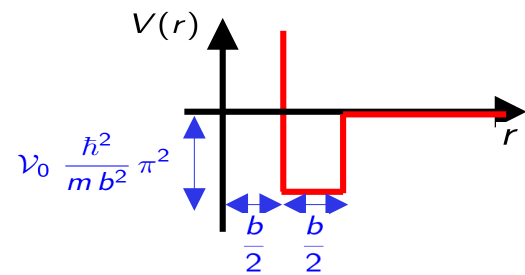
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# Part 3 : N-body problem

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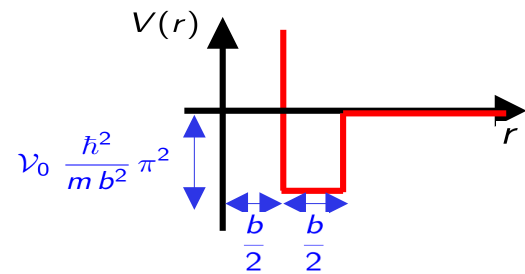
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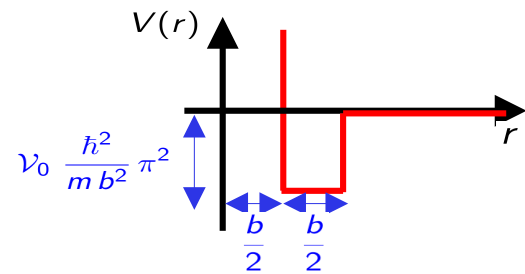
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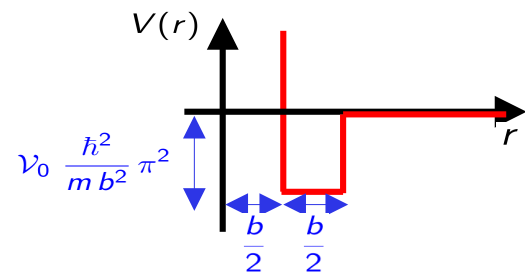
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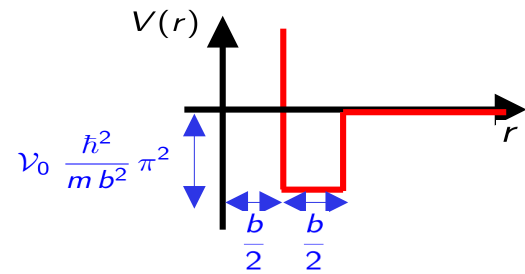
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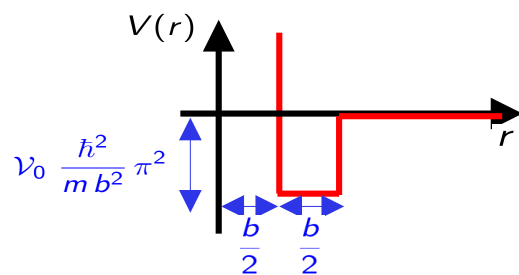
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finite for  $b \rightarrow 0$

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must compensate

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$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4) = \left( \frac{1}{r} - \frac{1}{a} \right) A(\vec{c}; \vec{r}_2, \vec{r}_4) + O(r)$$

$\vec{c} := \frac{\vec{r}_1 + \vec{r}_3}{2}$  fixed,  $(\vec{r}_2, \vec{r}_4)$  fixed

Justifications: Consider FRM for  $r \equiv r_{13} \sim b \ll r_{14}, r_{23}$  :

$$-\frac{\hbar^2}{m} \Delta_{\vec{r}} \psi + V(r) \psi - \frac{\hbar^2}{4m} \Delta_{\vec{c}} \psi - \frac{\hbar^2}{2m} (\Delta_{\vec{r}_2} + \Delta_{\vec{r}_4}) \psi + \underbrace{[V(r_{14}) + V(r_{23})]}_{\text{finite for } b \rightarrow 0} \psi + \sum_{i=1}^4 U(\vec{r}_i) \psi = E \psi$$

$\sim \frac{\hbar^2}{mb^2} \psi$        $\sim -\frac{\hbar^2}{mb^2} \psi$

must compensate

$$\Rightarrow -\frac{\hbar^2}{m} \Delta_{\vec{r}} \psi + V(r) \psi \approx 0 \Rightarrow \psi(\vec{r}_1, \dots, \vec{r}_4) \underset{r \ll b}{\approx} \underbrace{\phi_0(r)}_{\substack{\approx \\ r > b}} \times A$$

$$\approx \frac{1}{r} - \frac{1}{a}$$

$\Rightarrow$  ZRM

## Homogeneous gas

$$N_{\uparrow} = N_{\downarrow} \quad ( \text{“unpolarized” / “balanced”} )$$

$$\text{Volume } \mathcal{V}. \quad \text{Density } n = \frac{N}{\mathcal{V}}.$$

$$\mathcal{V} \rightarrow \infty, \quad N \rightarrow \infty \\ n \text{ fixed}$$

$$\text{interparticle distance } \sim d = n^{-1/3}$$

# Homogeneous gas

Zero-range limit:

$$b \ll d$$

$$b \ll \lambda_T \sim \frac{\hbar}{\sqrt{m k_B T}}$$

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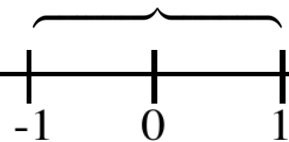
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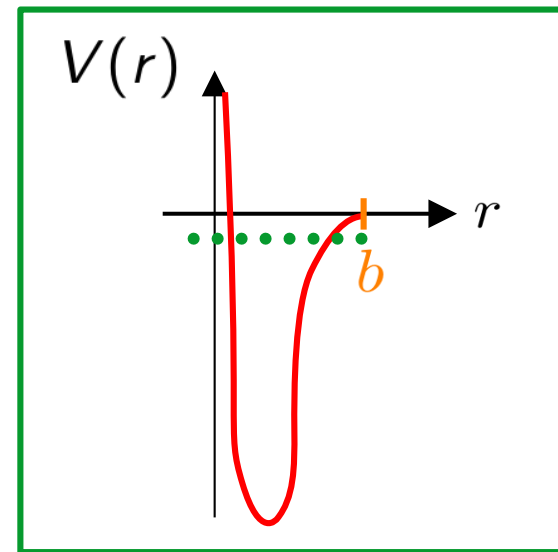
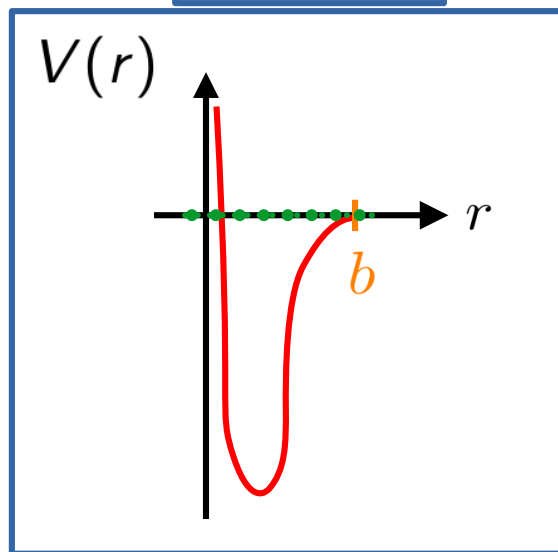
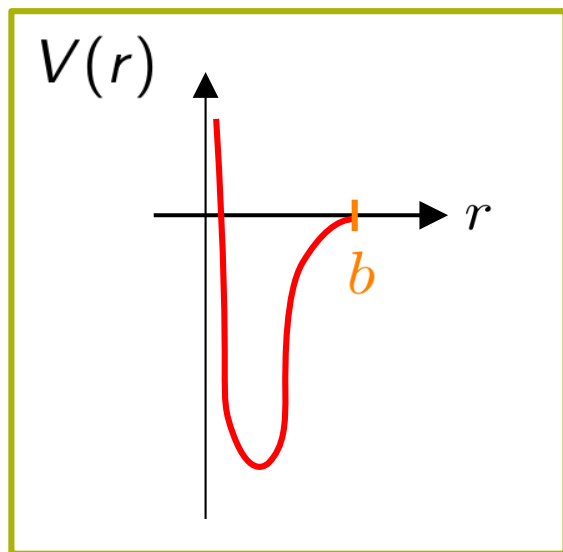
**BCS**

**strongly  
correlated  
regime**

**BEC of molecules**



unitary gas  
 $a = \infty$



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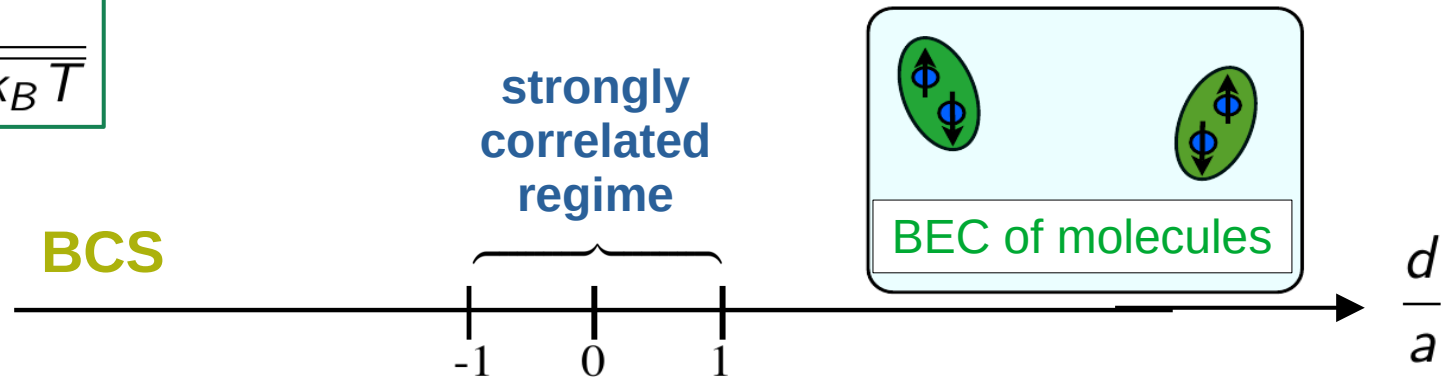
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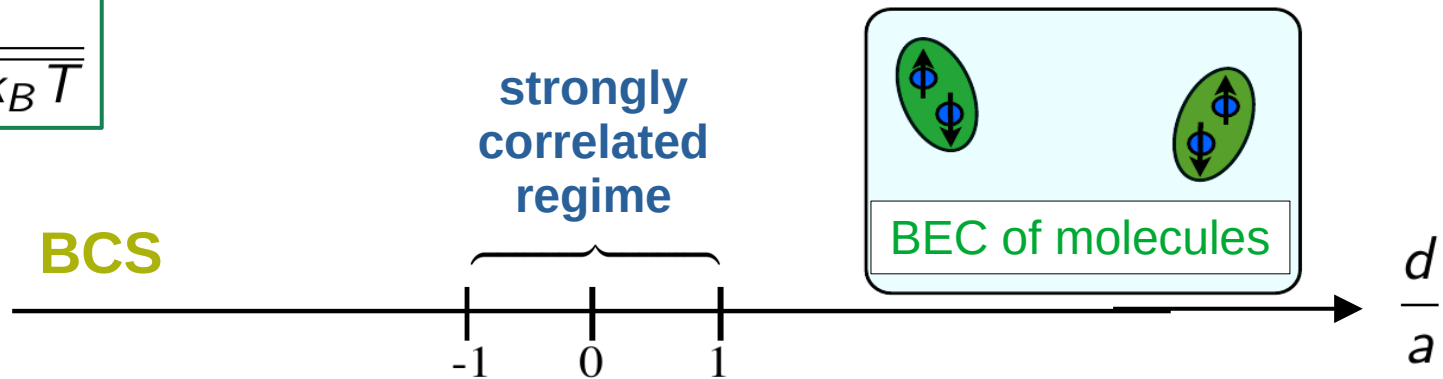
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$$T=0$$

Ideal Fermi gas

$$n_{\vec{k}, \uparrow} = n_{\vec{k}, \downarrow} = \begin{cases} 1, & k < k_F \\ 0, & k > k_F \end{cases}$$

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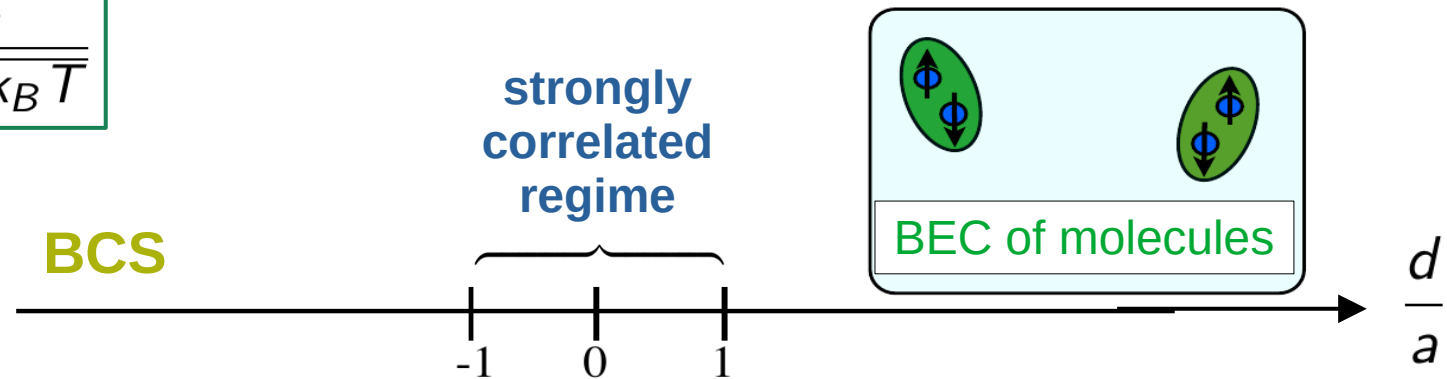
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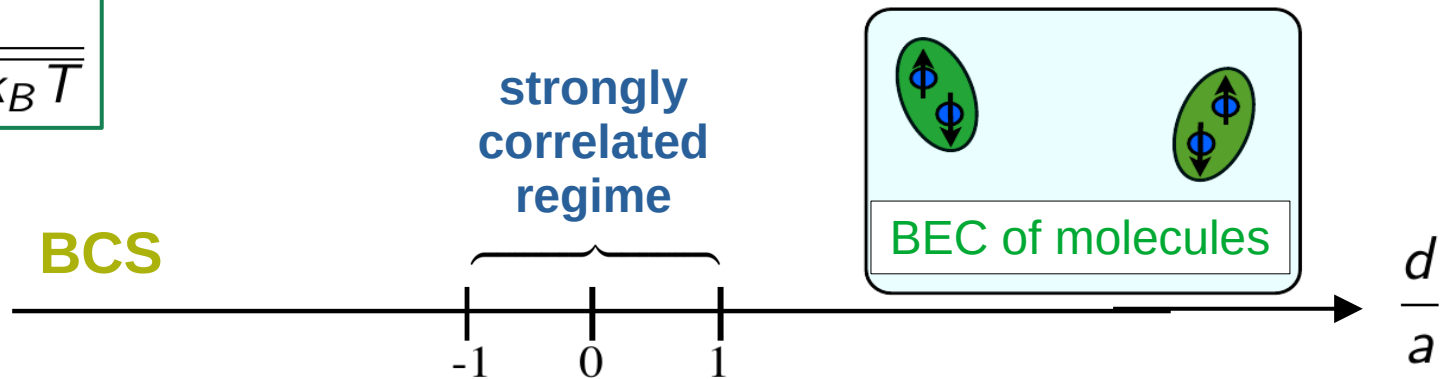
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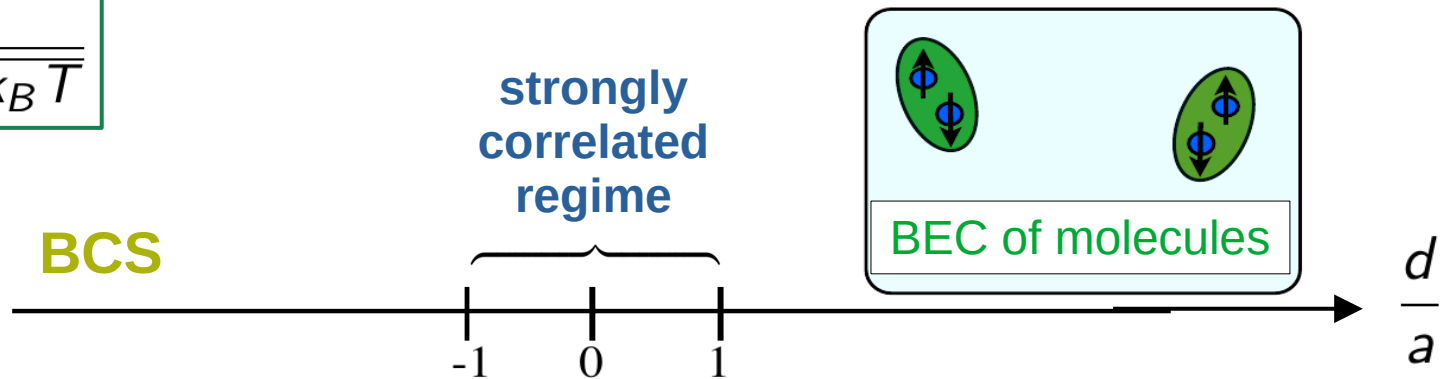
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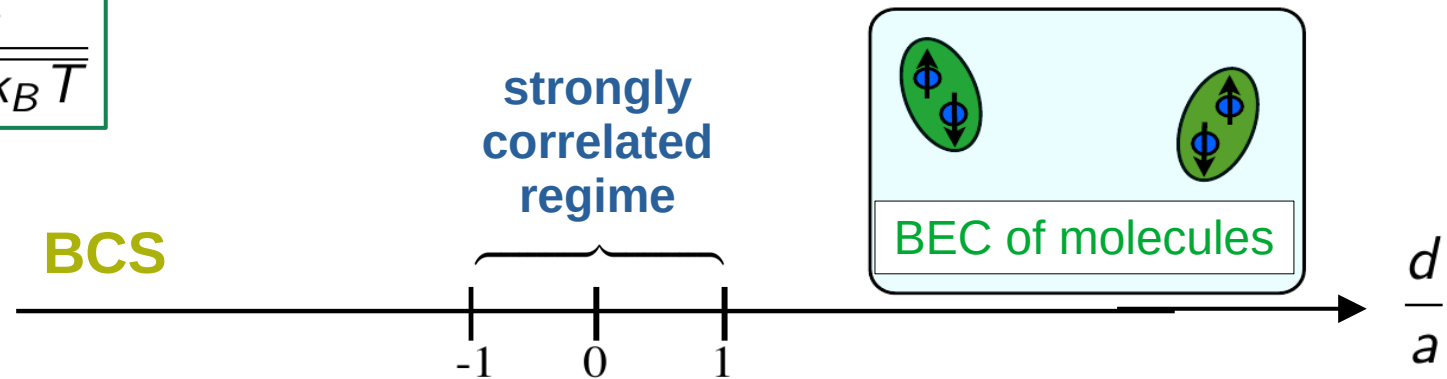
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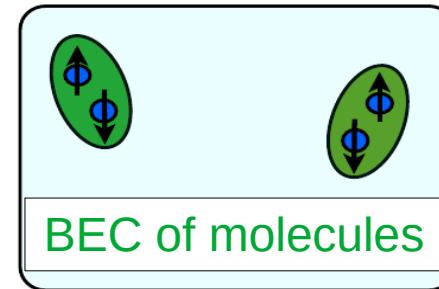
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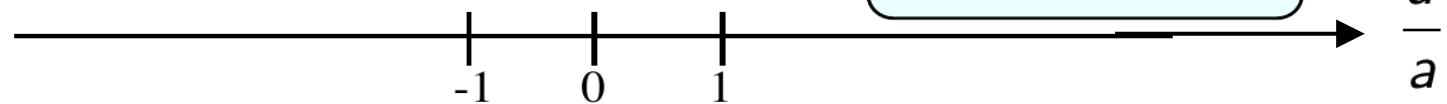
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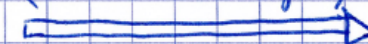
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(dimensional analysis)



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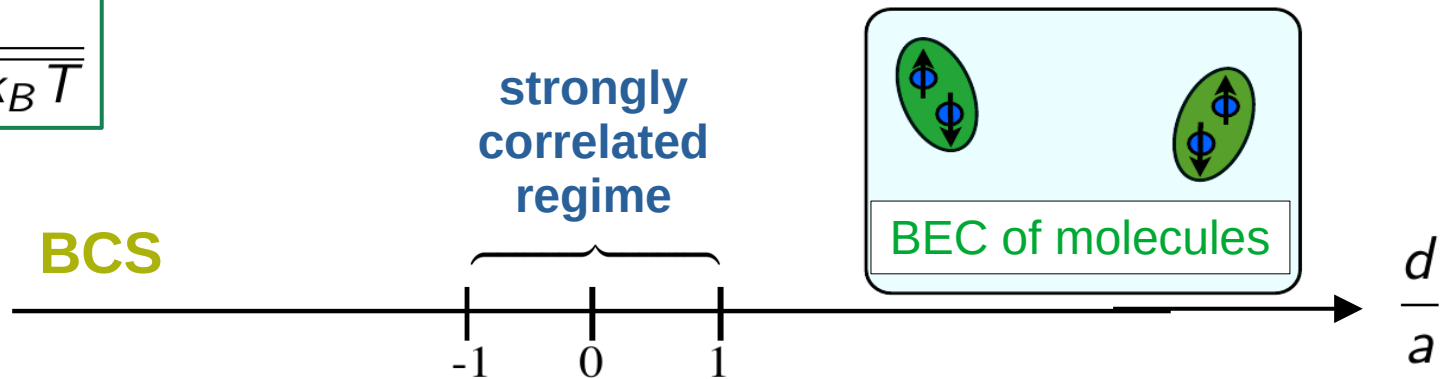
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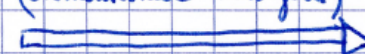
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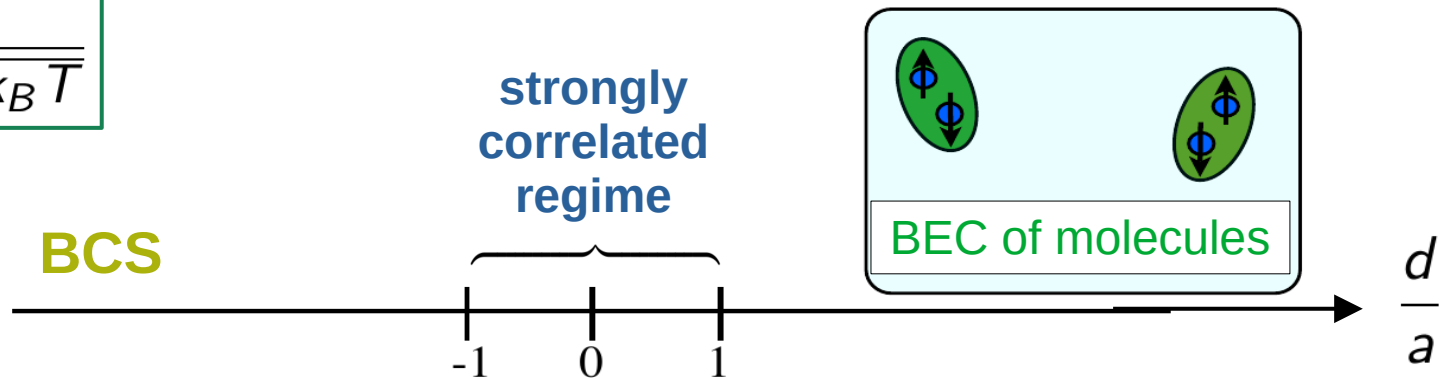
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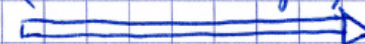
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For typical collisions in the gas:  $k \approx k_F \Rightarrow \sigma_k \approx \frac{4\pi}{k_F^2} \approx 1.3 d^2$

$\sigma_k \sim d^2$  like in a liquid