

Many-body open quantum systems: Dynamics and Mixing

Rahul Trivedi



MAX-PLANCK-INSTITUT
FÜR QUANTENOPTIK

Lecture outline

Monday (23.02.2026):

Dynamics:

- Basics
- Lattice-models, light cones and Lieb-Robinson bounds.
- Entanglement/separability.

Tuesday (24.02.2026):

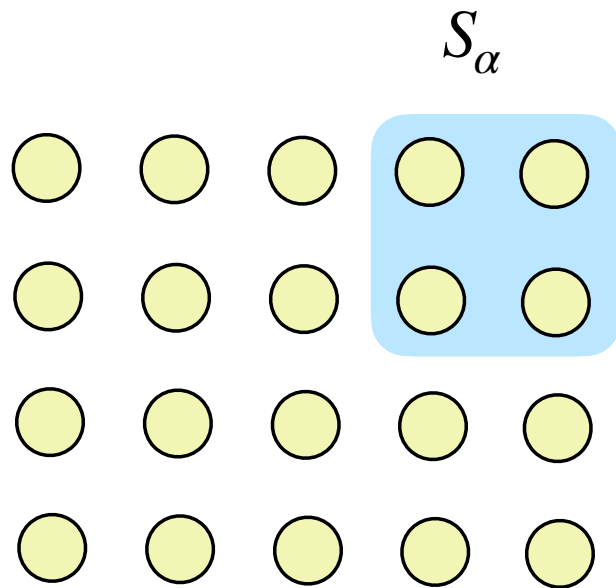
Steady states:

- Spectral Properties, Irreducibility.
- Mixing times:
 - Spectral gaps.
 - Modified Log-Sobolev Inequality.
- Rapid mixing and consequences.

Reference:

M. Wolf, Quantum Channels and operations: A guided Tour

Recap



$$\mathcal{L} = \sum_{\alpha} \mathcal{L}_{\alpha}$$

Dynamics of open systems

Describing operator spreading

Lieb-Robinson bounds.

Entanglement suppression

Percolation mappings.

Entanglement-breaking transitions.

Today: Steady states and mixing times.

Motivation: Important to understanding dissipative phases.

Recap: Classical probabilistic dynamics

Quantum Systems

State-space: \mathbb{C}^D

State: $\rho(t) = [\rho_{i,j}(t)]_{i,j \in \{1,2 \dots D\}}$

Dynamics: $\dot{\rho}(t) = \mathcal{L}\rho(t)$,
where \mathcal{L} is of Lindblad form.

Classical Systems

State-space: $\{1,2 \dots D\}$

State: $p(t) = [p_1(t), p_2(t) \dots p_D(t)]$

Dynamics: $\dot{p}(t) = Ap(t)$,
where $A_{i,j}$ = Rate of state $j \rightarrow i$,
and $A_{i,i} = -\sum_{j \neq i} A_{j,i}$

- **Classical dynamics as a special case of quantum master equation:**

$$\dot{\rho}(t) = \sum_{i \neq j} \mathcal{D}_{L_{i,j}} \rho(t) \text{ where } L_{i,j} = A_{i,j}^{1/2} |i\rangle\langle j|$$

More explicitly: $\dot{\rho}_{i,i}(t) = \sum_j A_{i,j} \rho_{j,j}(t)$ and $\dot{\rho}_{i,j}(t) = \frac{1}{2}(A_{i,i} + A_{j,j})\rho_{i,j}(t)$

i.e., *On-diagonal* elements follow classical Markov evolution, and *off-diagonal* elements decay exponentially.

Spectrum and steady-states

Lindbladian \mathcal{L} is generally a non-Hermitian operator.

- It always has at-least one zero eigenvalue: Correspond to steady-states/fixed points.

$$\mathcal{L}(\rho_\infty) = 0$$

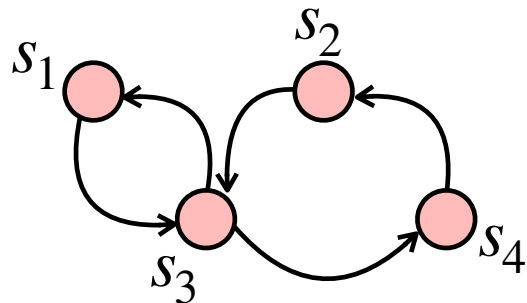
- All other eigenvalues: Either purely imaginary, or have a negative real part.

Our interest: Studying $e^{\mathcal{L}t}$ as $t \rightarrow \infty$.

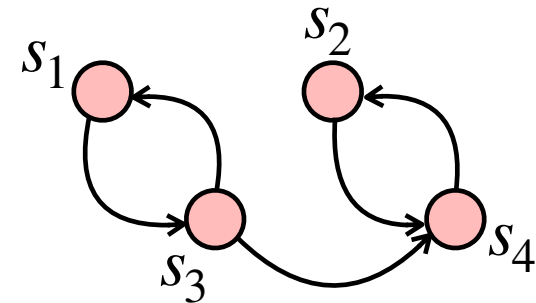
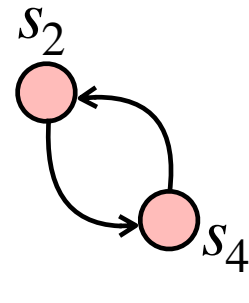
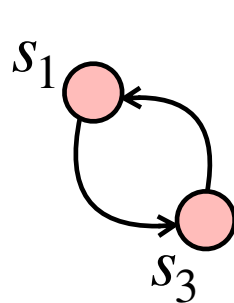
If \mathcal{L} has a unique steady-state ρ_∞ , then $\mathcal{E}_\infty(\rho) = \text{Tr}(\rho)\rho_\infty$

Irreducible Channels and Lindbladians

Classical probability: A Markov chain is irreducible if there is no proper subset of states such that the Markov chain never leaves that subset.



Irreducible Markov chain



Reducible Markov chains

Quantum channels: A channel $\mathcal{E} = \sum_i K_i(\cdot)K_i^\dagger$ is irreducible if there is no proper subspace $\mathcal{K} \subset \mathcal{H}$ such that

$$K_i \mathcal{K} \subseteq \mathcal{K} \quad \forall i$$

A Lindbladian \mathcal{L} is irreducible if $e^{\mathcal{L}t}$ is irreducible for some $t > 0$.

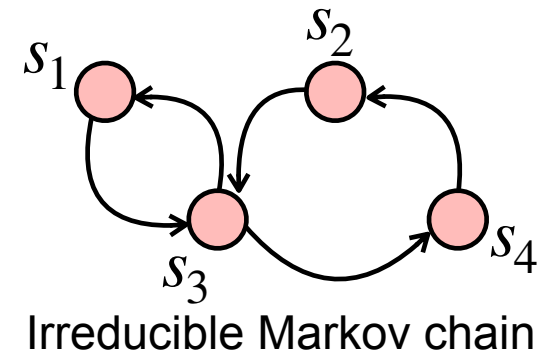
Checking irreducibility

Classical processes: A Markov process with rate matrix A is irreducible if for every two states (s_i, s_j) , there is a path $s_{k_1}, s_{k_2} \dots s_{k_n}$ connecting s_i and s_j with positive rates.

$$\text{i.e. } A_{s_i, s_{k_1}}, A_{s_{k_1}, s_{k_2}} \dots A_{s_{k_n}, s_j} > 0$$

Quantum processes: A Lindbladian with Hamiltonian H and jump operators $L_1, L_2 \dots L_n$ is irreducible if

$$\text{Alg} \left(iH + \frac{1}{2} \sum_{i=1}^n L_i^\dagger L_i, L_1, L_2 \dots L_n \right) = \text{All operators on } \mathcal{H}$$



Examples

Depolarizing:

$$\mathcal{L}(\rho) = \text{tr}(\rho) \otimes \frac{I}{2} - \rho: \text{ Jump operators } \sigma^x, \sigma^y, \sigma^z \implies \text{Irreducible.}$$

Dephasing:

$$\mathcal{L}(\rho) = \sigma^z \rho \sigma^z - \rho \implies \text{Not Irreducible.}$$

Dephasing with σ^x :

$$\mathcal{L}(\rho) = -i[\sigma^x, \rho] + \sigma^z \rho \sigma^z - \rho \implies \text{Irreducible.}$$

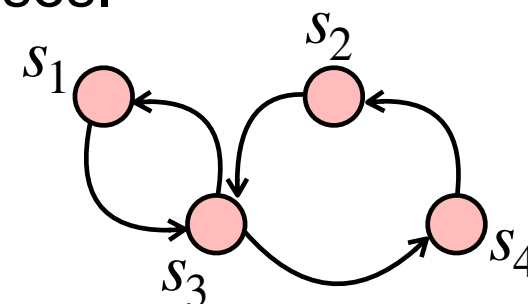
Amplitude damping:

$$\mathcal{L}(\rho) = \mathcal{D}_\sigma(\rho), \text{ where } \sigma = |0\rangle\langle 1| \implies \text{Not irreducible.}$$

Perron Frobenius theorem

Characterizes steady-states of irreducible Markov processes.

Classical processes: For an irreducible Markov process, the steady state probability distribution $p(\infty)$ is unique, and for all states s , $p_s(\infty) > 0$.



Irreducible Markov chain

Quantum Processes: For an irreducible Lindbladian, the steady state $\rho(\infty)$ is unique and $\rho(\infty) \succ 0$.

Important: A Lindbladian can have unique steady state without being irreducible. Example amplitude damping:

$$\mathcal{L}(\rho) = \mathcal{D}_\sigma(\rho), \text{ where } \sigma = |0\rangle\langle 1|$$

\implies Not irreducible but unique fixed point $|0\rangle\langle 0|$

Mixing times

Definition: A Lindbladian \mathcal{L} has mixing time $t_{mix}(\varepsilon)$ if for any initial state $\rho(0)$,

$$\|e^{\mathcal{L}t}(\rho(0)) - \rho(\infty)\|_1 \leq \varepsilon \text{ for all } t \geq t_{mix}(\varepsilon)$$

Important: Using 1-norm distance, since it quantifies how easy it is to distinguish two states with measurements (Holevo-Helstrom theorem).

Spectral gap γ :

\mathcal{L} has eigenvalues $0, -\lambda_1, -\lambda_2 \dots$ such that $\text{Re}(\lambda_i) > \gamma$:

$$e^{\mathcal{L}t}(\rho(0)) - \rho(\infty) = e^{-\gamma t} \sum_i e^{-(\lambda_i - \gamma)t} O_i$$

Physical expectation: For a fixed ε , $t_{mix}(\varepsilon) \sim 1/\gamma$.

Generally not true: In many-body models, $t_{mix}(\varepsilon)$ and $1/\gamma$ can have different scalings with system size n .

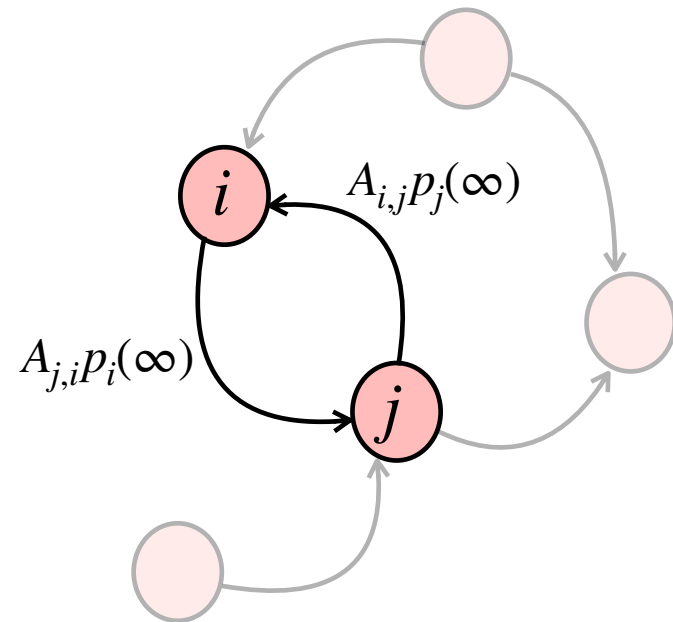
Znidaric (2015), Mori, Shirai (2020)

When to trust spectral gaps?

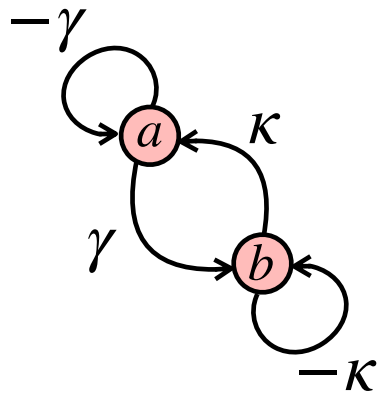
For quantum and classical processes with detailed balance.

Classical detailed balance: An irreducible classical process A is also detailed balance

$$\underbrace{A_{i,j}p_j(\infty)}_{\text{Probability current going from } j \rightarrow i} = \underbrace{A_{j,i}p_i(\infty)}_{\text{Probability current going from } i \rightarrow j}$$



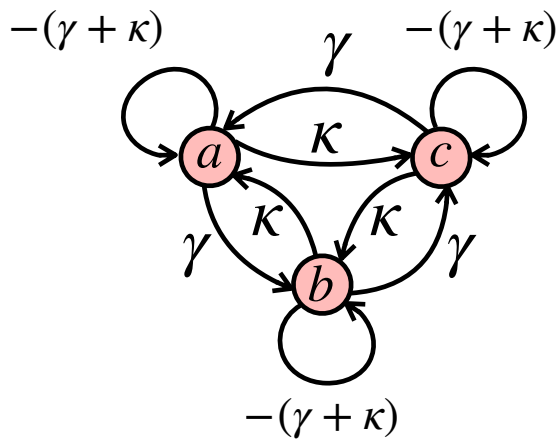
Classical Detailed Balance: Examples



$$p_a(\infty) = \frac{\kappa}{\kappa + \gamma} \quad p_b(\infty) = \frac{\gamma}{\kappa + \gamma}$$

$$\kappa \cdot p_b(\infty) = \gamma \cdot p_a(\infty)$$

Satisfies detailed balance



$$p_a(\infty) = p_b(\infty) = p_c(\infty) = \frac{1}{3}$$

$$\kappa \cdot p_b(\infty) \neq \gamma \cdot p_a(\infty)$$

Does not satisfy detailed balance.

Classical detailed balance \cong Gibb's process

- **Classical detailed balance:** $A_{i,j}p_j(\infty) = A_{j,i}p_i(\infty)$

- **Define frequencies ω_i :** $p_i(\infty) = \frac{1}{Z} \exp(-\omega_i)$

i.e. $p(\infty)$ can be interpreted as Gibb's state for $H = \sum_i \omega_i |i\rangle\langle i|$

- **Detailed balance condition:**

$$A_{i,j} \propto \exp(-\Delta\omega_{i,j}) = \exp(-(\omega_i - \omega_j))$$

i.e. Decay rate of transitioning from $i \rightarrow j$ follows a Boltzmann distribution as per the energy function $i \rightarrow \omega_i$.

Spectral gaps and Mixing times

Rate matrix A for classical detailed balance processes has real eigenvalues.

i.e. eigenvalues of $A = 0, -\lambda_2, -\lambda_3, \dots$, where $\lambda_2 \leq \lambda_3 \leq \lambda_4, \dots$

Proof:

$$\begin{aligned} A_{i,j}p_j(\infty) = A_{j,i}p_i(\infty) &\implies p_i^{-1/2}(\infty)A_{i,j}p_j^{1/2}(\infty) = p_j^{-1/2}(\infty)A_{j,i}p_i^{1/2}(\infty) \\ &\implies D_{p(\infty)}^{-1/2}AD_{p(\infty)}^{1/2} \text{ is symmetric, where } D_p = \text{Diag}(p) \\ &\implies D_{p(\infty)}^{-1/2}AD_{p(\infty)}^{1/2}, \text{ and thus } A, \text{ has real eigenvalues} \end{aligned}$$

Spectral gaps and Mixing times

For Markov processes with detailed balance: $t_{mix}(\varepsilon) \leq \frac{1}{2\lambda_2} \log\left(\frac{2}{\varepsilon^2 \min_s p_s(\infty)}\right)$

Important: For many-body systems, $\min p_s(\infty) \sim e^{-c_0 n}$, thus for fixed ε ,

$$t_{mix}(\varepsilon) \sim \mathcal{O}(\lambda_2^{-1}) + \mathcal{O}(n)$$

Quantum detailed balance

Recall: Classical detailed balance

$$A_{i,j}p_j(\infty) = A_{j,i}p_i(\infty) \text{ or}$$

$$D_{p(\infty)}^{-1/2} A D_{p(\infty)}^{1/2} = (D_{p(\infty)}^{-1/2} A D_{p(\infty)}^{1/2})^\dagger \text{ where } D_{p(\infty)} = \text{diag}(p(\infty))$$

Quantum detailed balance:

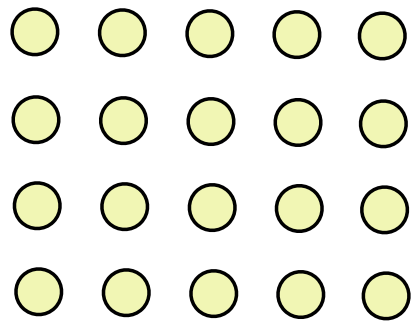
Consider a irreducible Lindbladian \mathcal{L} with fixed point $\rho(\infty) > 0$

For $s \in [0,1]$, define $\mathcal{D}_{\rho(\infty)}^{1/2} = \rho^{(1-s)/2}(\infty)(\cdot)\rho^{s/2}(\infty)$

Then, \mathcal{L} is detailed balance if

$$\mathcal{D}_{\rho(\infty)}^{-1/2} \mathcal{L} \mathcal{D}_{\rho(\infty)}^{1/2} = (\mathcal{D}_{\rho(\infty)}^{-1/2} \mathcal{L} \mathcal{D}_{\rho(\infty)}^{1/2})^\dagger$$

Quantum detailed balance and Gibb's processes



Environment at
inverse temperature β

Given System Hamiltonian: H

System-Bath coupling operators: S_α (e.g., $\sigma_i^x, \sigma_i^y, \sigma_i^z$)

Define: $S_\alpha(\omega)$ via

$$e^{iHt} S_\alpha e^{-iHt} = \sum_{\omega} S_\alpha(\omega) e^{-i\omega t}$$

Def: a Davies Lindbladian at temperature β

$$\mathcal{L} = \sum_{\alpha, \omega} \gamma_\alpha(\omega) \mathcal{D}_{S_\alpha(\omega)}$$

Where $\gamma_\alpha(\omega) > 0$ and $\gamma_\alpha(\omega) = e^{-\beta\omega} \gamma_\alpha(-\omega)$

Quantum detailed balance and Gibb's processes

- Consider $H = \sum_i E_i |E_i\rangle\langle E_i|$ where all Bohr-frequencies $E_i - E_j$ are non-degenerate.

$$e^{iHt} S_\alpha e^{-iHt} = \sum_\omega S_\alpha(\omega) e^{-i\omega t} \implies \begin{cases} S_\alpha(E_i - E_j) = \langle E_j | S_\alpha | E_i \rangle |E_j\rangle\langle E_i| \\ S_\alpha(E_j - E_i) = \langle E_i | S_\alpha | E_j \rangle |E_i\rangle\langle E_j| \end{cases}$$

Since by construction the rates $\gamma_\alpha(E_i - E_j) = e^{-\beta(E_i - E_j)} \gamma_\alpha(E_j - E_i)$
 $\rho(\infty) = Z^{-1} e^{-\beta H}$ i.e. Davies generator prepares the Gibb's state.

- If H is a geometrically local commuting Hamiltonian, then $S_\alpha(\omega)$ are geometrically local.

Alicki et al (2008), Kastaryano (2014)

- Recent progress in constructing "Davies-like" generators for non-commuting Hamiltonians with quasi-local jump operators.

Chen, Kastaryano, Gilyen (2025)

Mixing time of Quantum processes with detailed balance

For quantum processes with detailed balance: $t_{mix}(\varepsilon) \leq \frac{1}{2\lambda_2} \log\left(\frac{2}{\varepsilon^2 \lambda_{min}(\rho(\infty))}\right)$
Temme, Kastaryano et al (2010)

Spectral gap λ_2 known to be constant for several specific models:

- Davies generator of 1D Ising model, 2D Toric code, Kitaev Double model.
Alicki et al (2008) , Lucia (2021)
- Davies generator of any 1D commuting Hamiltonian.
Kastaryano (2014)
- Davies generator of 2D commuting Hamiltonians at high temperatures.
Kastaryano (2014)
- Davies generator of 1D non-commuting Hamiltonian.
C. F. Chen et al (2025)

However, in all cases $\lambda_{min}(\rho(\infty))$ is exponentially small in n .

$$t_{mix}(\varepsilon) \sim \mathcal{O}(\lambda_2^{-1}) + \mathcal{O}(n)$$

Getting to Rapid Mixing

Rapid mixing Lindbladians: A Lindbladian \mathcal{L} is rapid mixing if

$$\|e^{\mathcal{L}t}(\rho(0)) - \rho_\infty\|_1 \leq \text{poly}(n)e^{-\gamma t}$$

Rapid mixing Lindbladians:

- Mix in $\log(n)$ time.
- Local observables mix in $O(1)$ time.
- Robust to perturbations.
- Fixed point ρ_∞ has area-law as a mixed state.

Physical guess: Most Lindbladians are rapid mixing.

But very few sufficient conditions for a Lindbladian to be rapid-mixing?

Log-Sobolev inequalities

- Hamiltonian on n qubits with depolarizing noise:

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \kappa \sum_i \left(\text{Tr}_i(\rho) \otimes \frac{I_i}{2} - \rho \right)$$

- Fixed point: Unique and $\rho_\infty = (I/2)^{\otimes n}$
- Satisfies Quantum Detailed Balance,

$$\text{Spectral gap } \lambda_2 = \kappa \implies t_{mix} \leq \mathcal{O}(n/\kappa).$$

Probably an overestimate ! Does this model mix faster?

Log-Sobolev inequalities

- Use cross-entropy distance: $\mathbb{D}(\rho | \sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$

$$\|\rho - \sigma\|_1 \leq \sqrt{\mathbb{D}(\rho | \sigma)} \text{ and } \mathbb{D}(\rho | \sigma) \leq -\log(\lambda_{\min}(\sigma))$$

- Suppose $\rho(t) = e^{\mathcal{L}t}(\rho(0))$ and $\sigma = \rho(\infty)$, then study the rate of change of $\mathbb{D}(\rho(t) | \rho(\infty))$ as a function of t .

$$\frac{d}{dt} \mathbb{D}(\rho(t) | \rho(\infty)) = -\text{Tr}[\mathcal{L}(\rho(t))(\log \rho(\infty) - \log \rho(t))]$$

If $\exists \alpha > 0 : \text{Tr}[\mathcal{L}(\rho)(\log \rho(\infty) - \log \rho)] \geq \alpha \mathbb{D}(\rho | \rho(\infty))$

Then $\frac{d}{dt} \mathbb{D}(\rho(t) | \rho(\infty)) \leq -\alpha \mathbb{D}(\rho(t) | \rho(\infty))$

$$\implies \mathbb{D}(\rho(t) | \rho(\infty)) \leq \mathbb{D}(\rho(0) | \rho(\infty)) e^{-\alpha t}$$

**Log-Sobolev
Inequality**

α : Log-Sobolev
Constant

Temme (2012)

LSI: Depolarizing noise

- Hamiltonian on n qubits with depolarizing noise:

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \kappa \sum_i \left(\text{Tr}_i(\rho) \otimes \frac{I_i}{2} - \rho \right)$$

- Depolarizing noise satisfies Log-Sobolev inequality with $\alpha = \kappa$:

$$\mathbb{D}(\rho(t) \mid \frac{I}{2^n}) \leq \mathbb{D}(\rho(0) \mid \frac{I}{2^n}) e^{-\kappa t}$$

Aharonov (1996), A-Muller Hermes (2015)

- Furthermore, $\mathbb{D}(\rho(0) \mid \frac{I}{2^n}) \leq n$, and $\|\rho(t) - \frac{I}{2^n}\|_1 \leq \sqrt{\mathbb{D}(\rho(t) \mid \frac{I}{2^n})}$

$$\|\rho(t) - \frac{I}{2^n}\|_1 \leq \sqrt{n} \exp(-\kappa t)$$

Examples Rapid Mixing

Known rapid mixing processes

Davies generators of 1D commuting Hamiltonians.

Bardet, Capel et al (2023)

Davies generator of 2D Toric code.

Stengele et al (2025)

Parent Lindbladian of injective tensor network states.

Baruah et al RT (2025)

Summary

Steady states of Lindbladians can be characterized with algebraic conditions.

Spectral gap is a “somewhat” faithful indicator of mixing time only for models with detailed balance.

Log-sobolev constant gives a more faithful measure of the “gap” and trustable mixing time bounds.