Periodically driven systems

video: YouTube (bluedwarf1127)

video: YouTube (Harvard Nat Sci)

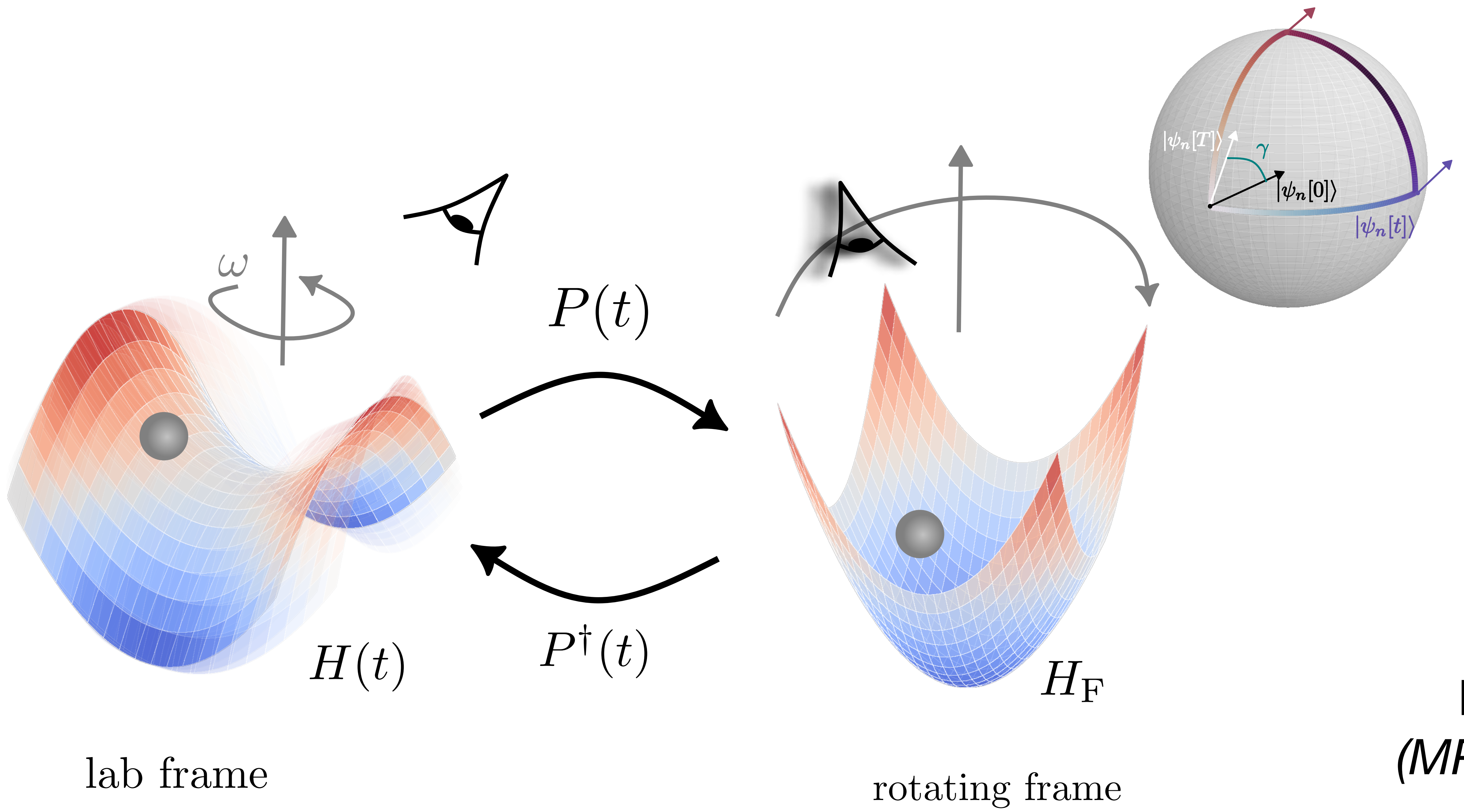
High-frequency periodic drives
can change drastically
the fundamental properties of physical systems

MAX PLANCK INSTITUTE
FOR THE PHYSICS OF COMPLEX SYSTEMS

mpipks

Geometric Floquet theory

Paul M Schindler
(MPI PKS)



Étienne Poliquin
(MPI PKS, Montreal U)

Schindler & MB, PRX 15, 031037 (2025)

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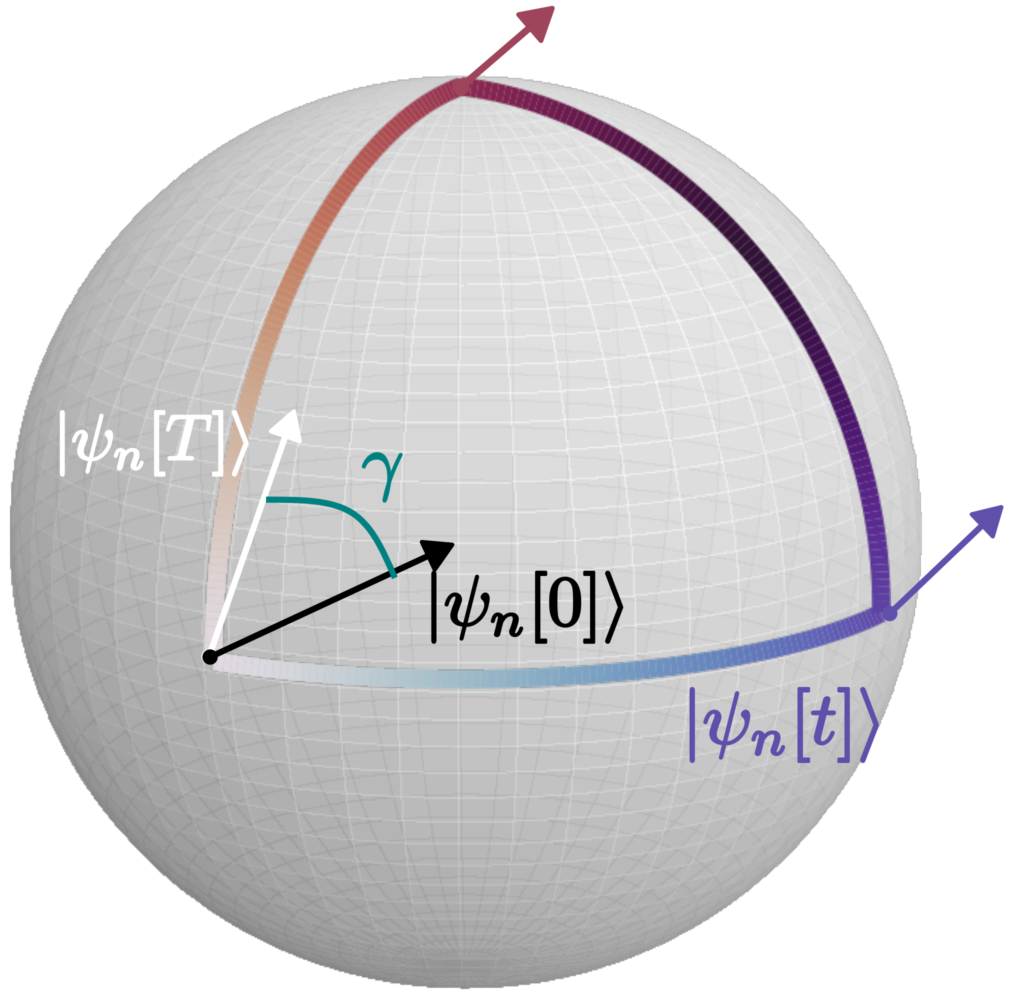
Marín Bukov

mpipks (Dresden)

Geometric Floquet theory

(take-home messages)

- Floquet theory follows from the adiabatic theorem
  - alternative decomposition of dynamics: geometric & dynamical phases



Geometric Floquet theory

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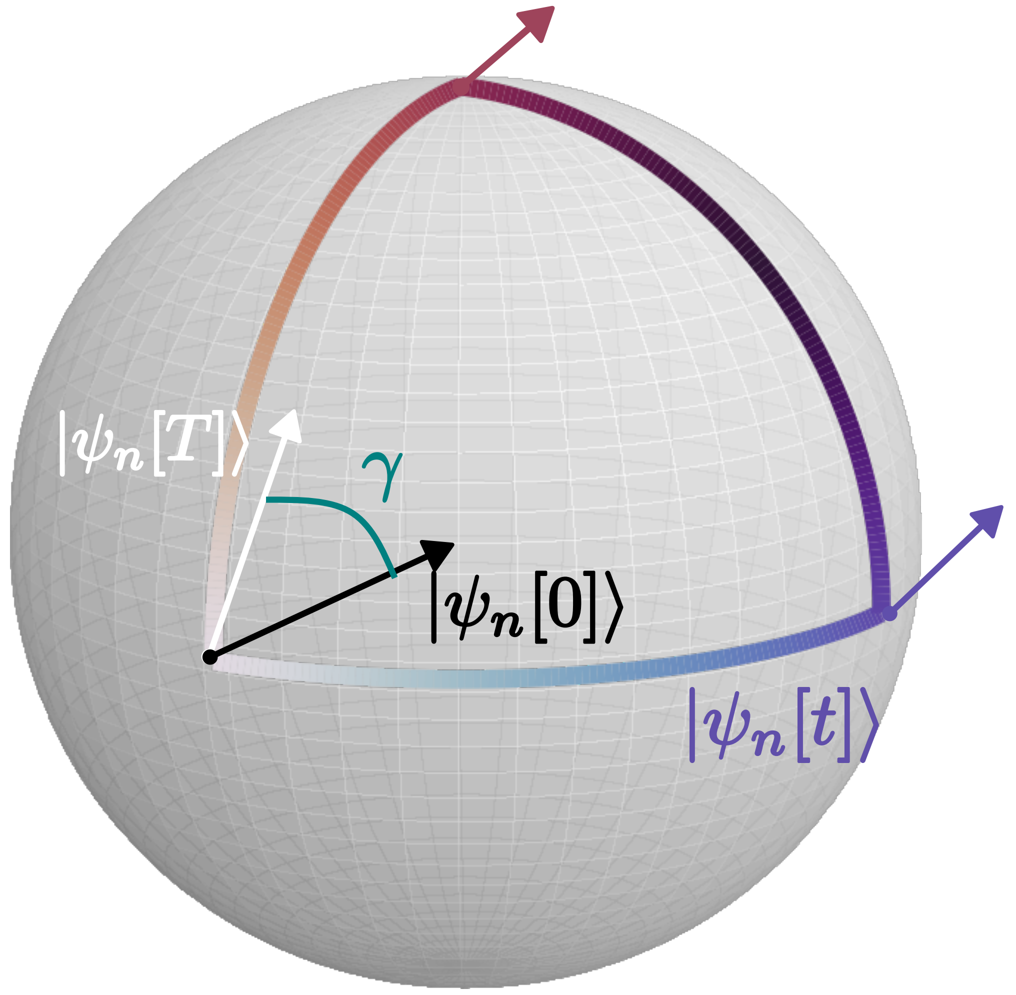
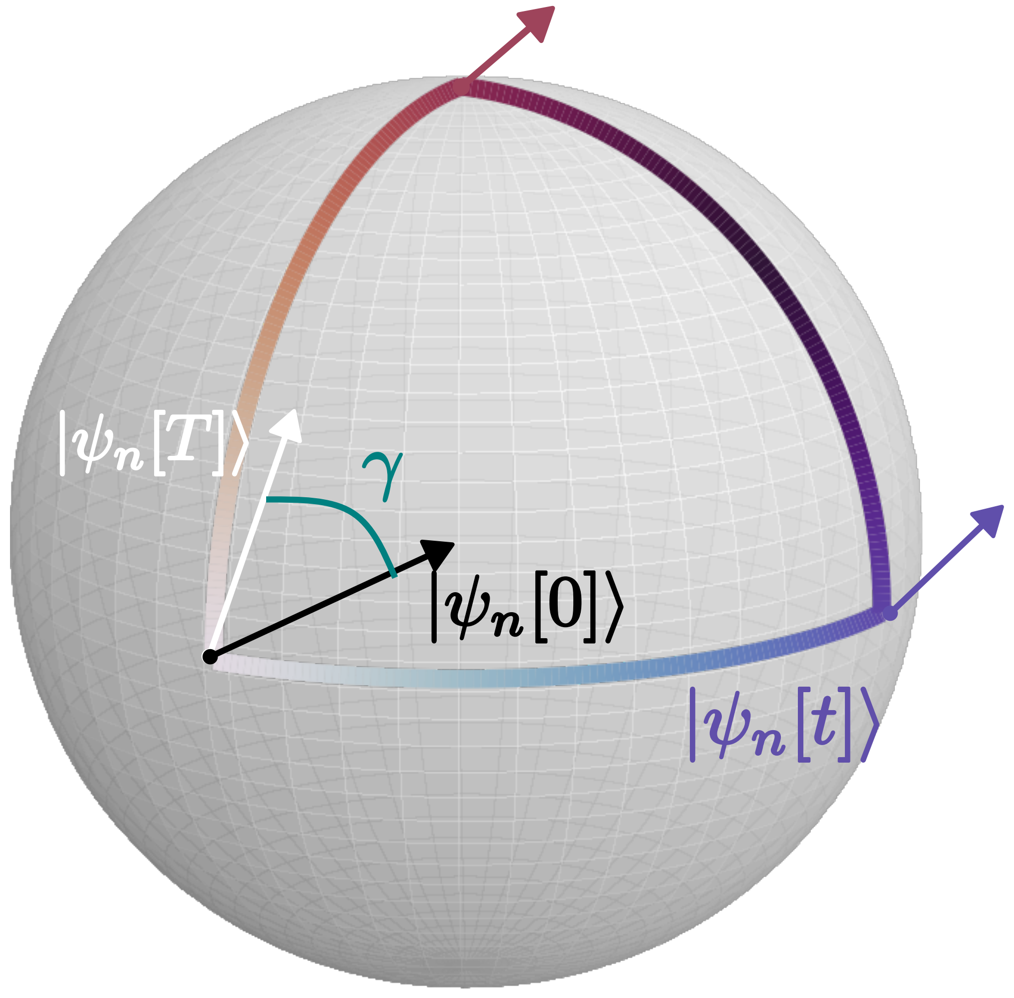


- **Floquet theory follows from the adiabatic theorem**
  - alternative decomposition of dynamics: geometric & dynamical phases

- **dynamical phase defines uniquely a Floquet *ground* state**
  - guaranteed by parallel-transport gauge and the adiabatic limit

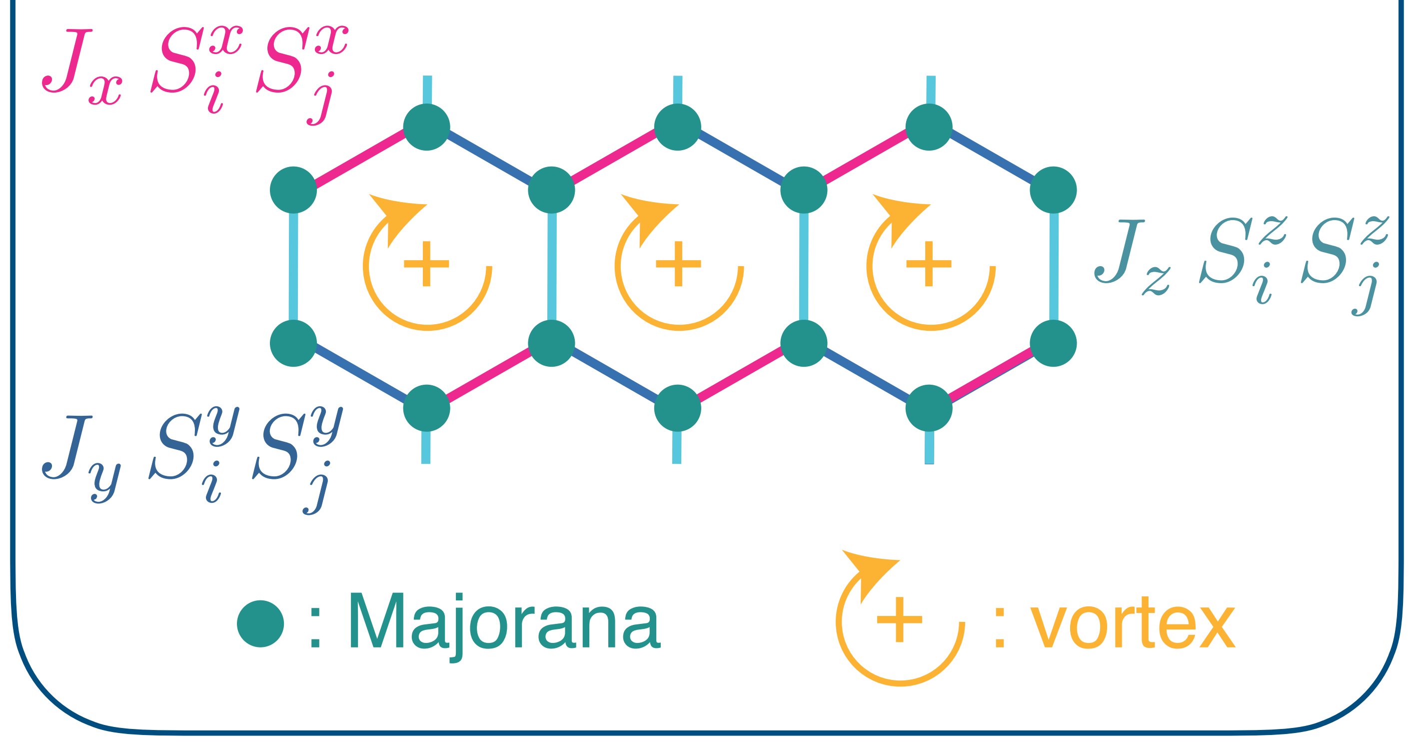
Geometric Floquet theory

(take-home messages)

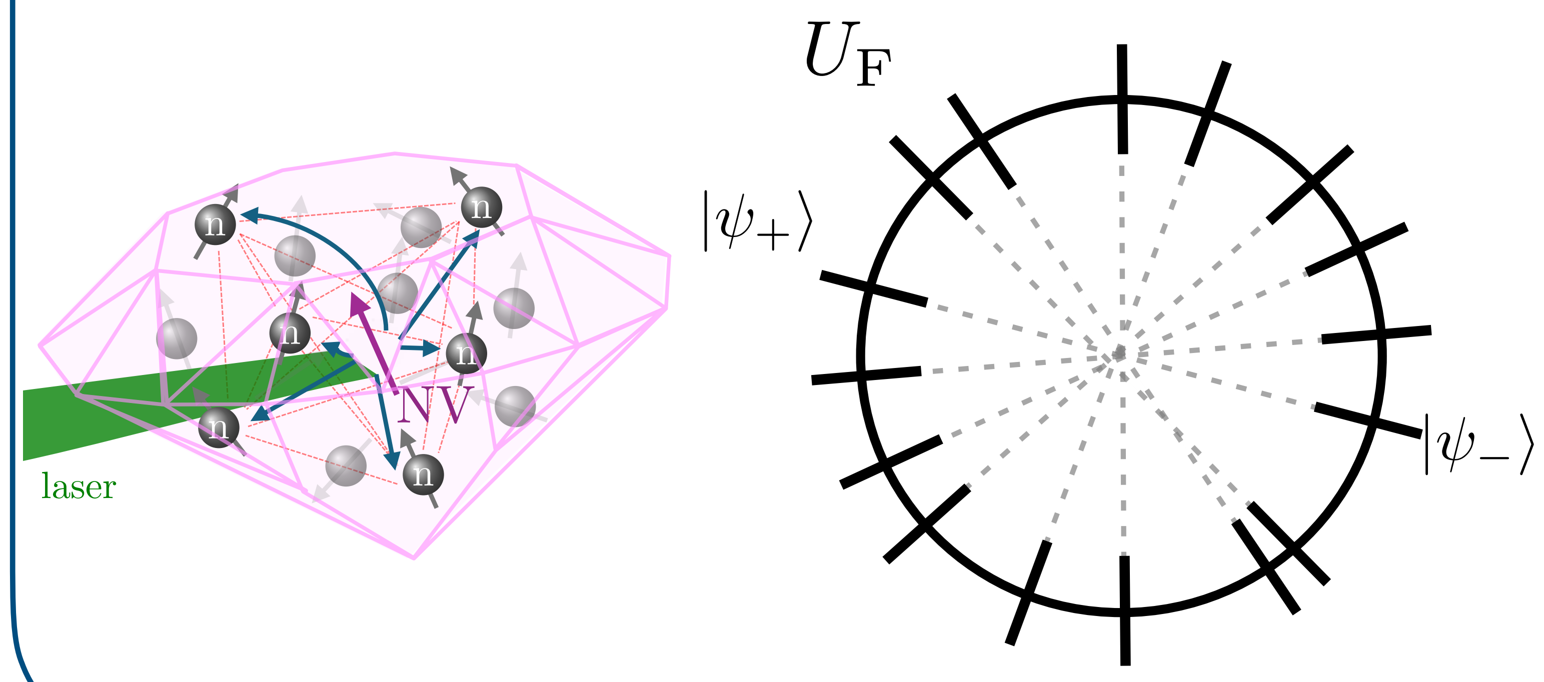


- **Floquet theory follows from the adiabatic theorem**
  - alternative decomposition of dynamics: geometric & dynamical phases
- **dynamical phase defines <u>uniquely</u> a Floquet *ground* state**
  - guaranteed by parallel-transport gauge and the adiabatic limit
- **geometric phase captures inherently nonequilibrium phenomena**

Floquet topological order

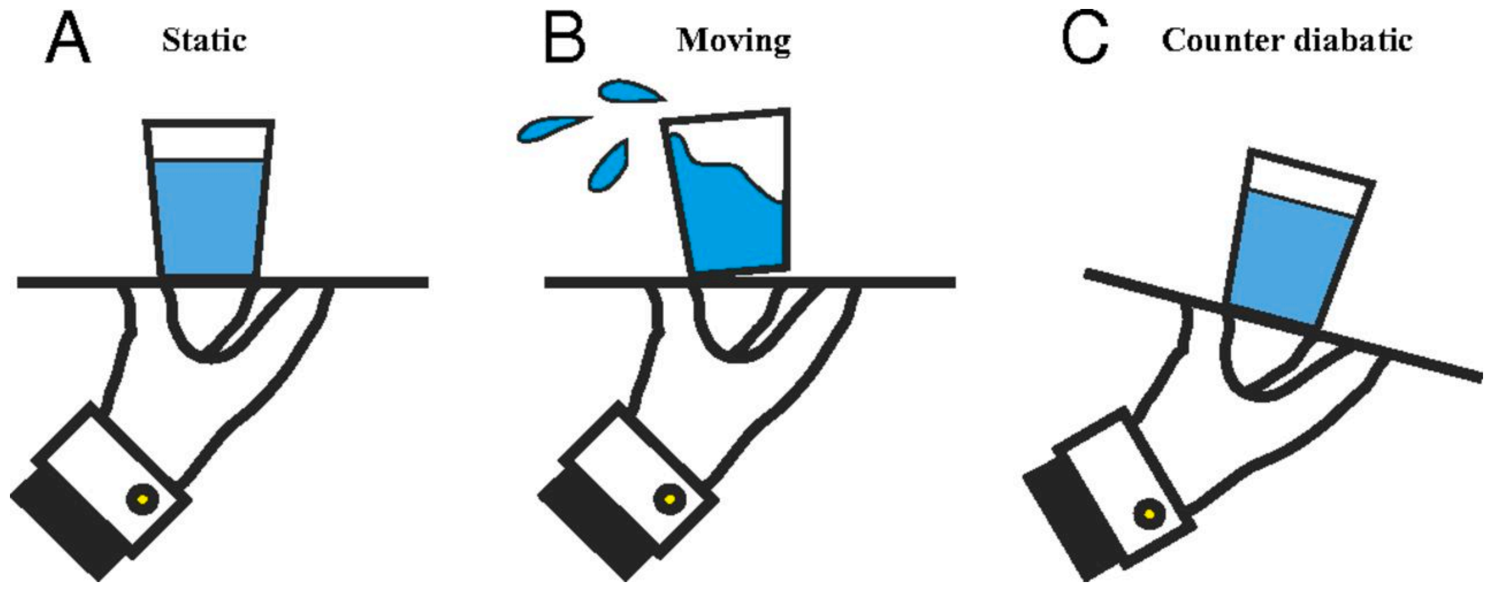


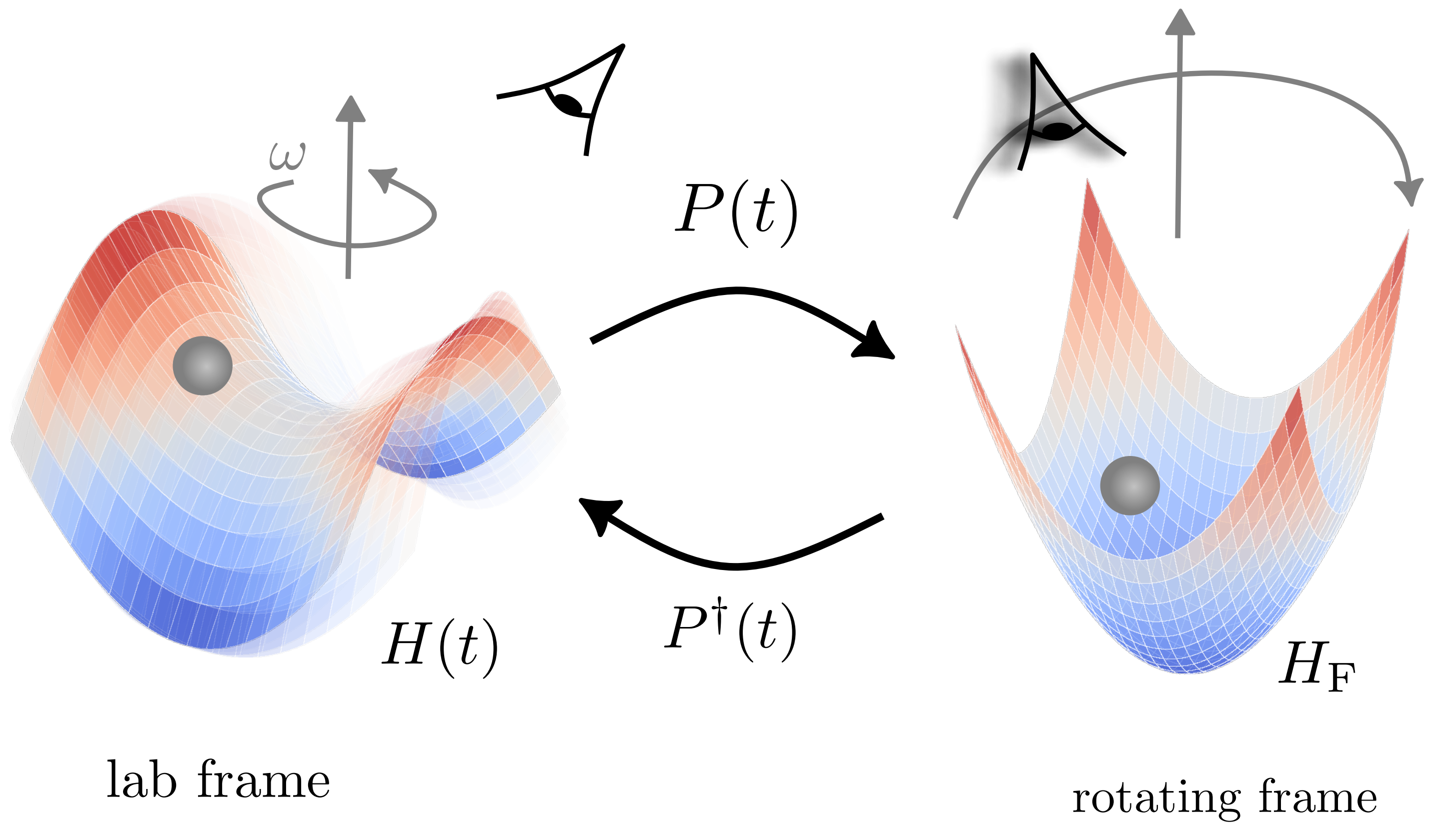
discrete time crystals



Outline

- **Floquet theory from counterdiabatic driving**
  - geometric Floquet decomposition
  - *unambiguous* Floquet ground state
- **Anomalous chiral spin liquids**
  - Floquet topological order
- **Time crystals**
  - nonequilibrium eigenstate order
- **Heating in kicked Ising chain**
  - locality of average energy





Schindler & MB, PRX 15, 031037 (2025)

Floquet theorem as a special case of the Adiabatic theorem

- **Floquet's theorem:**

$$H_{\text{lab}}(t) = H_{\text{lab}}(t+T)$$

$$U_{\text{lab}}(t,0) = \underset{\substack{\uparrow \\ \text{micromotion}}}{P(t,0)}\ \exp(-it\,\underset{\substack{\uparrow \\ \text{quasienergy}}}{H_F[0]})$$

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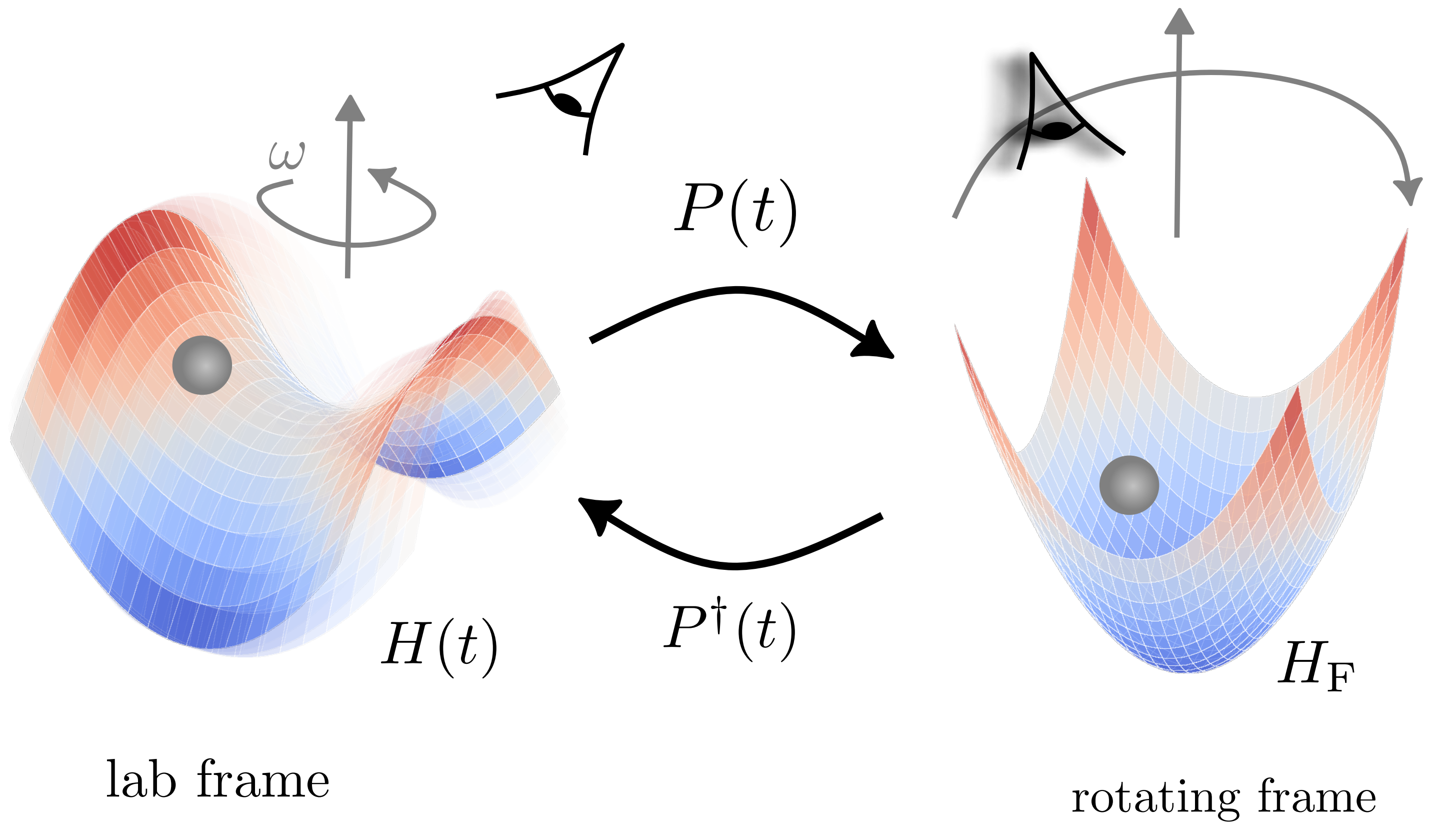
- **rotating frame**



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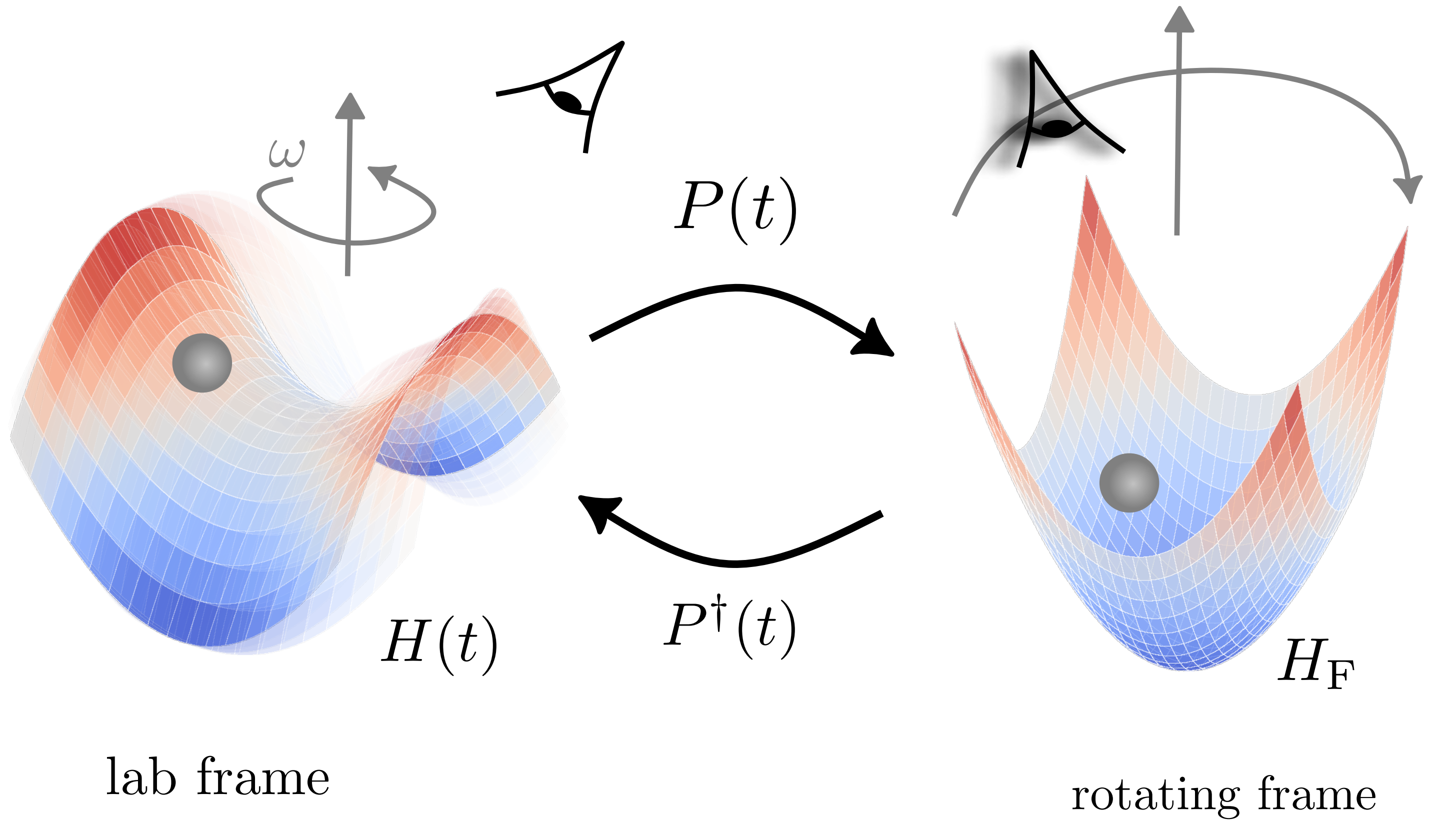


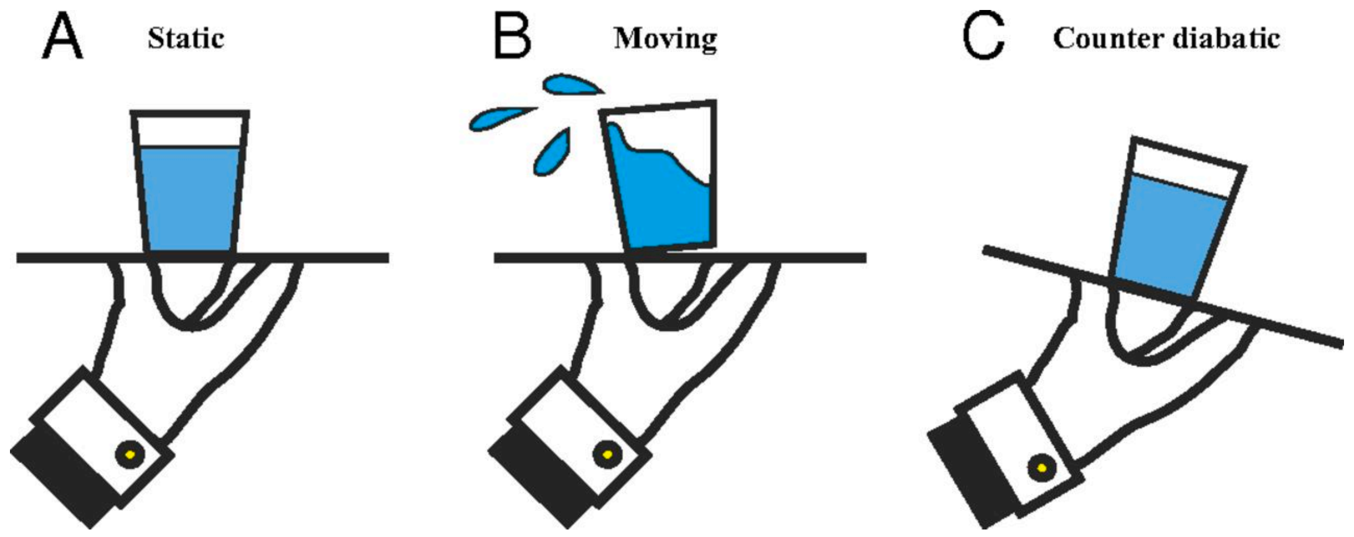
- **Adiabatic theorem (CD driving):**

$$H^{\text{CD}}_{\text{lab}}(t) = H_{\text{ctrl}}(\lambda(t)) + \dot{\lambda}\mathcal{A}_K(\lambda(t))$$

$$U_{\text{lab}}(t,0) = \mathcal{T}\exp\left(-i\int_0^t \mathcal{A}_K(s)\mathrm{d}s\right)\ \exp(-it\,\Phi(t,0))$$

geometric phase dynamical 'phase'

- **co-moving frame**



Sels and Polkovnikov, PNAS 114 (2017)

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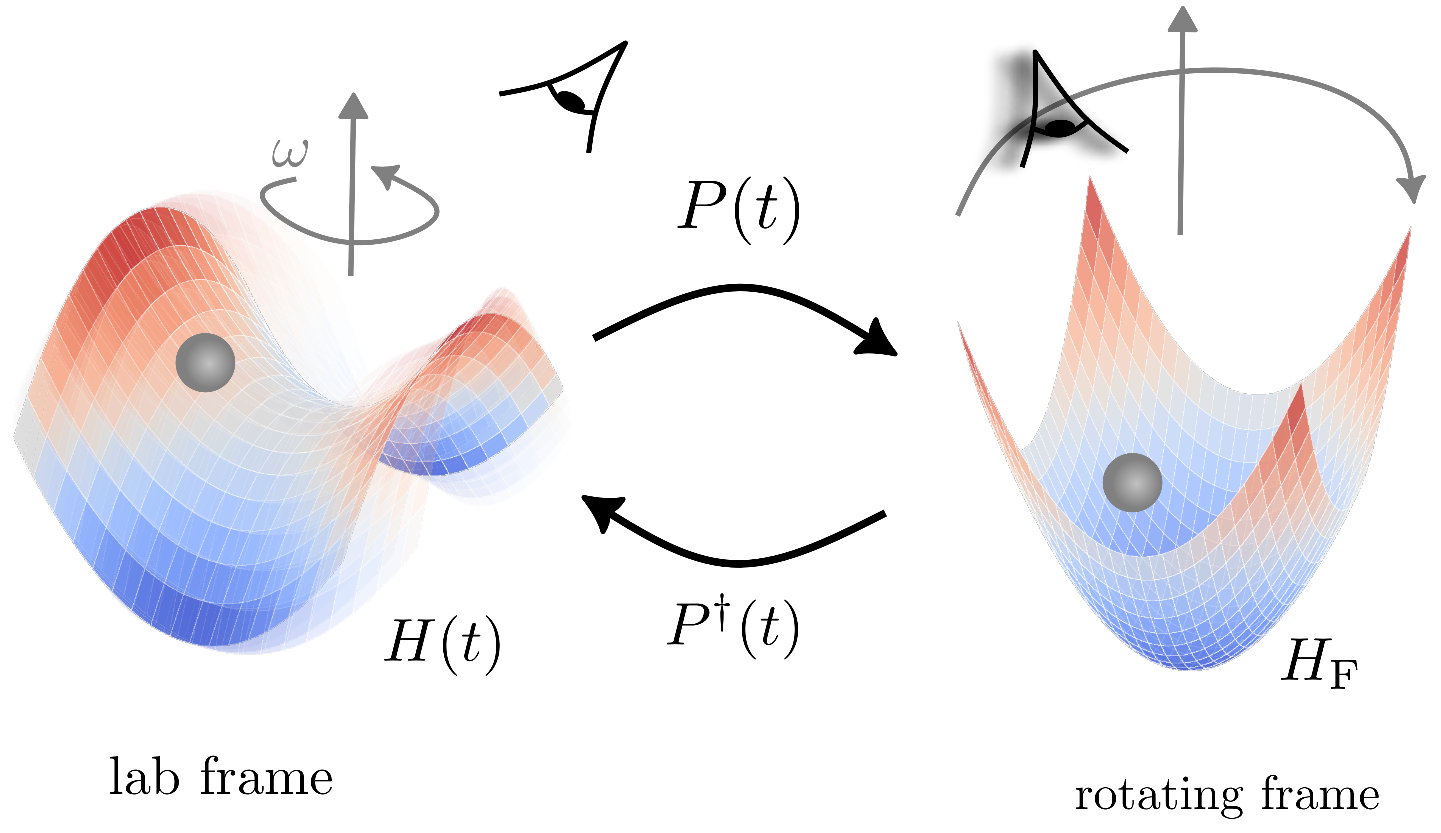

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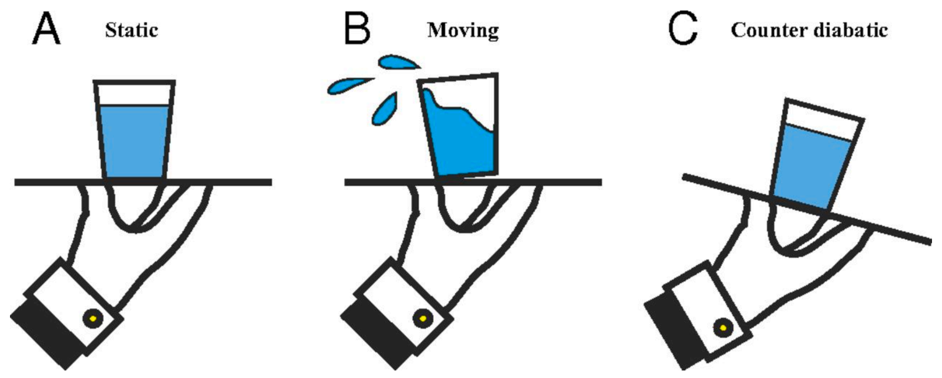

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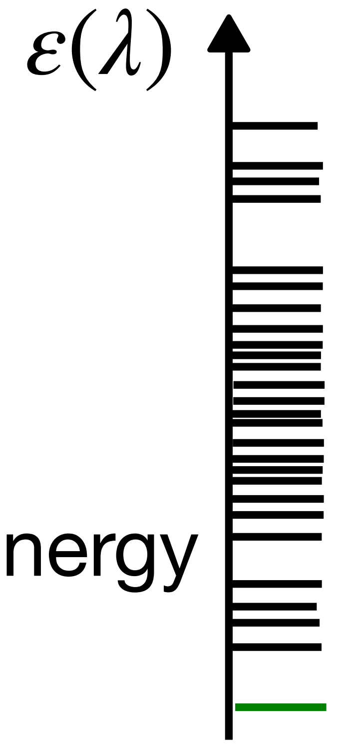

- ✦ **dynamical 'phase'** $\phi_n(t) = \int_0^t \mathrm{d}s\ \varepsilon(\lambda(s)) \in \mathbb{R}$

  can be *unambiguously* unfolded/unwound: comes from energy

- ✦ **geometric phase** $\gamma_n(t) = \int_{\lambda(0)}^{\lambda(t)} \mathrm{d}\lambda\ \langle n[\lambda]|i\partial_\lambda|n[\lambda]\rangle \in [0,2\pi)$

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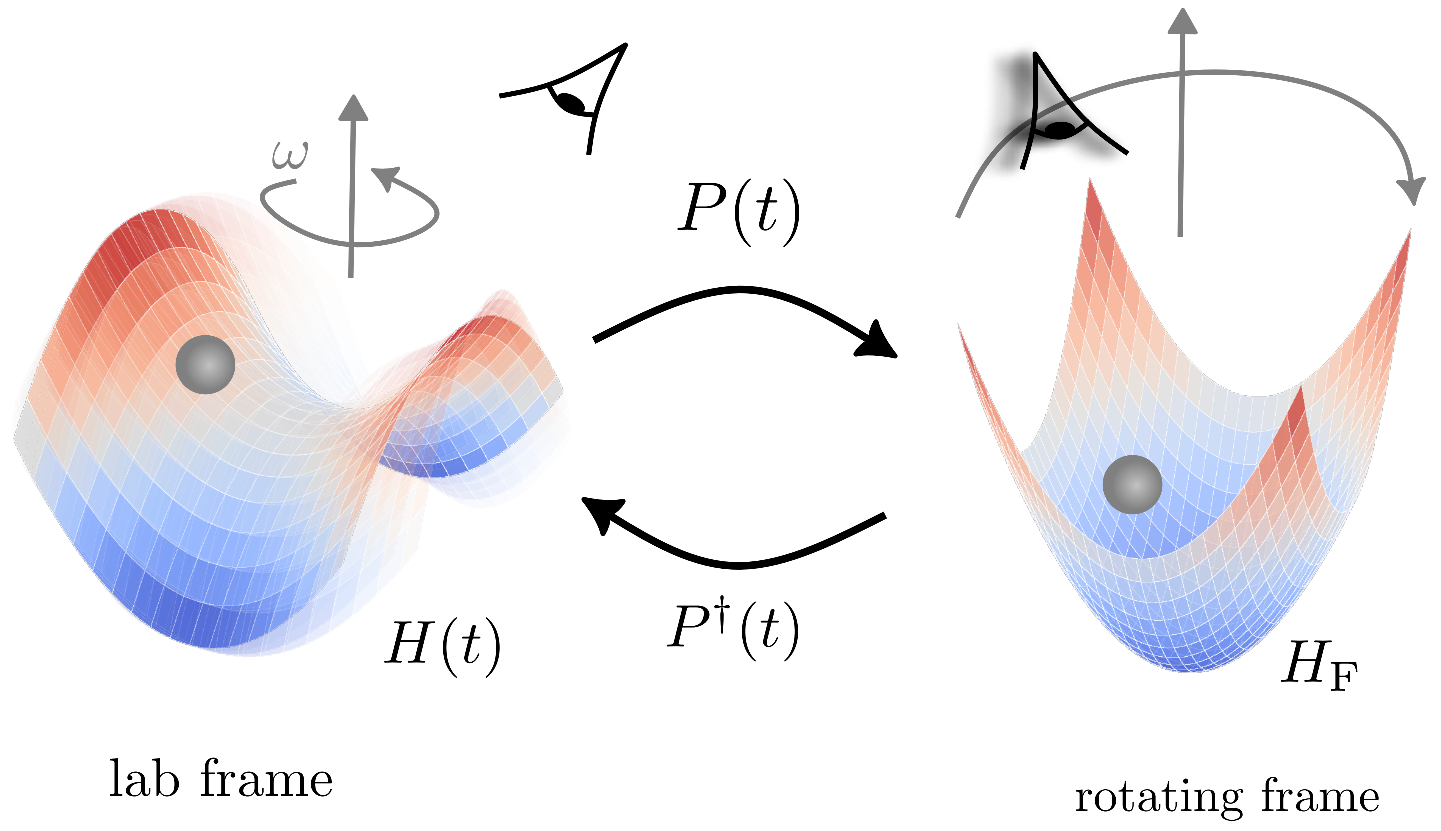

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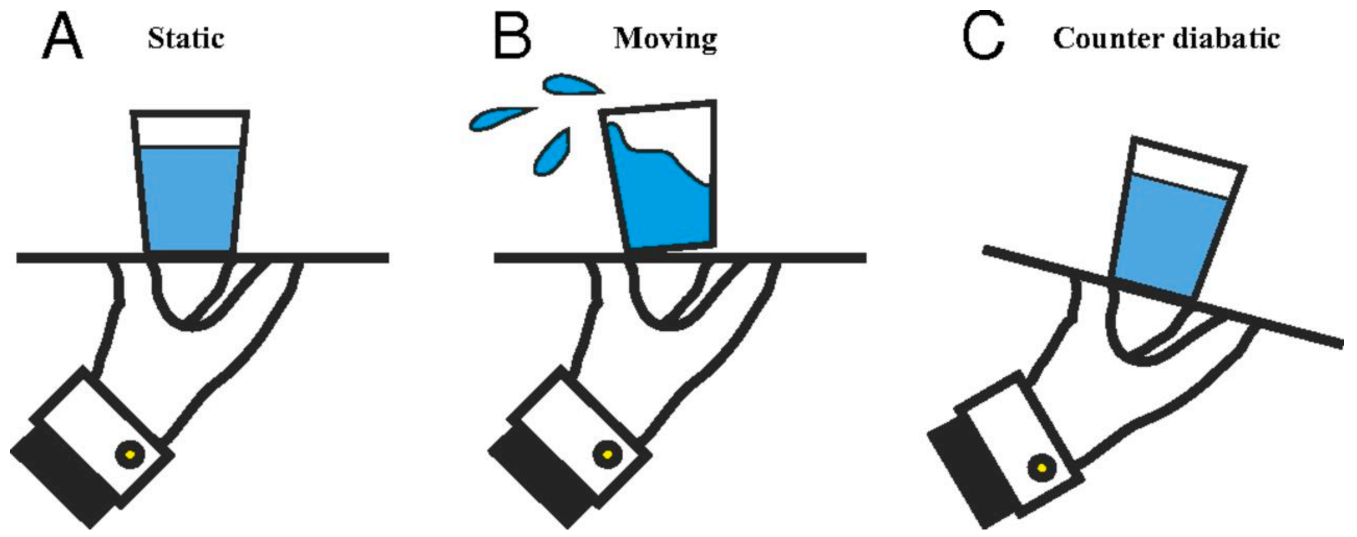

Sels and Polkovnikov, PNAS 114 (2017)

Floquet rotating frame = waiter's co-moving frame (up to a gauge)

Floquet theory as a shortcut to adiabaticity

- **Floquet's theorem:** $/ \quad P(t)\,(\,\cdot\,)\,P^\dagger(t)$

$$H_F[0] = P^\dagger(t)H(t)P(t) - P^\dagger(t)i\partial_t P(t)$$

$$|n_F[t]\rangle = P(t)\,|n_F[0]\rangle$$

$$H_F[t] = P(t)H_F[0]P^\dagger(t)$$

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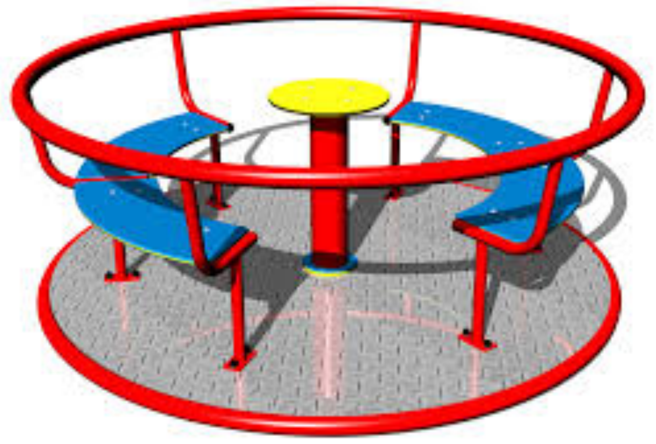
$\mathscr{A}_F(t) = i\partial_t P(t)P^\dagger(t)$:

gauge potential w.r.t. time / phase of drive



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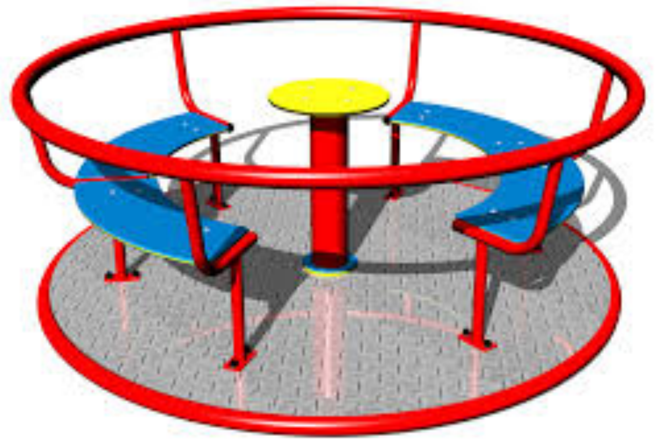

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$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

$H(t)$ is the CD Hamiltonian for $H_F[t]$?

$$H_{\text{CD}}(\lambda) = H_{\text{ctrl}}(\lambda) + \dot{\lambda}\mathscr{A}_\lambda$$

$$\lambda \mathrel{\hat{=}} t$$

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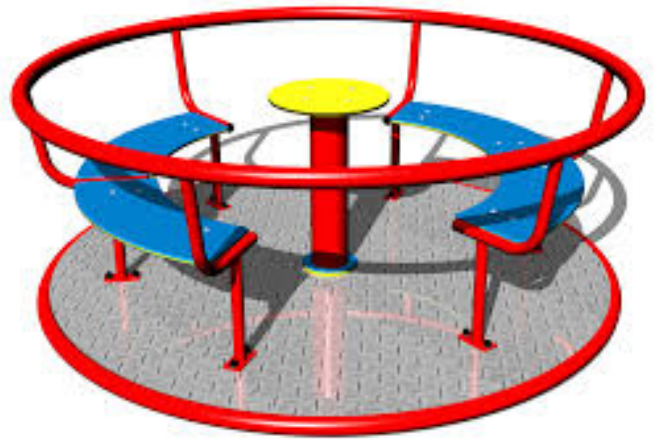

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recall definition (CD driving): evolved e'state $=$ $e^{i\,\text{phase}}$ instantaneous e'state

- check: $|n_F(t)\rangle = \mathscr{T}e^{-i\int_0^t \mathrm{d}sH(s)}|n_F(0)\rangle = P(t)e^{-itH_F}|n_F(0)\rangle = e^{-it\varepsilon_F^{(n)}}P(t)|n_F[0]\rangle = e^{-it\varepsilon_F^{(n)}}|n_F[t]\rangle$

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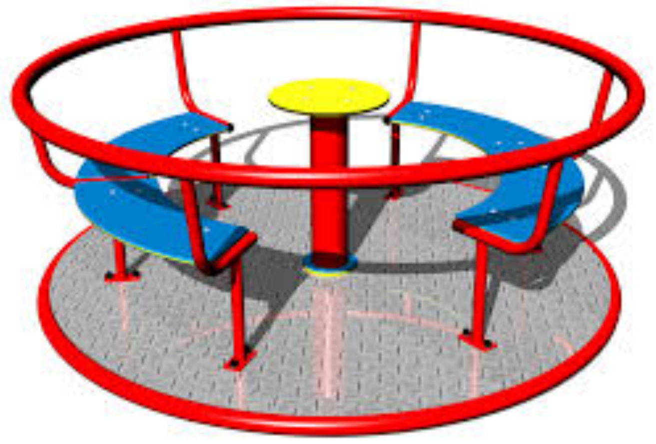

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- **hidden relation between counterdiabatic driving and Floquet physics**
  - provides a *geometric* angle at Floquet theory

Geometric Floquet Theory

AGP $\mathscr{A}$ not unique: U(1) gauge freedom

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

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- periodically driven system: $H(t)$
- evolution operator:

$$U(T{,}0) = \mathscr{T}\exp\left(-i\int_0^T H(t)\mathrm{d}t\right)$$

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Periodic gauge: $\mathscr{A}_F(t)$

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micromotion

quasienergy

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geometric phase

dynamical ‘phase’

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micromotion

quasienergy

$$P(T) = \mathbf{1}$$

$$P(t+T) = P(t)$$

- micromotion: $P(t) = \mathscr{T}\exp\left(-i\int_0^t \mathscr{A}_F(s)\mathrm{d}s\right)$

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geometric phase dynamical ‘phase’

$\mathscr{W}(T) \neq \mathbf{1}$

$\mathscr{W}(t+T) \neq \mathscr{W}(t)$

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micromotion quasienergy

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- quasienergy: $H_F[0] = \sum_n \varepsilon_F^{(n)} |n_F[0]\rangle\langle n_F[0]|$

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- **A**verage **E**nergy: $\text{Æ}(T,0) = \sum_n \text{æ}_n(T,0)|n_F[0]\rangle\langle n_F[0]|$
  - eigenstates: Floquet states
  - e’energies: $\text{æ}_n(T,0) = \frac{1}{T}\int_0^T \mathrm{d}t\,\langle n_F[t]|H(t)|n_F[t]\rangle$

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$$\boxed{\varepsilon_F^{(n)} = T^{-1}\gamma_n + æ_n}$$

Parallel-transport gauge: $\mathscr{A}_K(t)$

$$U(T,0) = \mathcal{T}\exp\left(-i\int_0^T \underbrace{\mathscr{A}_K(t)}_{\text{geometric phase}}\mathrm{d}t\right)\ \exp(-iT\underbrace{Æ(T,0)}_{\text{dynamical ‘phase’}})$$

$\mathscr{W}(T) \neq 1$

$\mathscr{W}(t+T) \neq \mathscr{W}(t)$

- Wilson line: $\mathscr{W}(t) = \mathcal{T}\exp\left(-i\int_0^t \mathscr{A}_K(s)\mathrm{d}s\right)$
- **A**verage **E**nergy: $Æ(T,0) = \sum_n æ_n(T,0)|n_F[0]\rangle\langle n_F[0]|$
  - ✦ eigenstates: Floquet states
  - ✦ e’energies: $æ_n(T,0) = \frac{1}{T}\int_0^T \mathrm{d}t\ \langle n_F[t]|H(t)|n_F[t]\rangle$

- ◉ periodically driven system: $H(t)$
- ◉ evolution operator:

$$U(T,0) = \mathcal{T}\exp\left(-i\int_0^T H(t)\mathrm{d}t\right)$$

Geometric Floquet Theory

$H(t) = H_F[t] + \mathscr{A}_F(t)$

AGP $\mathscr{A}$ not unique: U(1) gauge freedom

<u>Periodic gauge:</u> $\mathscr{A}_F(t)$

$$U(T,0) = \mathcal{T}\exp\left(-i\int_0^T \mathscr{A}_F(t)\mathrm{d}t\right)\exp(-iTH_F[0])$$

micromotion quasienergy

$P(T) = 1$

$P(t+T) = P(t)$

- micromotion: $P(t) = \mathcal{T}\exp\left(-i\int_0^t \mathscr{A}_F(s)\mathrm{d}s\right)$
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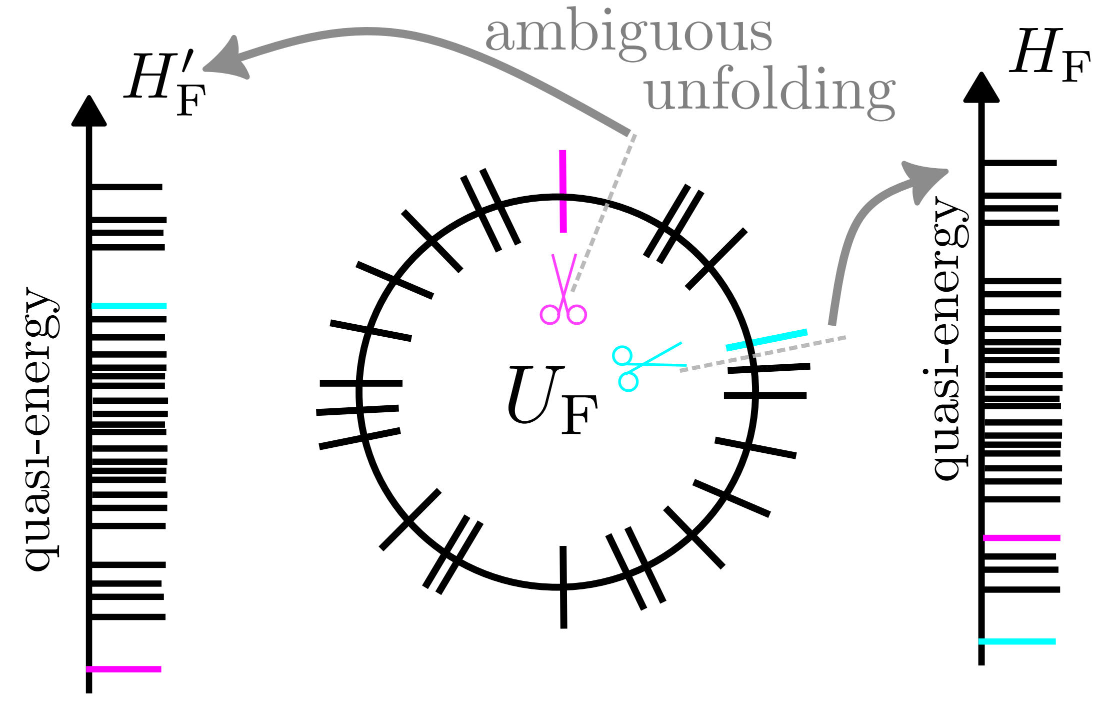

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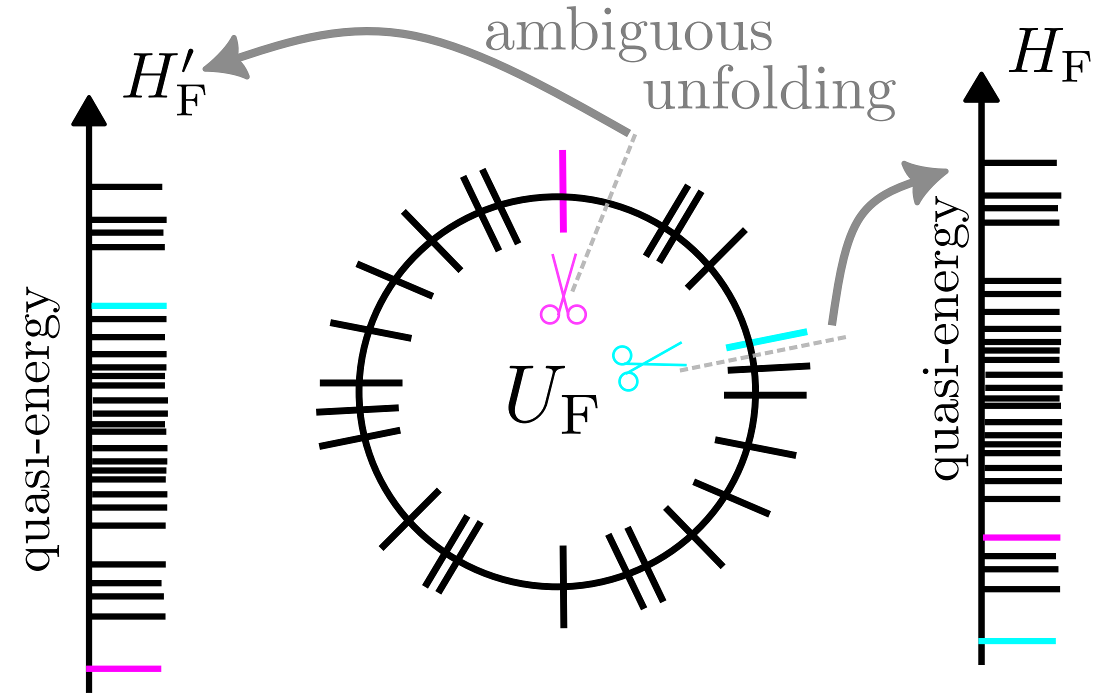

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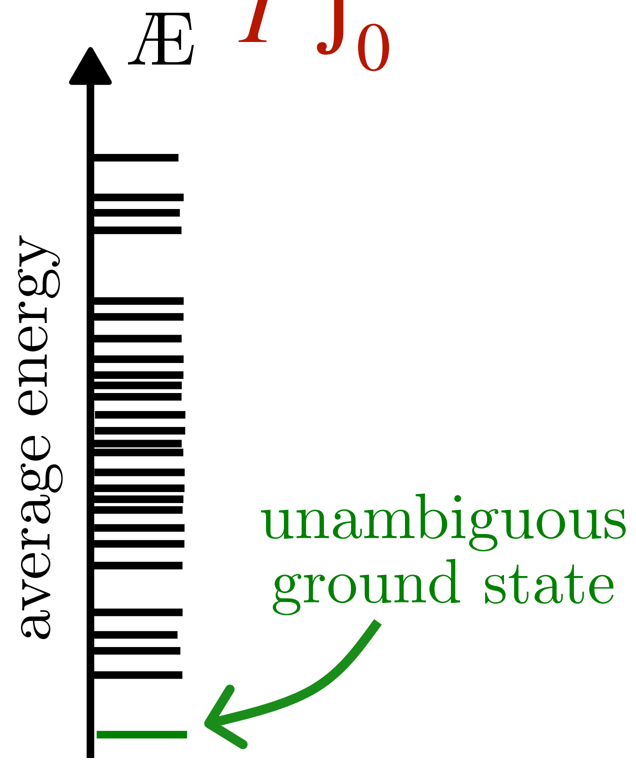

no folding: $H(t)$ extensive

★ *sorts Floquet states !*

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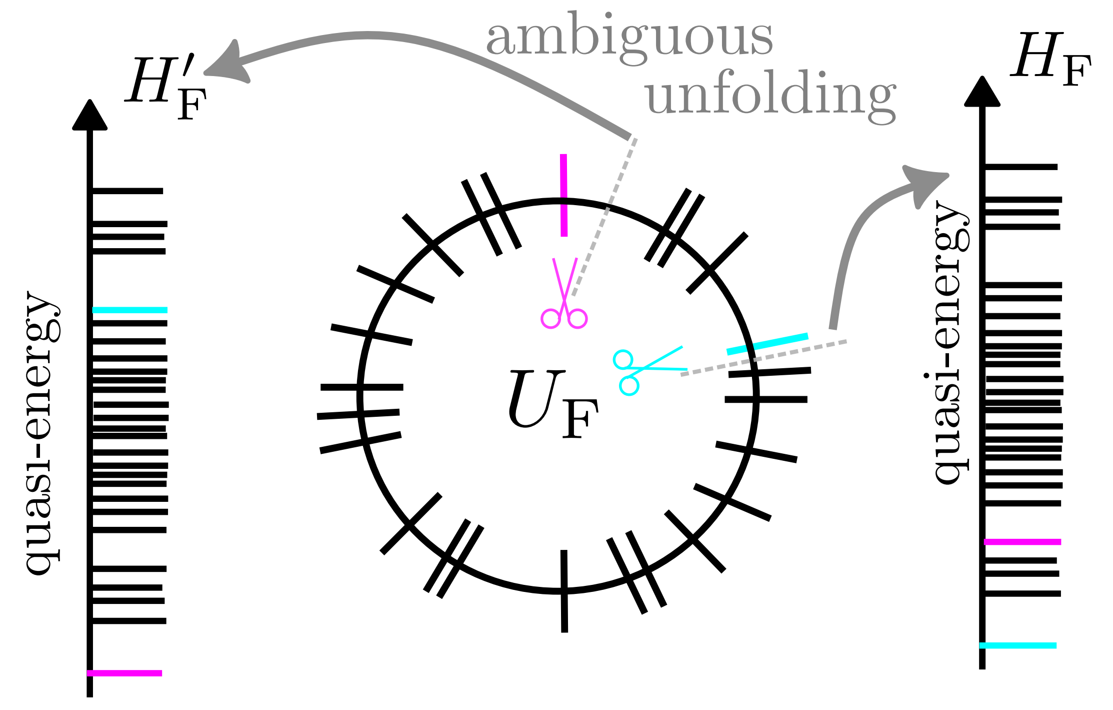

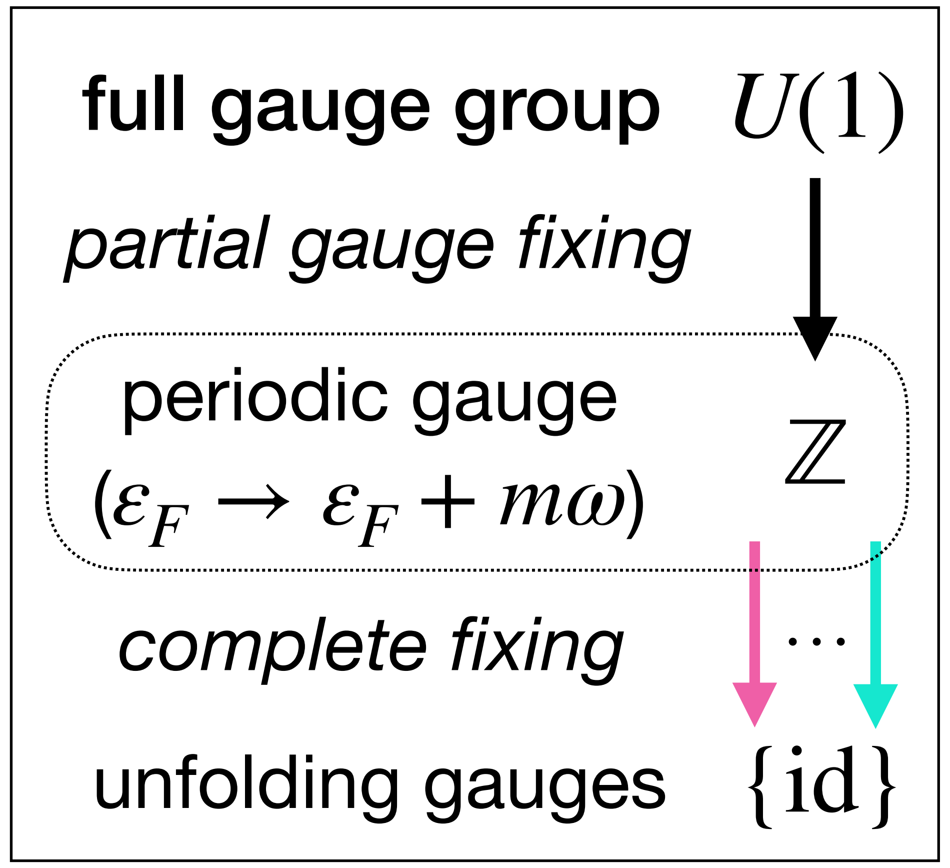
full gauge group $U(1)$

partial gauge fixing

periodic gauge $(\varepsilon_F \to \varepsilon_F + m\omega)$ $\mathbb{Z}$

complete fixing $\cdots$

unfolding gauges $\{\mathrm{id}\}$

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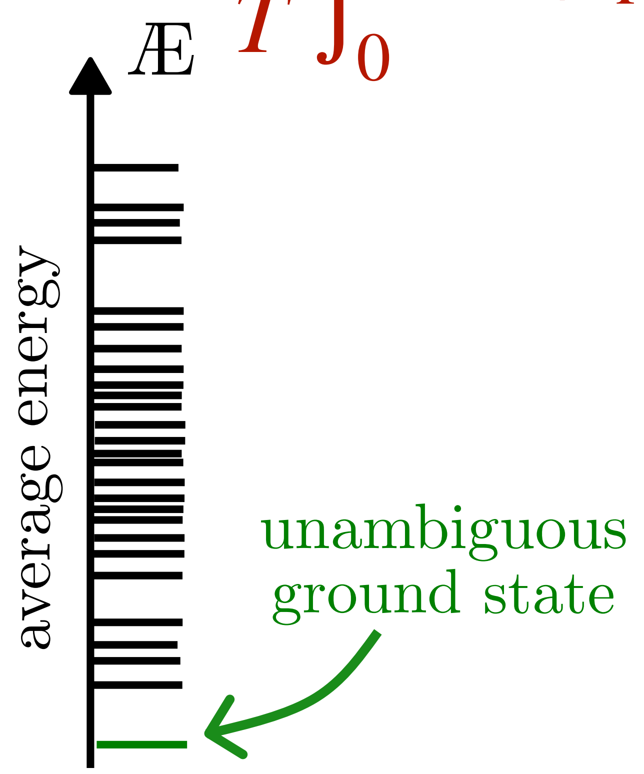

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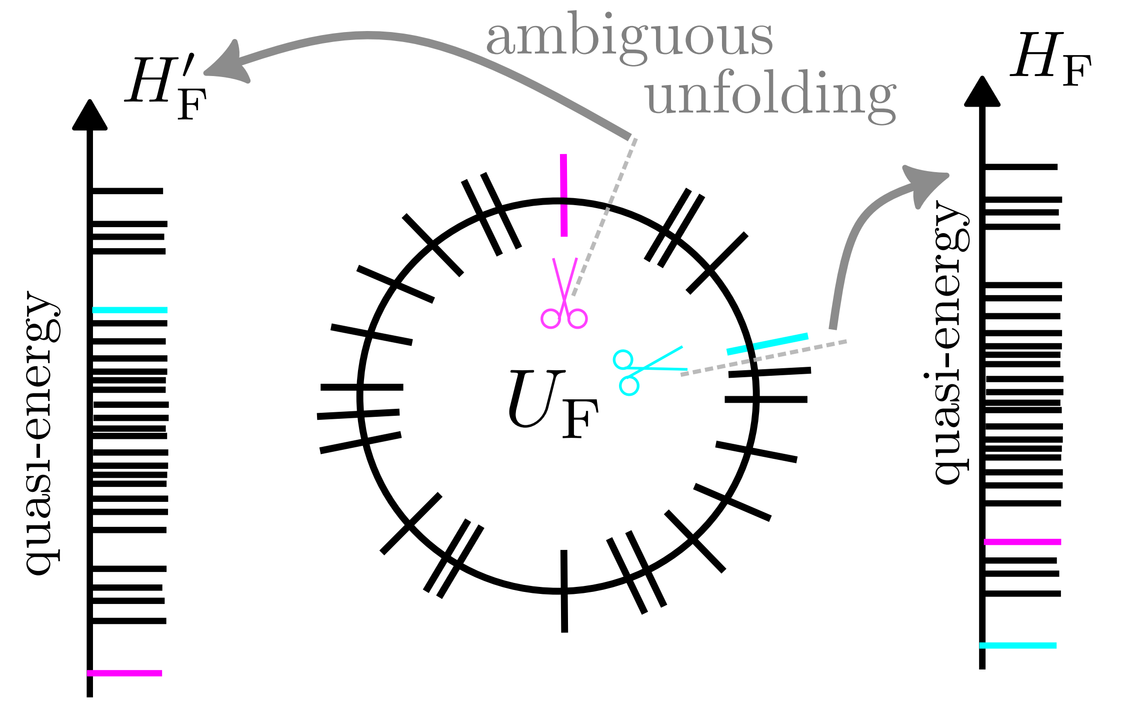

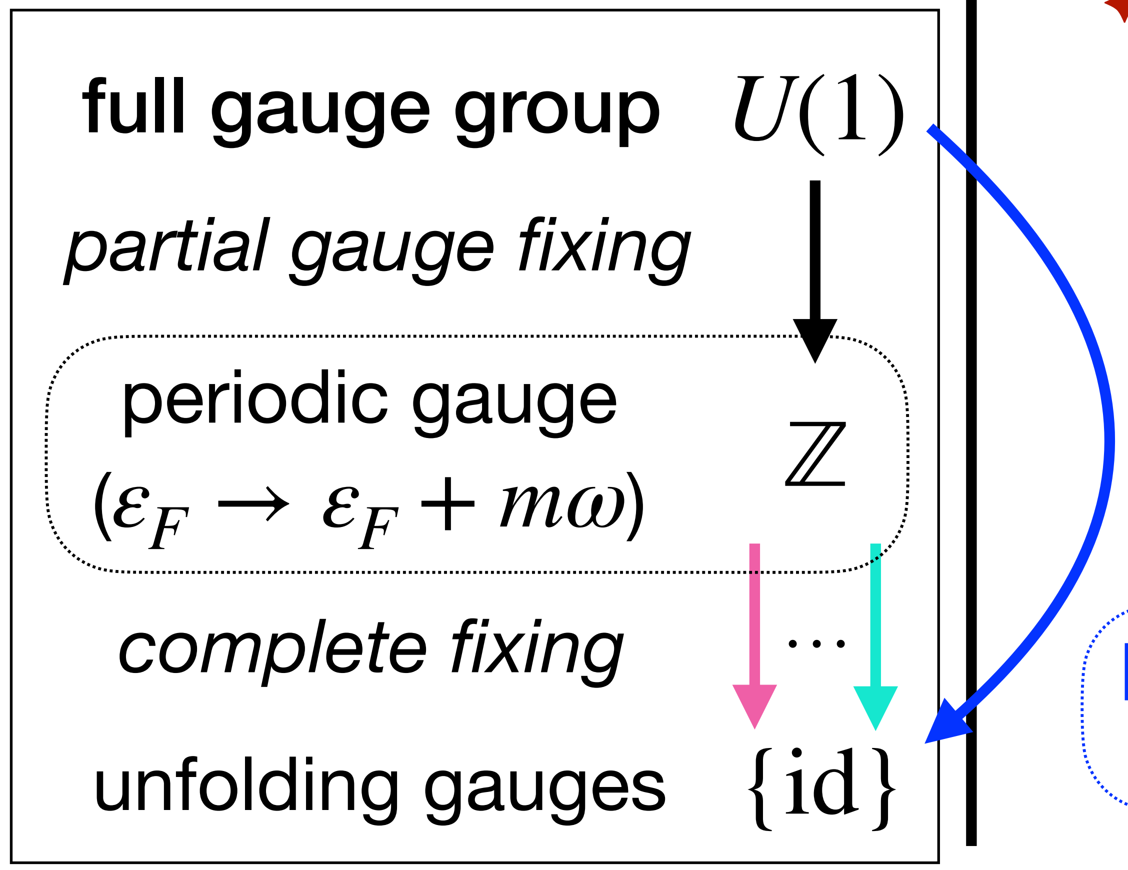
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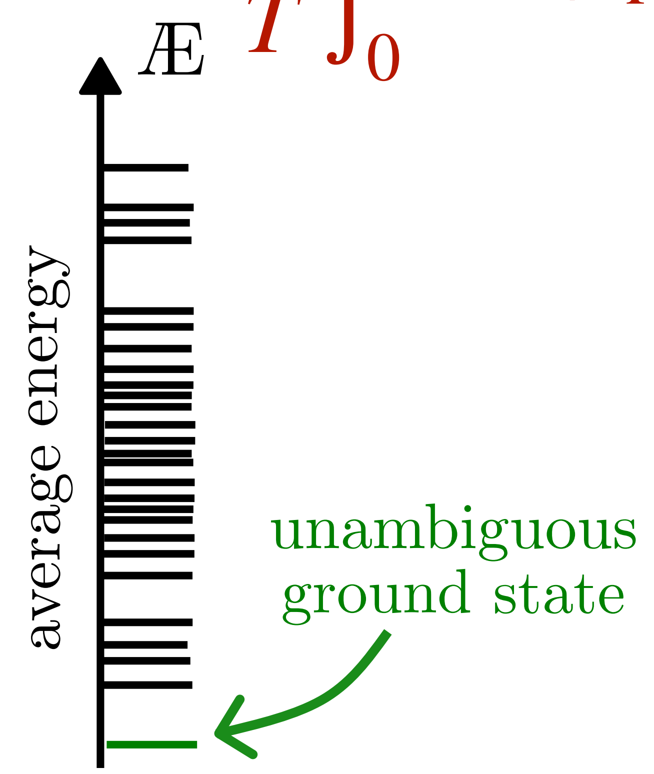
complete gauge fixing

parallel-transport gauge (unique)



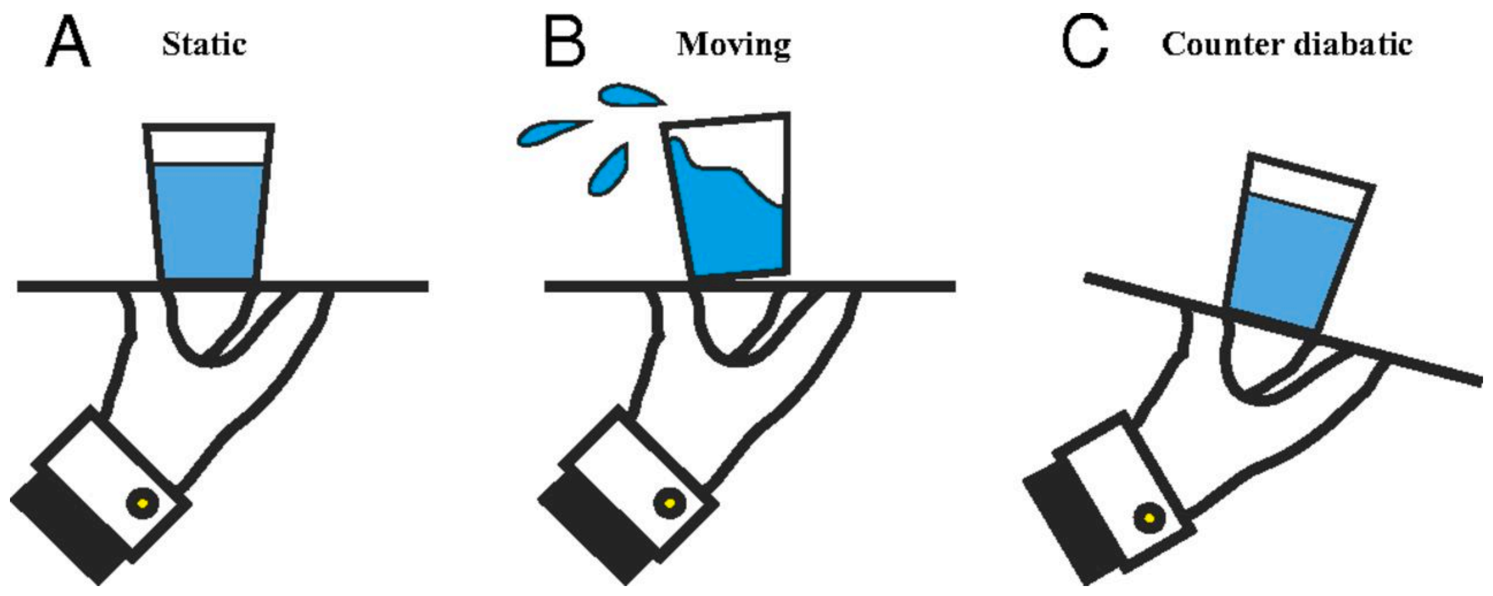
no folding: $H(t)$ extensive

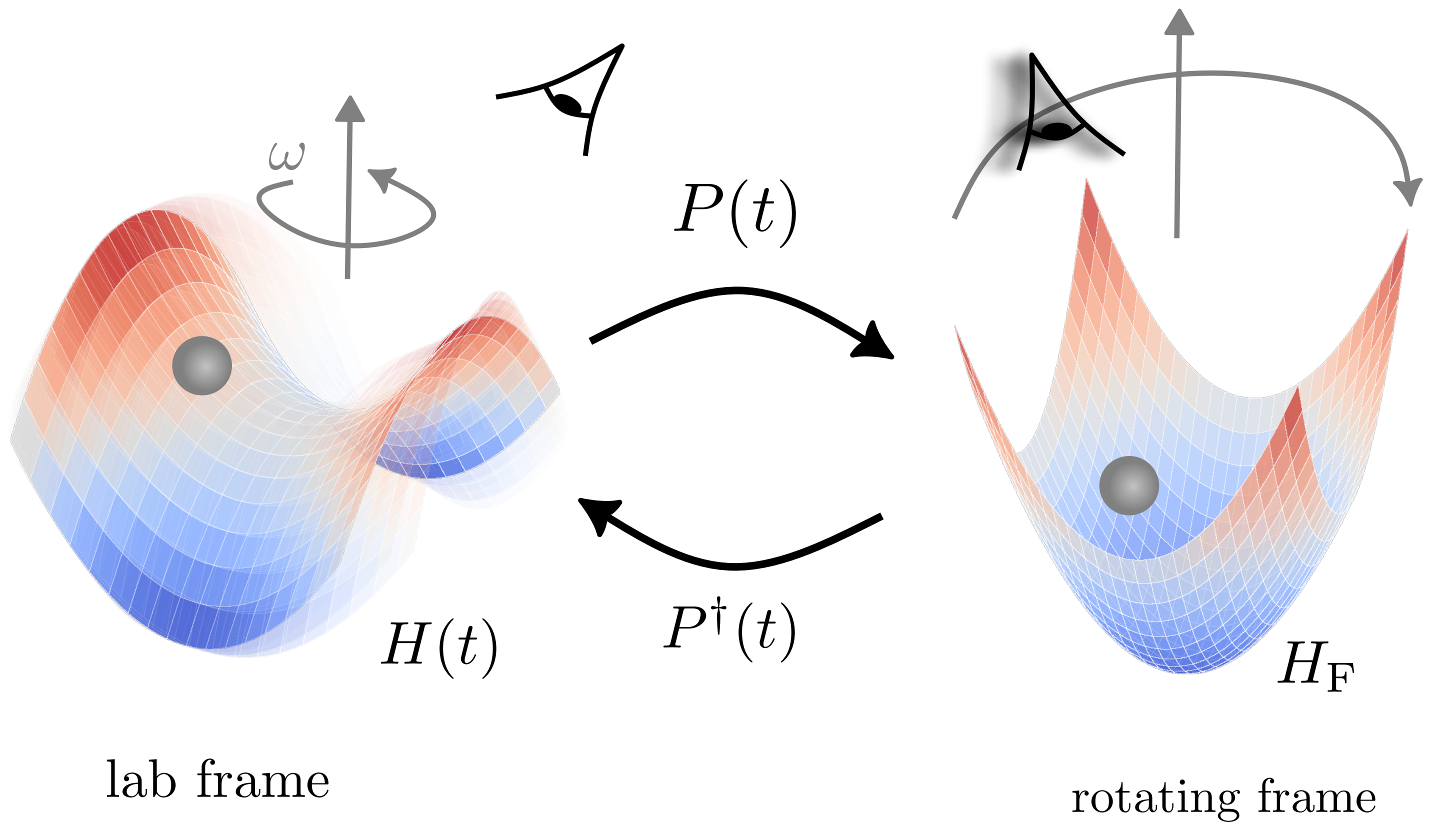
★ *sorts Floquet states !*

Outline

✓ Floquet theory from counterdiabatic driving
  - geometric Floquet decomposition
  - *unambiguous* Floquet ground state

- **Anomalous chiral spin liquids**
  - Floquet topological order
- **Time crystals**
  - nonequilibrium eigenstate order
- **Heating in kicked Ising chain**
  - locality of average energy



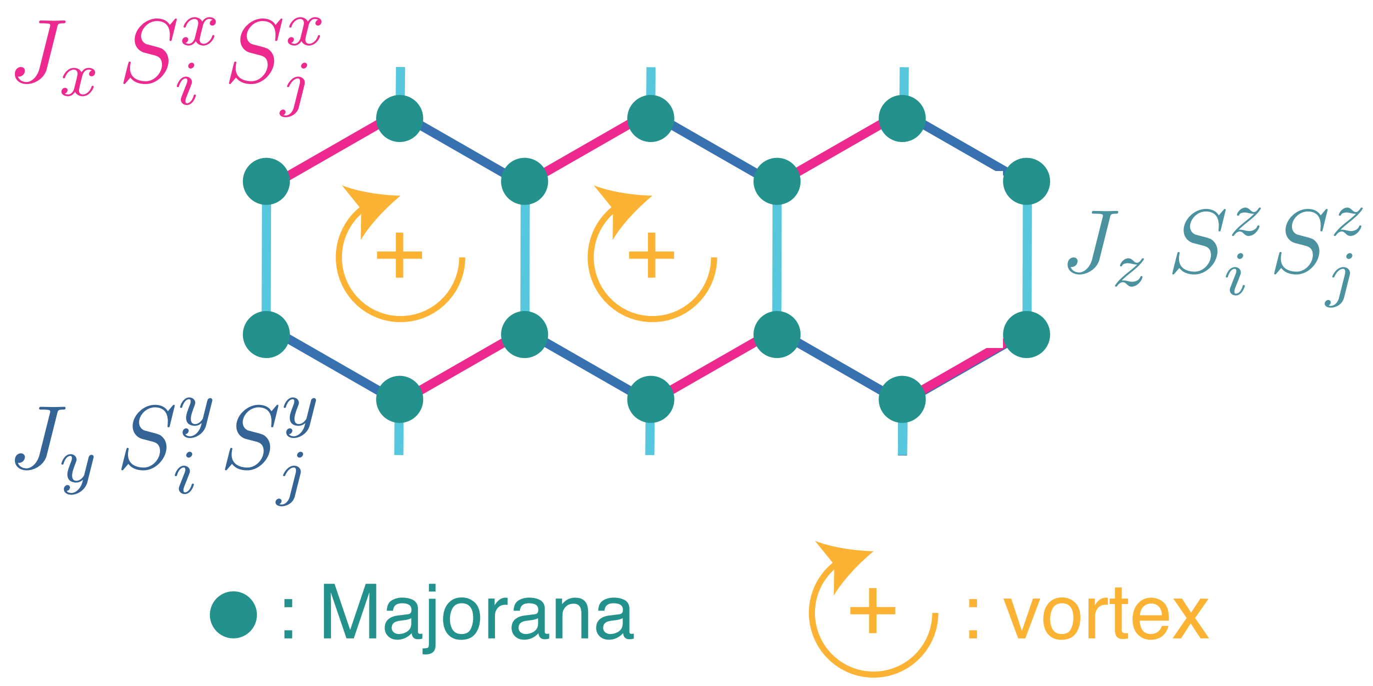


Schindler & MB, PRX 15, 031037 (2025)

Kitaev's honeycomb model

- **Hamiltonian:**

$H =$



$$= \sum_{\langle ij\rangle_\alpha} J^\alpha_{ij} S^\alpha_j S^\alpha_j$$

Kitaev, Annals of Physics 321, 2 (2006)

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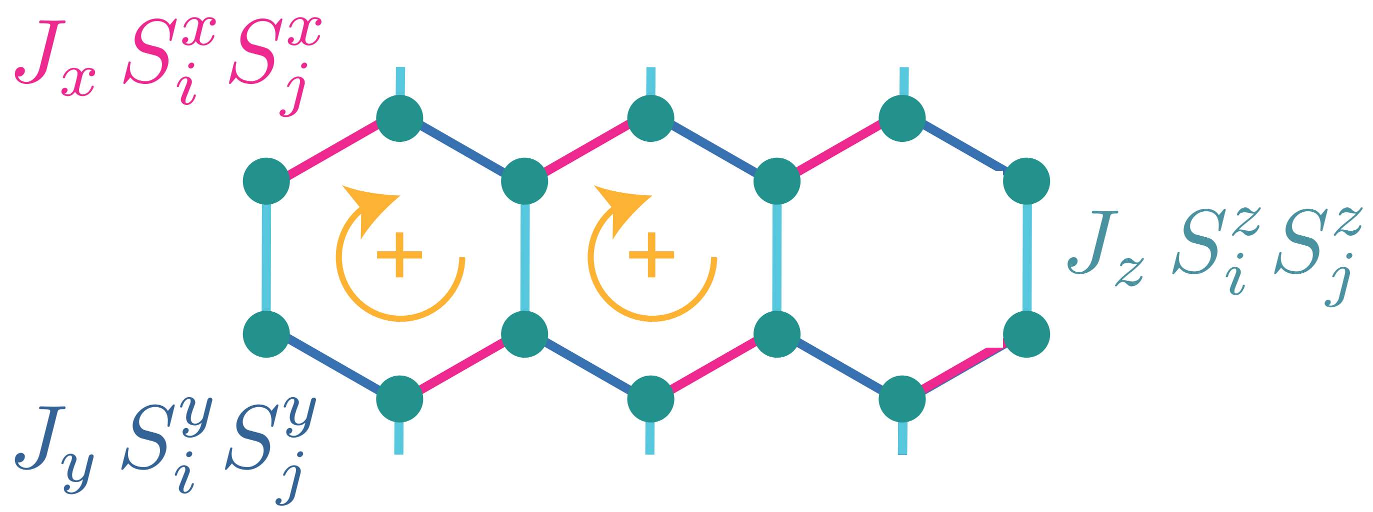
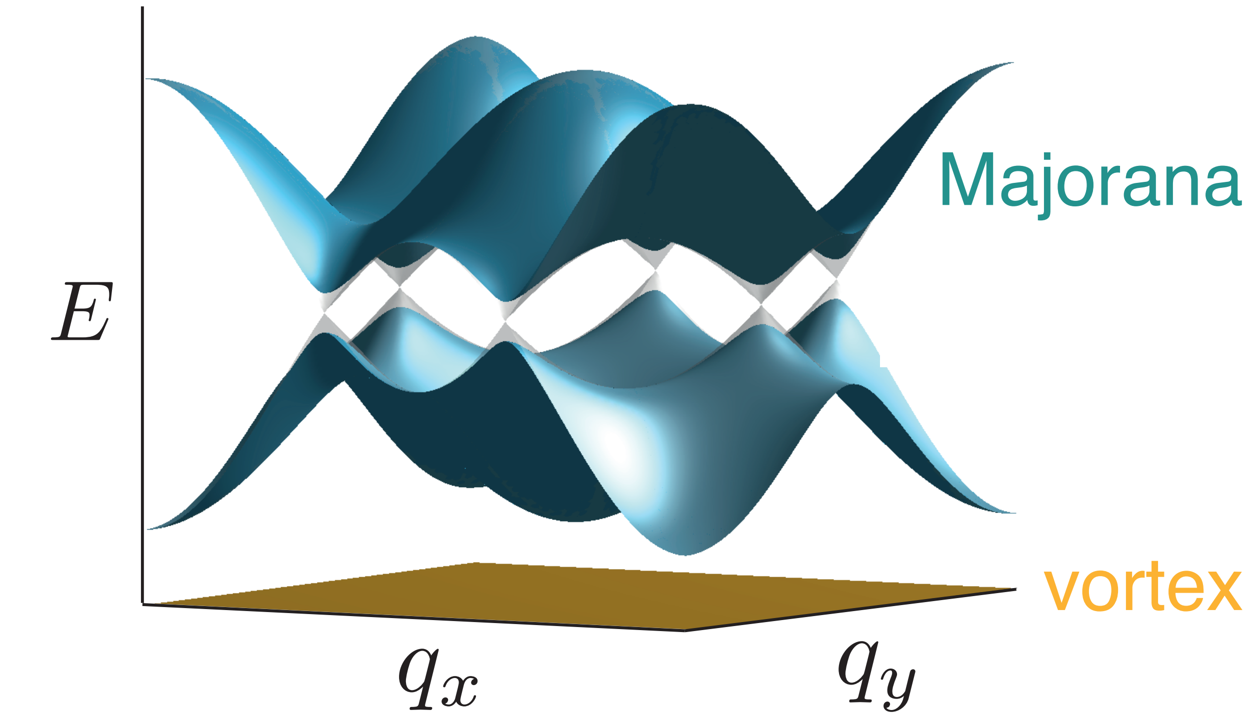


$$= \sum_{\langle ij\rangle_\alpha} J_{ij}^{\alpha} S_j^{\alpha} S_j^{\alpha}$$

  - fractional excitations

● : Majorana   ⊕ : vortex

dispersion relation (zero-flux sector)



Kitaev, Annals of Physics 321, 2 (2006)

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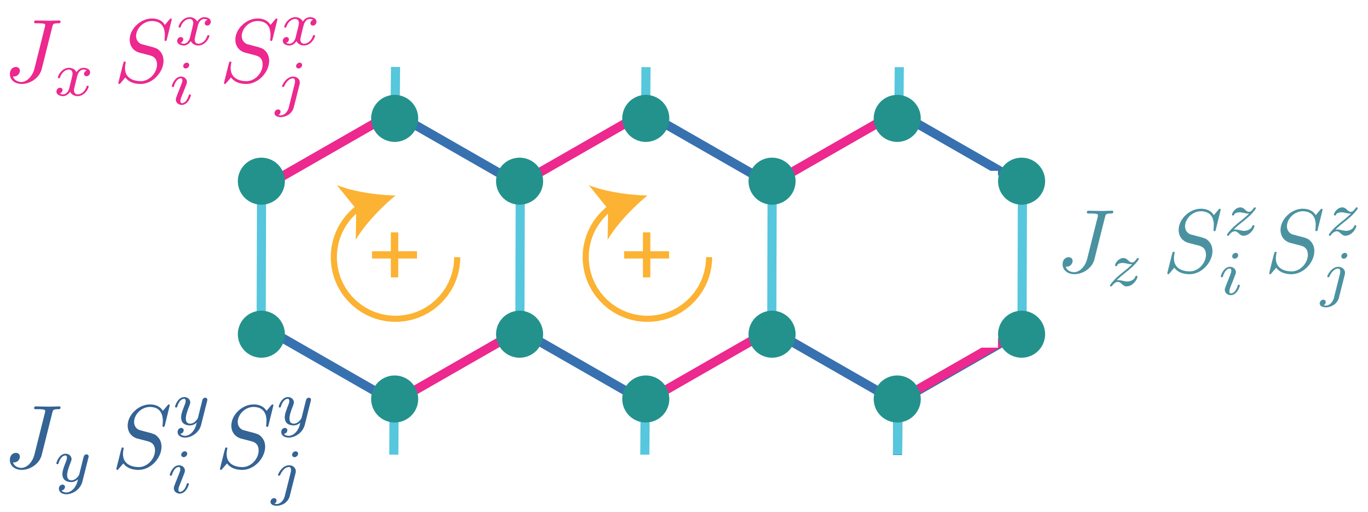

$$= \sum_{\langle ij\rangle_\alpha} J_{ij}^\alpha S_j^\alpha S_j^\alpha$$

  - fractional excitations ● : Majorana ⊕ : vortex
  - extensive number of conserved flux operators $W_p = S_1^x S_2^y S_3^z S_4^x S_5^y S_6^z$
    - work in fixed flux sector: integrability $[H, W_p] = 0$

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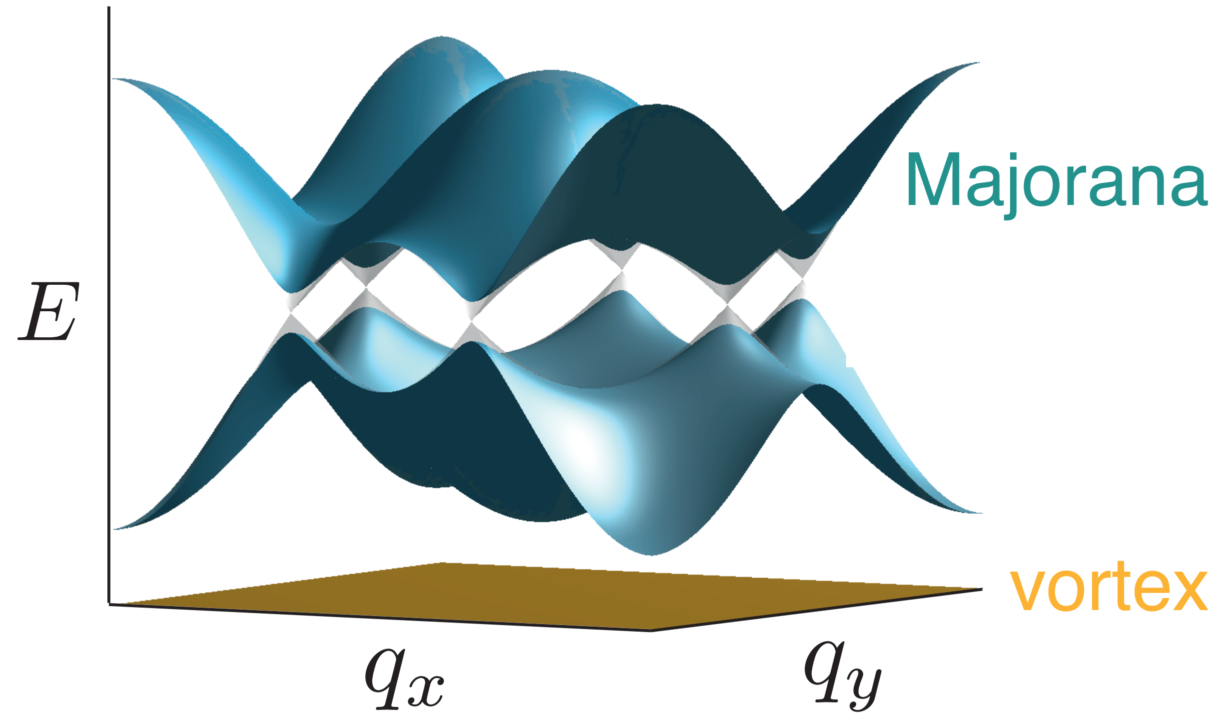

Kitaev, Annals of Physics 321, 2 (2006)

Kitaev's honeycomb model

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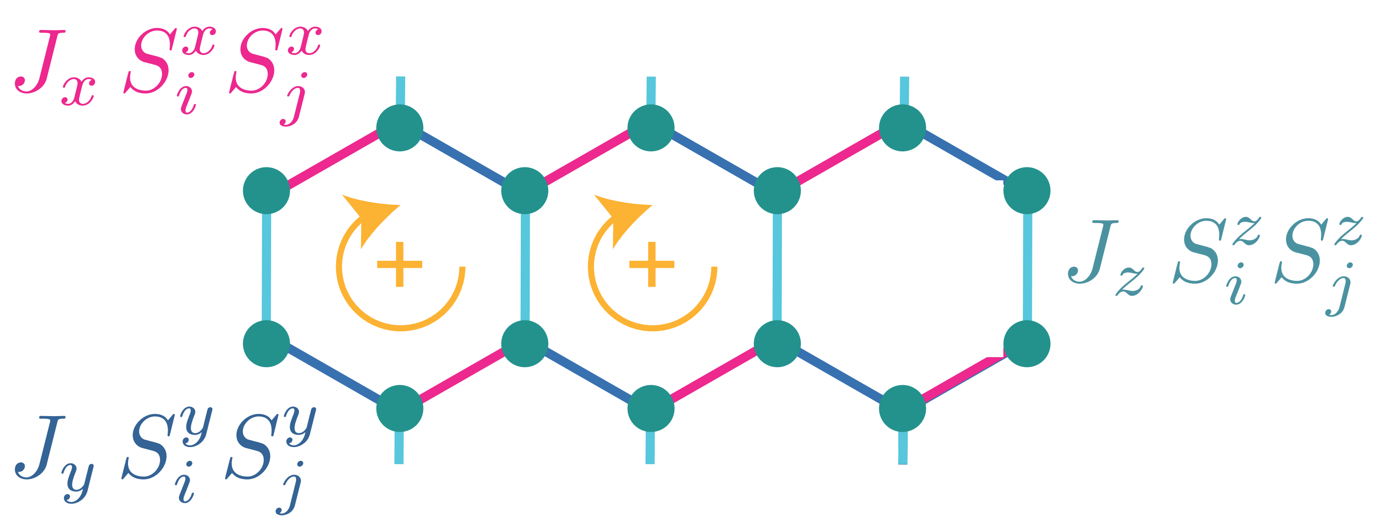

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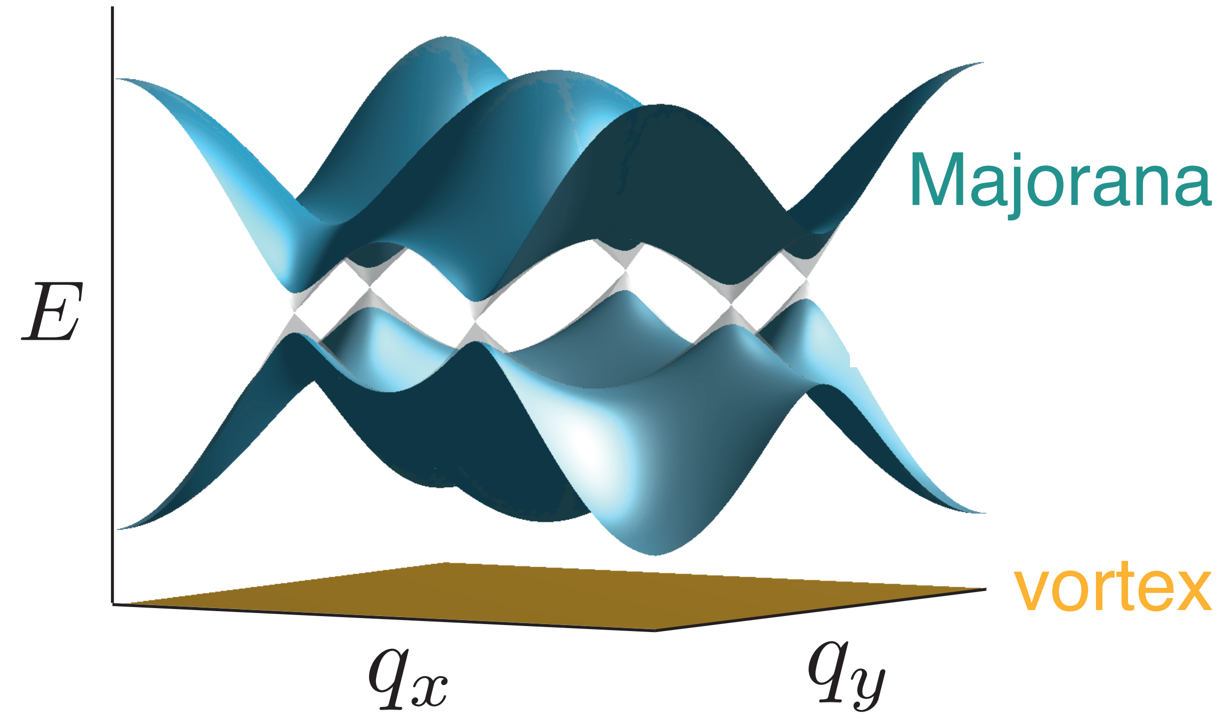
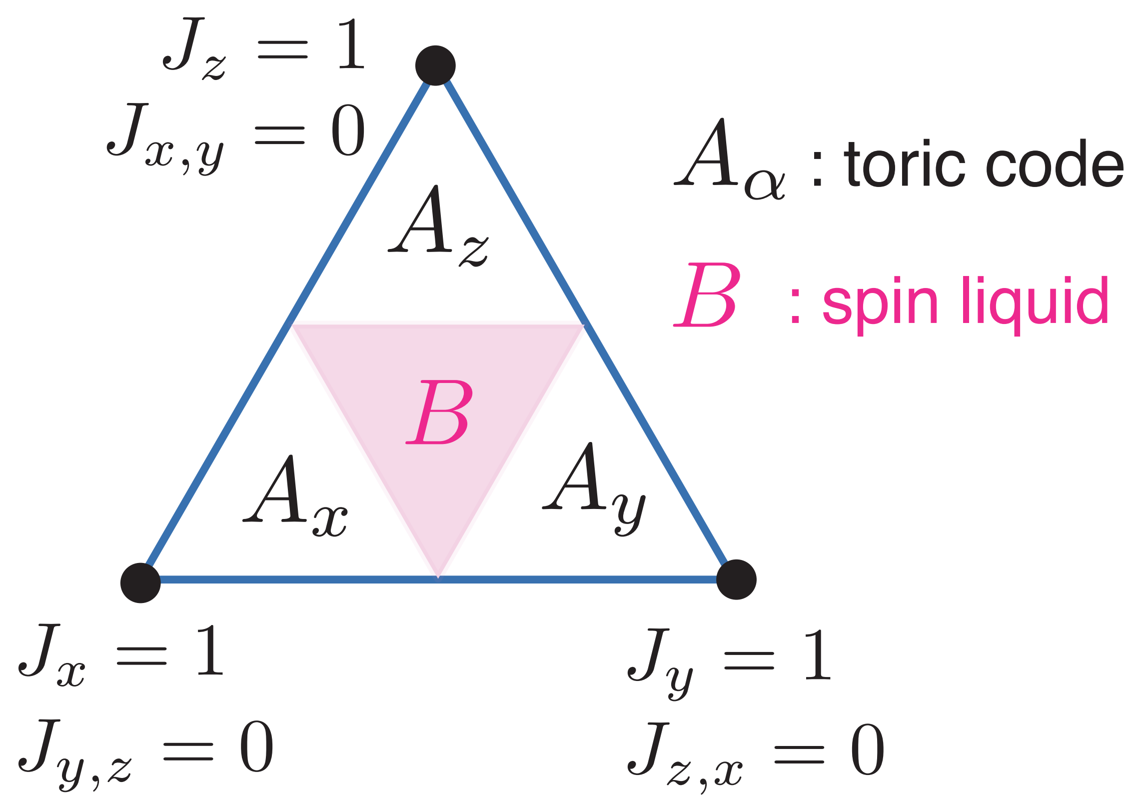


- **ground state phase diagram**
  - gapped toric code phases A
    - applications in quantum error correction
  - gapless spin liquid phase B
    - excitations: abelian anyons



Kitaev, Annals of Physics 321, 2 (2006)

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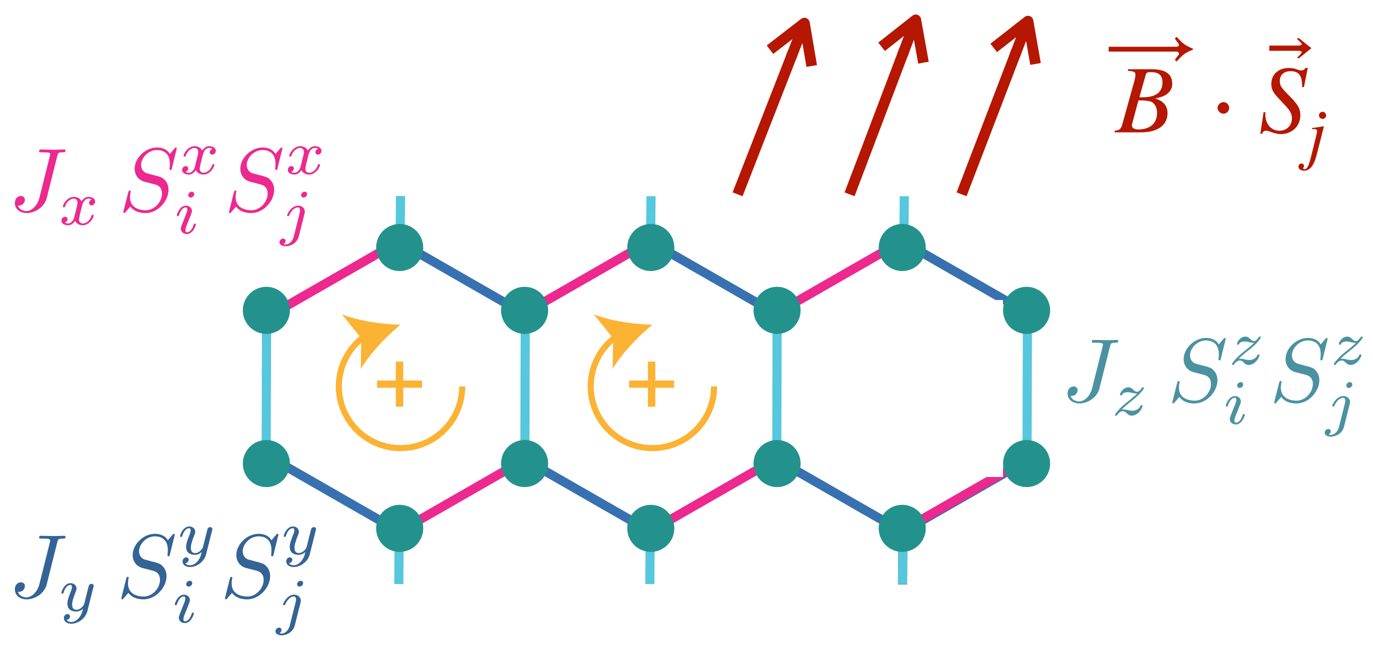
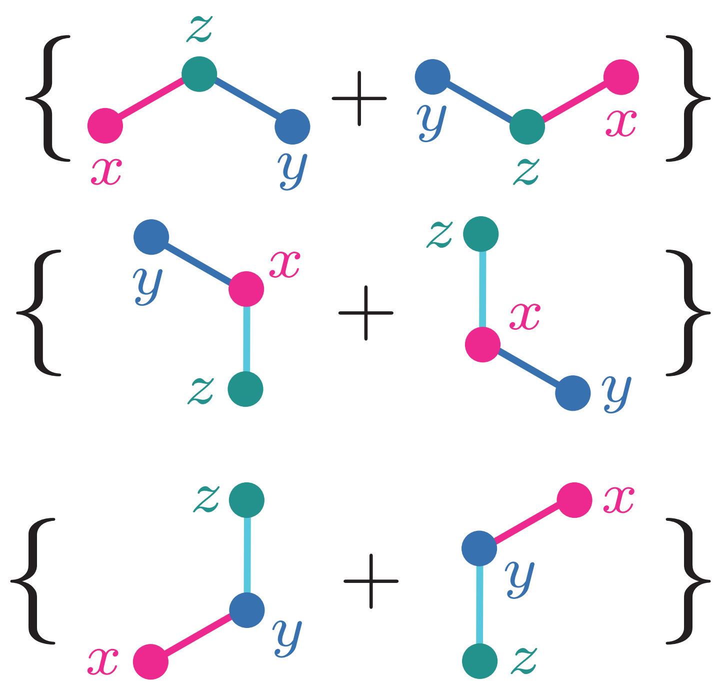


$\vec{B} \cdot \vec{S}_j$ external field

$$\approx \sum_{\langle ij\rangle_\alpha} J_{ij}^\alpha S_j^\alpha S_j^\alpha + h \sum_{[ijk]_{\alpha\beta\gamma}} S_i^\alpha S_j^\beta S_k^\gamma$$



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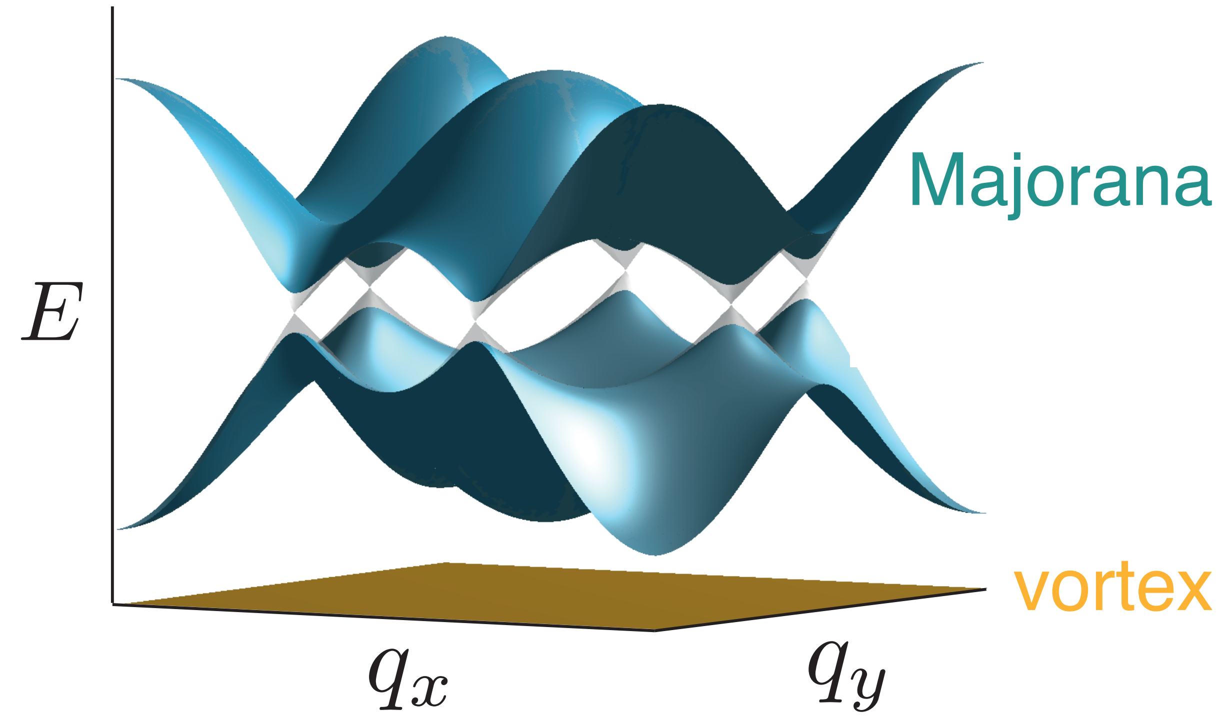

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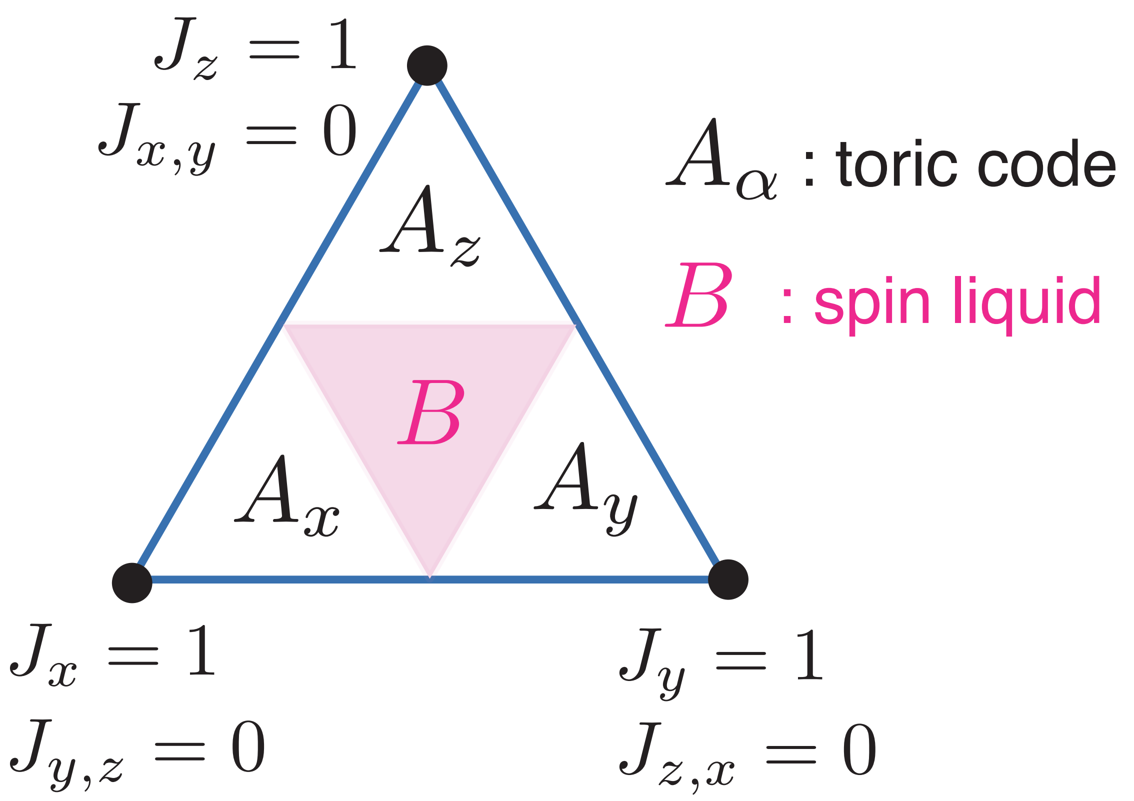

Marín Bukov

Kitaev, Annals of Physics 321, 2 (2006)

Kitaev’s honeycomb model

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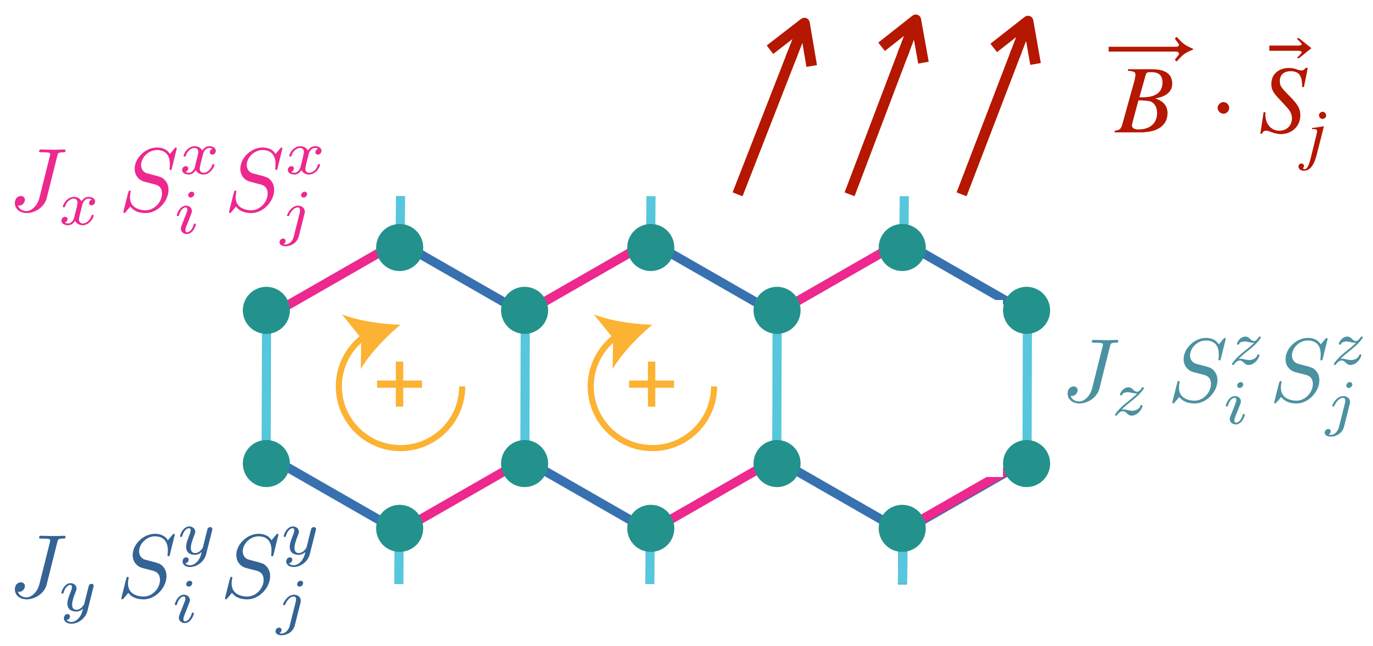
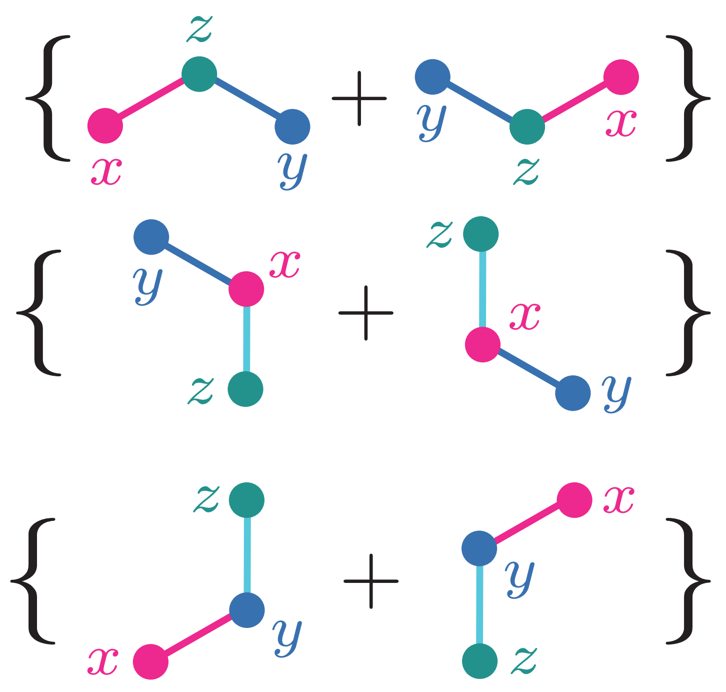


$$H = \quad \approx \sum_{\langle ij\rangle_\alpha} J^\alpha_{ij} S^\alpha_j S^\alpha_j + h \sum_{[ijk]_{\alpha\beta\gamma}} S_i^\alpha S_j^\beta S_k^\gamma$$



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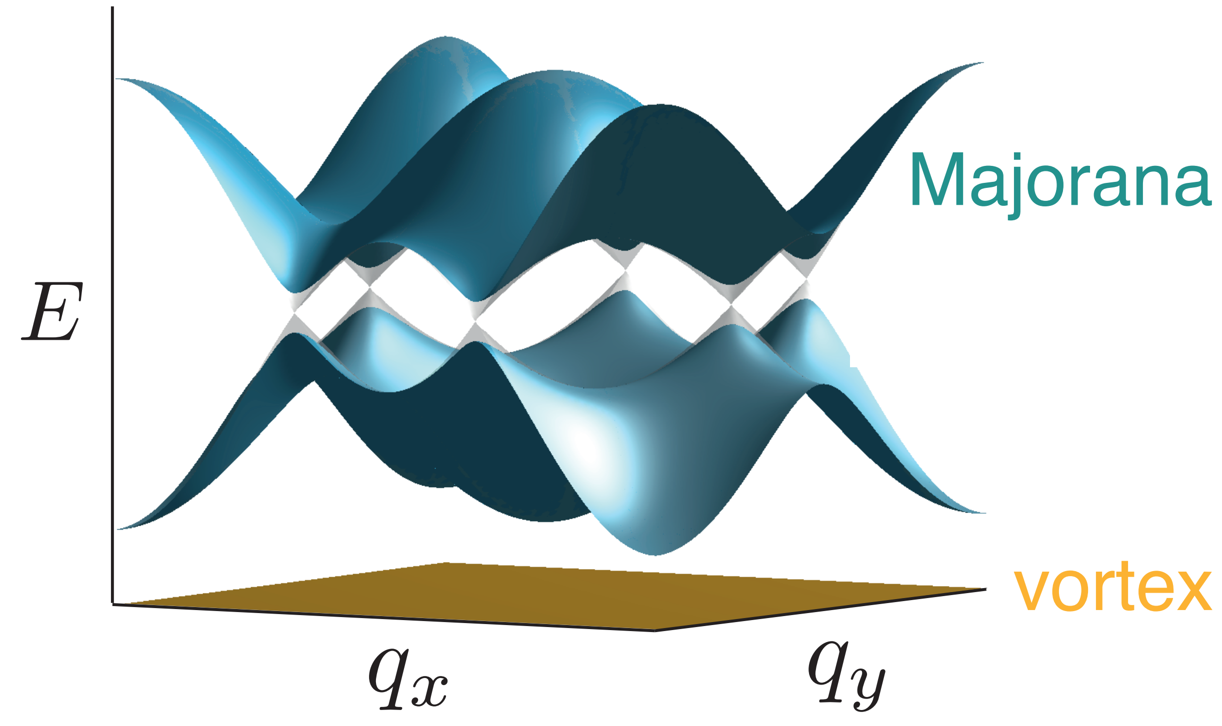

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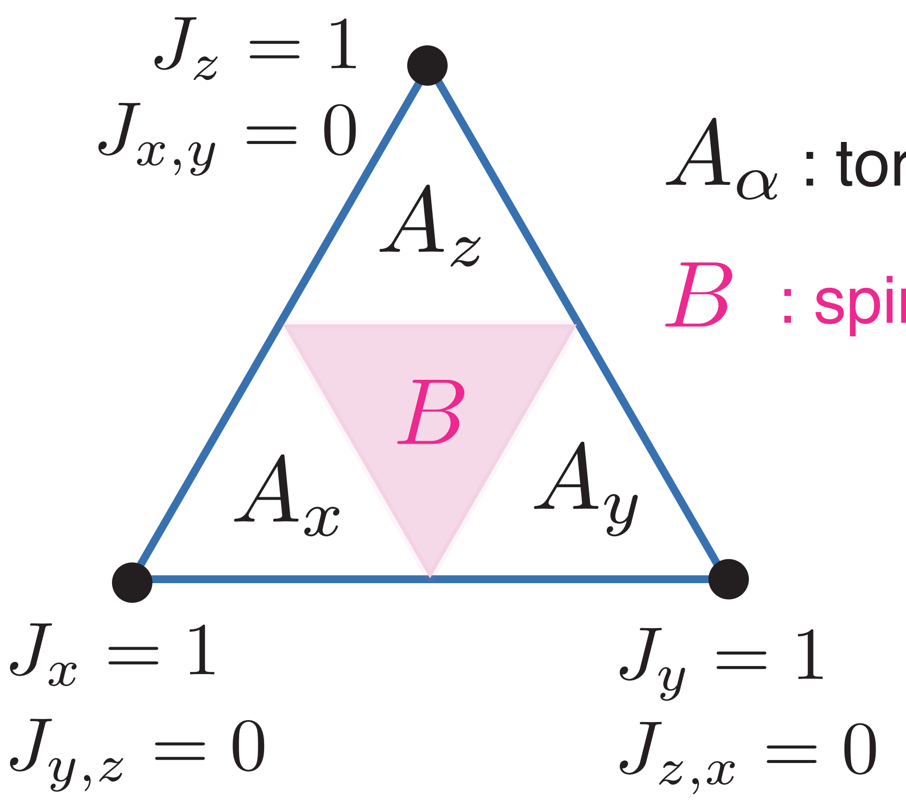


A_α : toric code

B : spin liquid

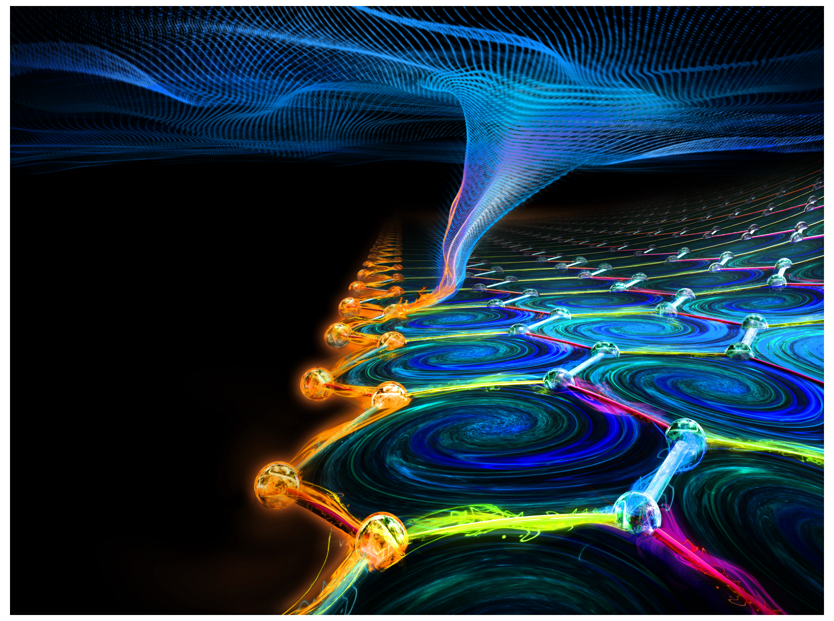


image: Nathan Goldman

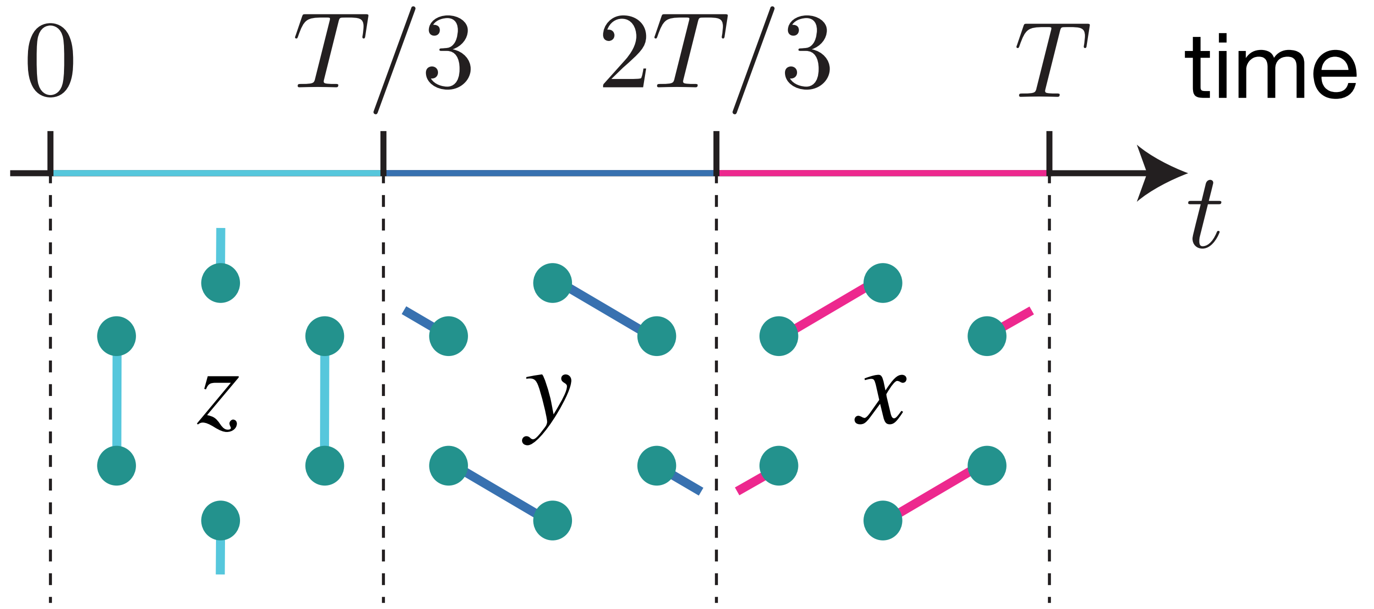
Marín Bukov

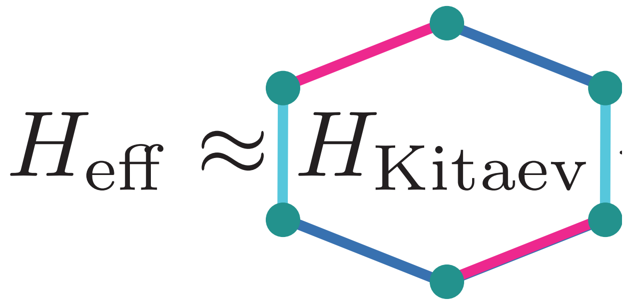
Kitaev, Annals of Physics 321, 2 (2006)

Floquet realization of Kitaev's honeycomb model

- apply 3-step periodic switching of bond couplings
  - high-frequency regime: $\omega_D = 2\pi/T$

$$U_F = e^{iTJ/3 \sum_{x-\text{links}} S_i^x S_j^x} \times e^{iTJ/3 \sum_{y-\text{links}} S_i^y S_j^y} \times e^{iTJ/3 \sum_{z-\text{links}} S_i^z S_j^z}$$





time average

Sun, Aidelsburger, Goldman, MB, PRXQ 4, 020329 (2023)
Kalinowski et al, PRX 13, 031008 (2023)

Kitaev, Annals of Physics 321, 2 (2006) **Po et al, PRB 96, 245116 (2017)**

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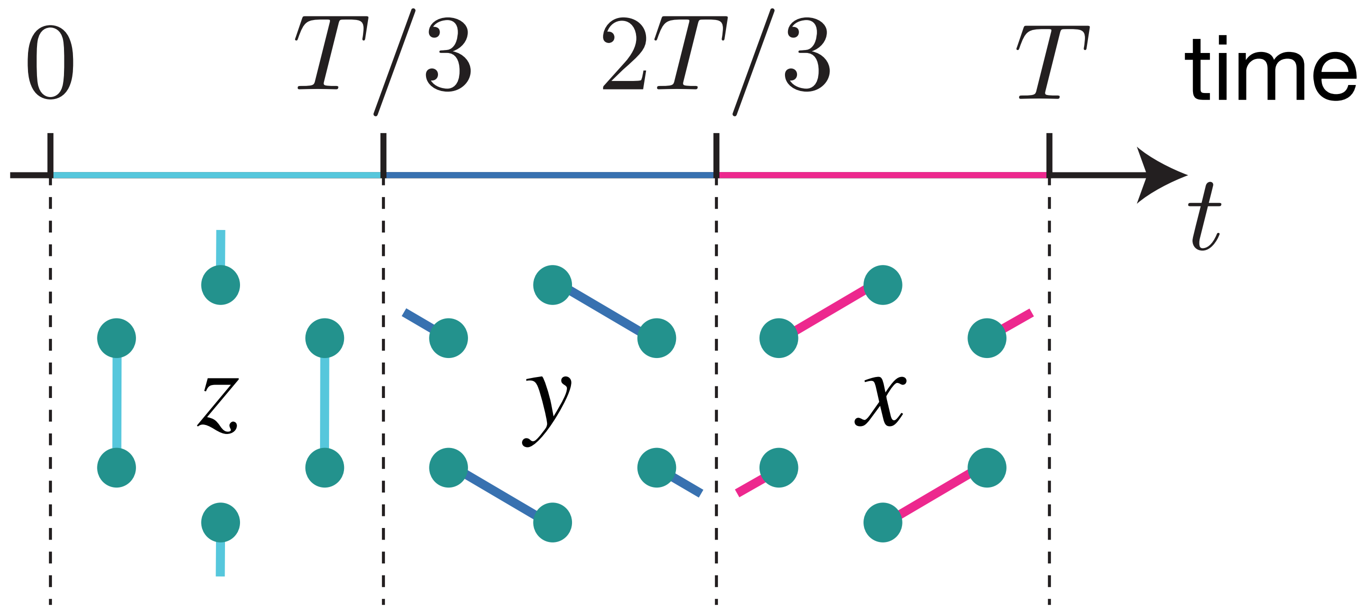
  - high-frequency regime: $\omega_D = 2\pi/T$

$$H_{\text{eff}} \approx H_{\text{Kitaev}} + \frac{\pi}{3\omega_D} \sum_{[ijk]_{\alpha\beta\gamma}} J_\alpha J_\gamma S_i^\alpha S_j^\beta S_k^\gamma$$

time average

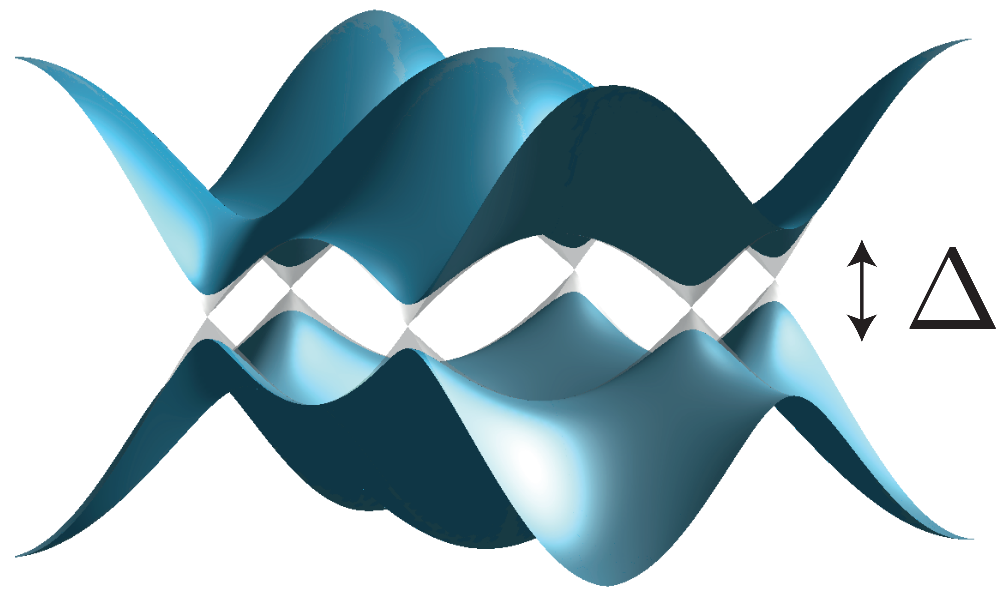
leading ω_D^{-1} correction



Sun, Aidelsburger, Goldman, MB, PRXQ 4, 020329 (2023)

Kalinowski et al, PRX 13, 031008 (2023)

- step drive breaks time-reversal & opens up chiral Majorana gap $\Delta \sim \omega_D^{-1}$ ($\Rightarrow$ non-abelian anyons)



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Po et al, PRB 96, 245116 (2017)

Floquet realization of Kitaev’s honeycomb model

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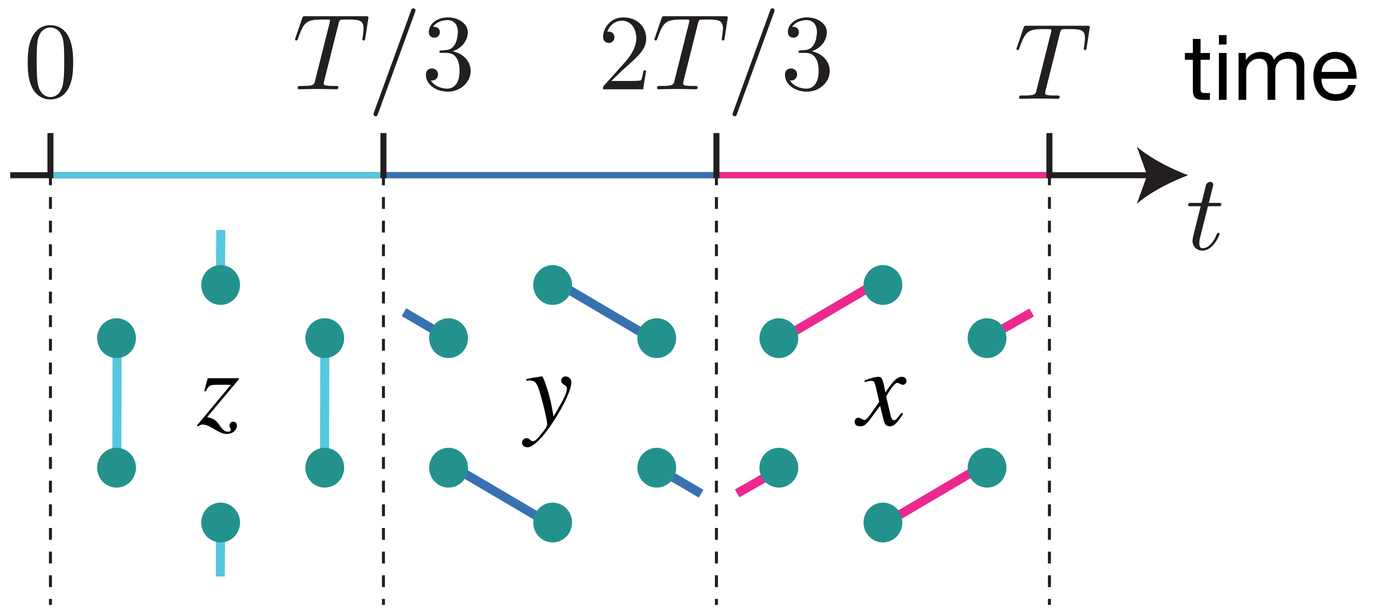

Sun, Aidelsburger, Goldman, MB, PRXQ 4, 020329 (2023)
Kalinowski et al, PRX 13, 031008 (2023)

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time average   leading ω_D^{-1} correction

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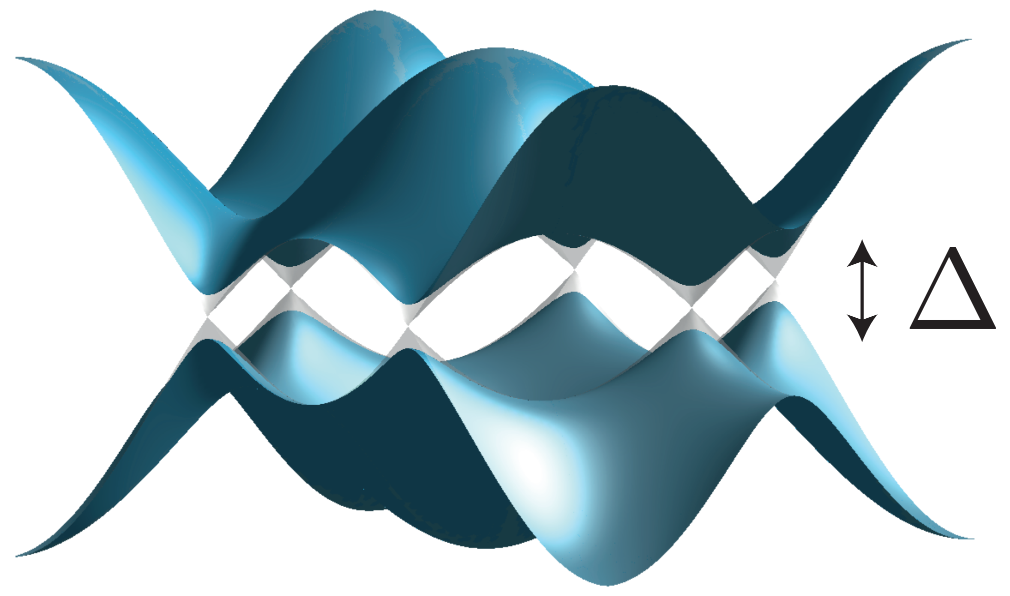
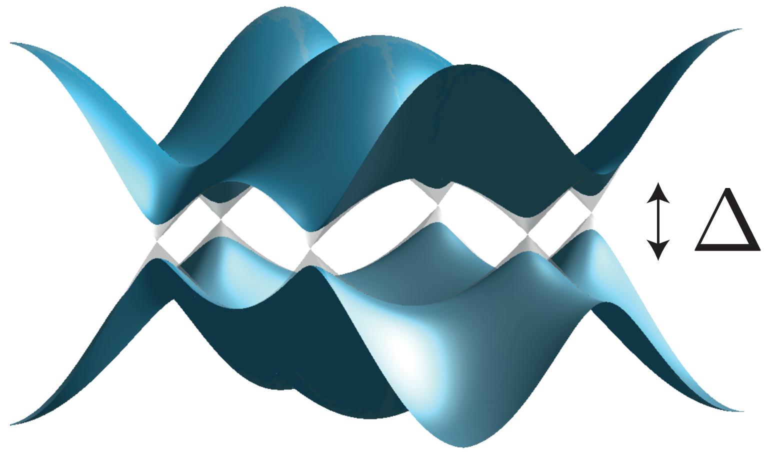
    - model is integrable at any drive period T



Kitaev, Annals of Physics 321, 2 (2006)   **Po et al, PRB 96, 245116 (2017)**



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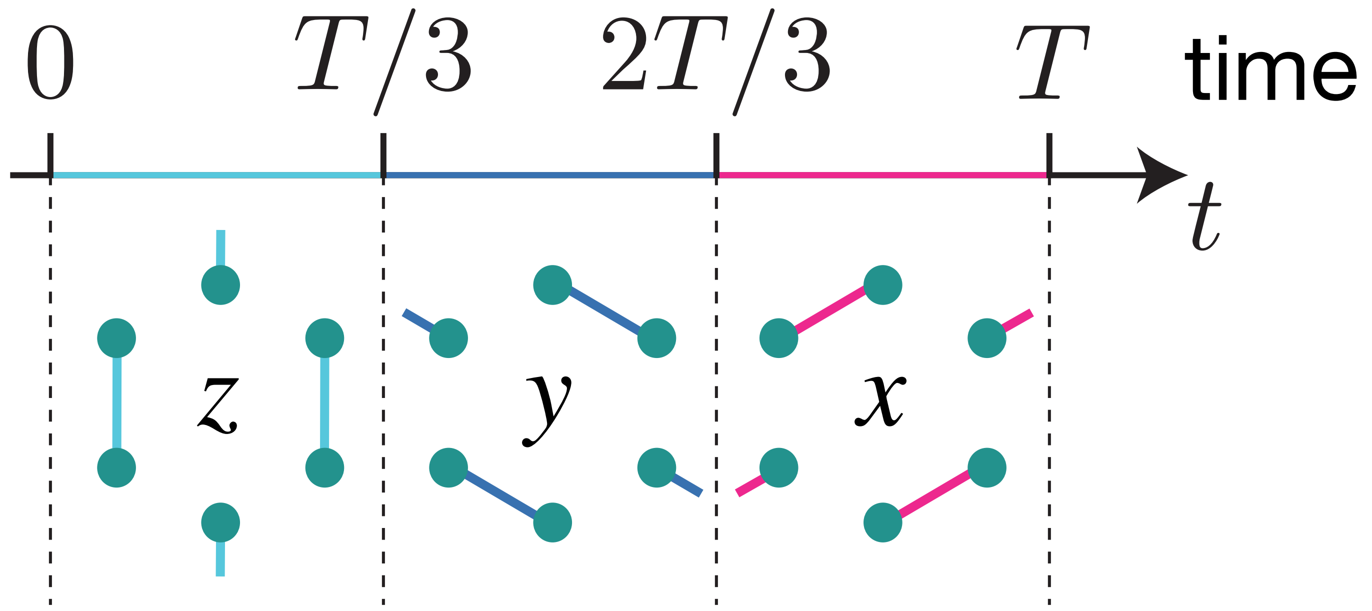

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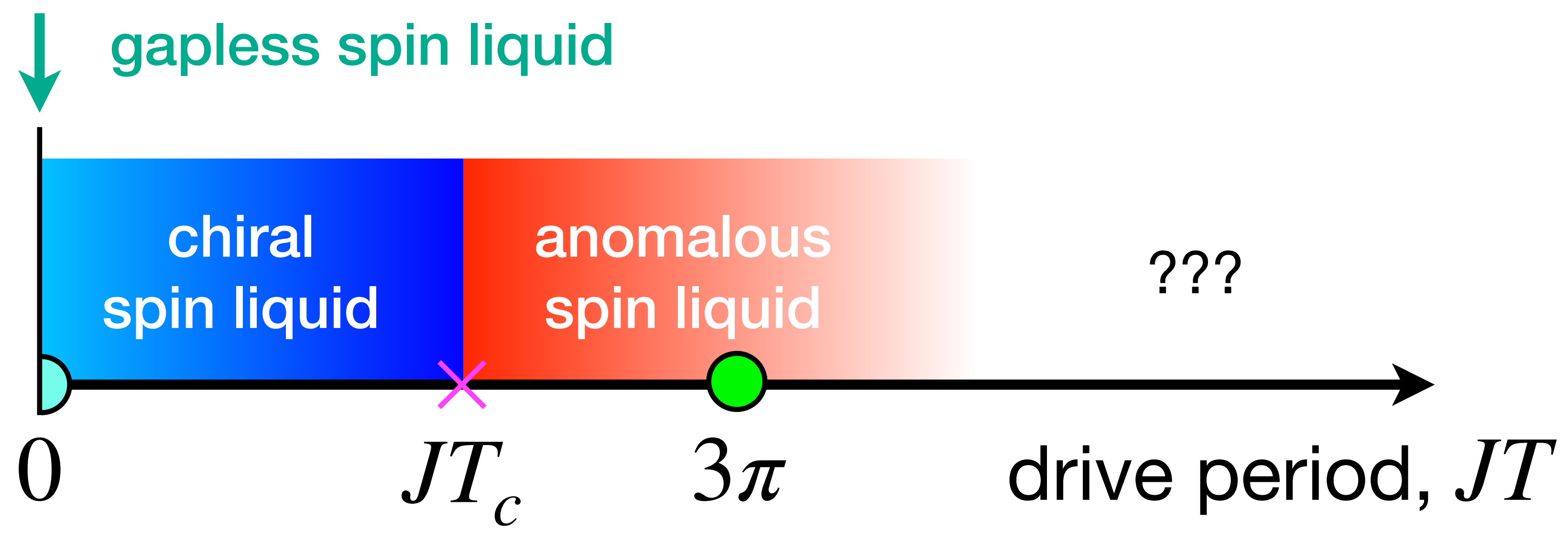
  - **intermediate frequency:** Floquet topological order
    - exactly solvable point $JT/3 = \pi$

phase diagram



Kitaev, Annals of Physics 321, 2 (2006) **Po et al, PRB 96, 245116 (2017)** **Mambrini et al, arXiv:2512.23418**

experts

Anomalous Floquet chiral spin liquid phase

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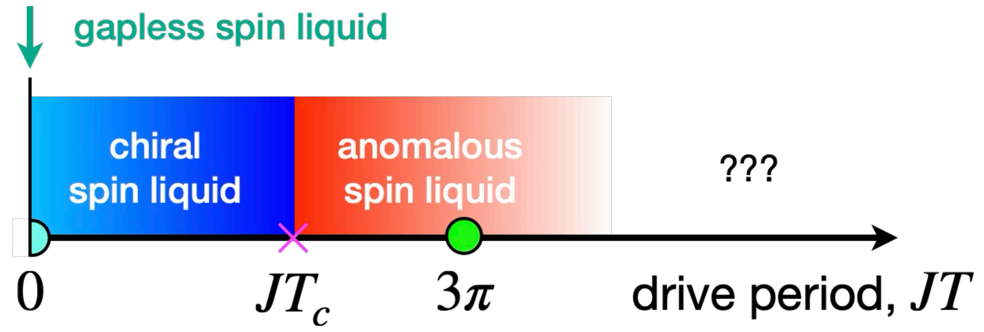
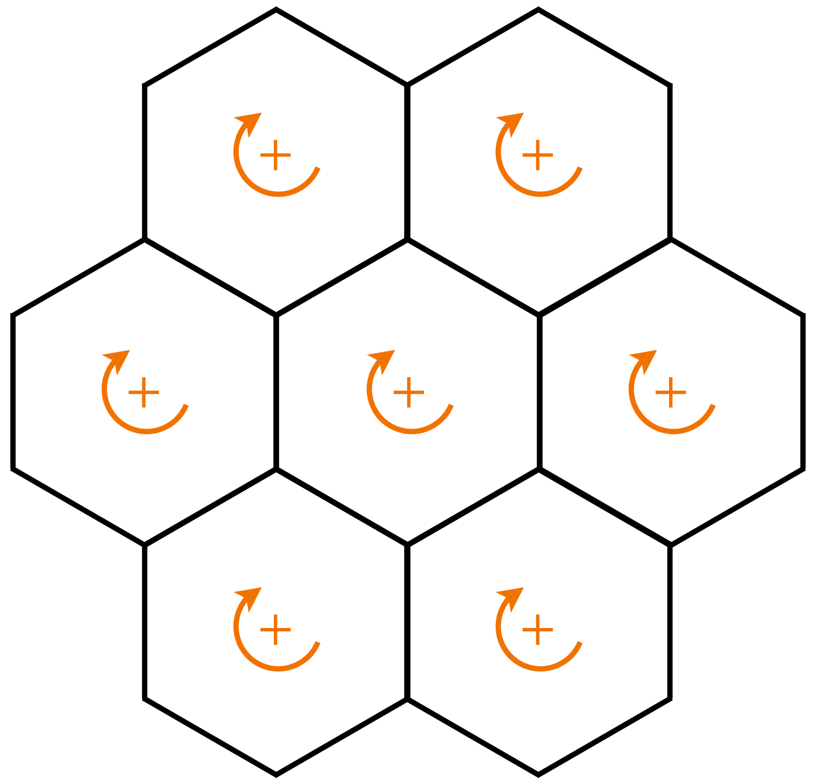
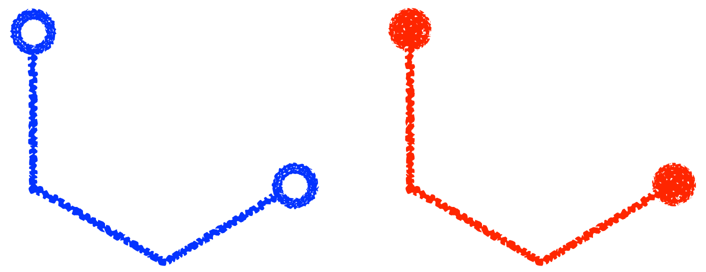


- characteristic feature:



flux state



unoccupied / occupied
paired Majorana state
(complex fermion ψ)

Po et al, PRB 96, 245116 (2017)

experts

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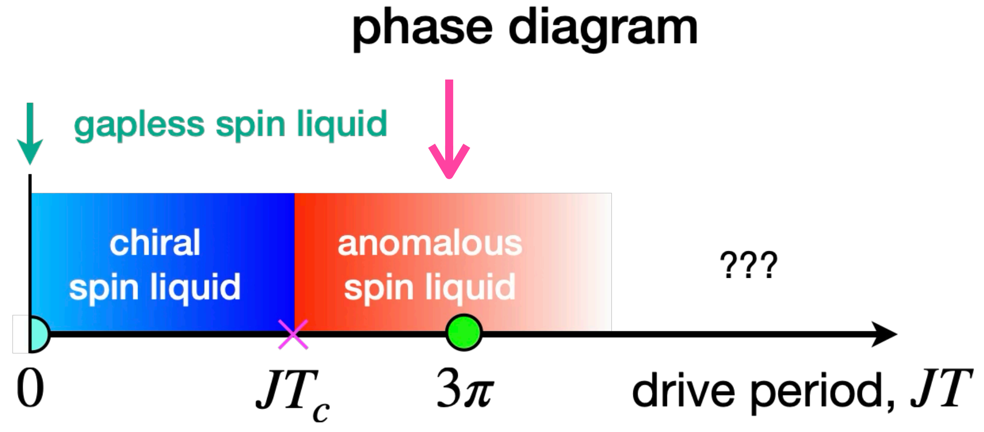

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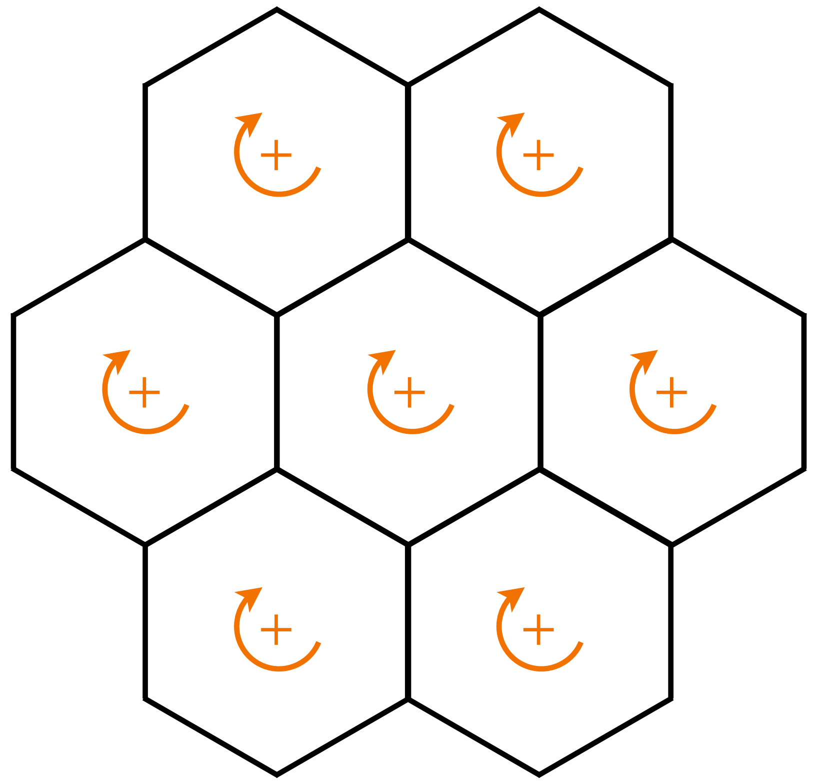

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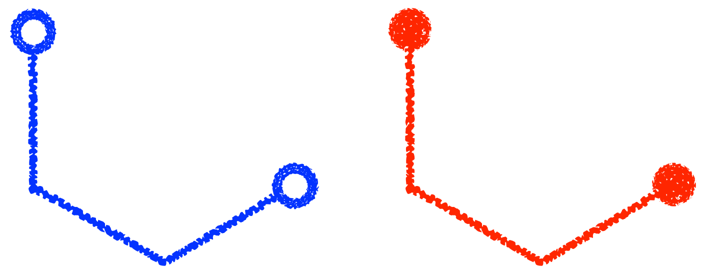

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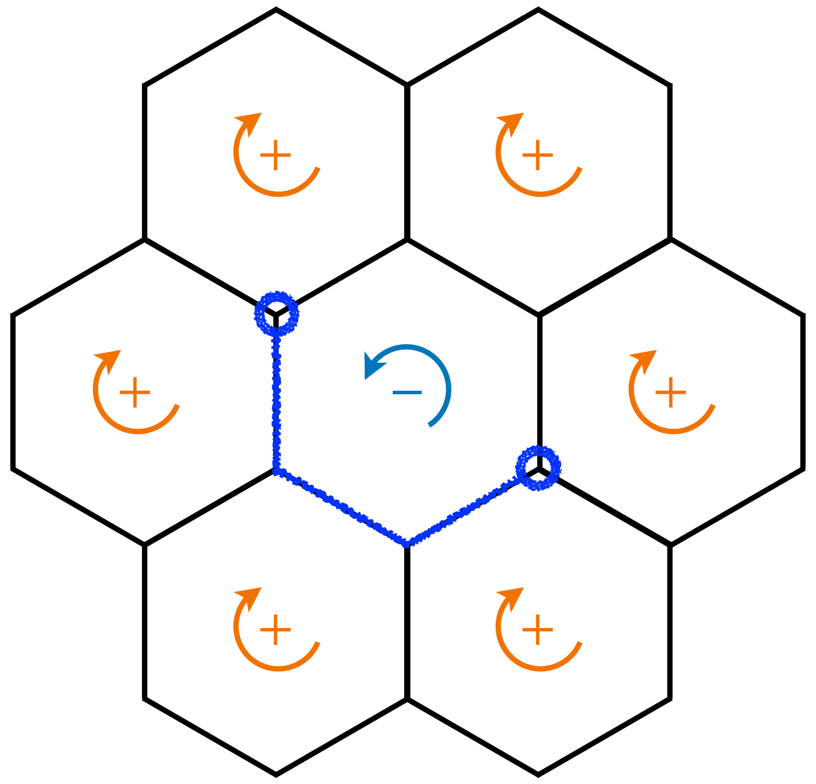
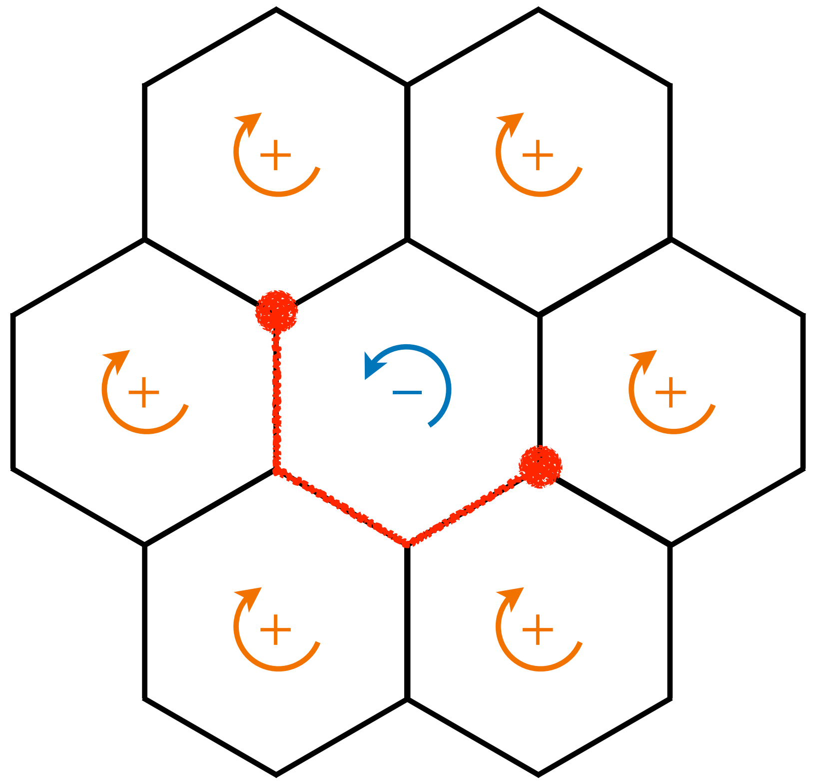


magnetic anyon: $|\, m\rangle$



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Po et al, PRB 96, 245116 (2017)

experts

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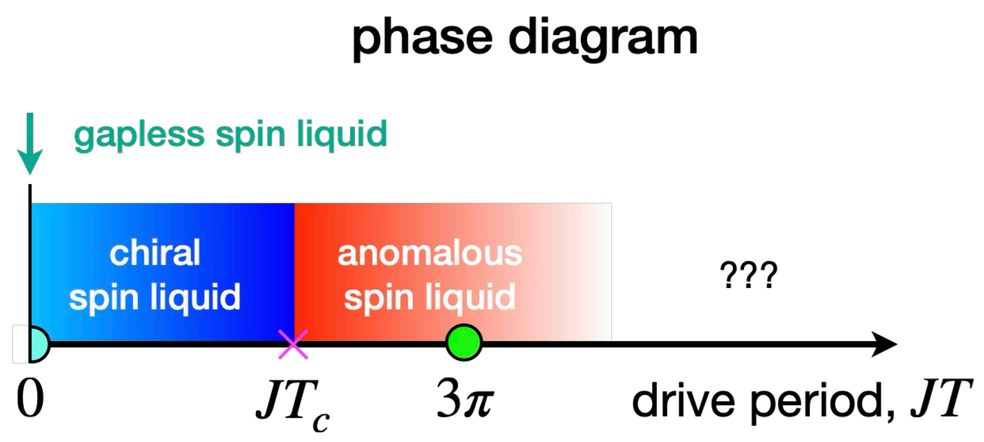

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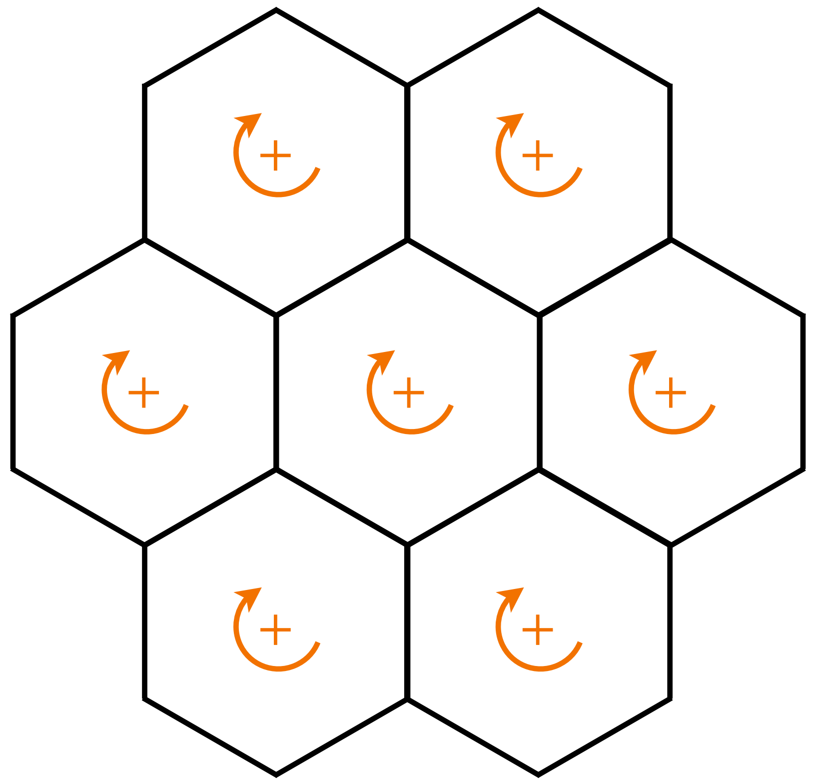
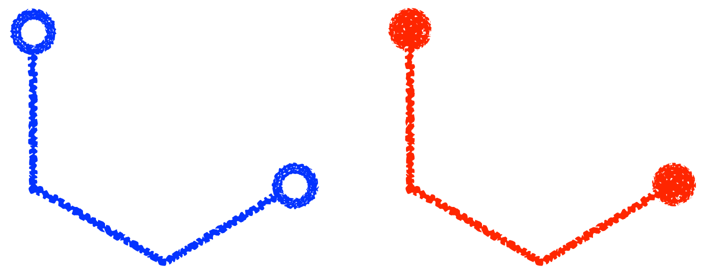


flux state



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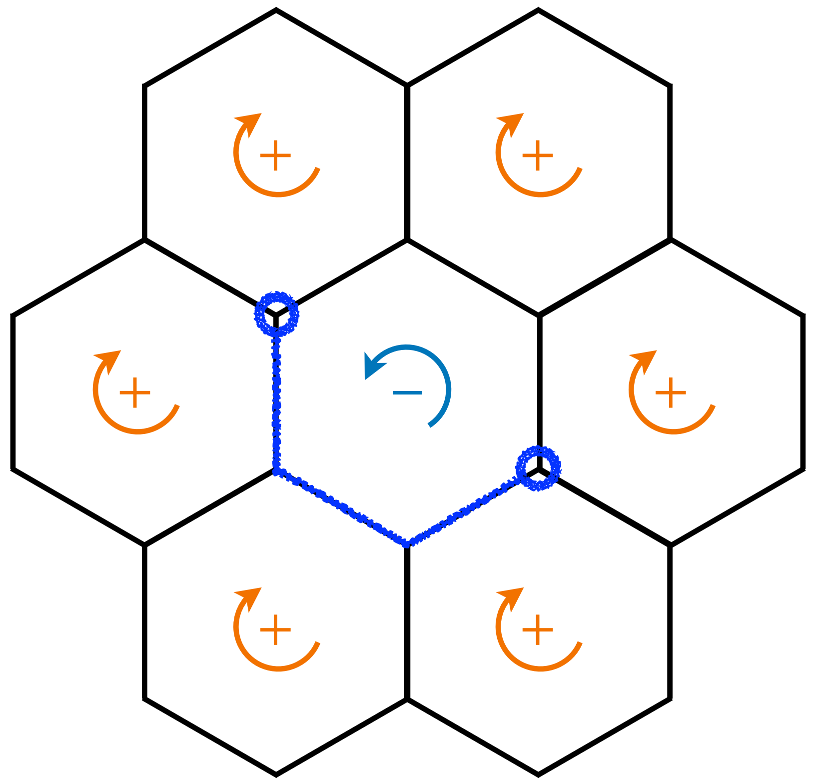

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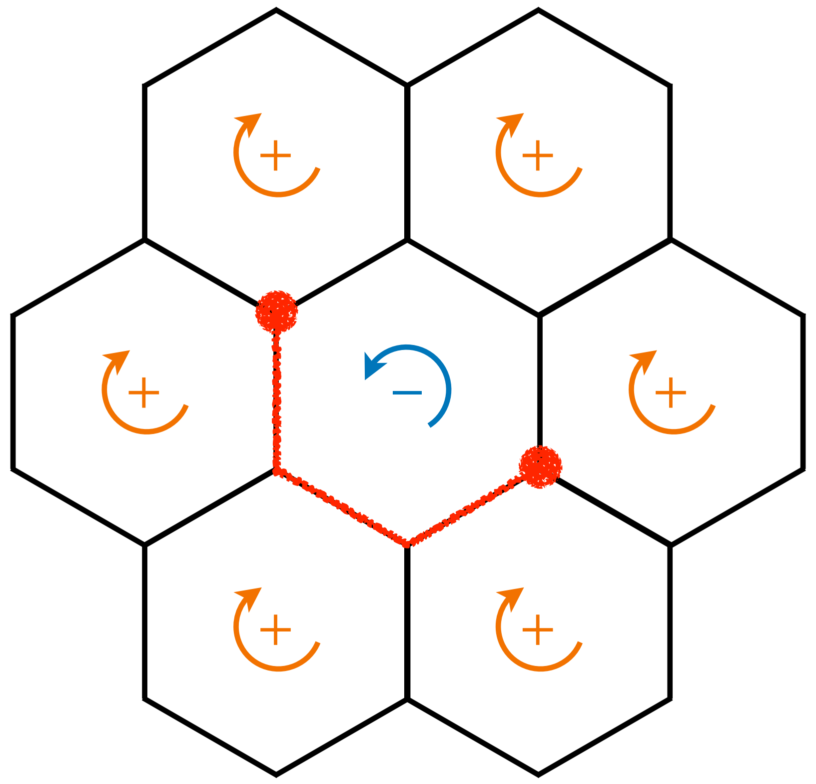

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Po et al, PRB 96, 245116 (2017)

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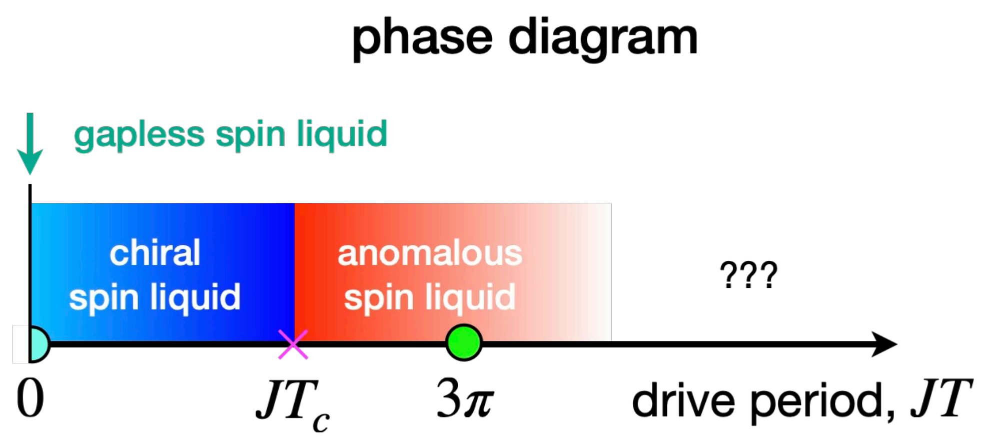

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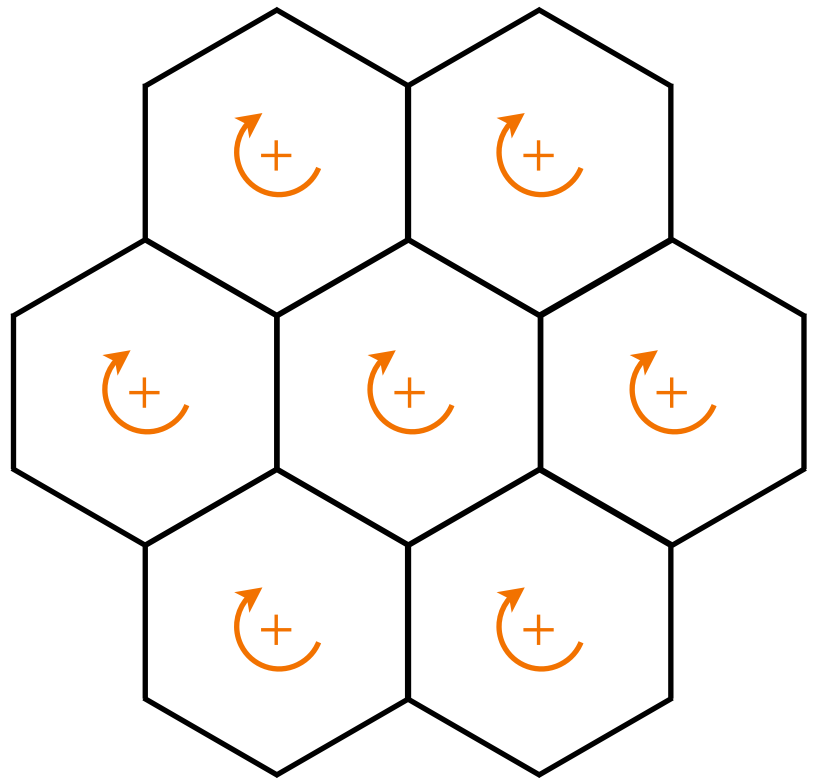

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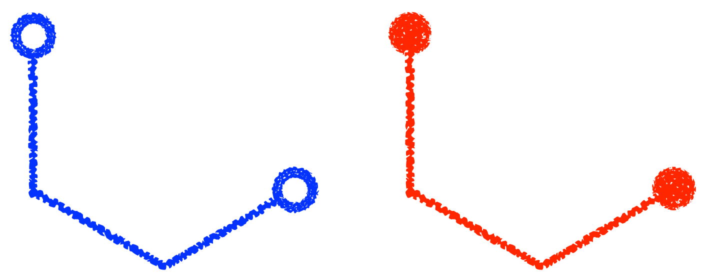

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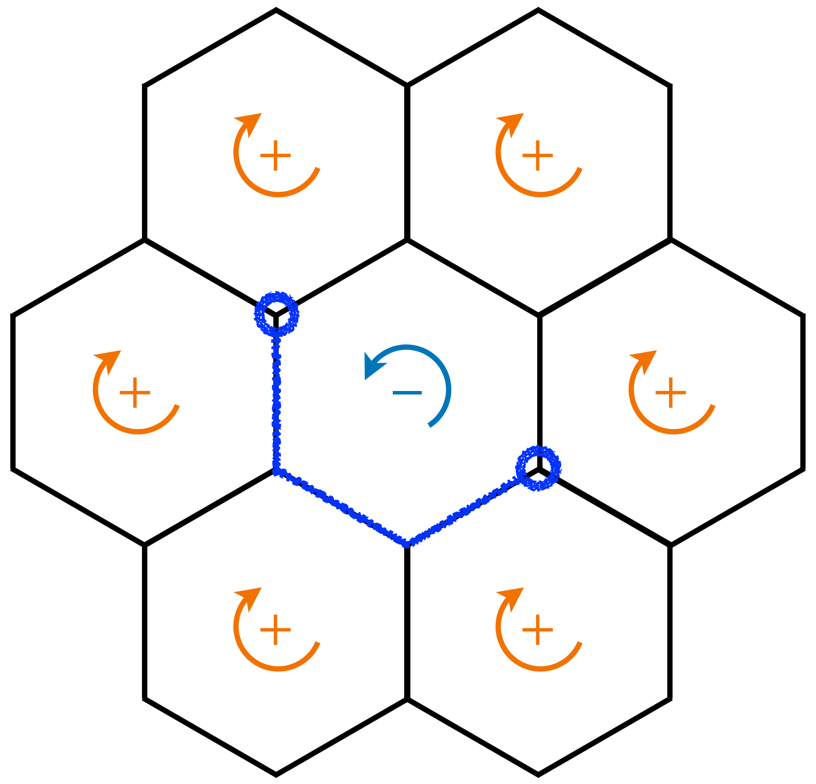

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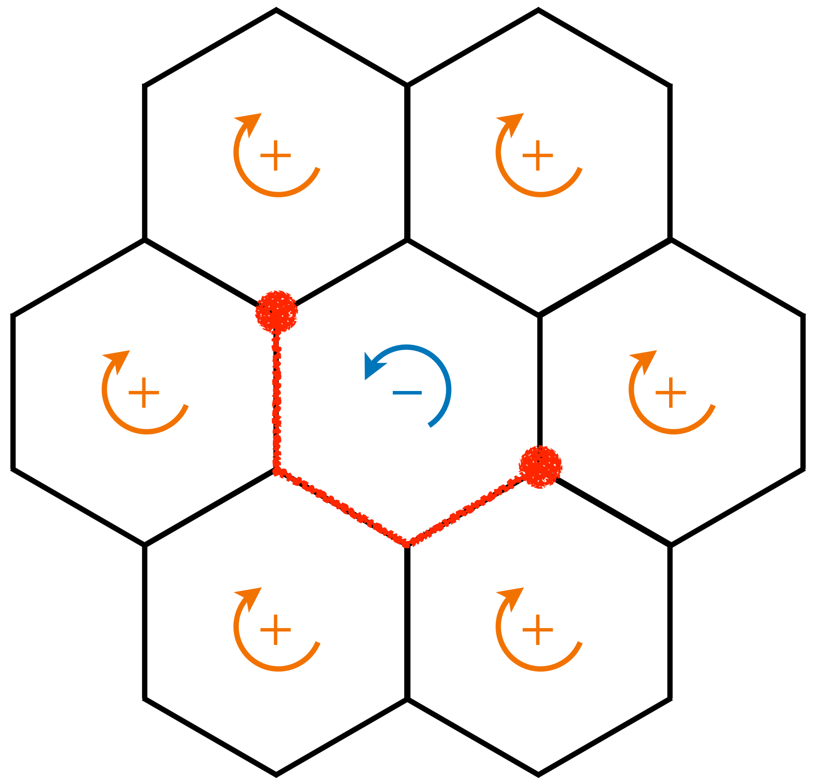

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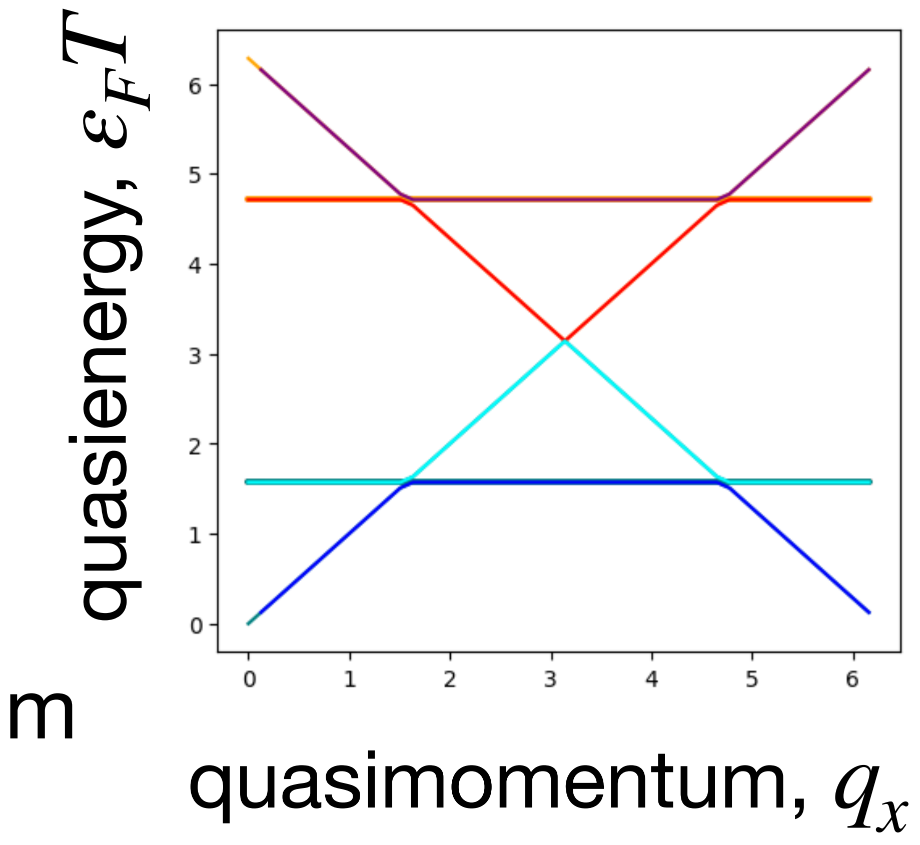

Po et al, PRB 96, 245116 (2017)

Rudner et al, PRX 3, 031005 (2013)

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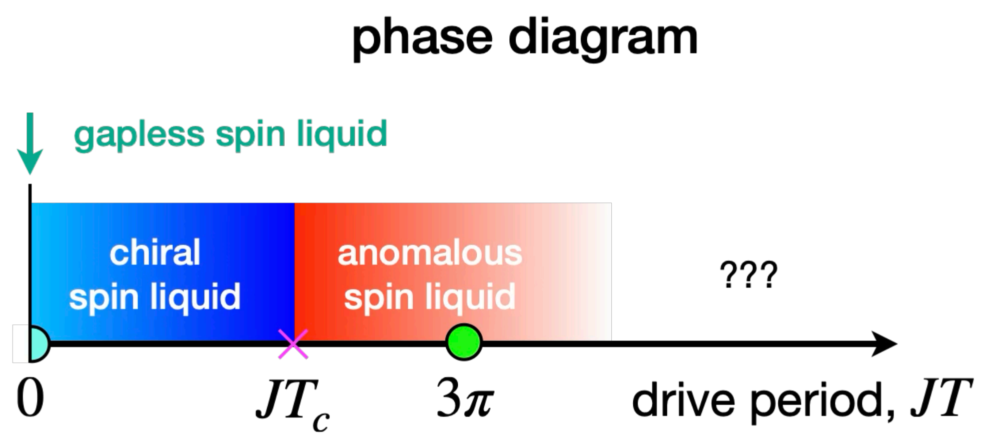

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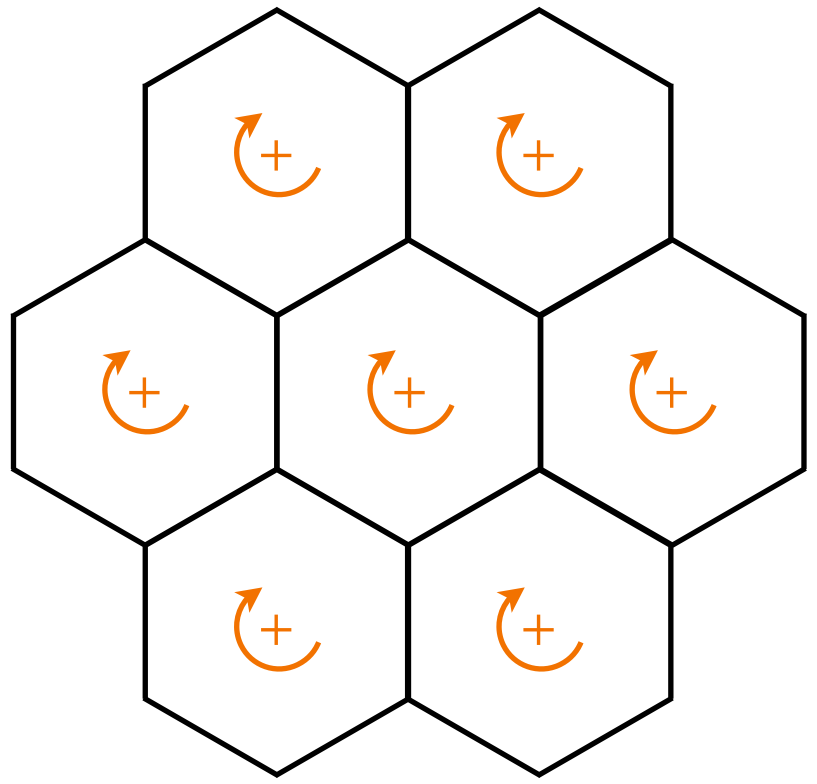

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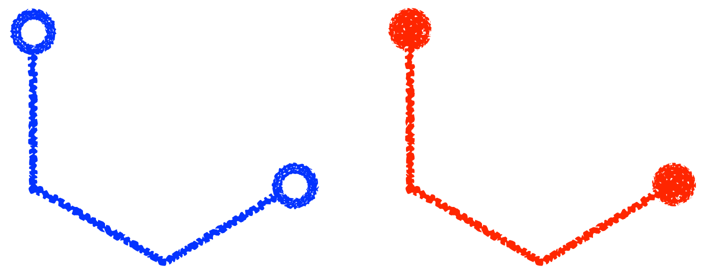

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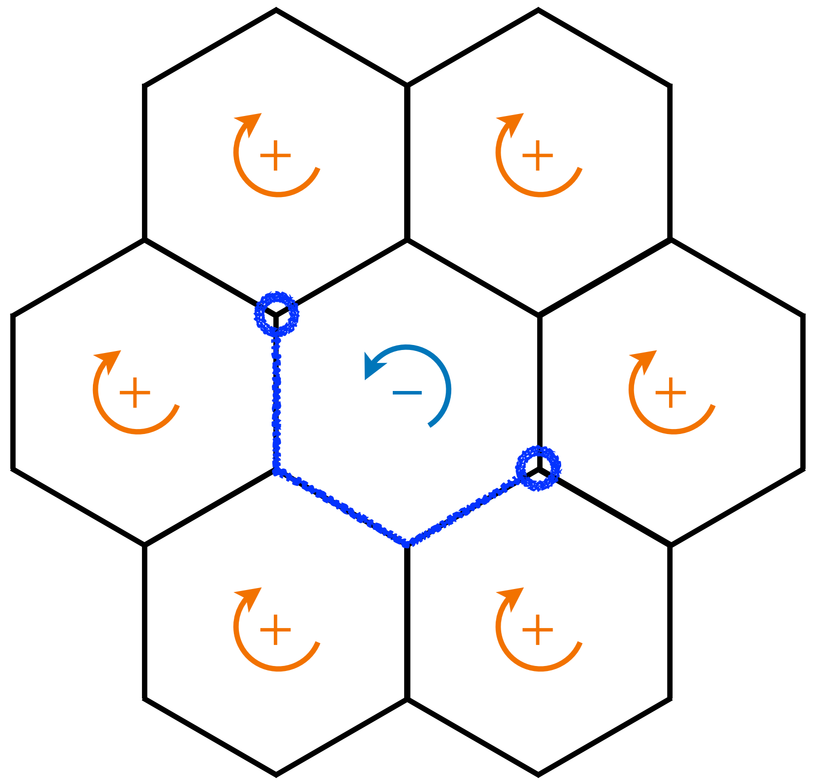

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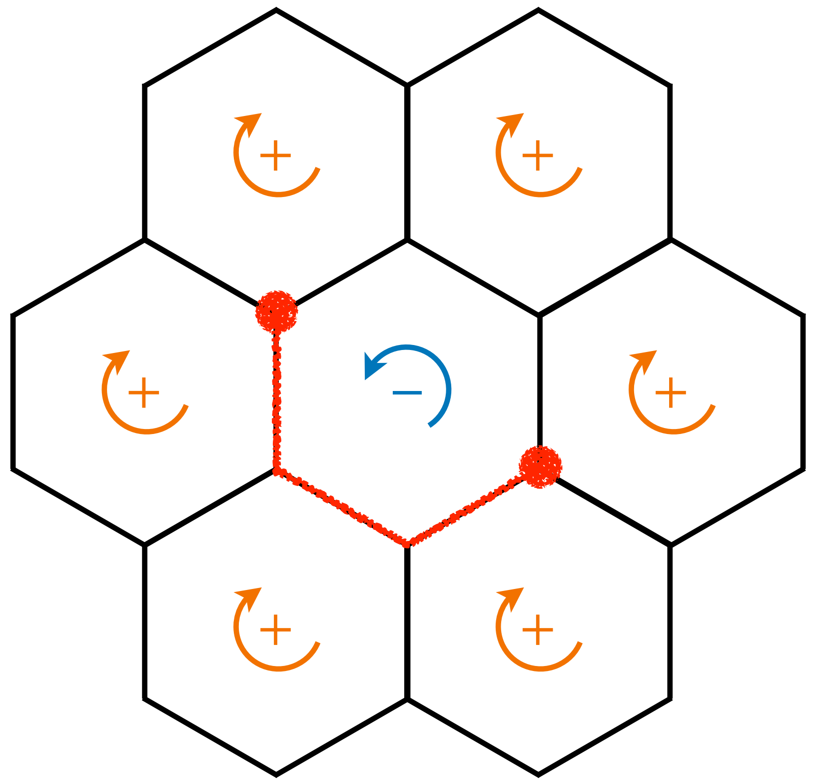

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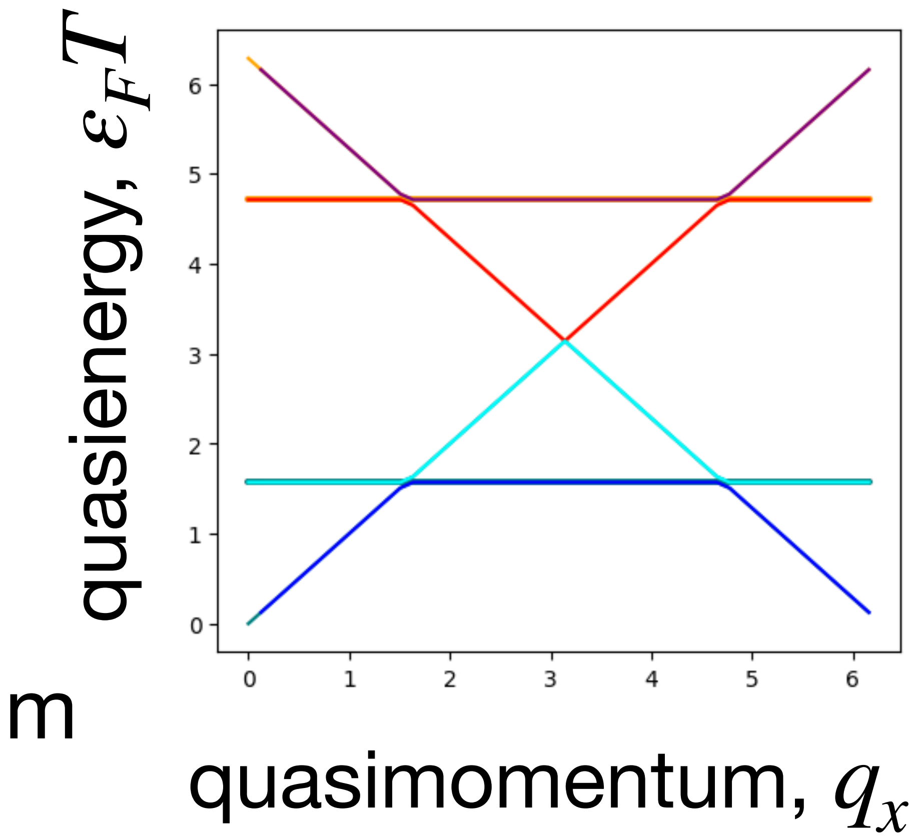
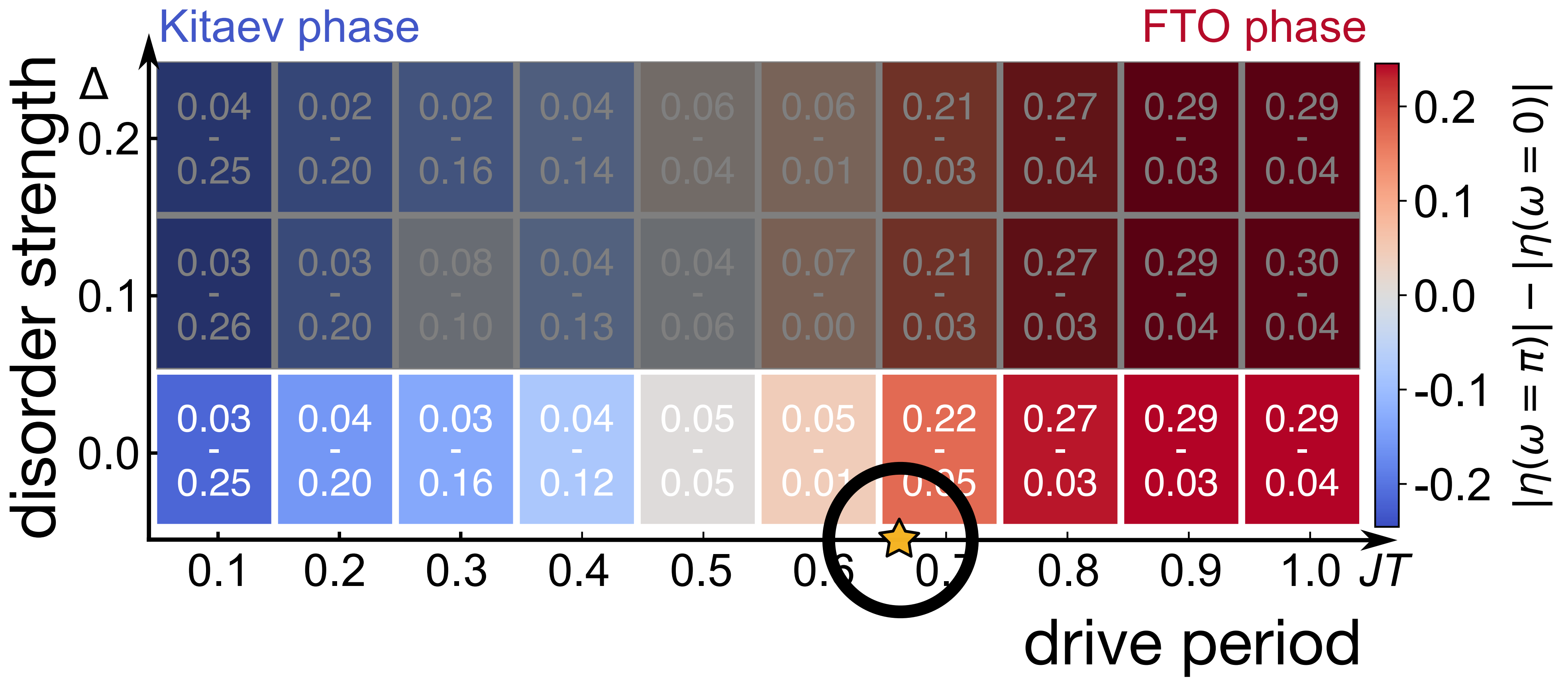




Δ \ JT	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.2	0.04 - 0.25	0.02 - 0.20	0.02 - 0.16	0.04 - 0.14	0.06 - 0.04	0.06 - 0.01	0.21 - 0.03	0.27 - 0.04	0.29 - 0.03	0.29 - 0.04
0.1	0.03 - 0.26	0.03 - 0.20	0.08 - 0.10	0.04 - 0.13	0.04 - 0.06	0.07 - 0.00	0.21 - 0.03	0.27 - 0.03	0.29 - 0.04	0.30 - 0.04
0.0	0.03 - 0.25	0.04 - 0.20	0.03 - 0.16	0.04 - 0.12	0.05 - 0.05	0.05 - 0.01	0.22 - 0.[illegible]5	0.27 - 0.03	0.29 - 0.03	0.29 - 0.04

Marín Bukov

Po et al, PRB 96, 245116 (2017)

Will et al. arXiv:2501.18461

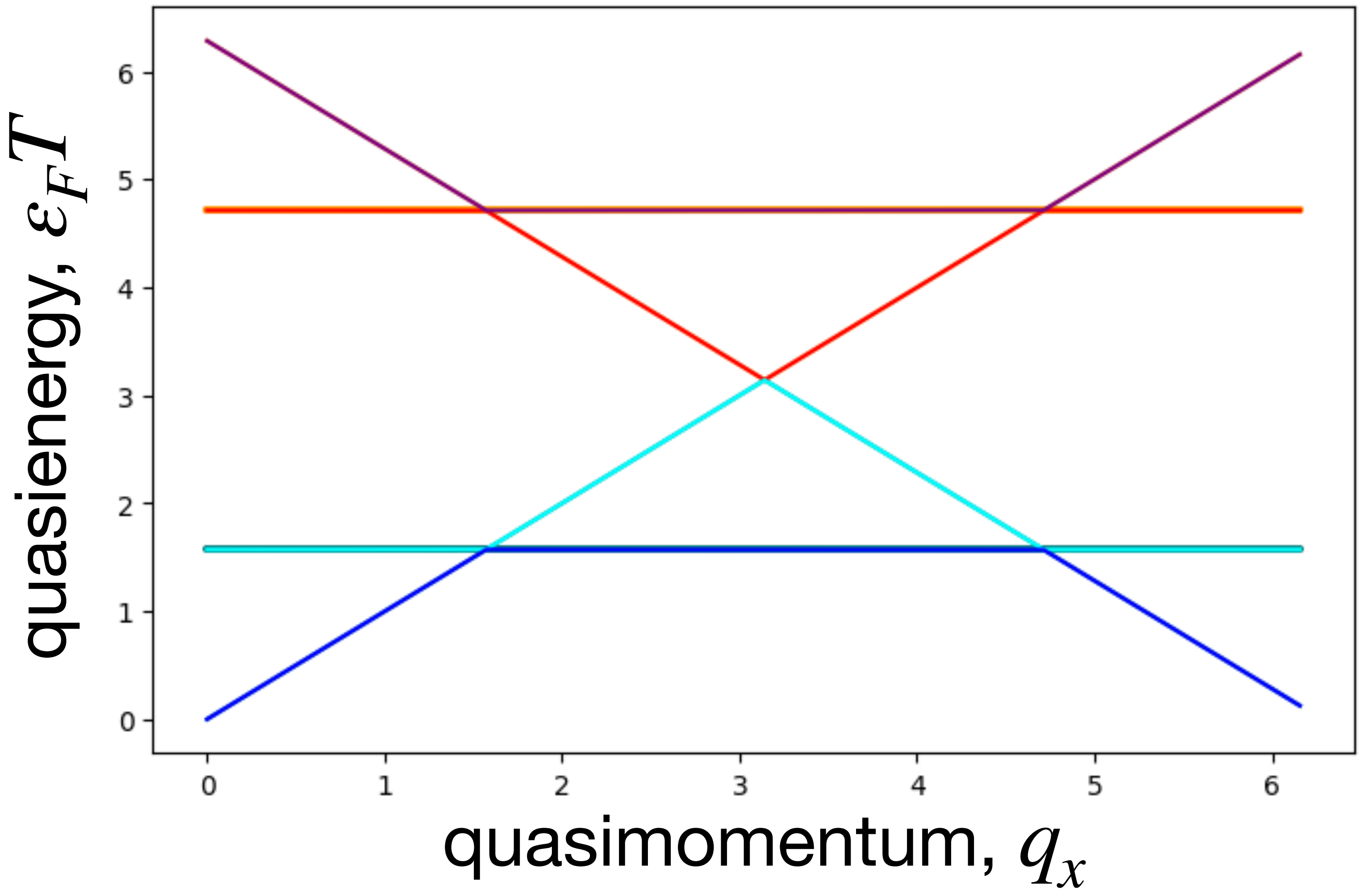
mpipks (Dresden)

Geometric origins of Floquet topological order

$$W_p = 0$$
$$JT/3 = \pi$$

- geometric Floquet theory:
  - Majorana dispersion

$$\boxed{\varepsilon_F^{(n)} = æ_n + T^{-1}\gamma_n}$$



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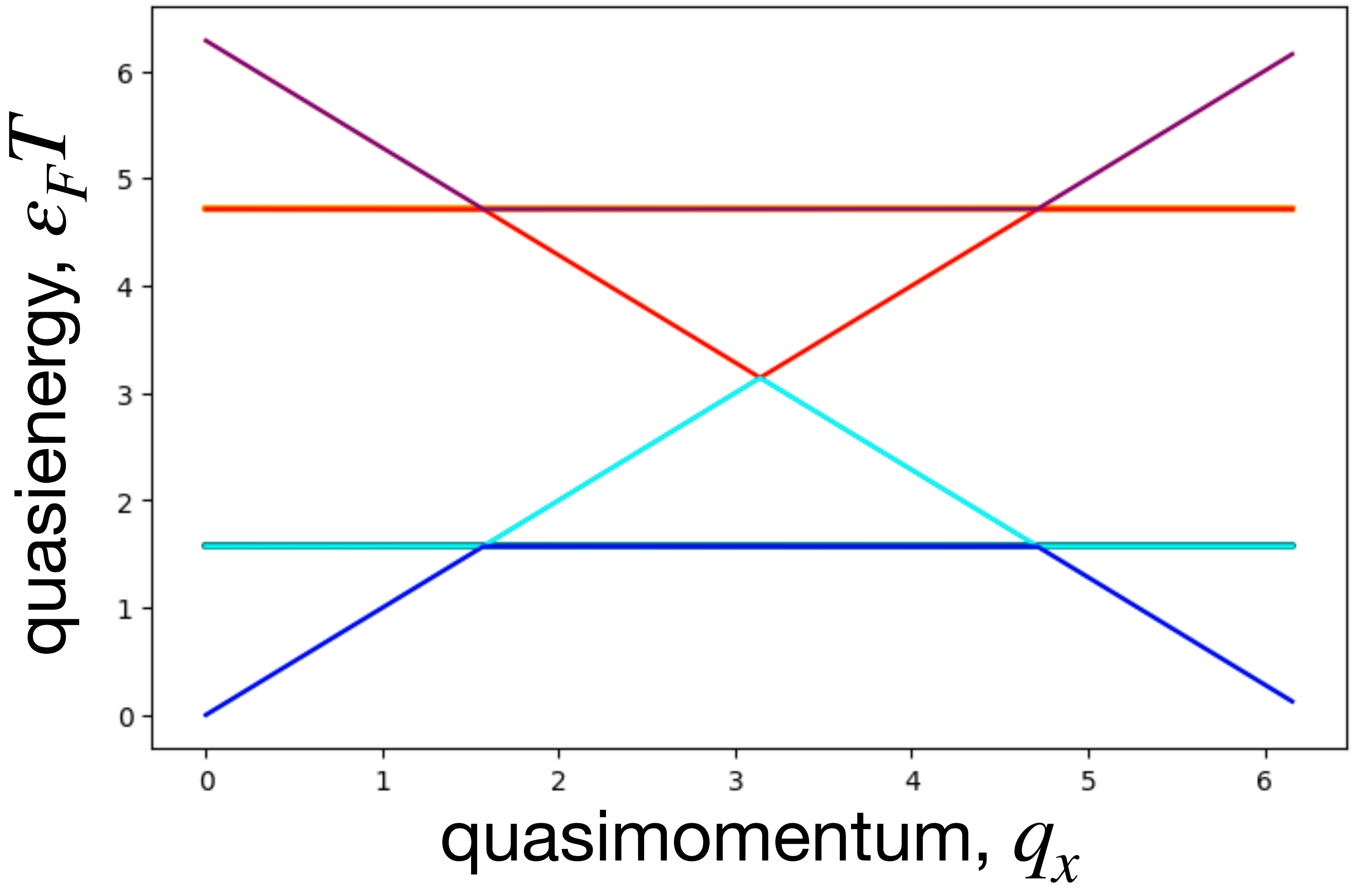

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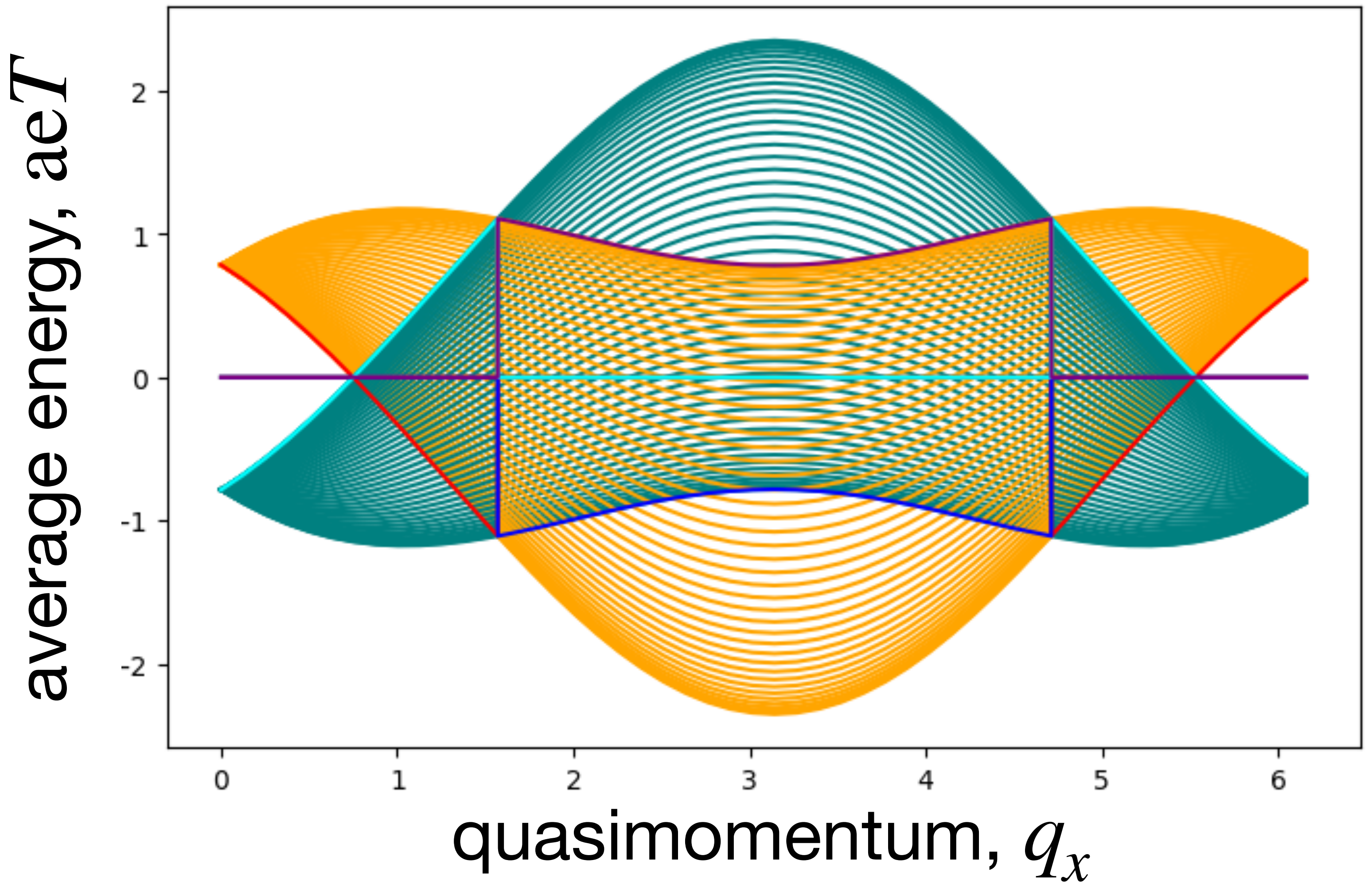

+

geometric phase, γ

quasimomentum, q_1

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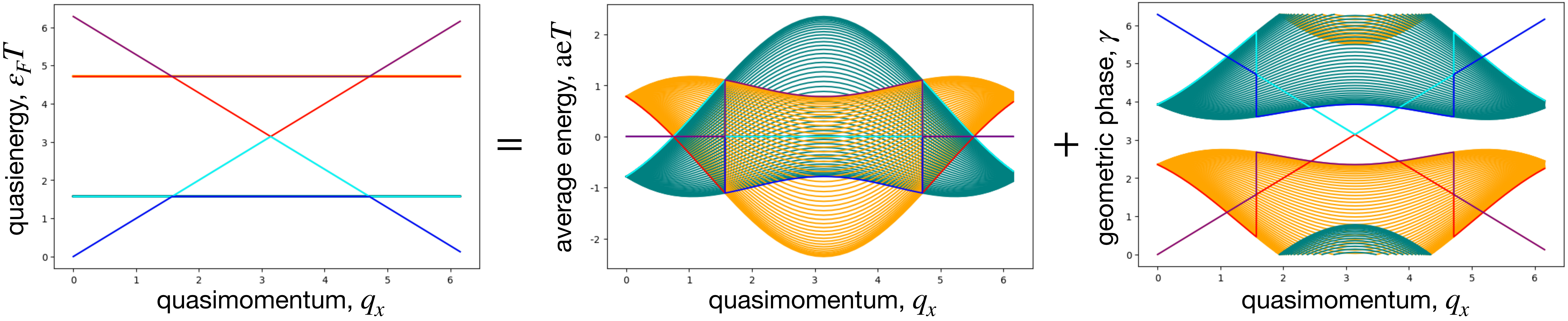

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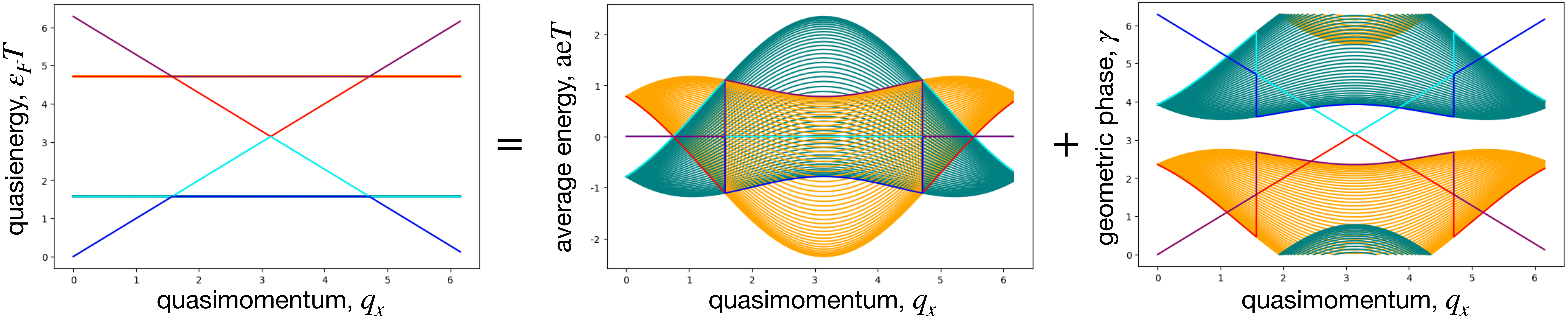

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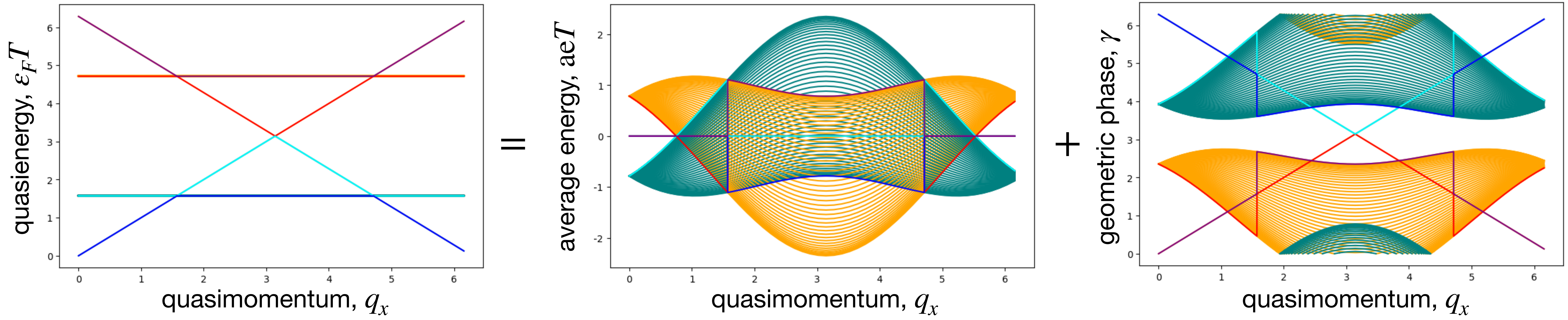

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Wilczek and Zee, PRL 52, 2111 (1984)

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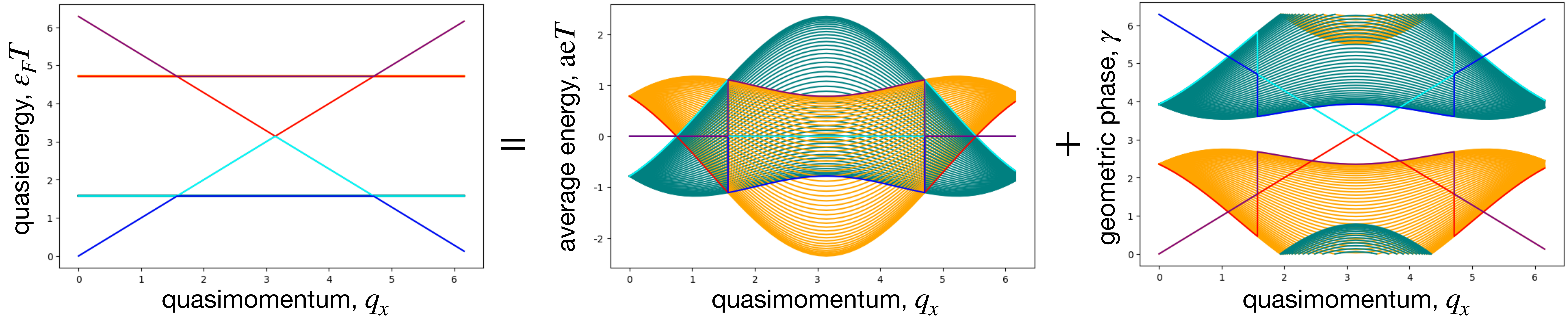

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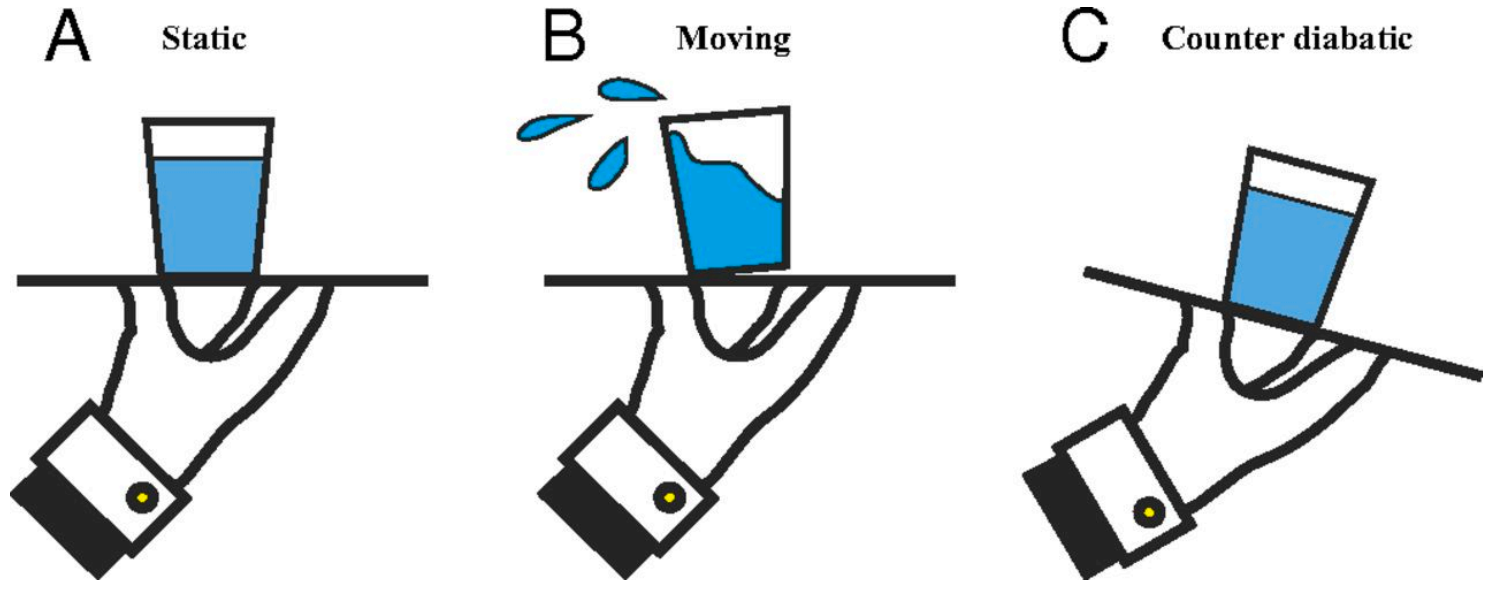
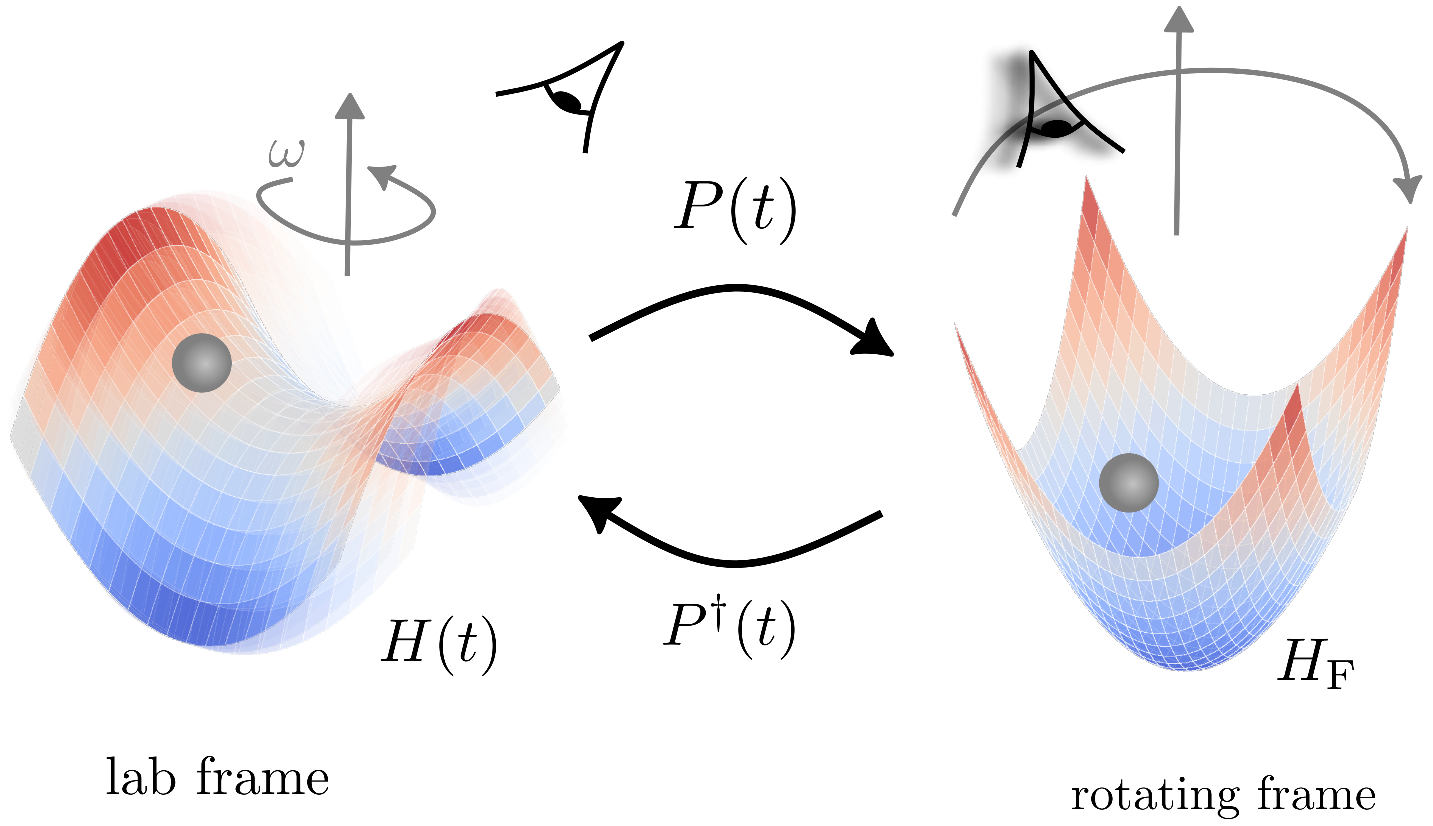
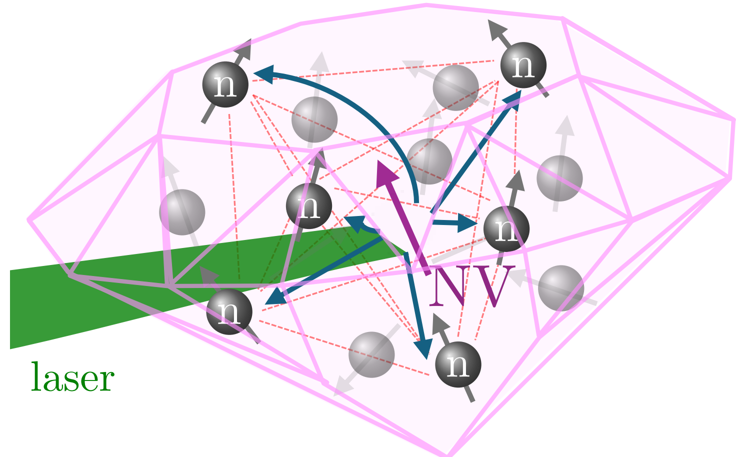
Outline

- ✓ Floquet theory from counterdiabatic driving
  - geometric Floquet decomposition
  - *unambiguous* Floquet ground state
- ✓ Anomalous chiral spin liquids
  - Floquet topological order
- **Time crystals**
  - nonequilibrium eigenstate order
- **Heating in kicked Ising chain**
  - locality of average energy





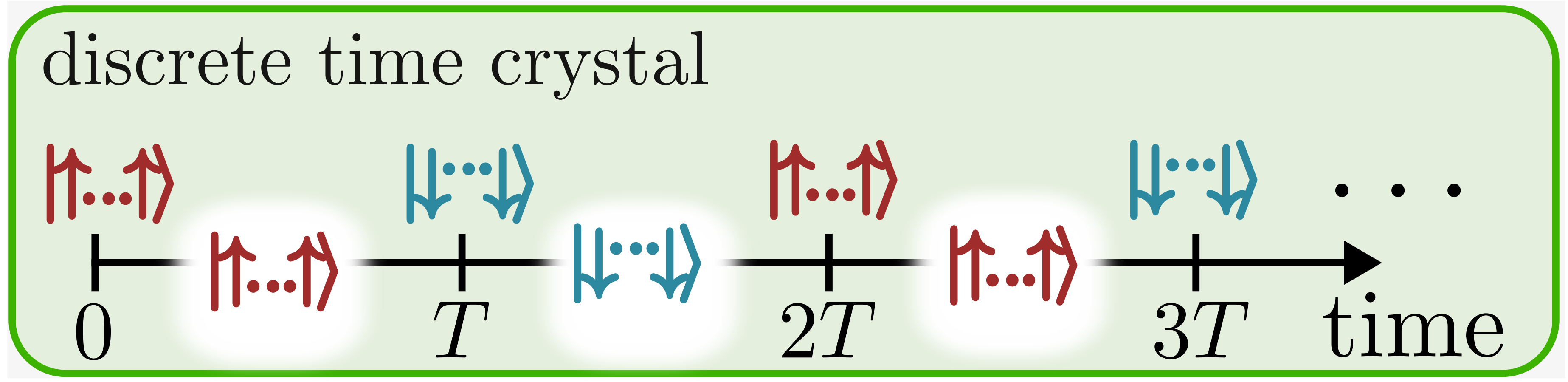
Schindler & MB, PRX 15, 031037 (2025)



Discrete time crystals

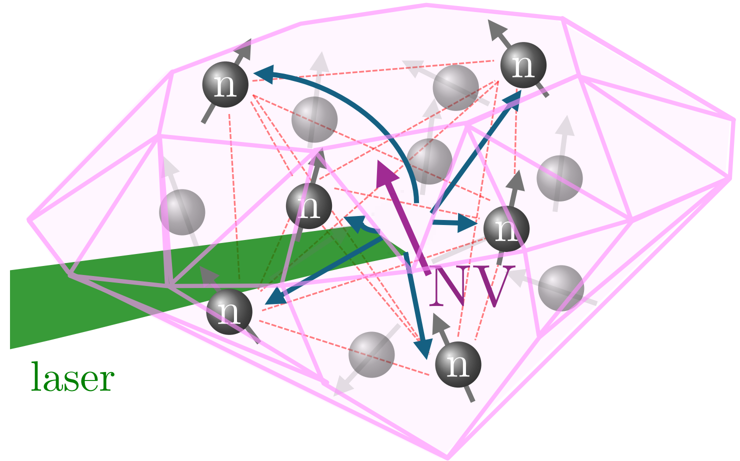
- **evolution operator** $U_F(\theta_x) = e^{-iTH^z} e^{-i\theta_x H^x}$ $\quad H^z = \sum_n J_n Z_n Z_{n+1} \quad H^x = \frac{1}{2}\sum_{n=1}^{L} X_n$

$|\uparrow\cdots\uparrow\rangle$



nonequilibrium phase of matter without equilibrium counterpart

Sacha et al., Rep Prog Phys 81, 016401 (2017) **Khemani et al., arXiv:1910.10745** **Else et al., Ann Rev Cond Mat Phys 11, 467 (2020)**



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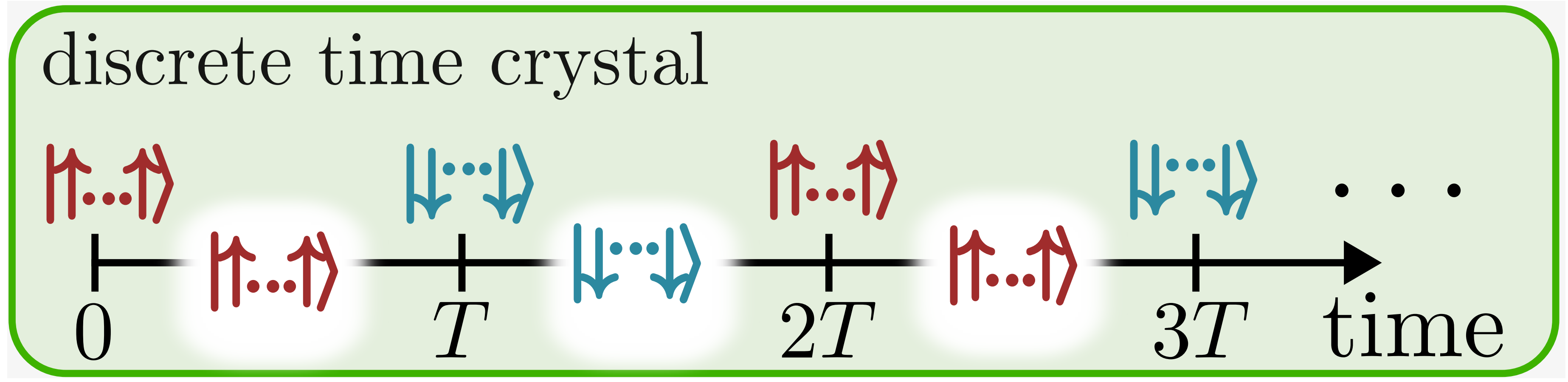

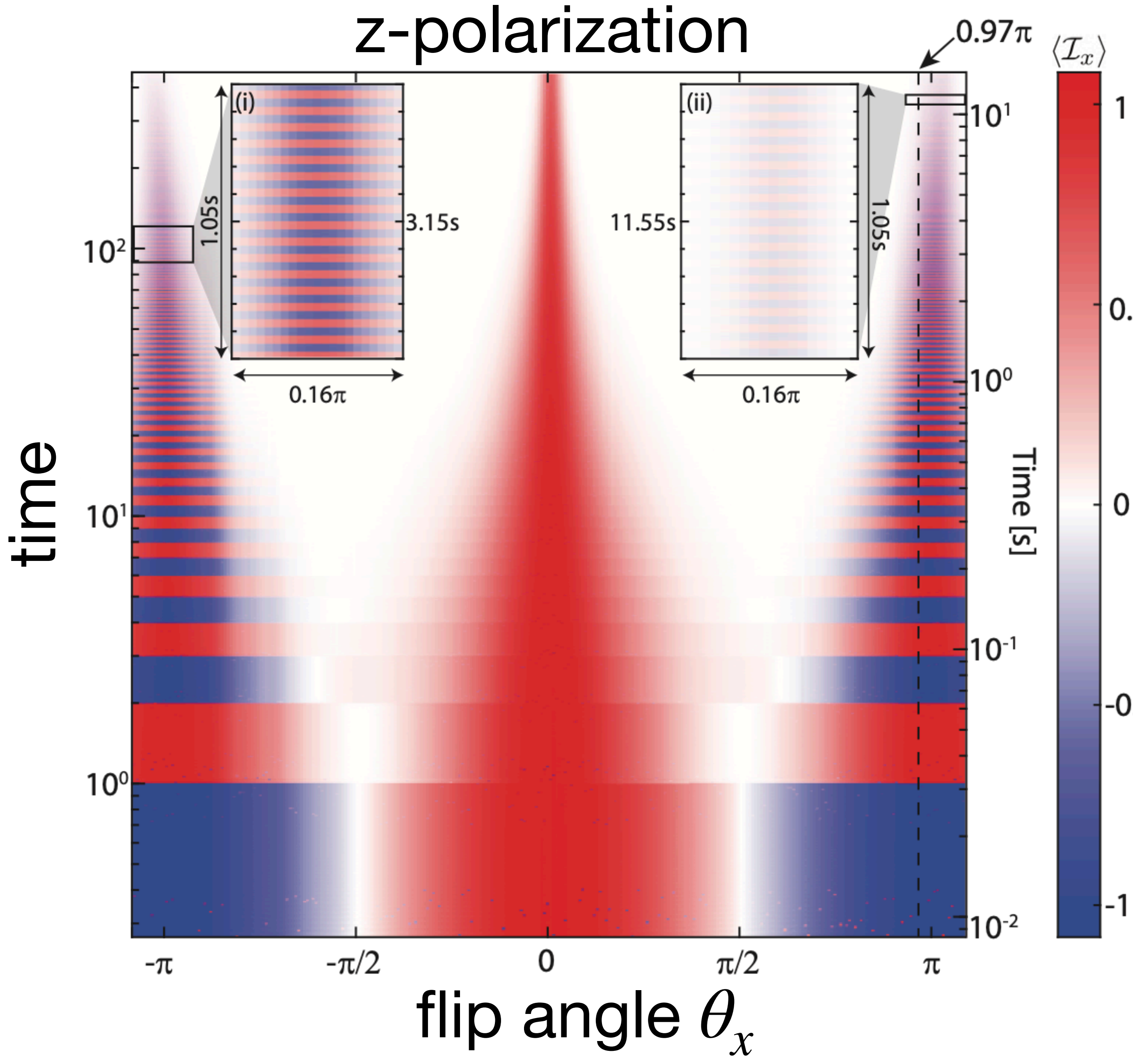

Beatrez, …, MB, Ajoy, Nat Phys '22

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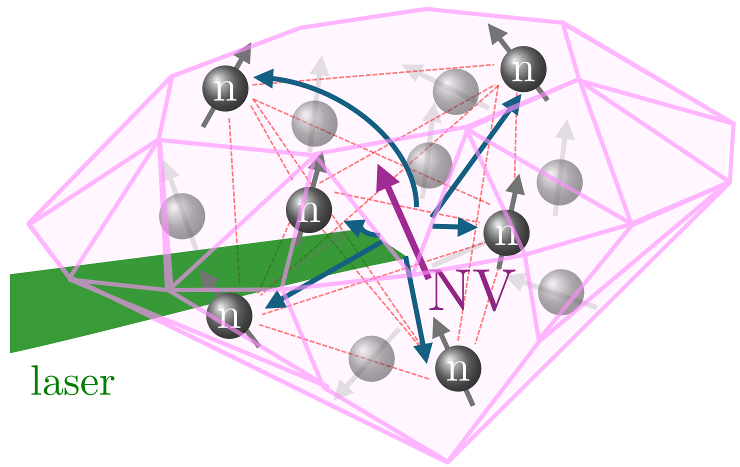
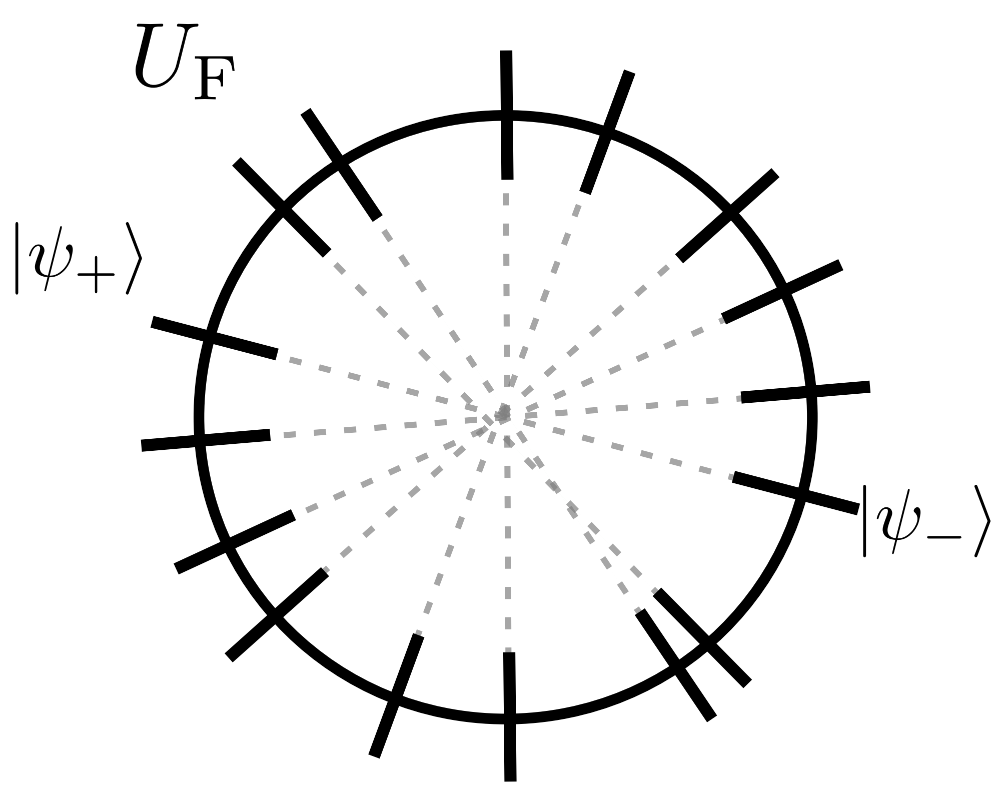


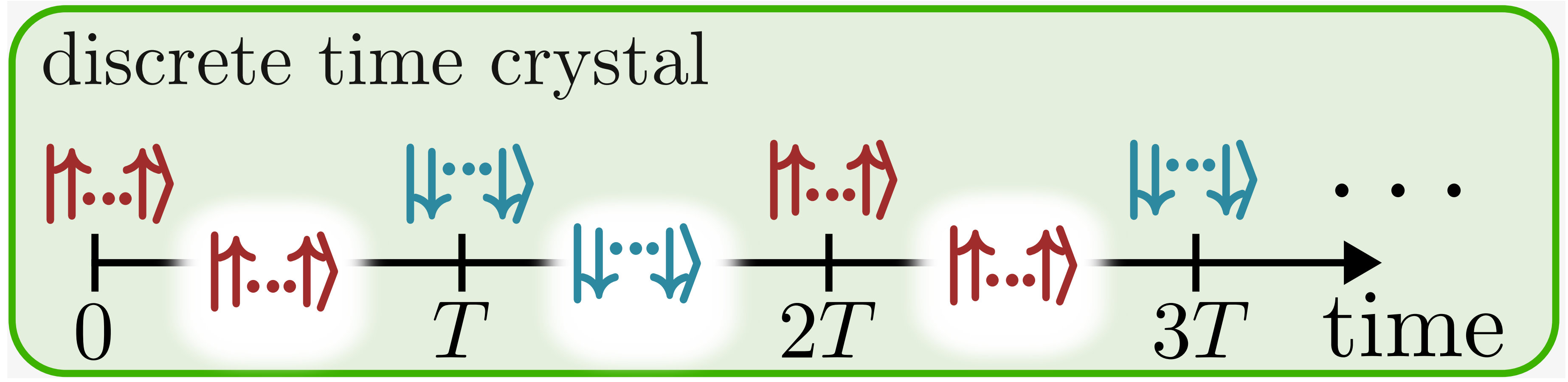
Discrete time crystals

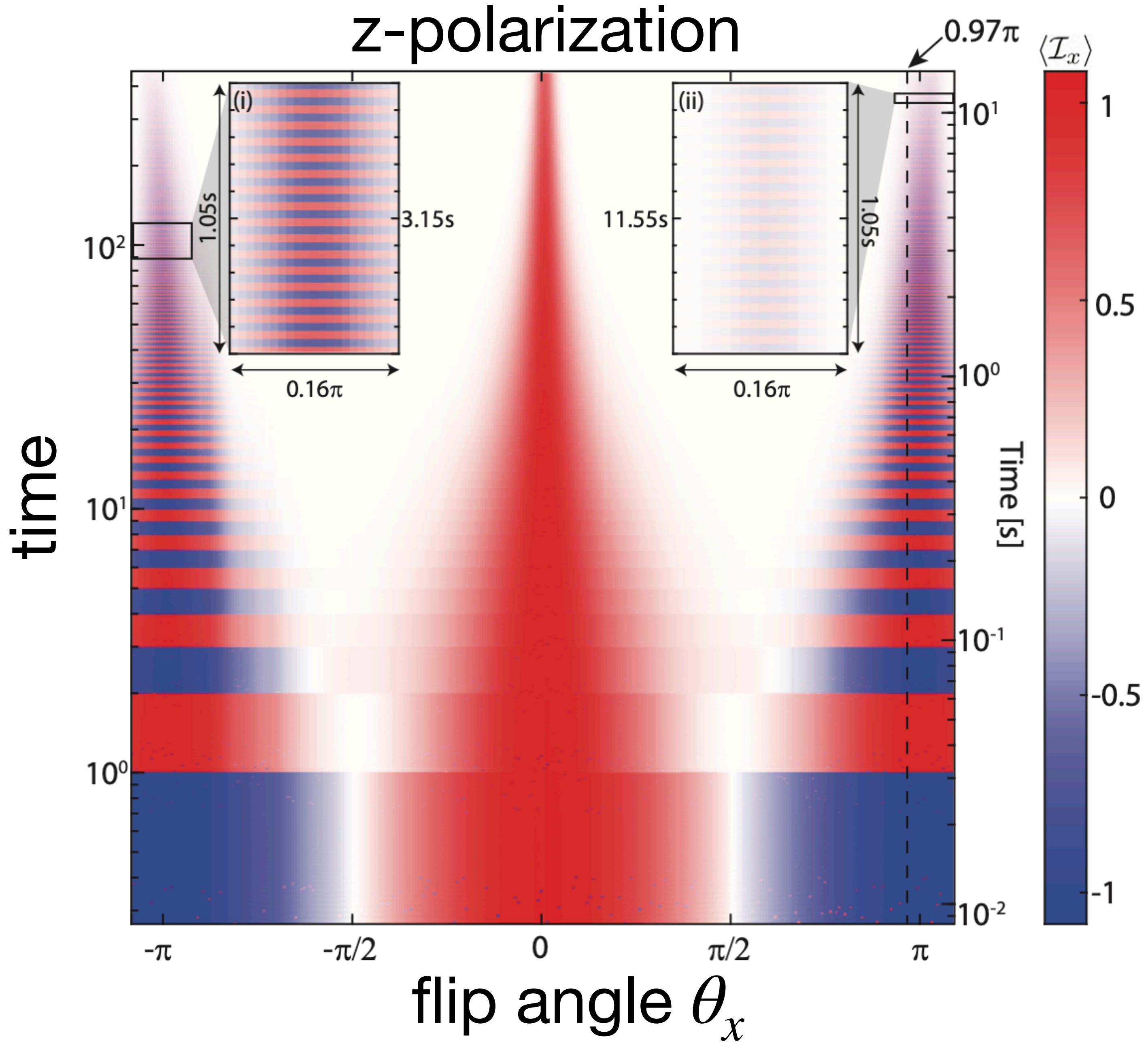
- **evolution operator** $U_F(\theta_x) = e^{-iTH^z} e^{-i\theta_x H^x}$ $\quad H^z = \sum J_n Z_n Z_{n+1} \quad H^x = \frac{1}{2}\sum_{n=1}^{L} X_n \quad H^z|n\rangle = \varepsilon_n |n\rangle$
  - pairing of Floquet states ($\mathbb{Z}_2$ symmetry)

$$U_F(\theta_x = \pi)\,|n_F^{\pm}\rangle = \pm\, e^{-iT\varepsilon_n}\,|n_F^{\pm}\rangle$$

    - π-splitting in q'energy spectrum
    - difference in paired q'energies: $\Delta\varepsilon_F T = \pm\,\pi$

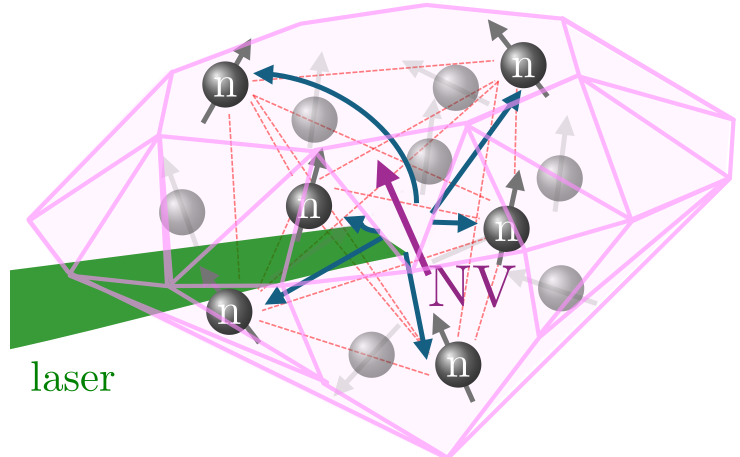






Beatrez, …, MB, Ajoy, Nat Phys '22

Sacha et al., Rep Prog Phys 81, 016401 (2017) **Khemani et al., arXiv:1910.10745** **Else et al., Ann Rev Cond Mat Phys 11, 467 (2020)**



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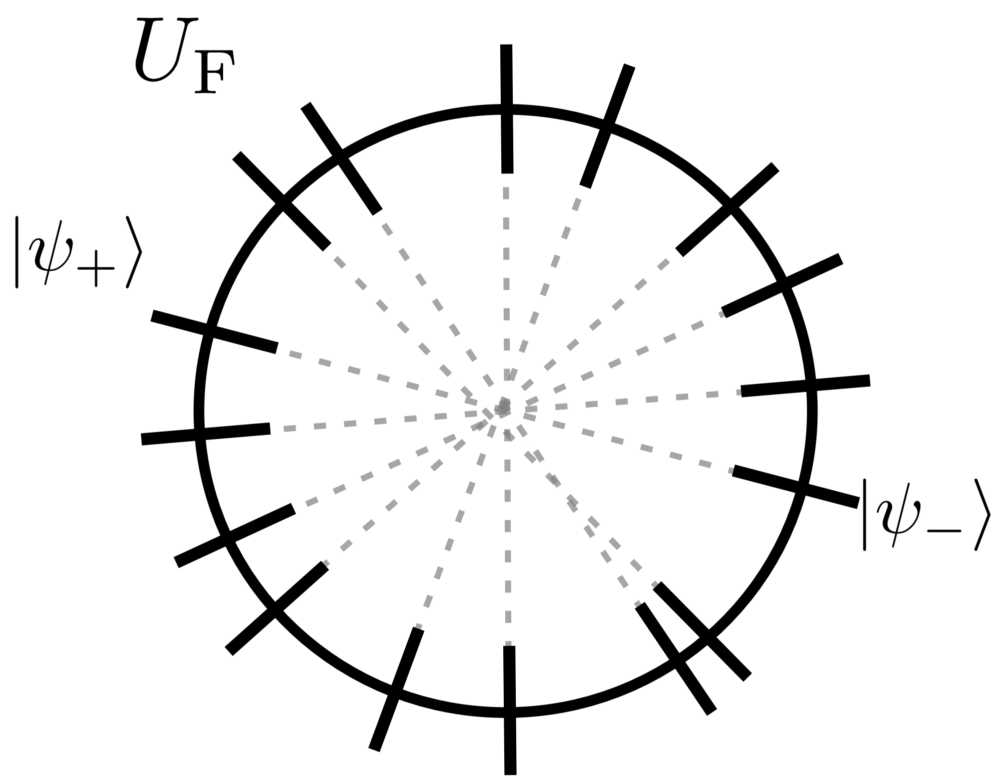

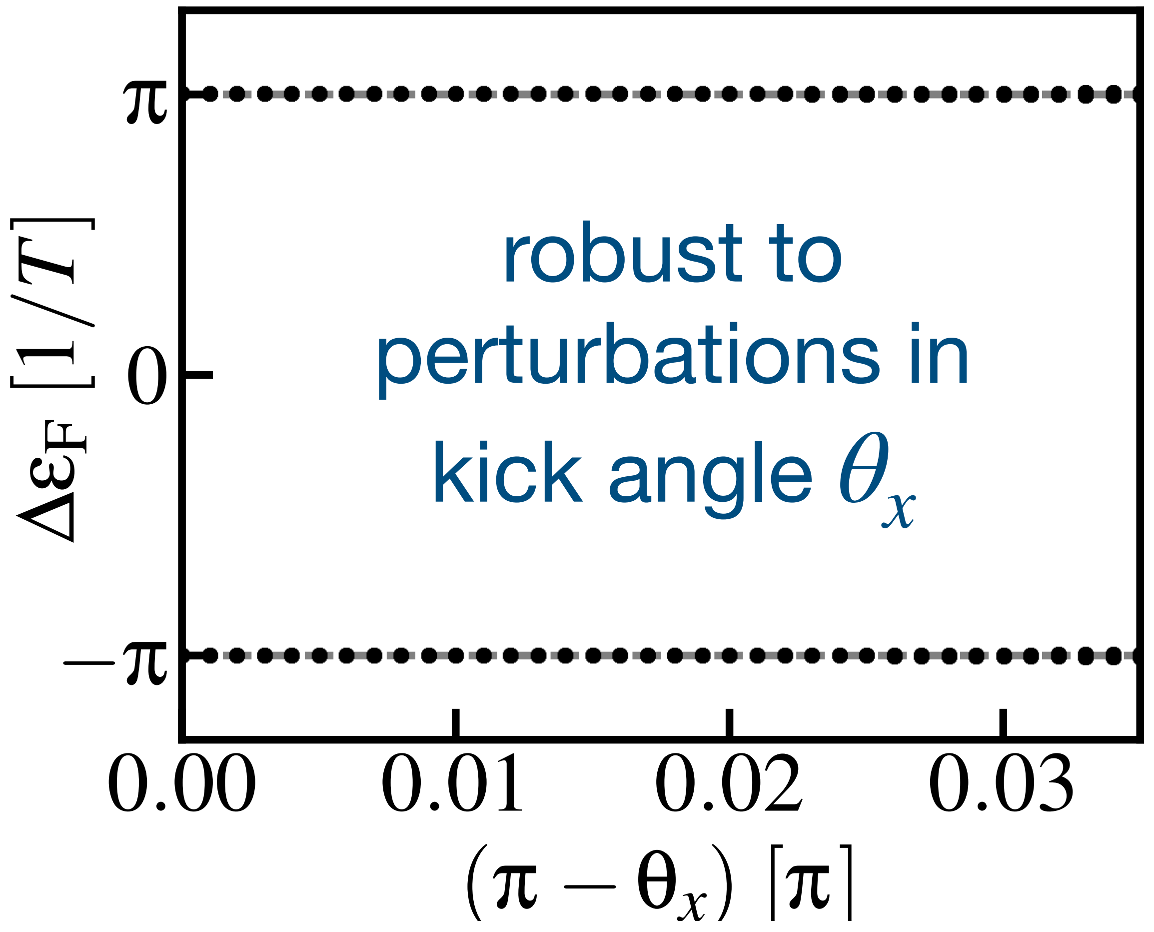

robust to perturbations in kick angle θ_x

  - pairing of Floquet states ($\mathbb{Z}_2$ symmetry)

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    - difference in paired q'energies: $\Delta\varepsilon_F T = \pm\,\pi$

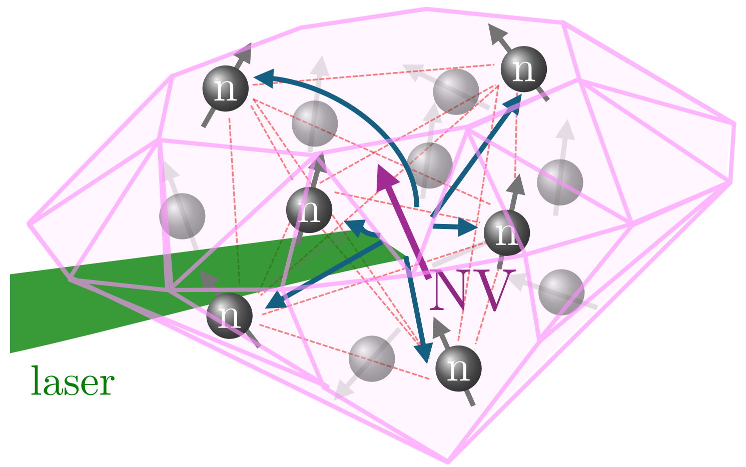
  - symmetry breaking protects against perturbations in kick angle θ_x

time crystals = π-pairing of quasienergies + symmetry breaking (no static counterpart)

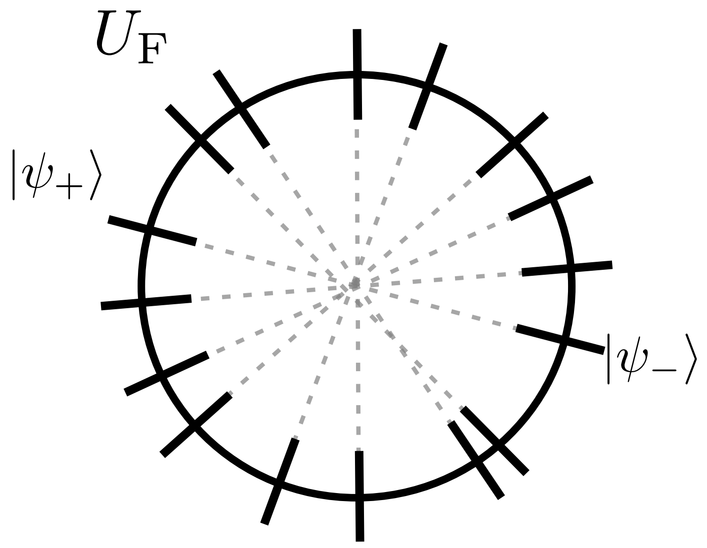
Sacha et al., Rep Prog Phys 81, 016401 (2017)

Khemani et al., arXiv:1910.10745

Else et al., Ann Rev Cond Mat Phys 11, 467 (2020)



Discrete time crystals



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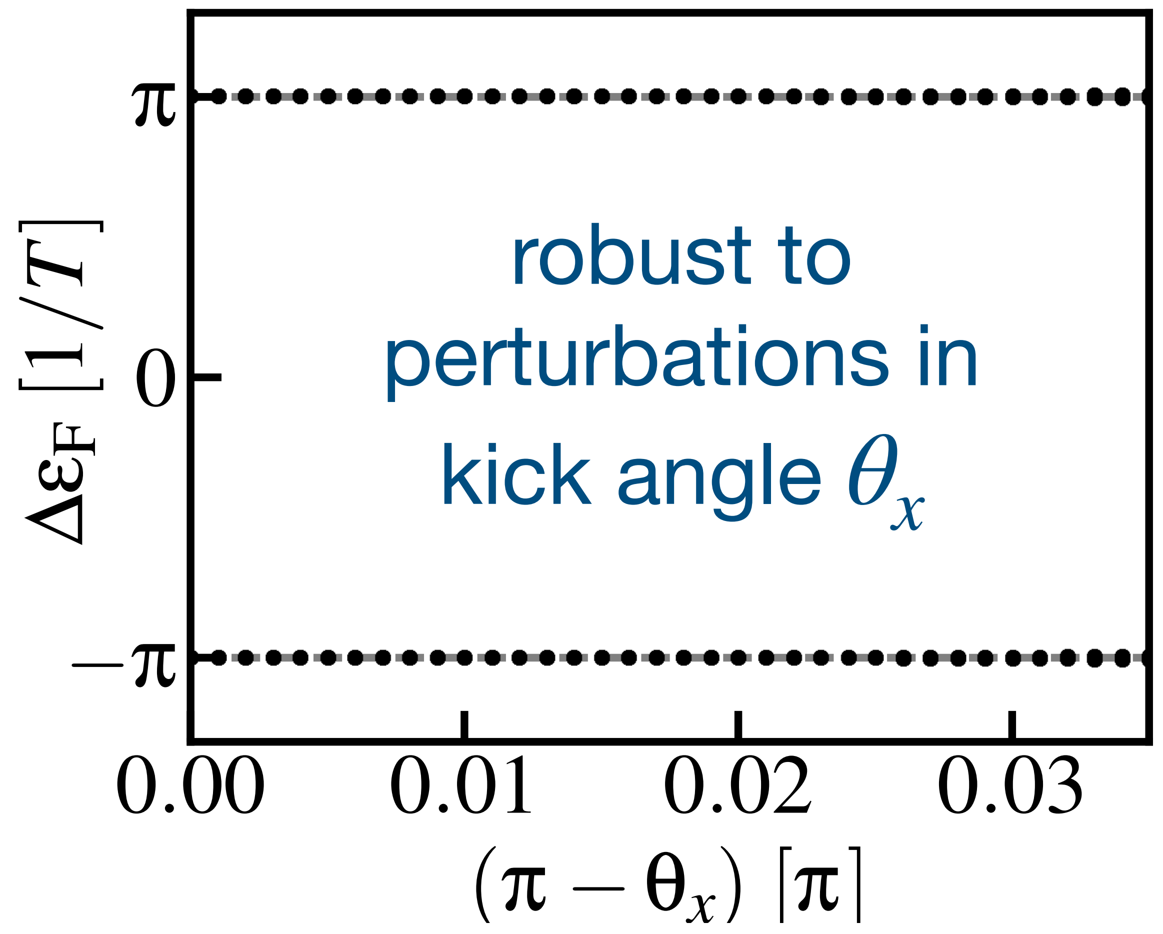
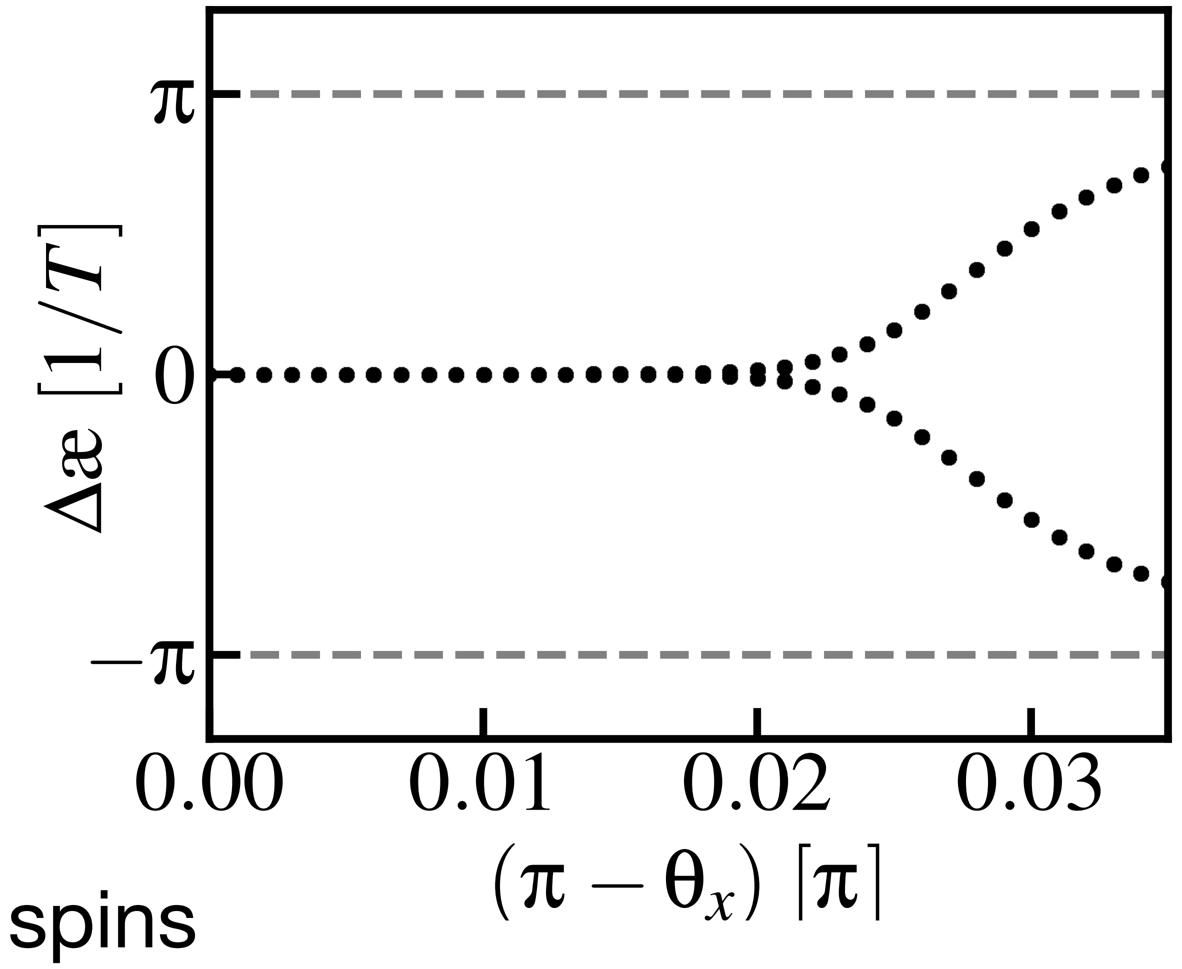
$$\Delta\varepsilon_F^{(n)} = T^{-1}\Delta\gamma_n(T) + \Delta\, æ_n(T)$$

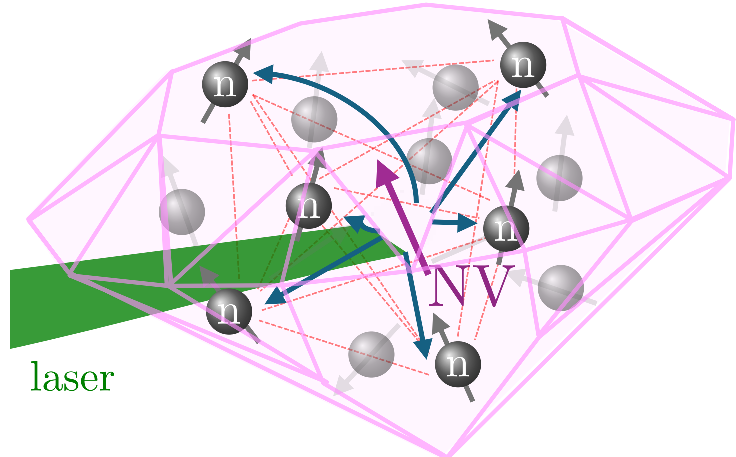


robust to perturbations in kick angle θ_x

average energy



$L = 10$ spins



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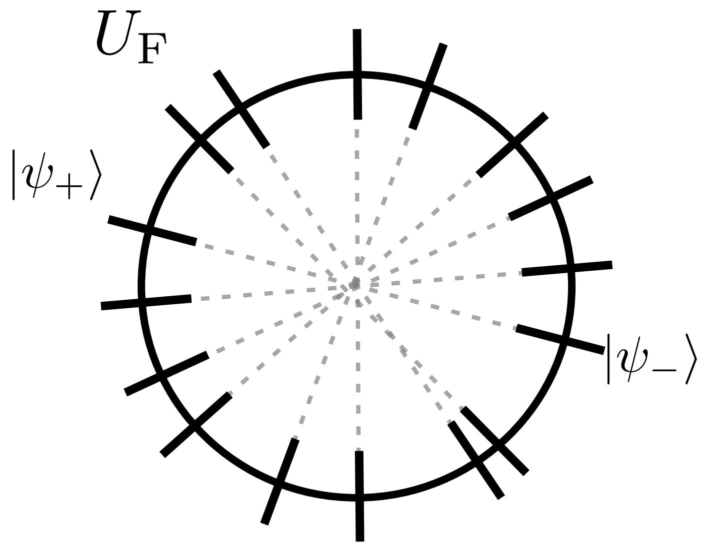

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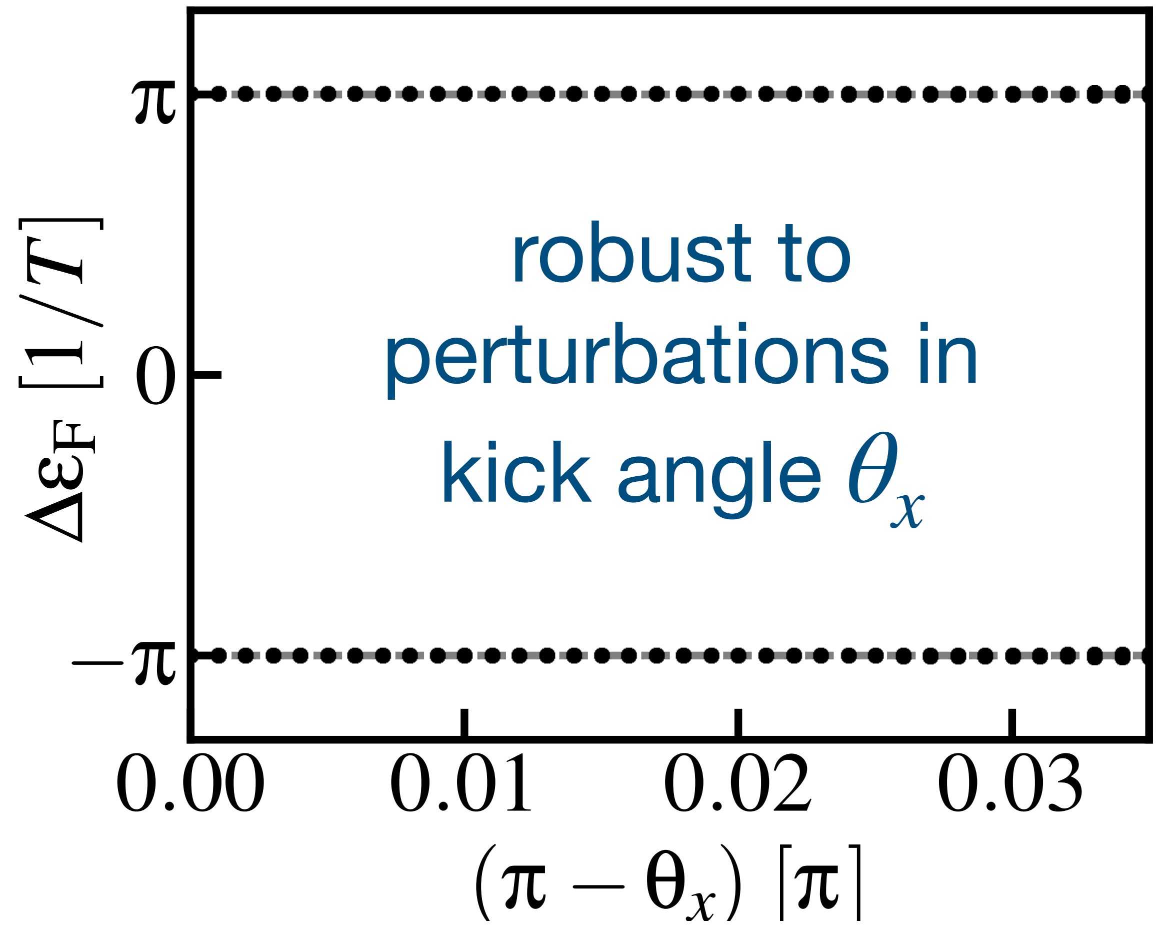

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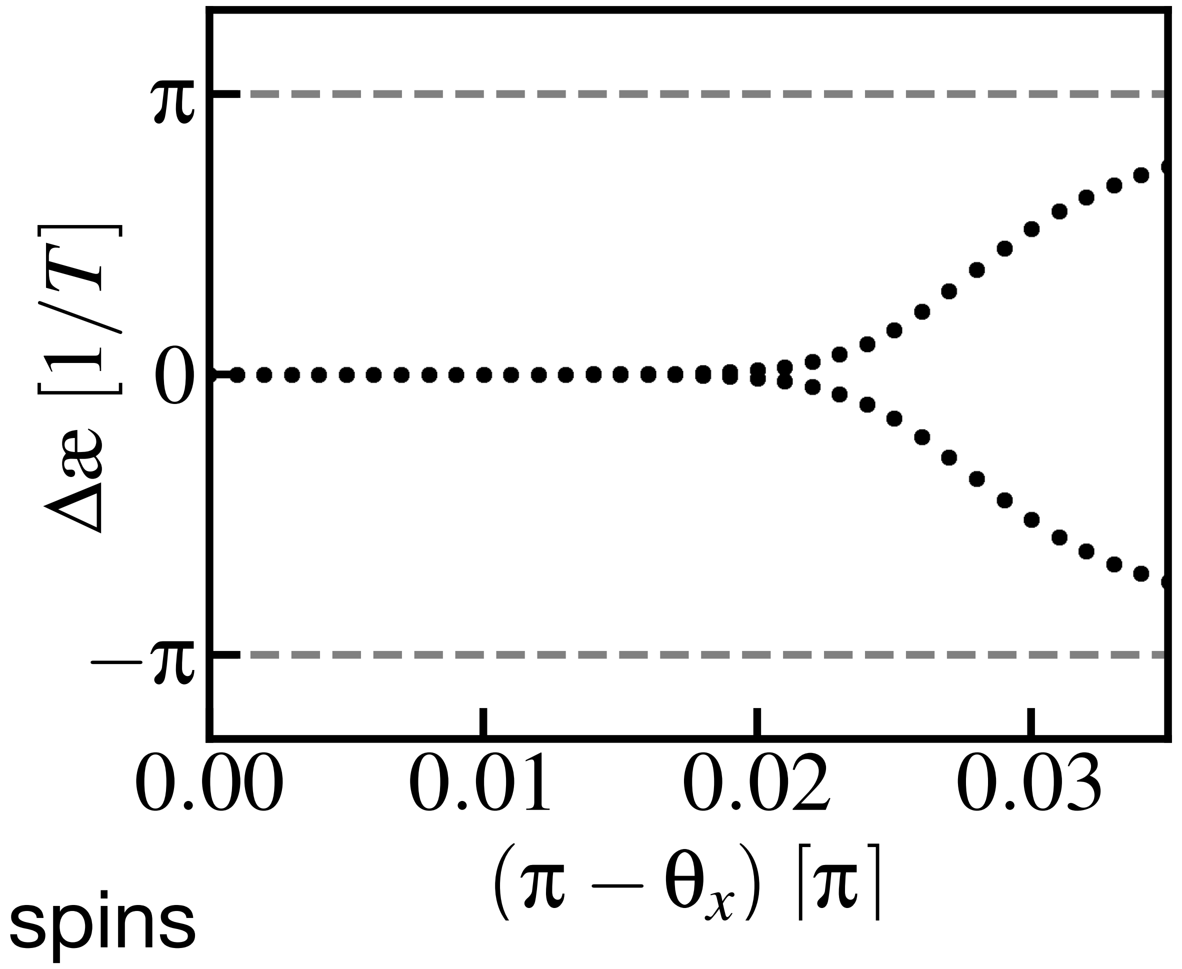

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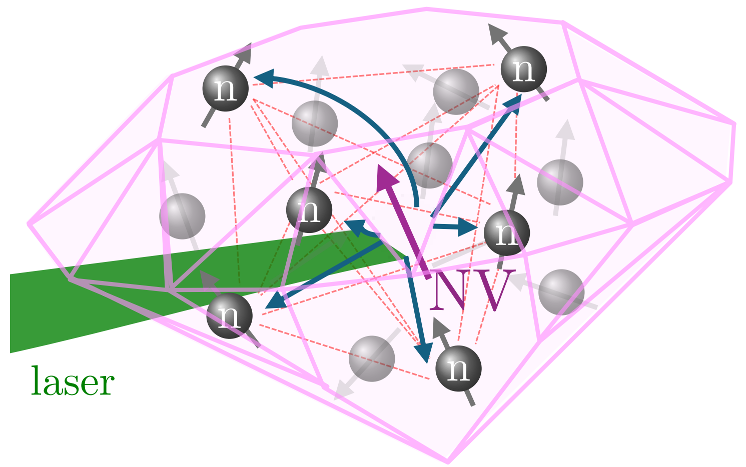

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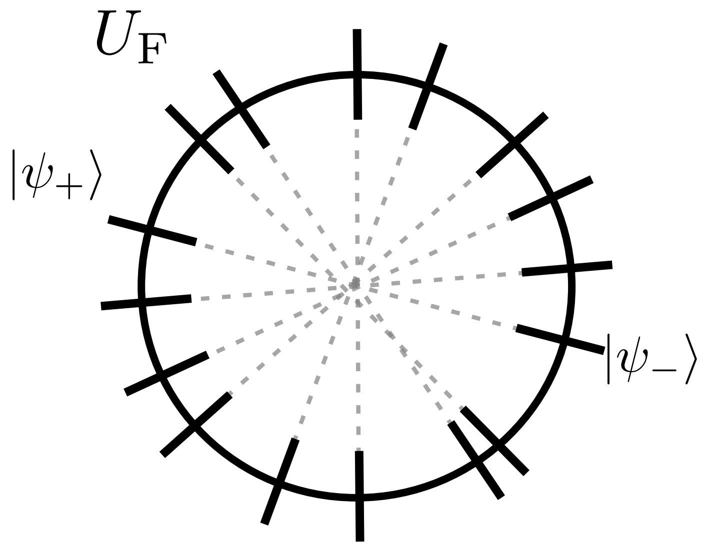

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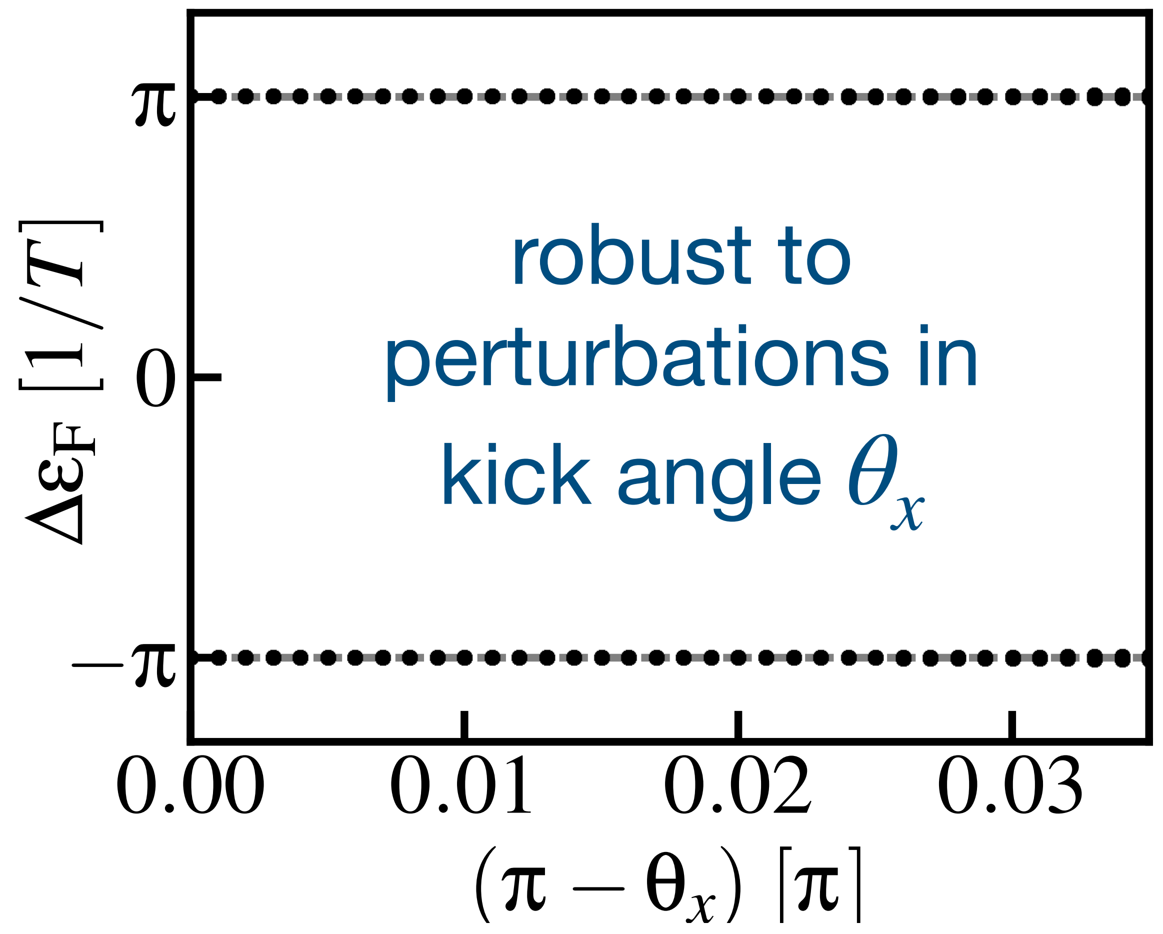

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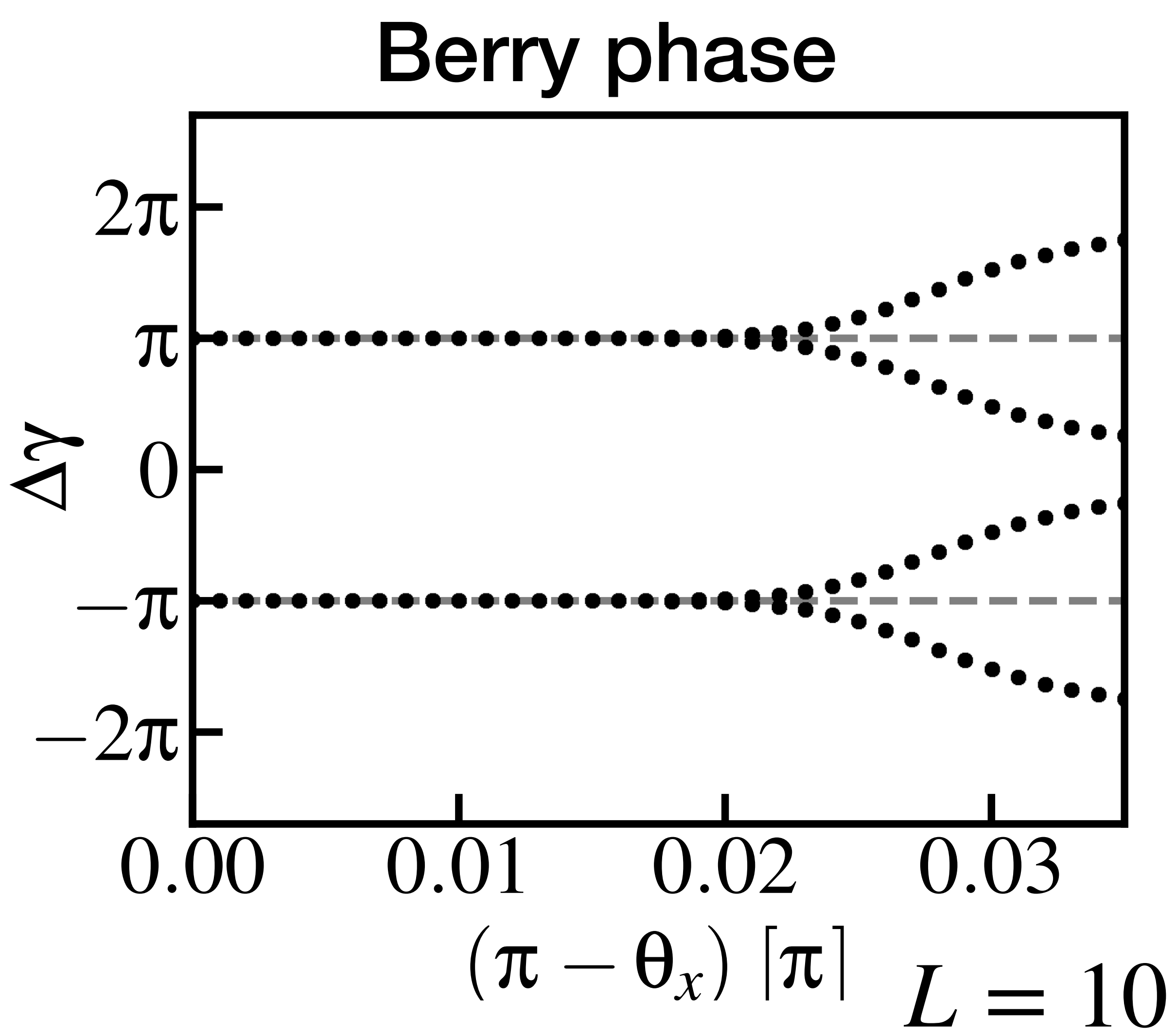

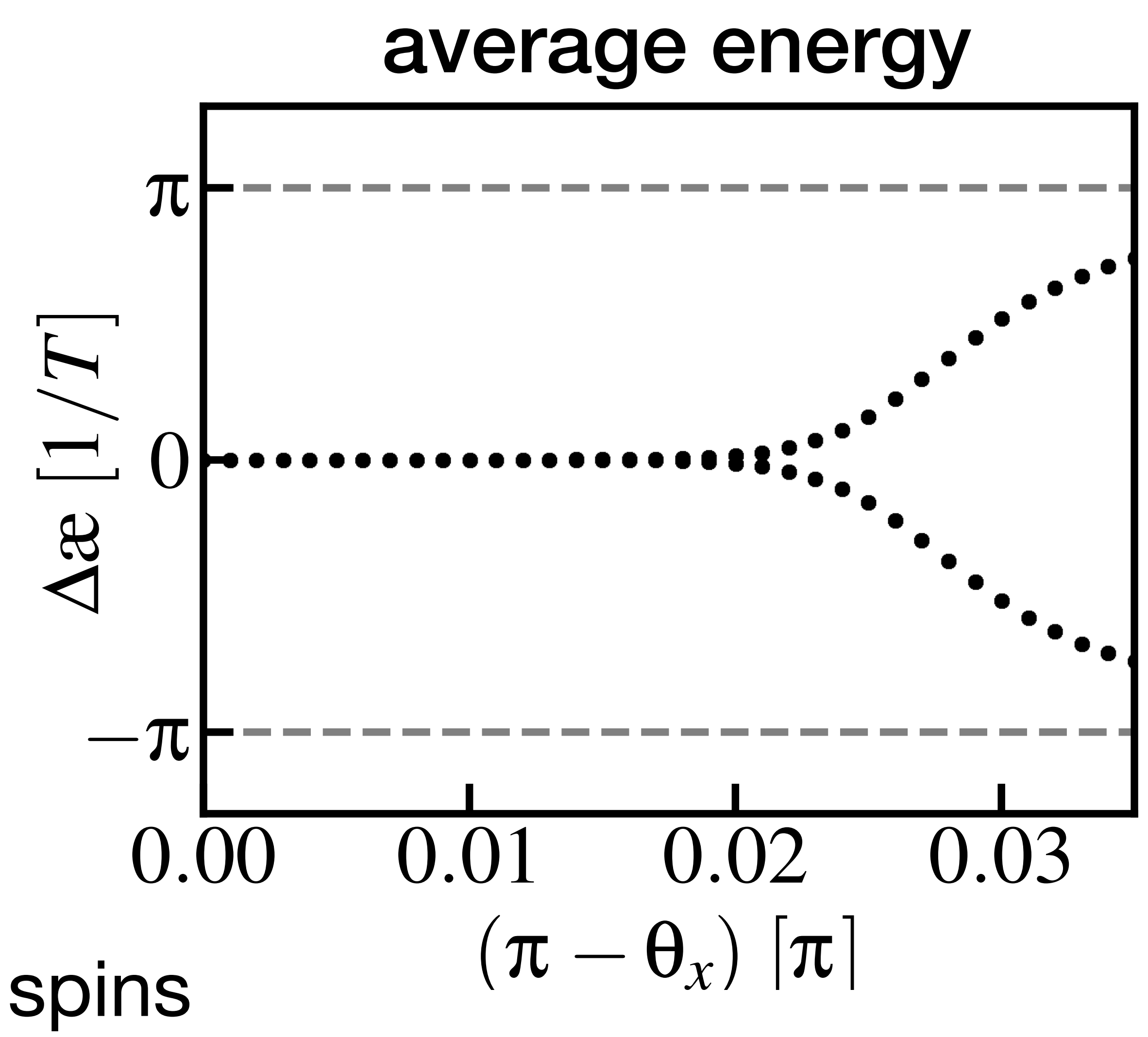

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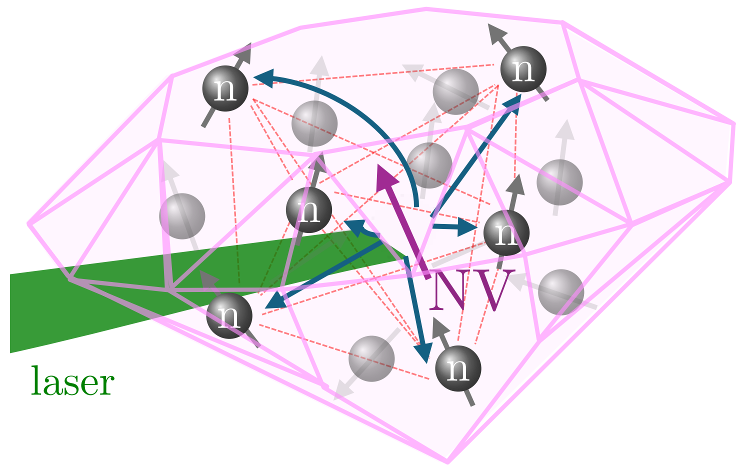

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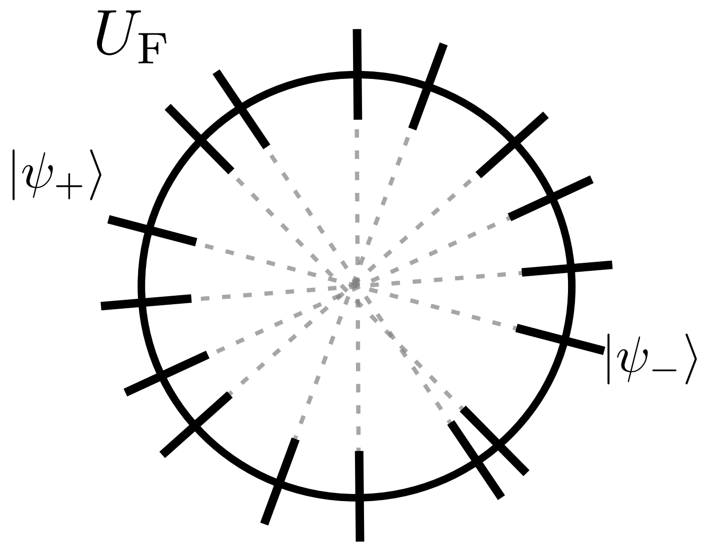

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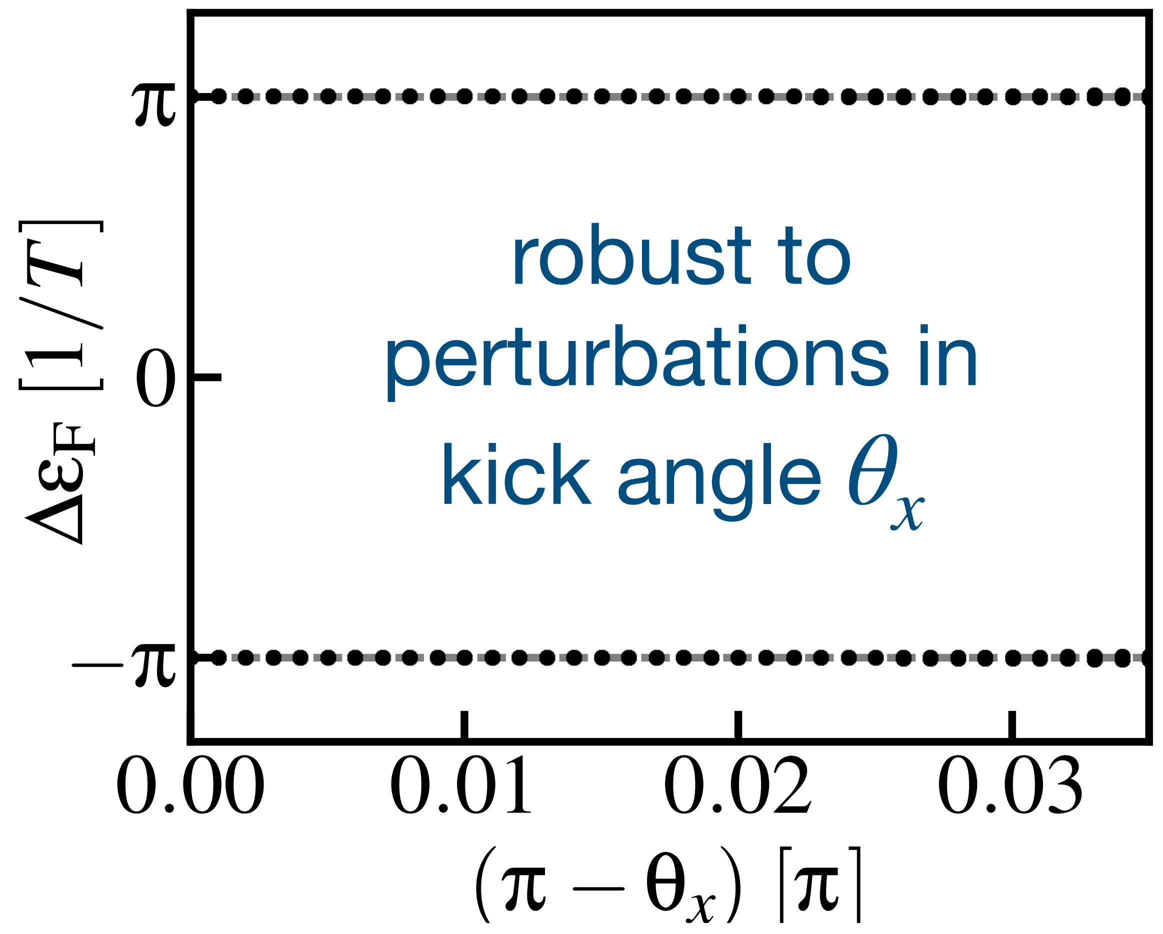

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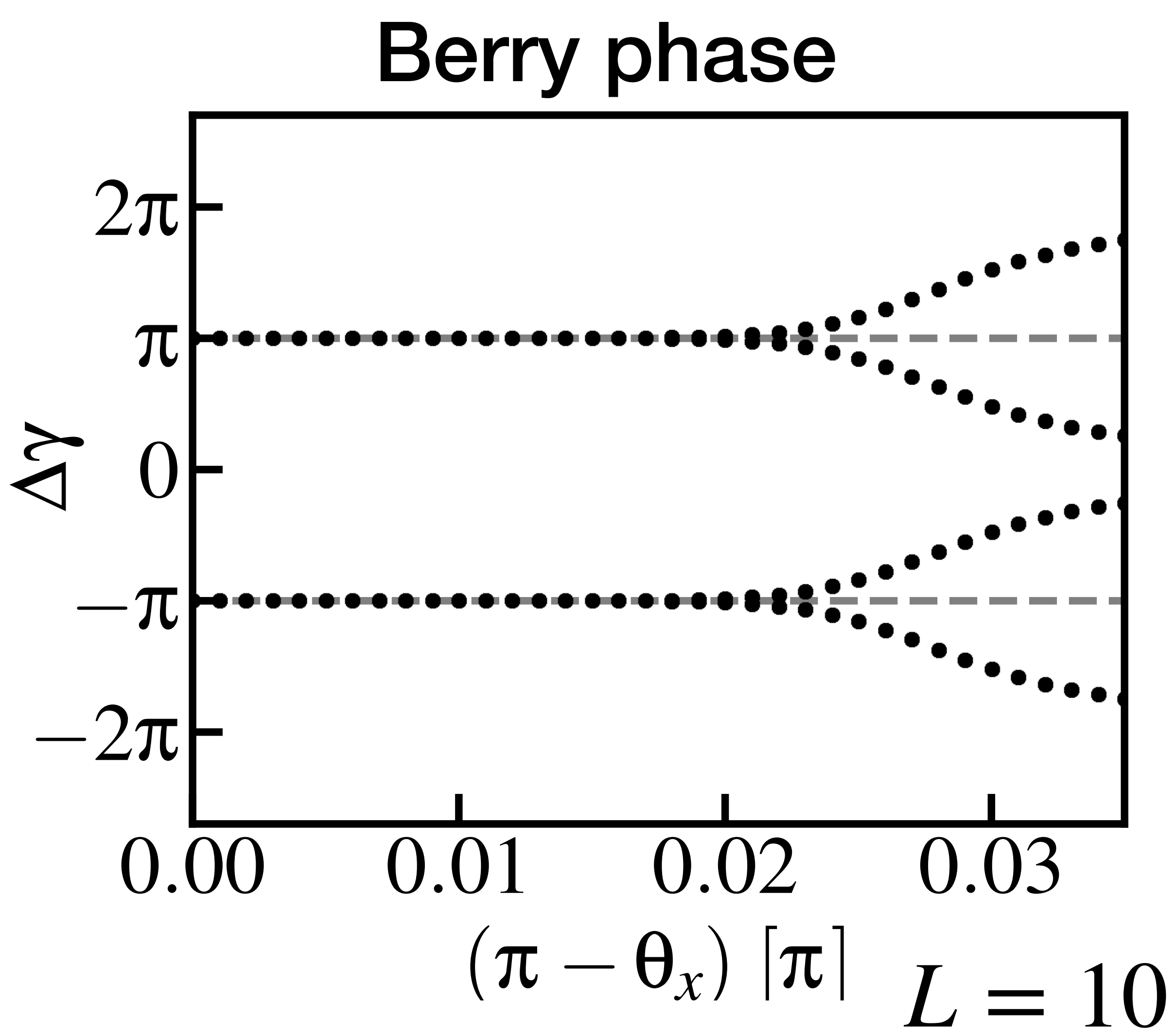

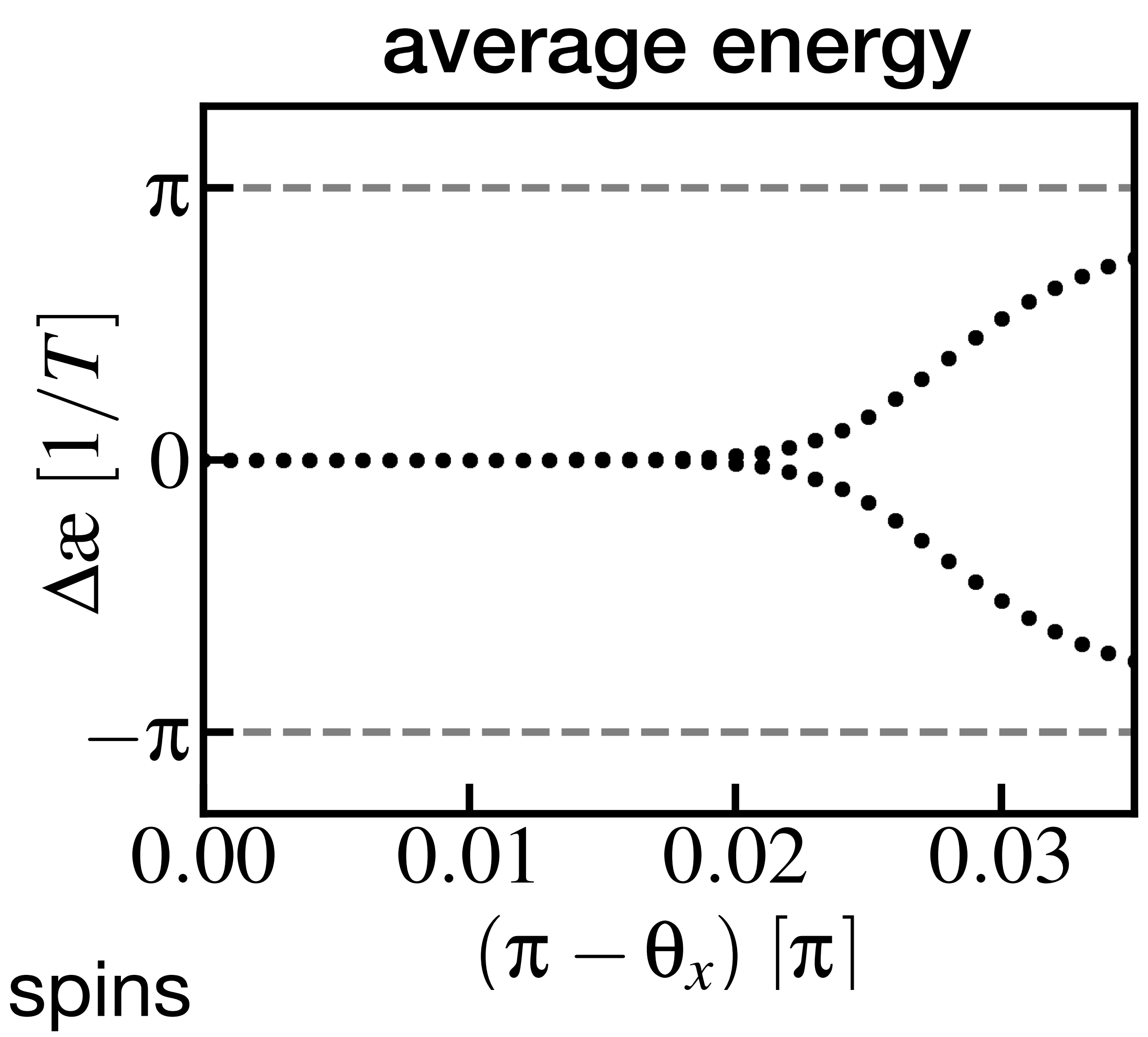

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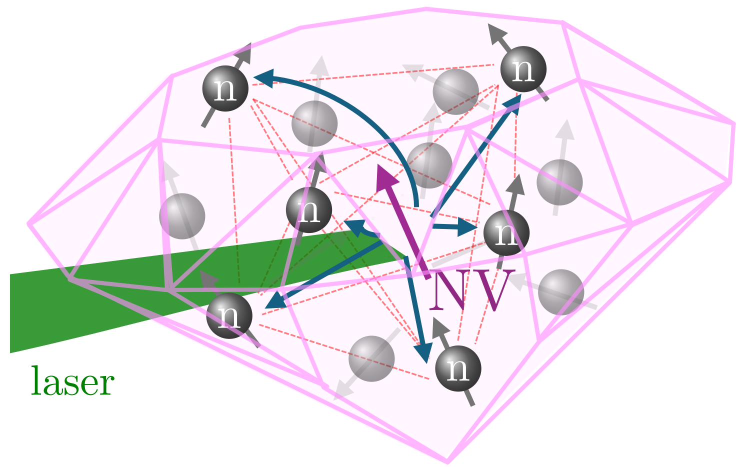

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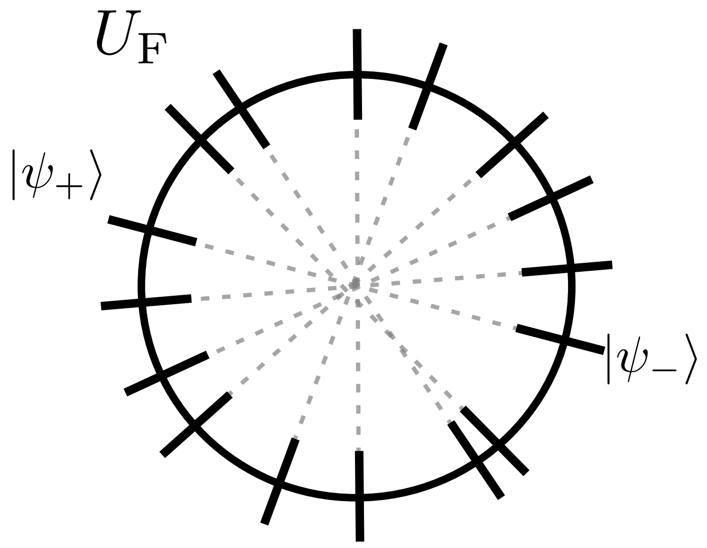

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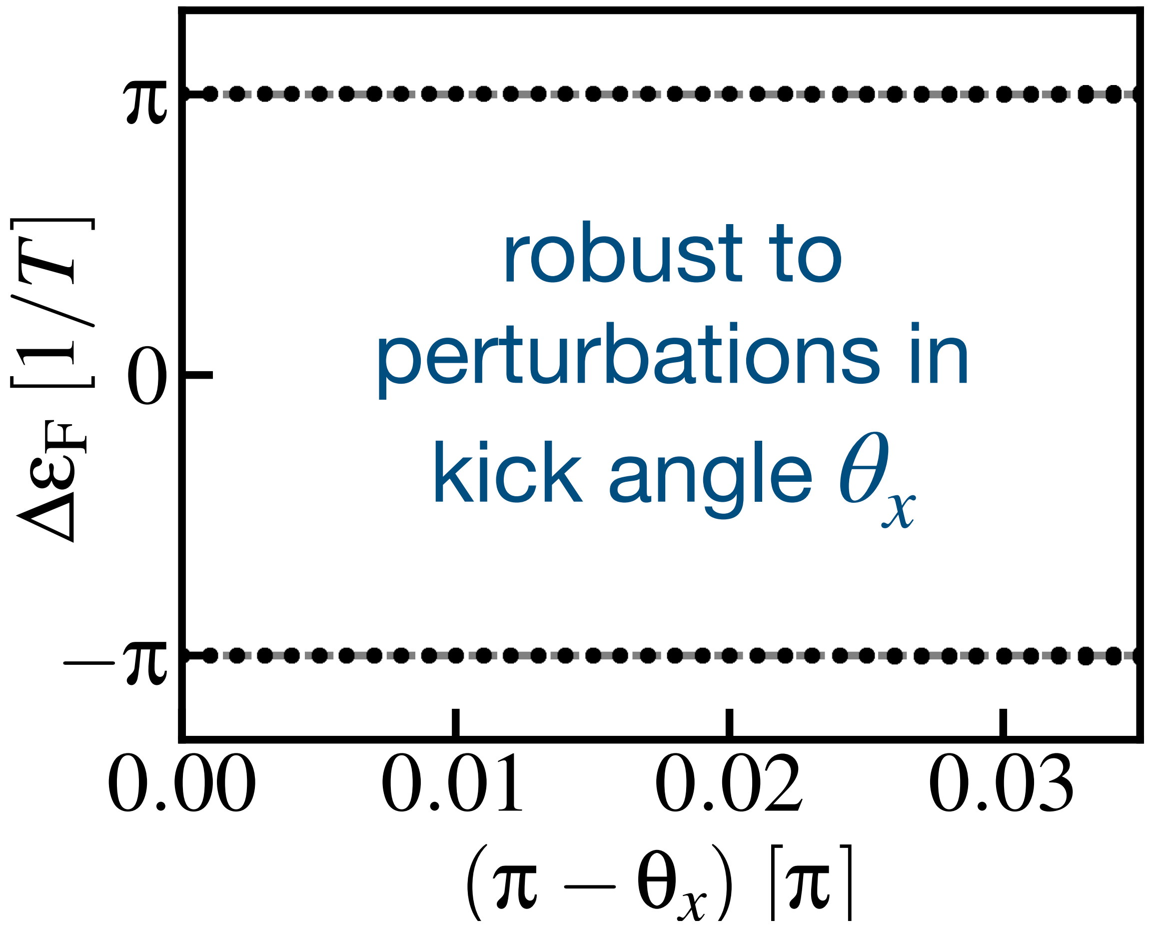
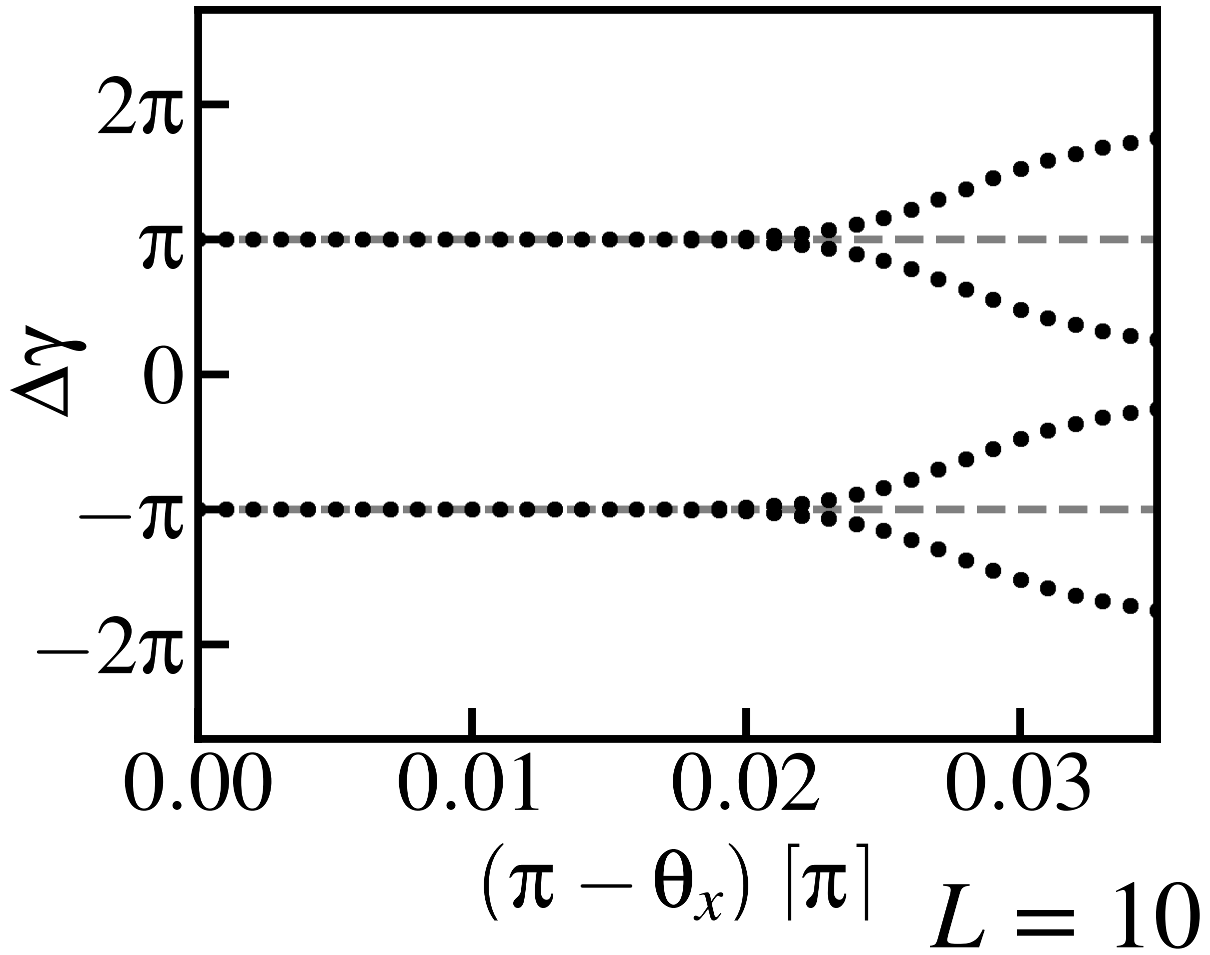
❖ **inherently nonequilibrium phenomena have geometric origin?**



robust to perturbations in kick angle θ_x

Berry phase



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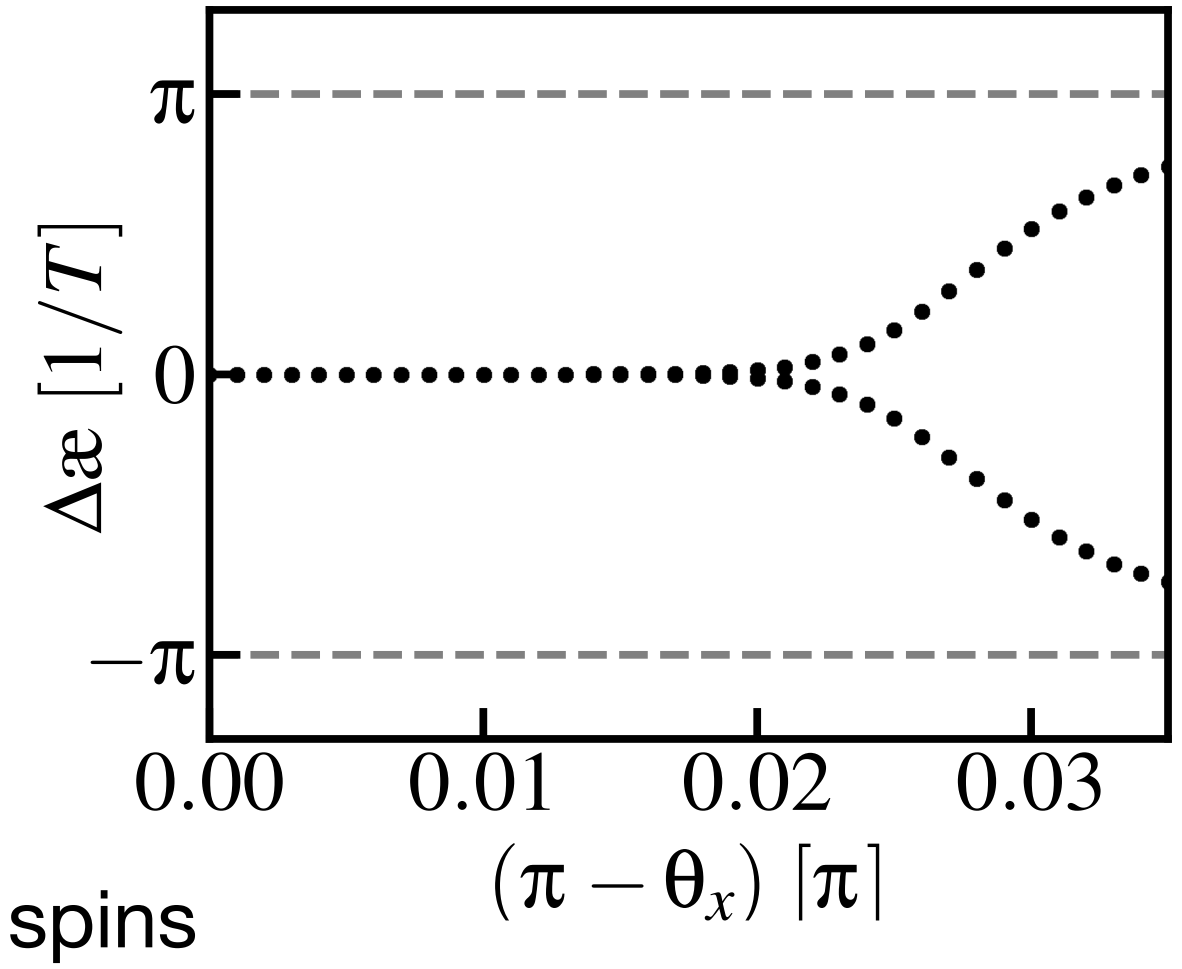


$L = 10$ spins

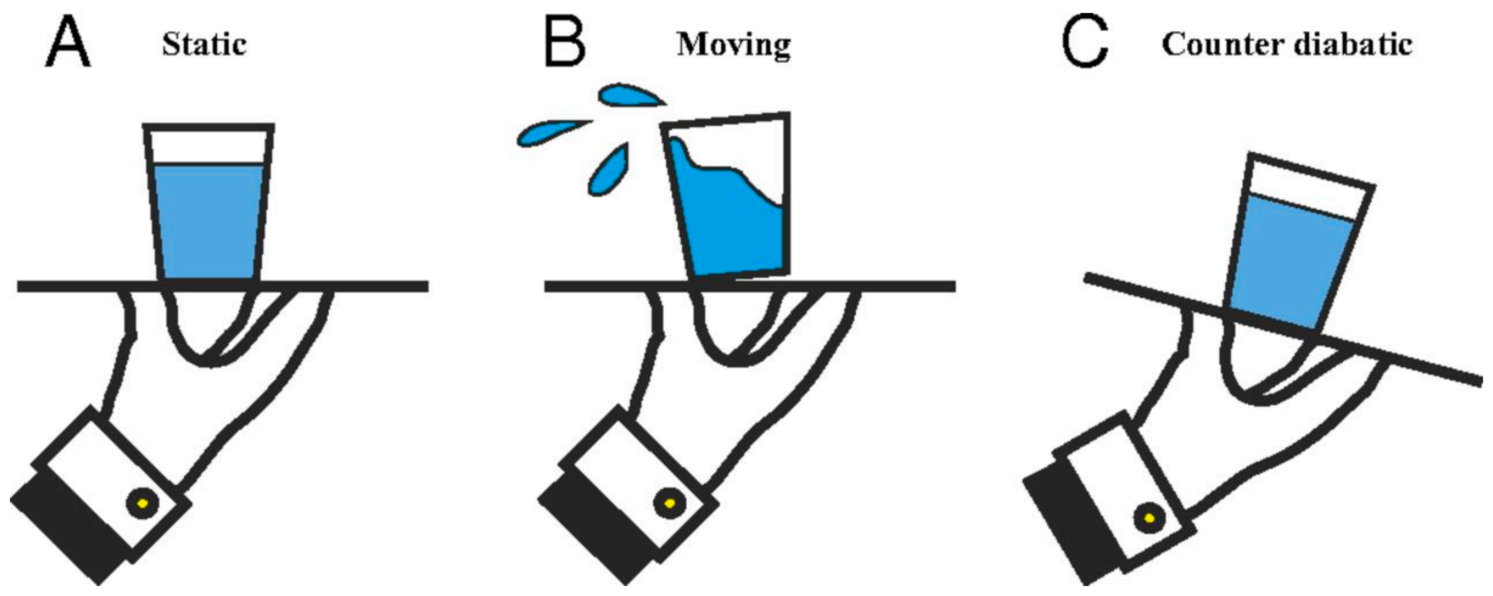
Outline

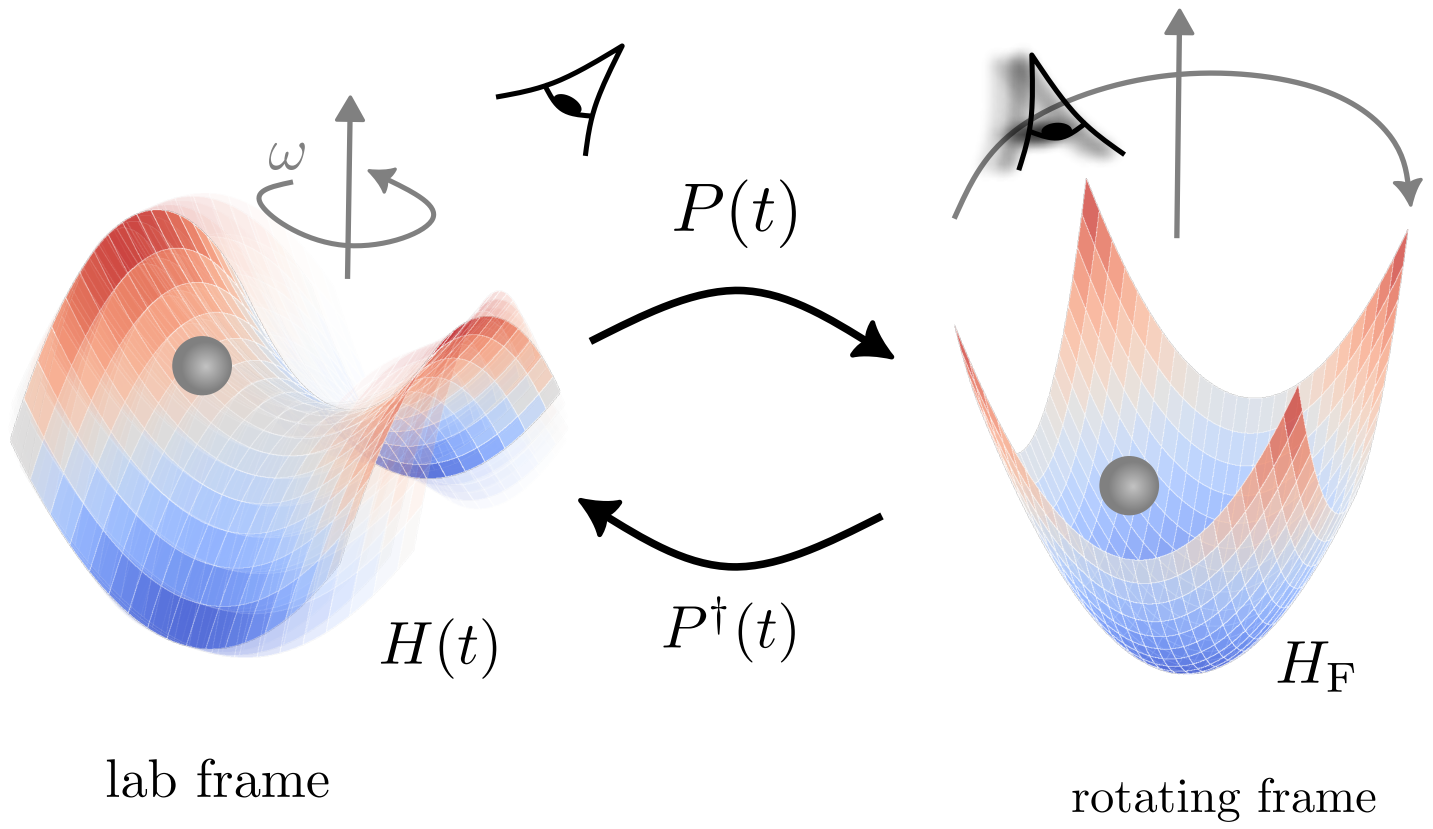
✓ Floquet theory from counterdiabatic driving
  - geometric Floquet decomposition
  - *unambiguous* Floquet ground state

✓ Anomalous chiral spin liquids
  - Floquet topological order

✓ Time crystals
  - nonequilibrium eigenstate order

- **Heating in kicked Ising chain**
  - locality of average energy





Schindler & MB, PRX 15, 031037 (2025)

Heating in kicked Ising spin chain

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$$H^z = -\sum_{n=1}^{L} \frac{J}{4} Z_n Z_{n+1} + \frac{h}{2} Z_n \qquad H^x = -\frac{g}{2}\sum_{n=1}^{L} X_n$$

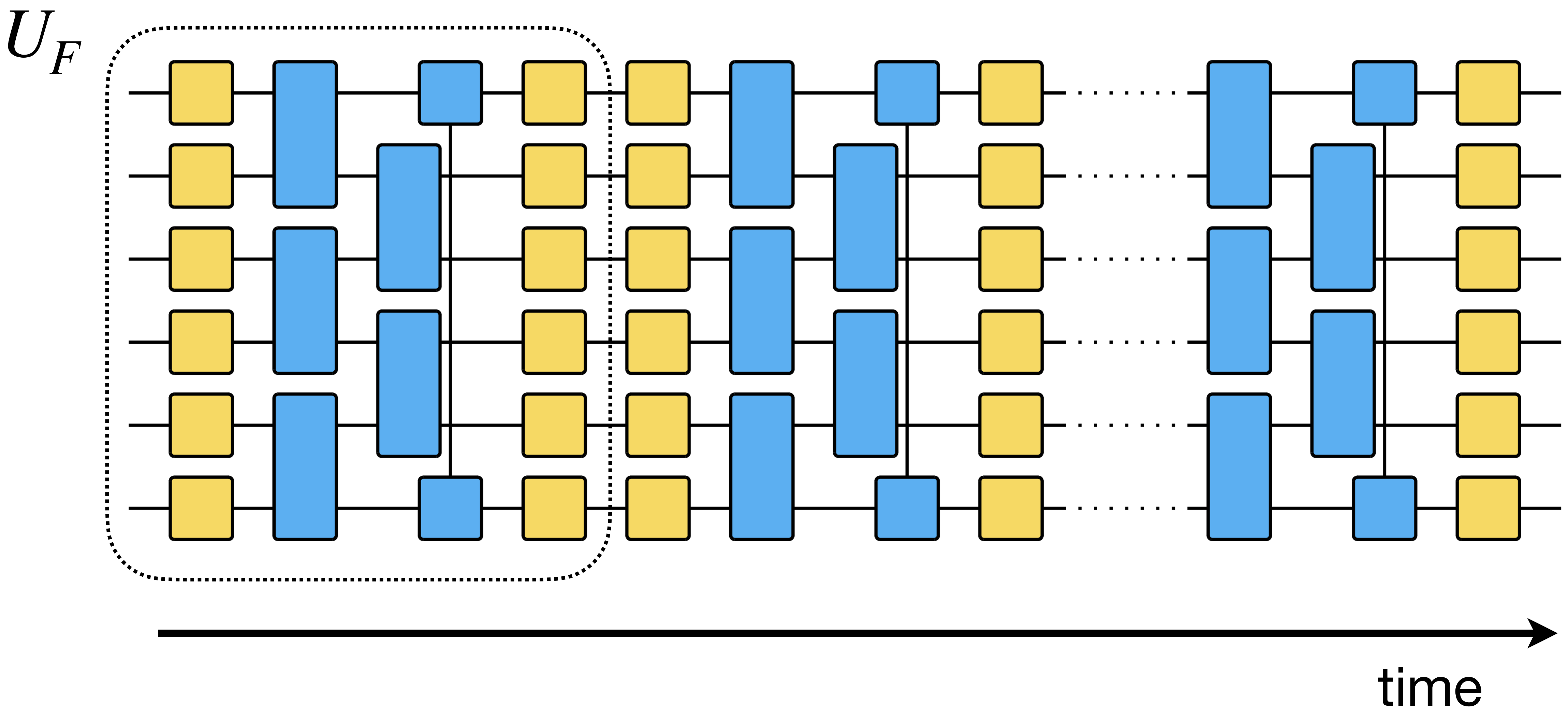

image: Schmitt & Turkeshi

Ikeda et al., PRL 133, 030401 (2024)

Heyl et al., Sci Adv 5 8342 (2019)

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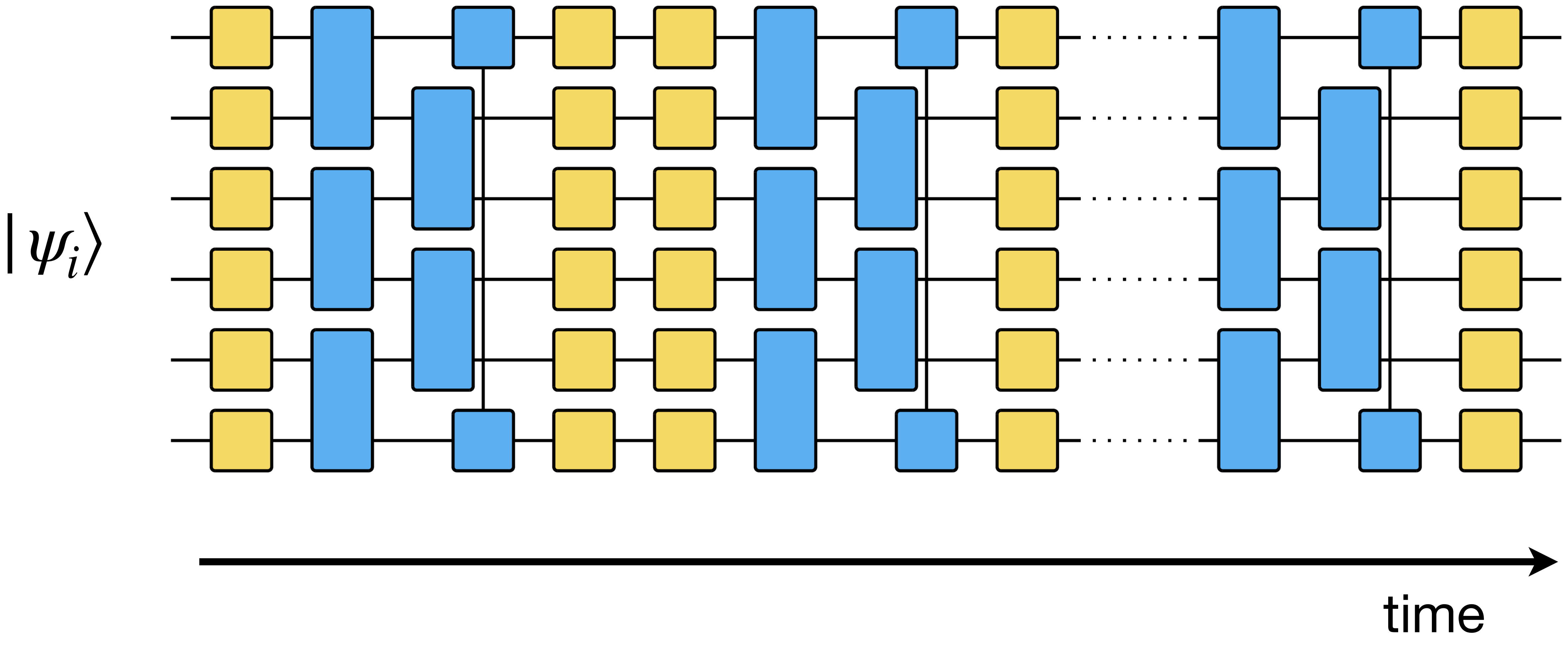

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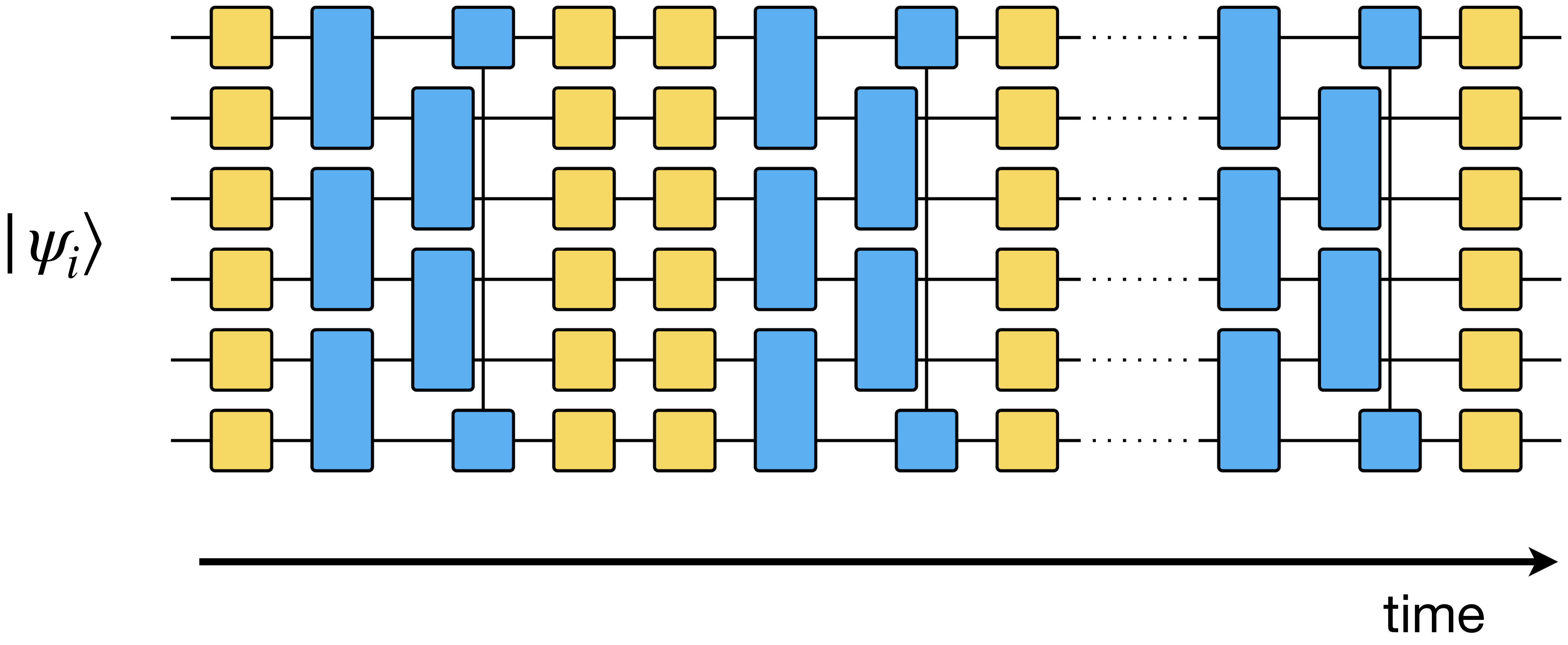

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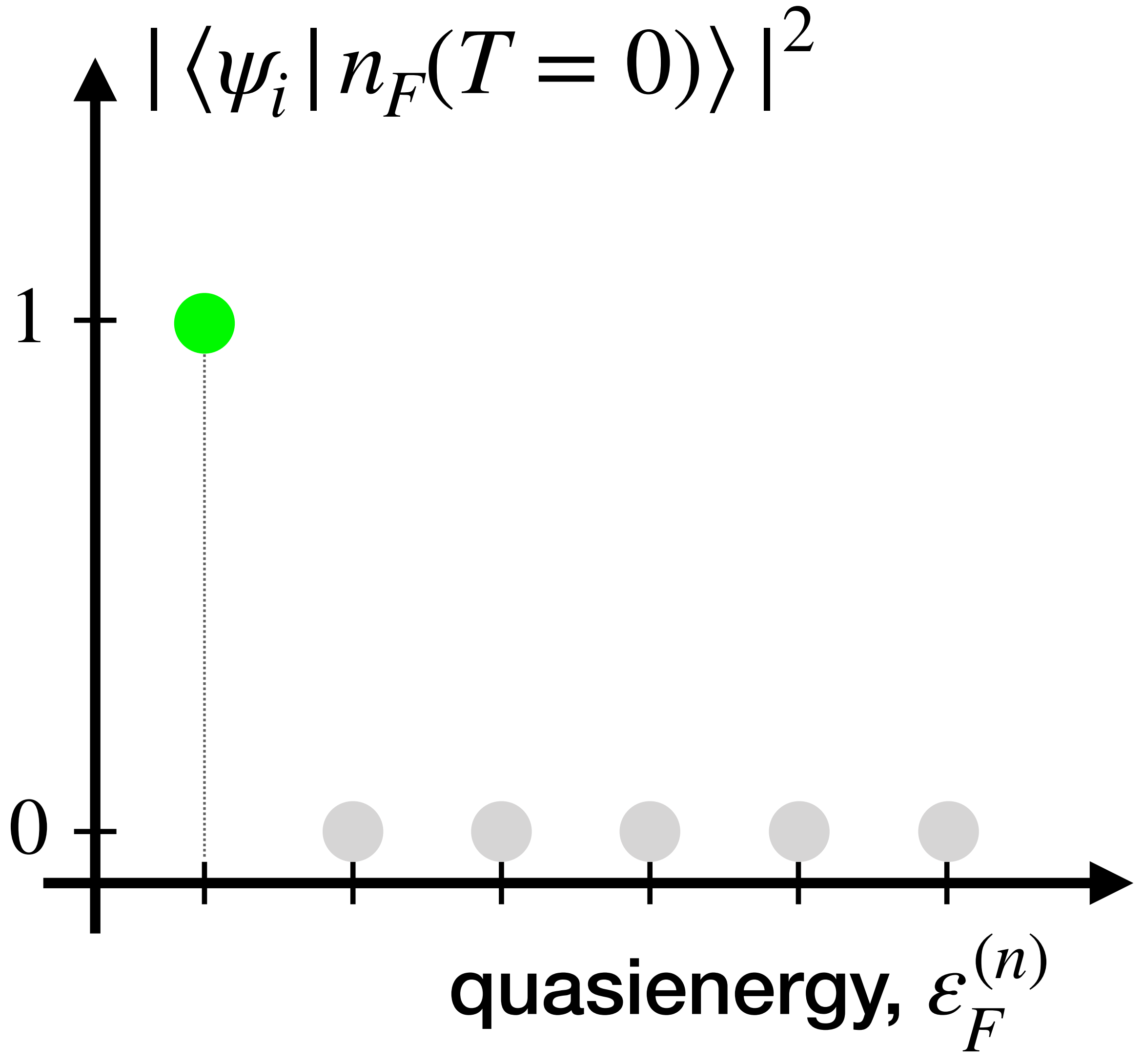

spread-out distribution = heating

Ikeda et al., PRL 133, 030401 (2024)

Heyl et al., Sci Adv 5 8342 (2019)

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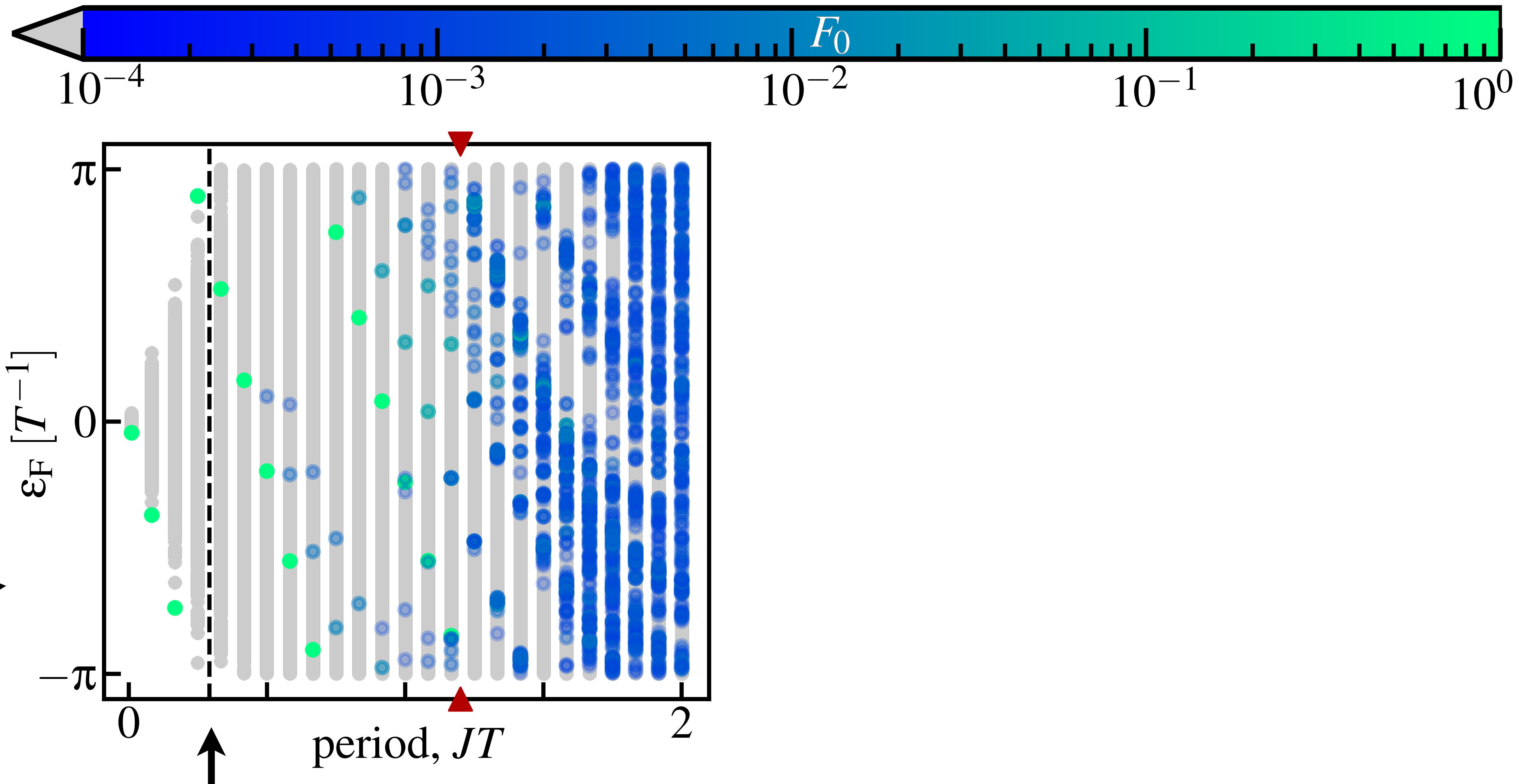
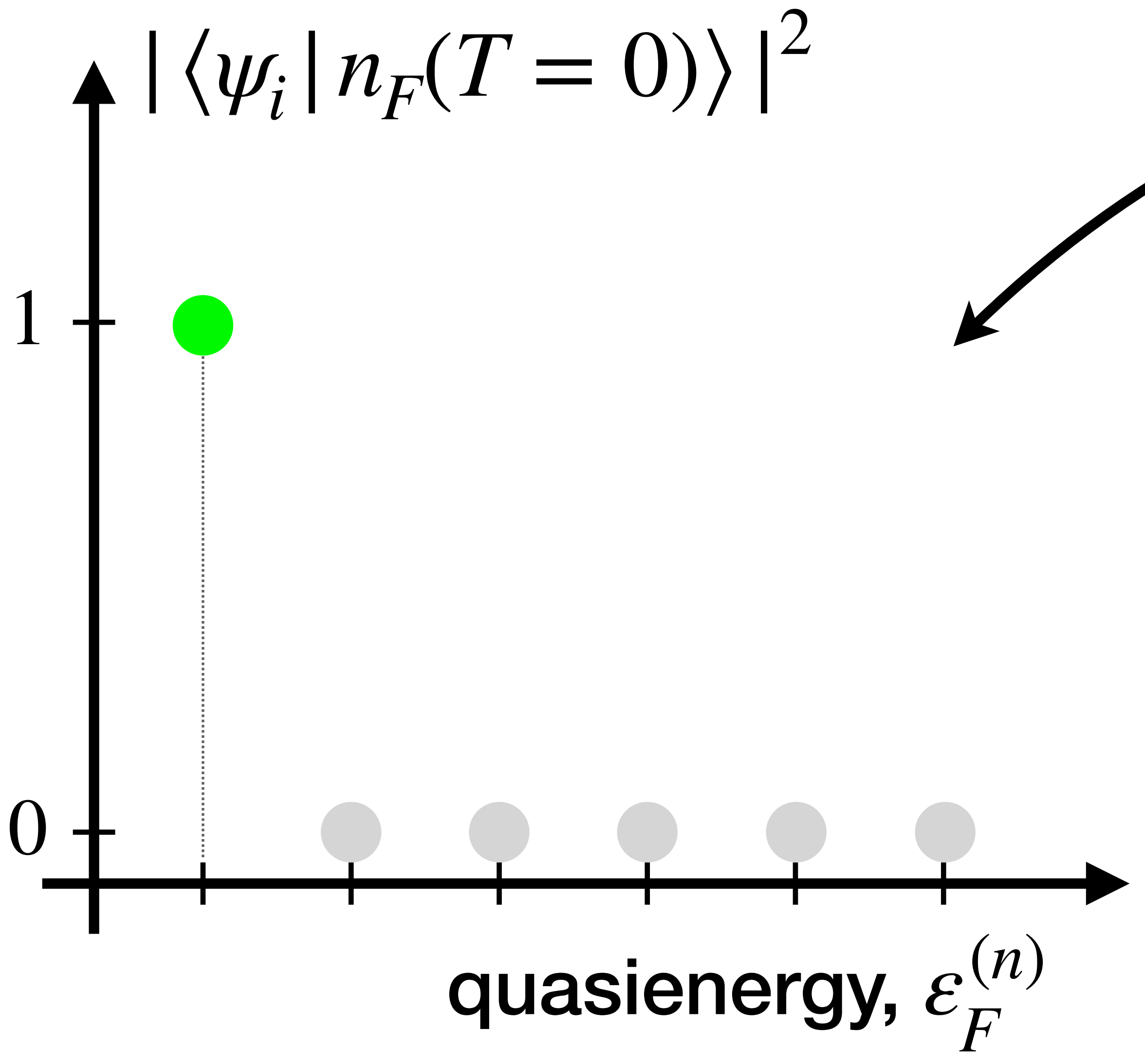


spectral folding



top view

$L = 16$ spins

spread-out distribution = heating

Ikeda et al., PRL 133, 030401 (2024)

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- distribution over average energy
  - no folding problems
  - low-lying excitations

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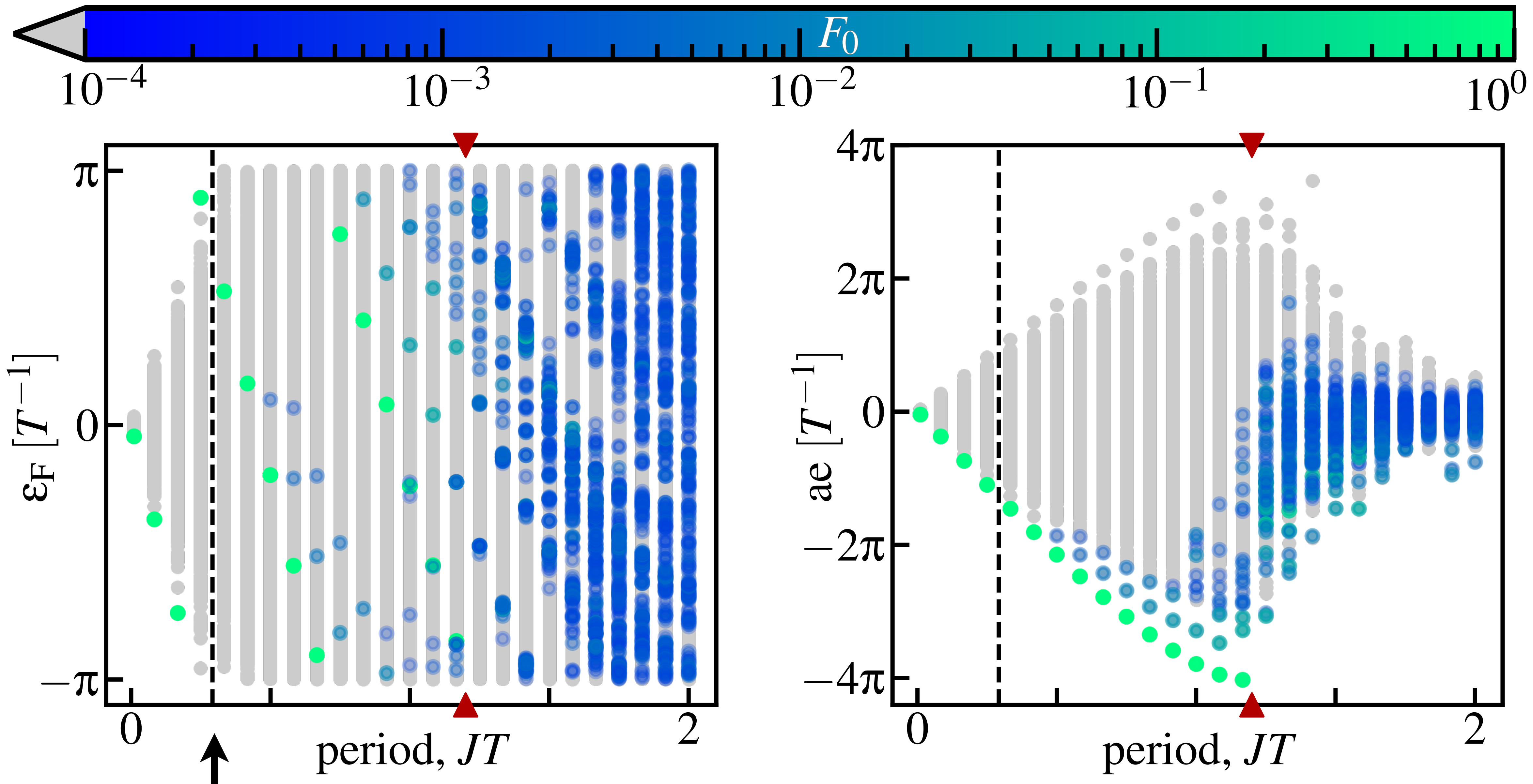

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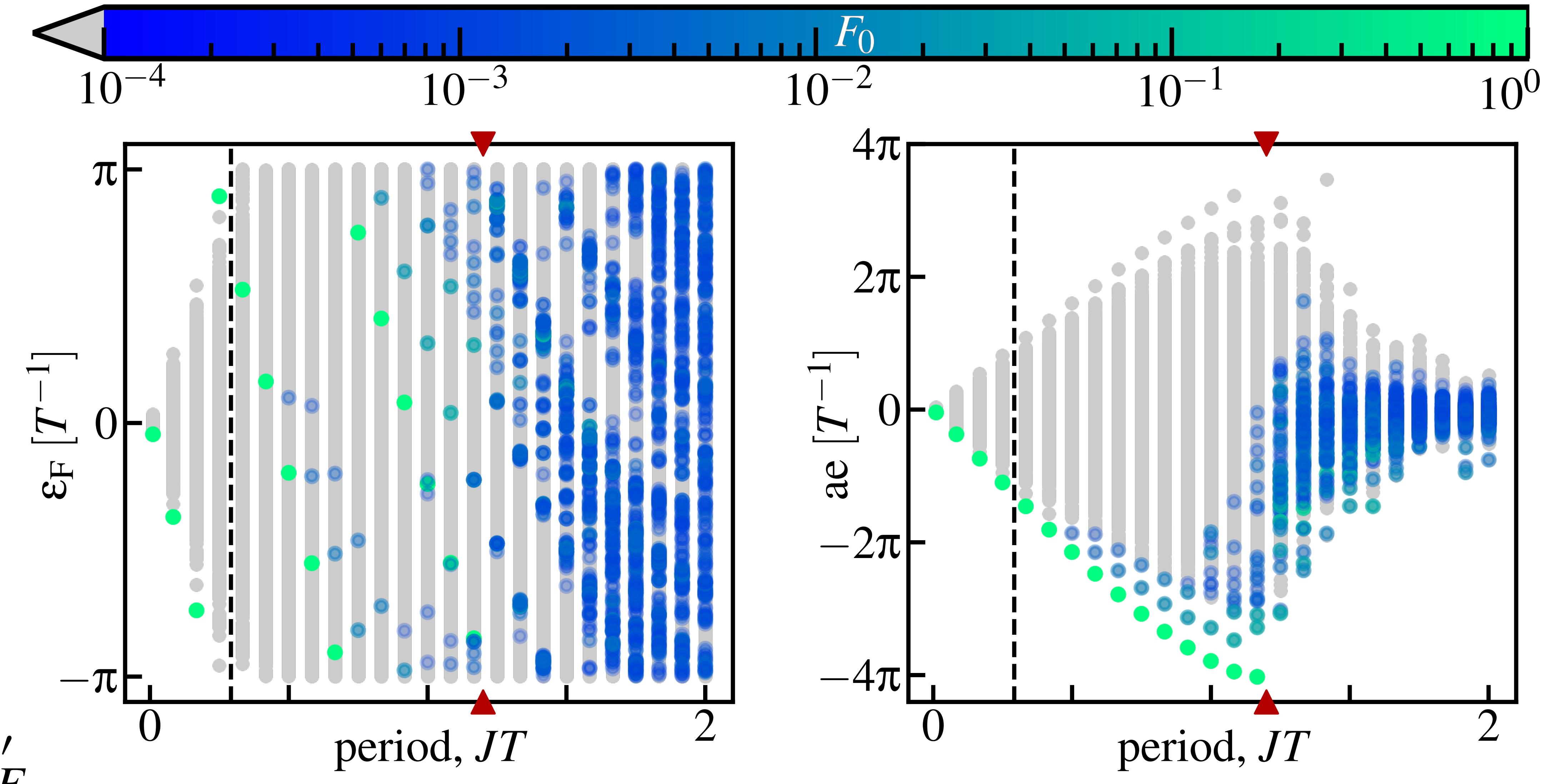
- **locality** H_F **vs. Æ**

H_F is non-local!

$$H_F = \sum \varepsilon_F^{(n)} |n_F\rangle\langle n_F| \longrightarrow \sum \varepsilon_F^{(n)} |n_F\rangle\langle n_F| + \omega |m_F\rangle\langle m_F| = H'_F$$



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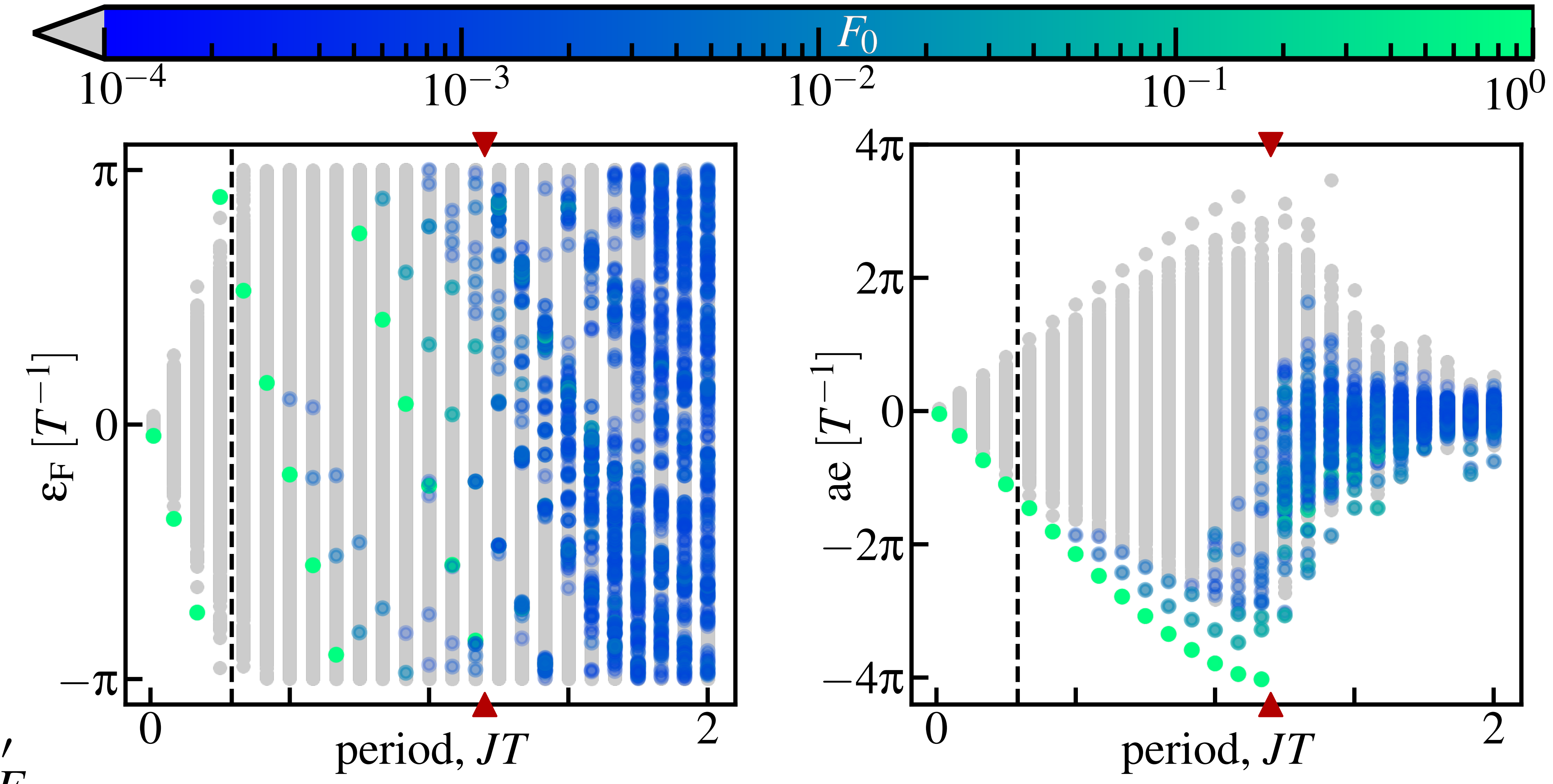
- find operator weight on all 2-body Pauli strings:

$$\mathcal{O}_{\text{approx}} = \sum_{\alpha,\beta} o_\alpha \sigma^i + o_{\alpha\beta} \sigma^\alpha \sigma^\beta \qquad \sigma^\alpha \in \{X, Y, Z, I\}$$



$L = 16$ spins

$$\frac{\|\mathcal{O}_{\text{approx}}\|}{\|\mathcal{O}_{\text{exact}}\|}$$

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- locality H_F vs. Æ

H_F is non-local!

$$H_F = \sum \varepsilon_F^{(n)} |n_F\rangle\langle n_F| \longrightarrow \sum \varepsilon_F^{(n)} |n_F\rangle\langle n_F| + \omega |m_F\rangle\langle m_F| = H'_F$$

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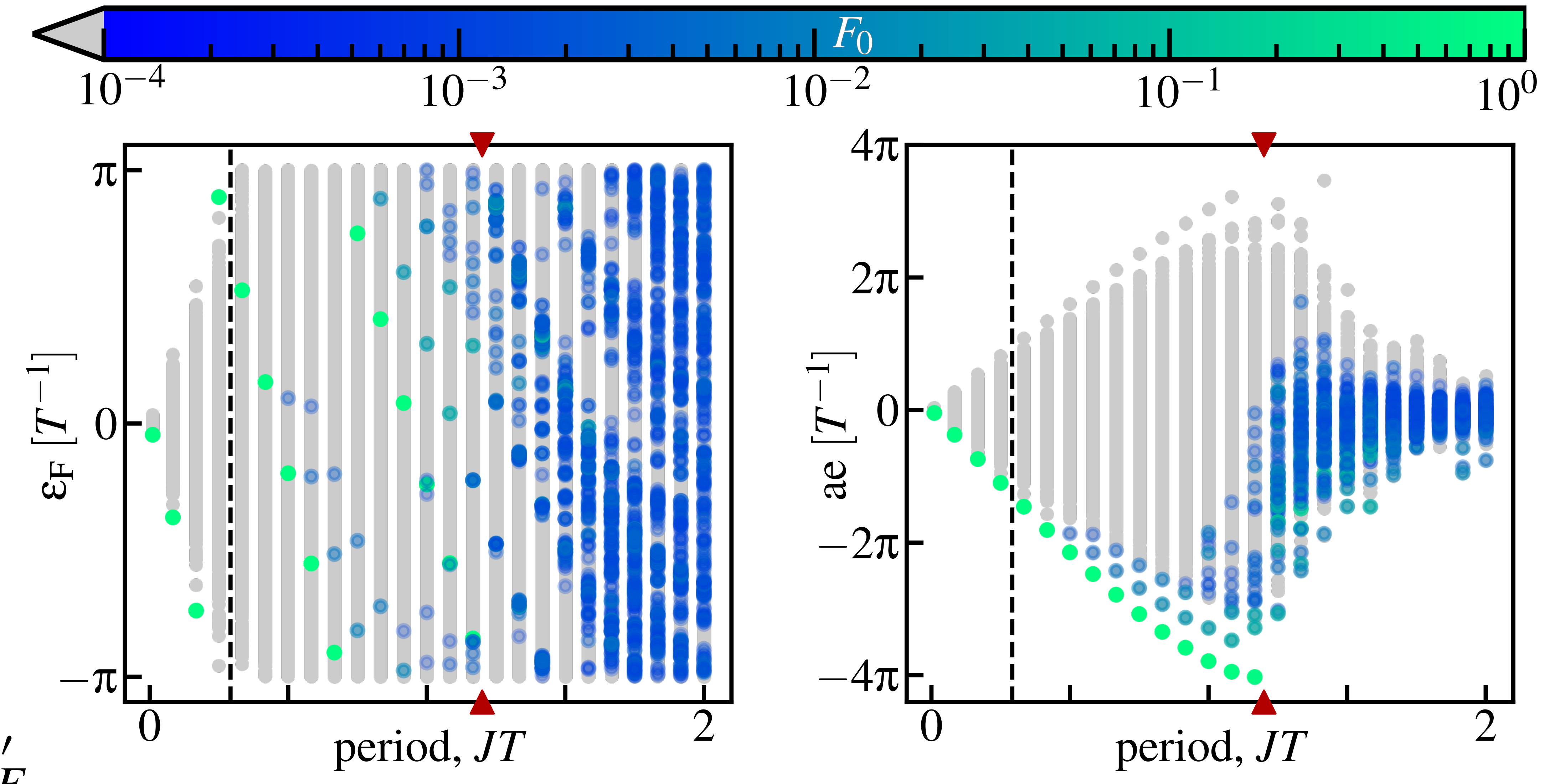
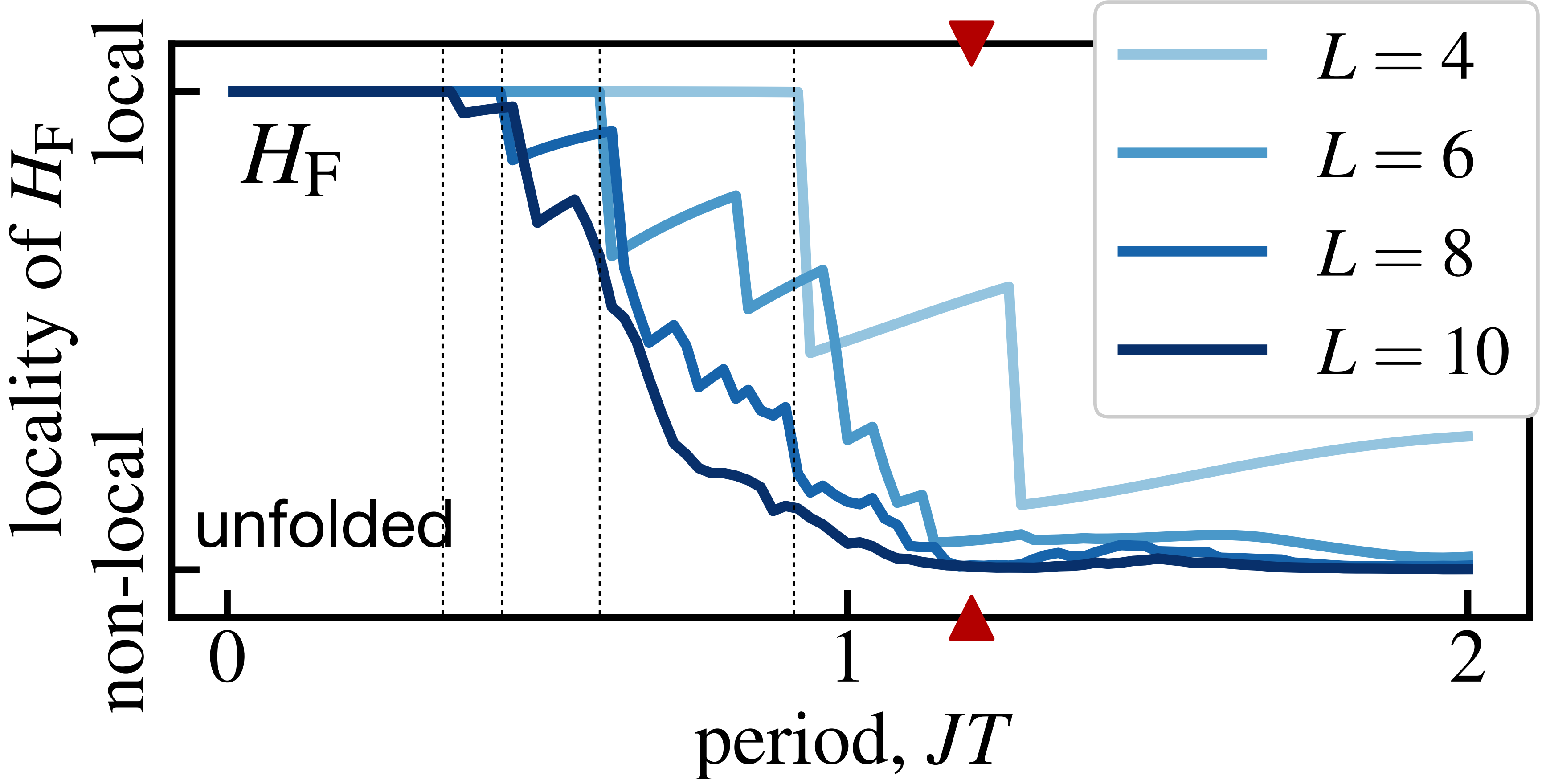


$L = 16$ spins

$$\frac{\|\mathcal{O}_{\text{approx}}\|}{\|\mathcal{O}_{\text{exact}}\|}$$



Heating in kicked Ising spin chain

$$H^z = -\sum_{n=1}^{L} \frac{J}{4} Z_n Z_{n+1} + \frac{h}{2} Z_n$$

$$H^x = -\frac{g}{2} \sum_{n=1}^{L} X_n$$

- **evolution operator** $U_F = e^{-i\frac{T}{4}H^z} e^{-i\frac{T}{2}H^x} e^{-i\frac{T}{4}H^z}$
  - initial state: ground state of $H^z + H^x$
  - decompose in Floquet basis $|\langle\psi_i|n_F\rangle|^2$
- **distribution over average energy**
  - no folding problems
  - low-lying excitations
- **locality** H_F **vs. Æ**

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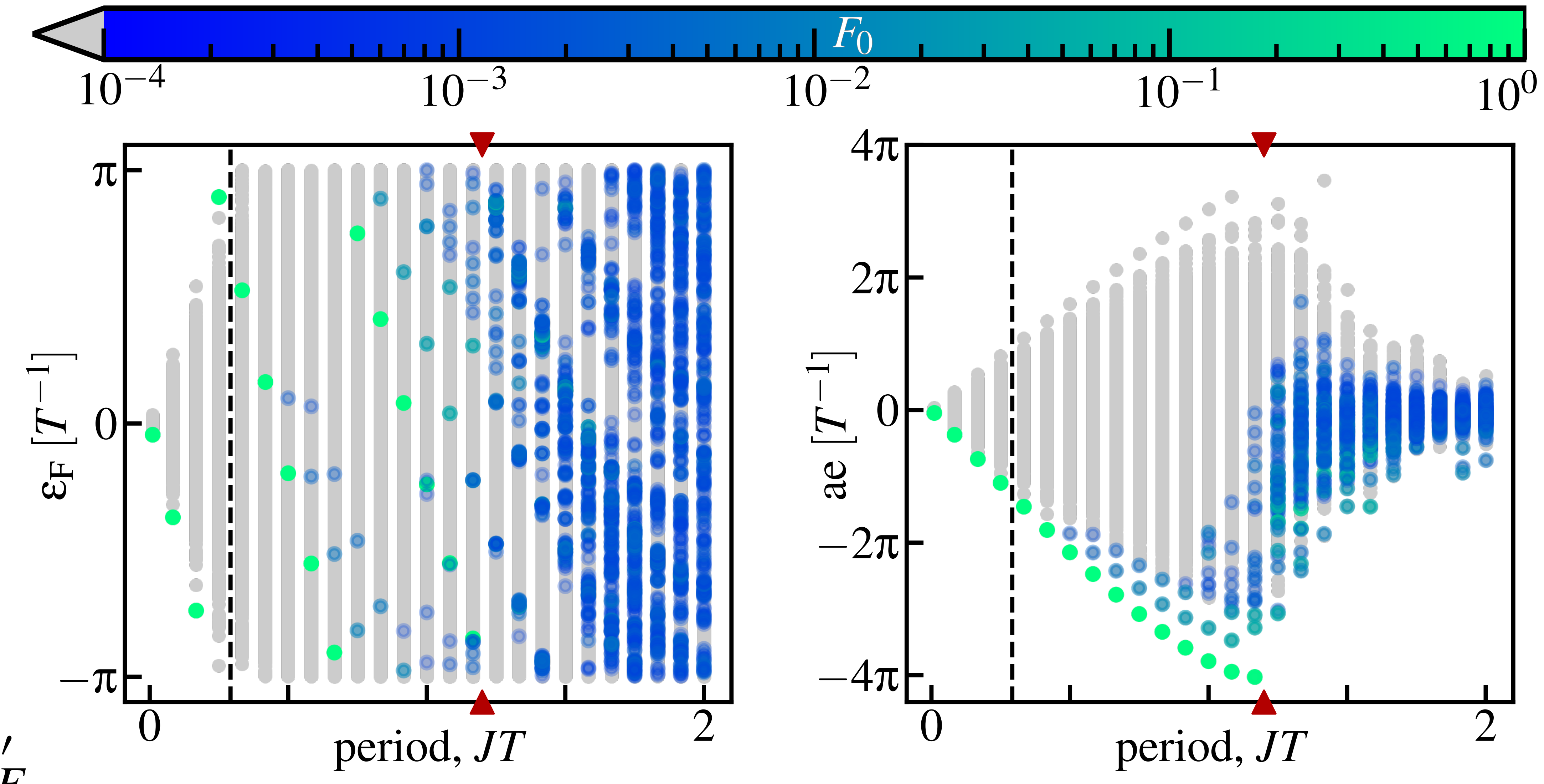

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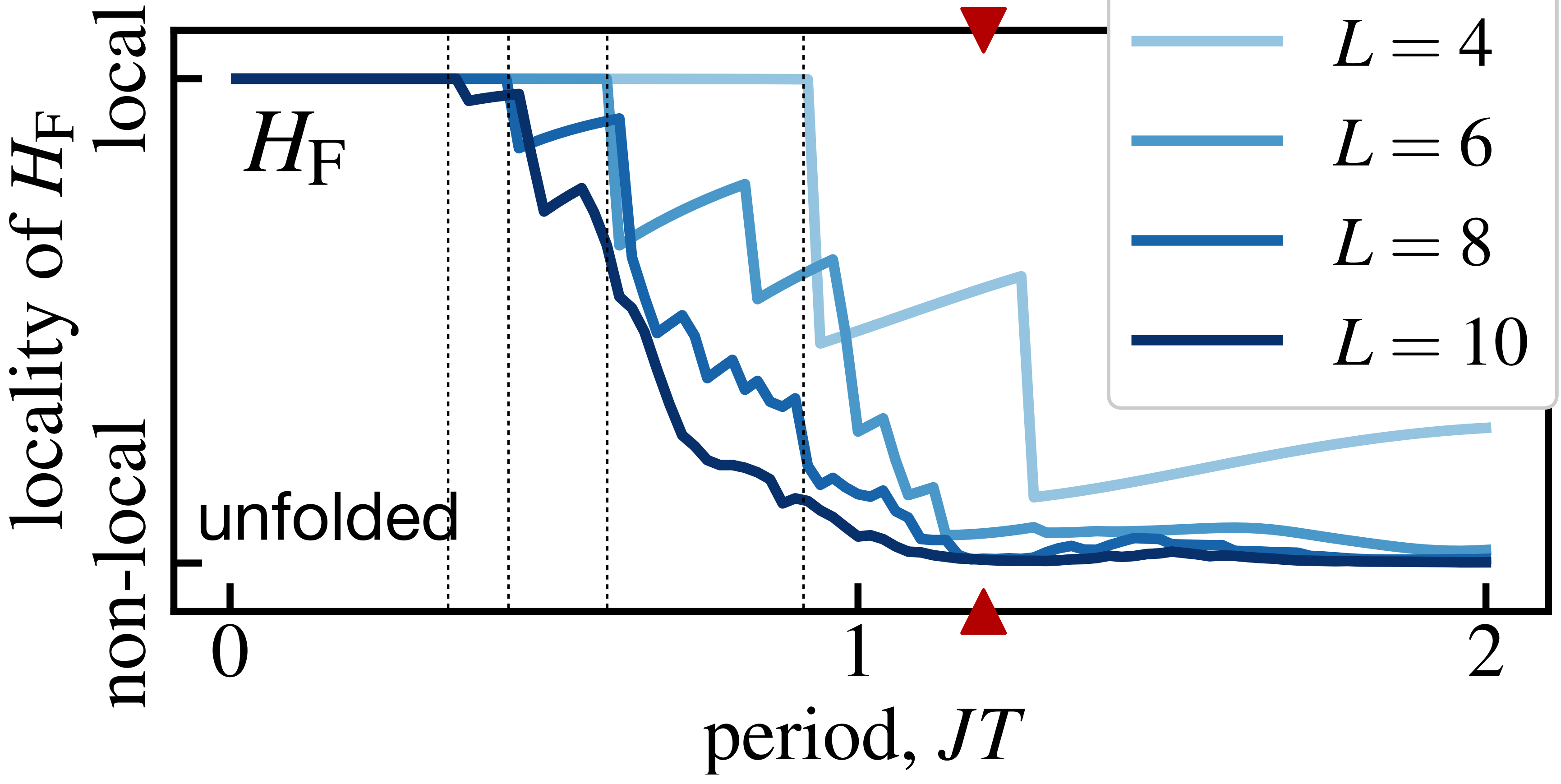

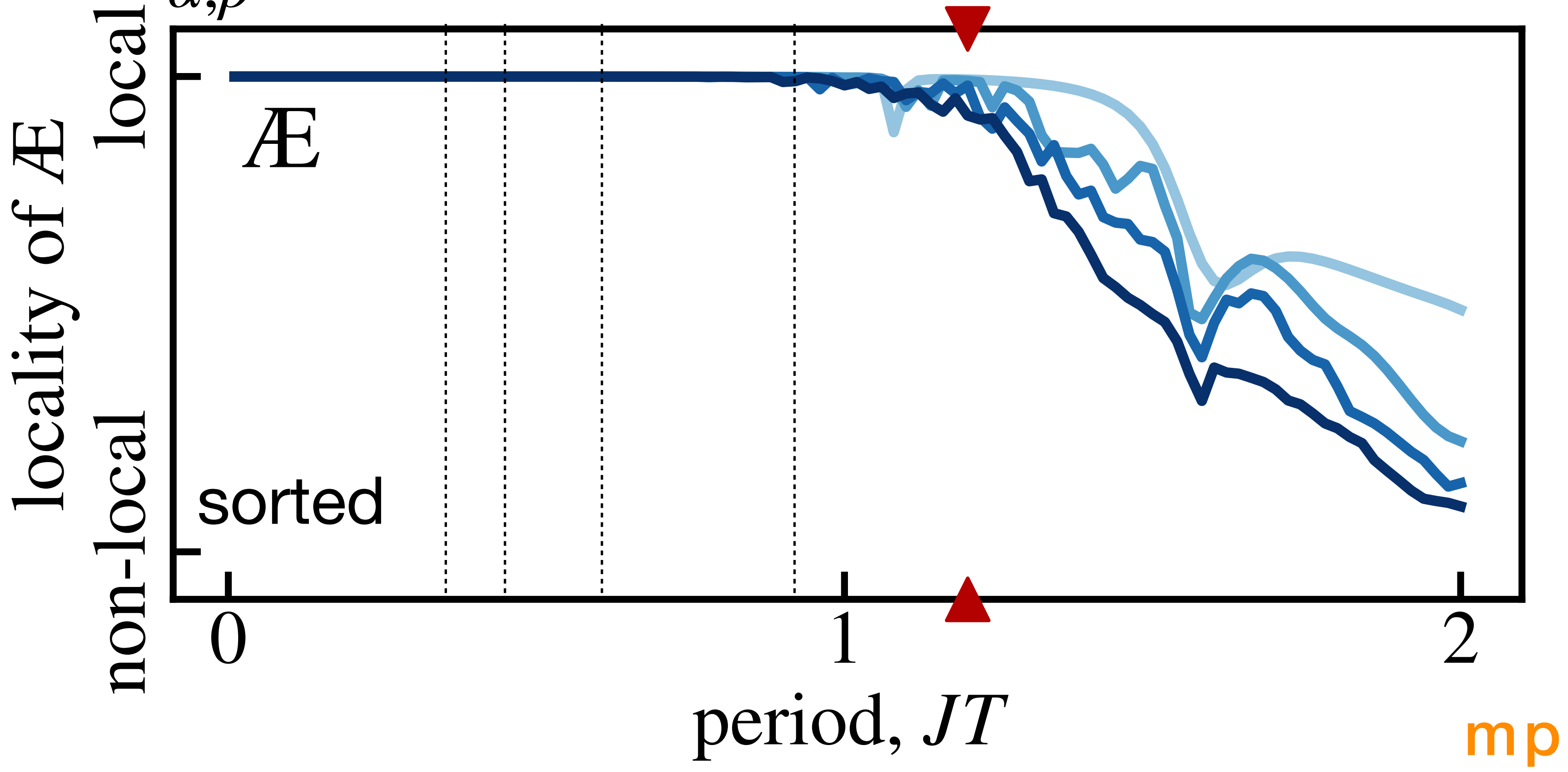

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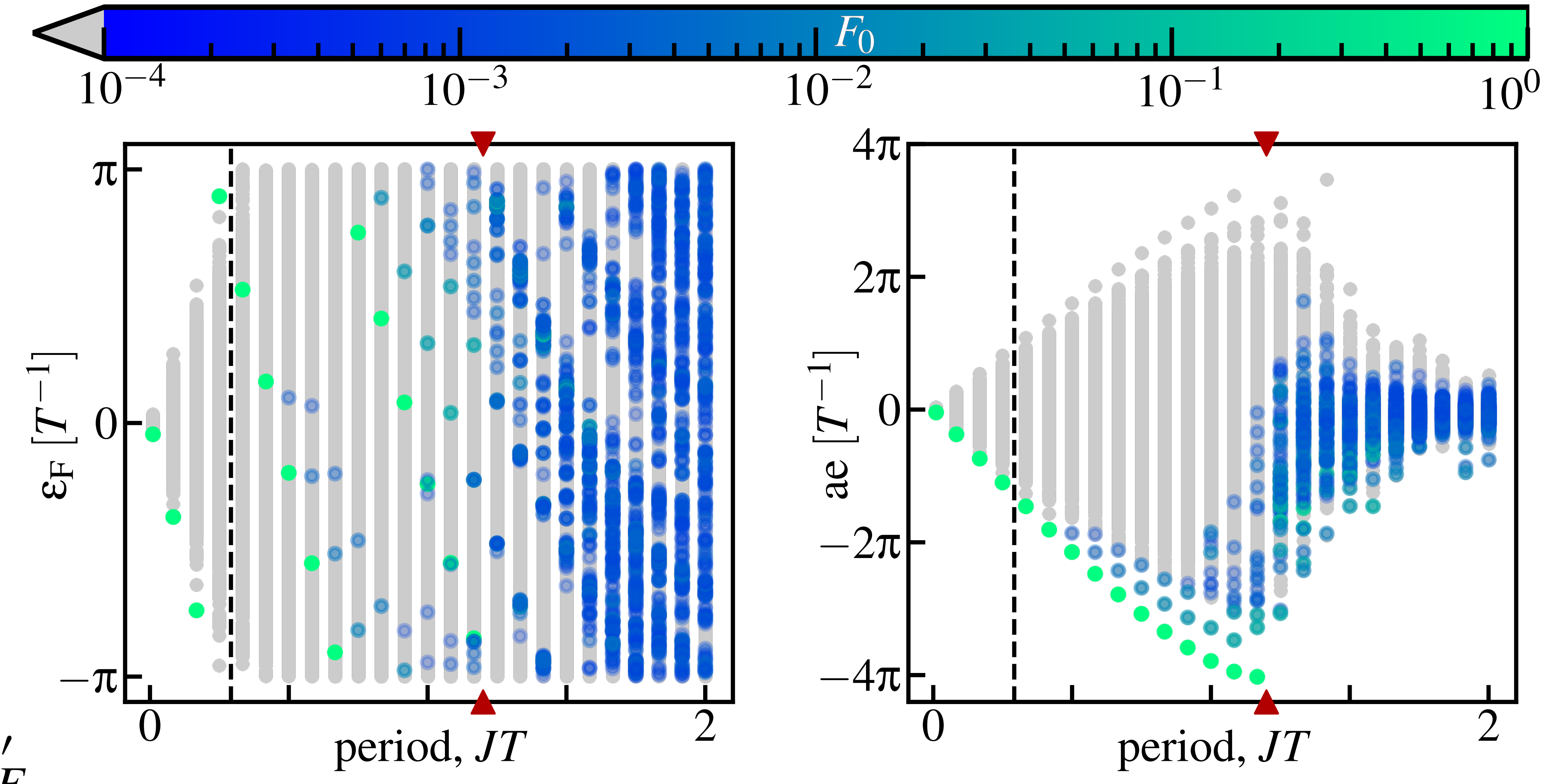

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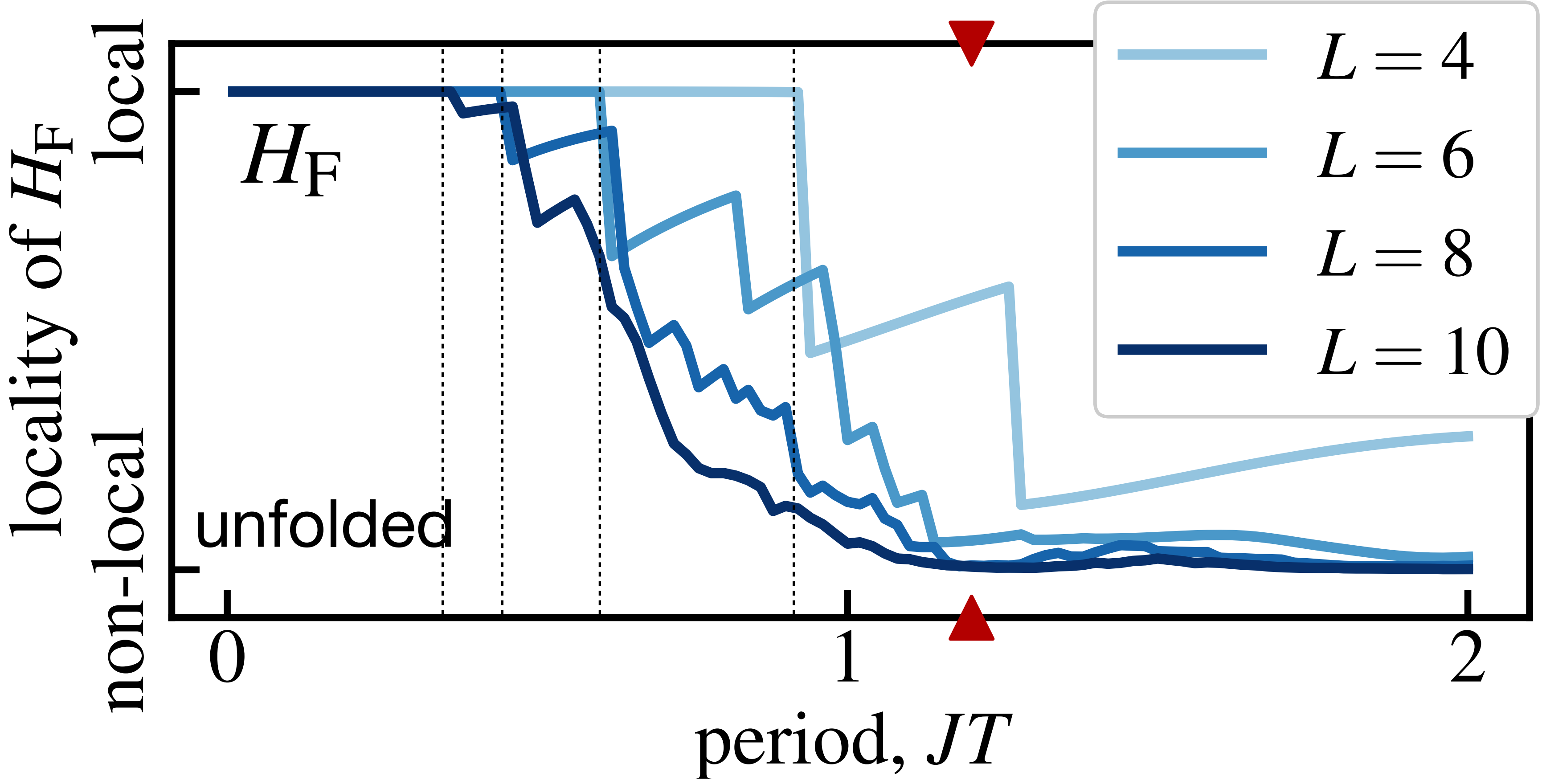

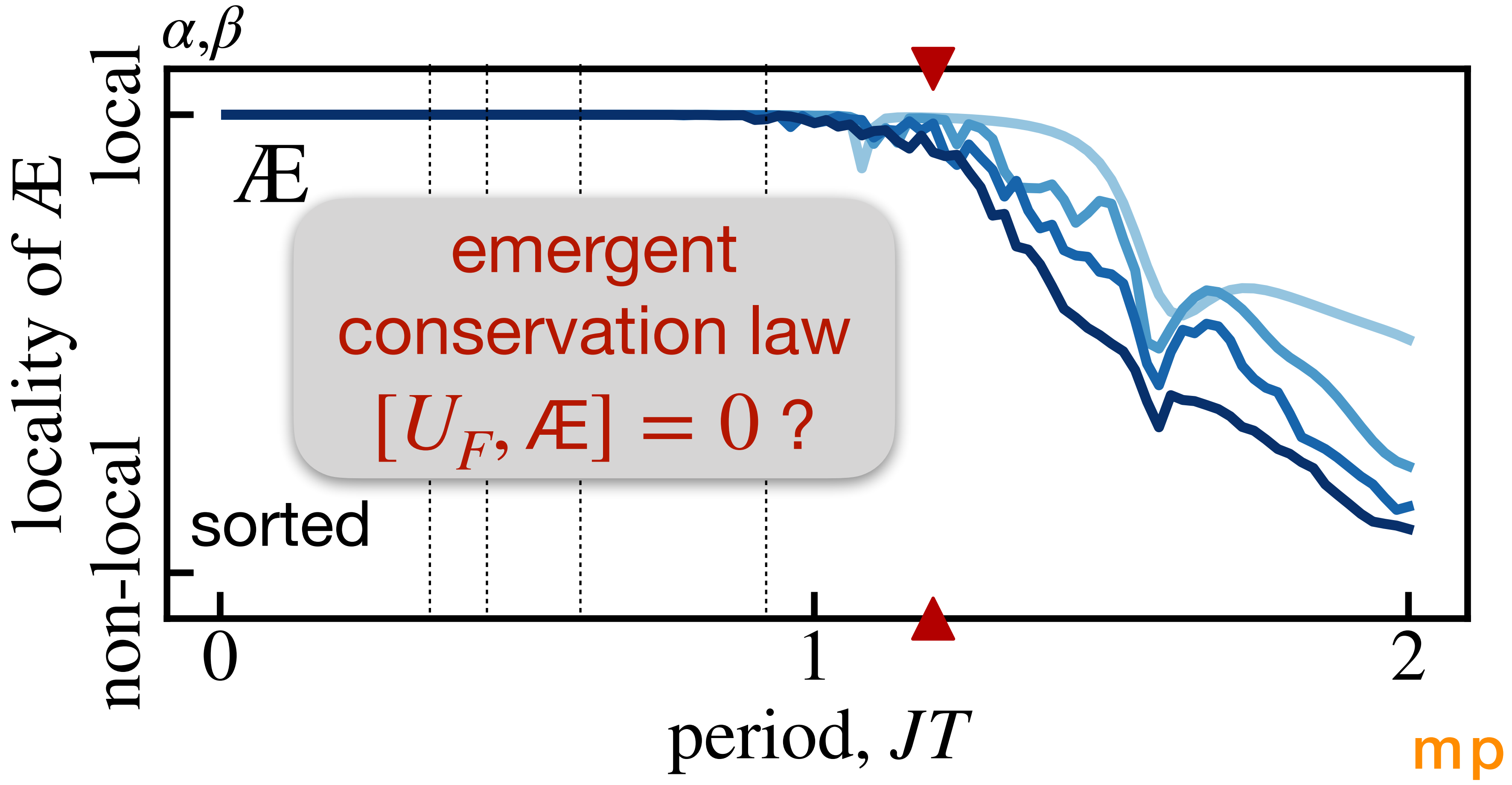

emergent conservation law $[U_F, Æ] = 0$?

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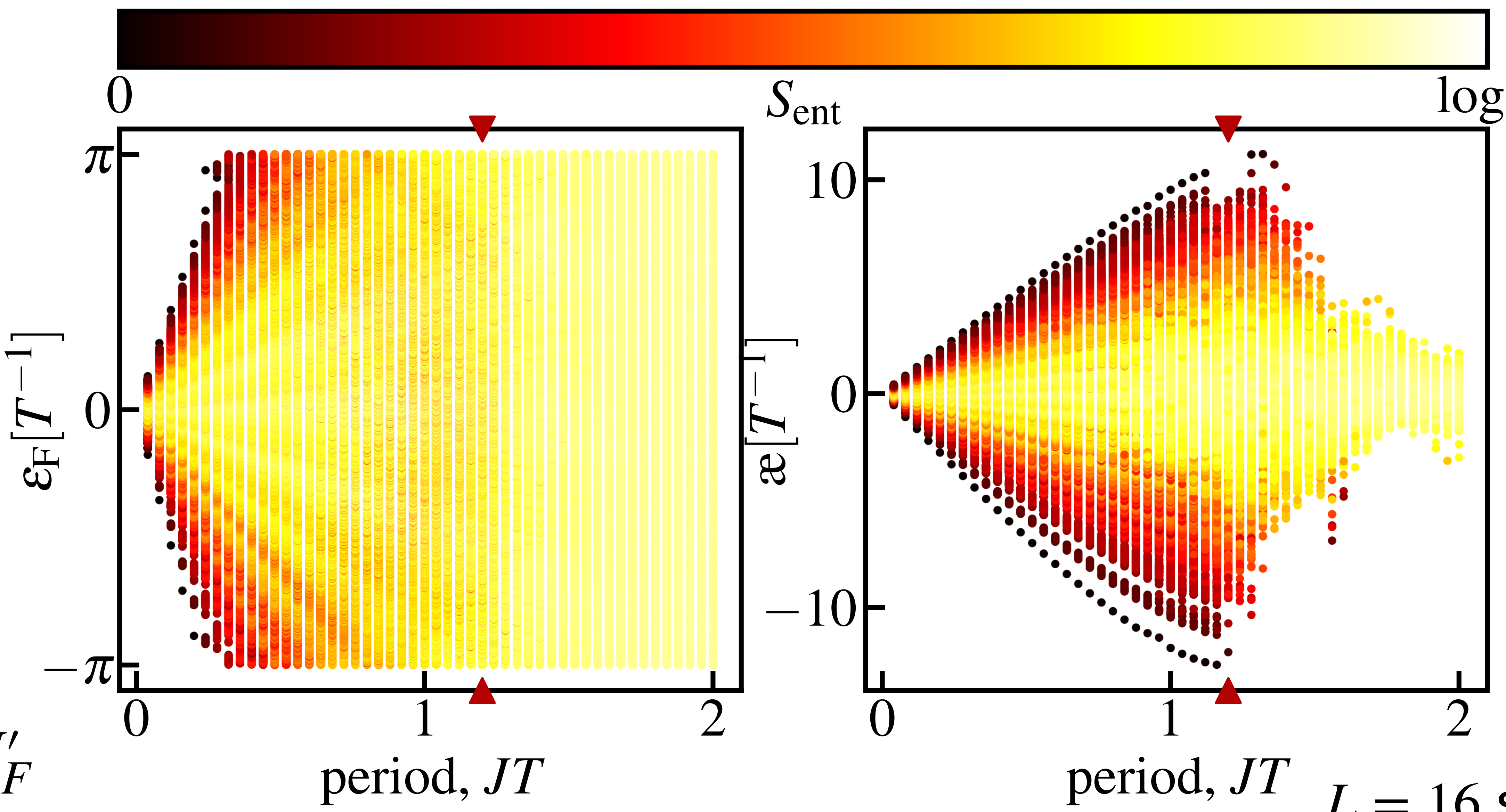

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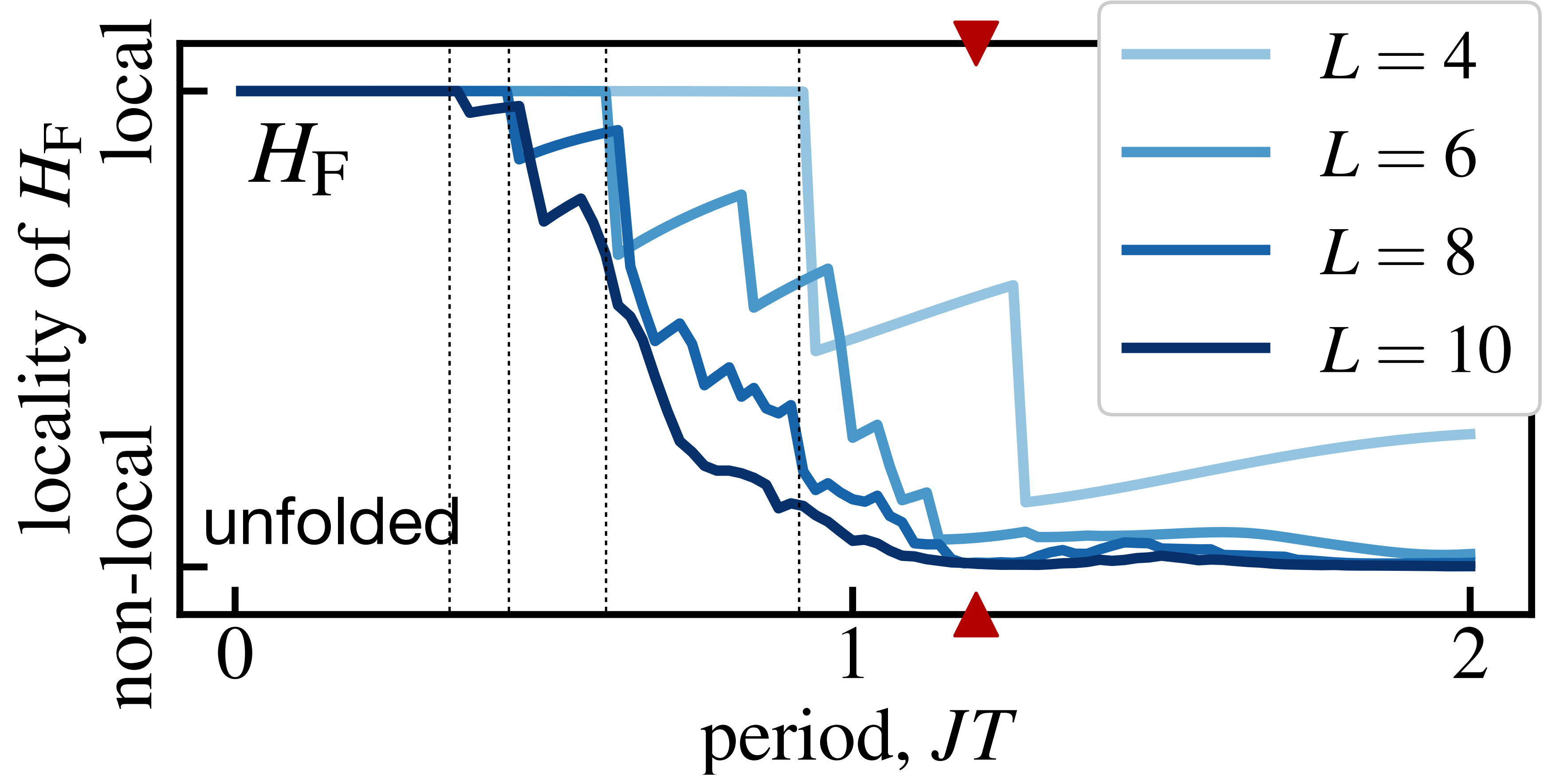
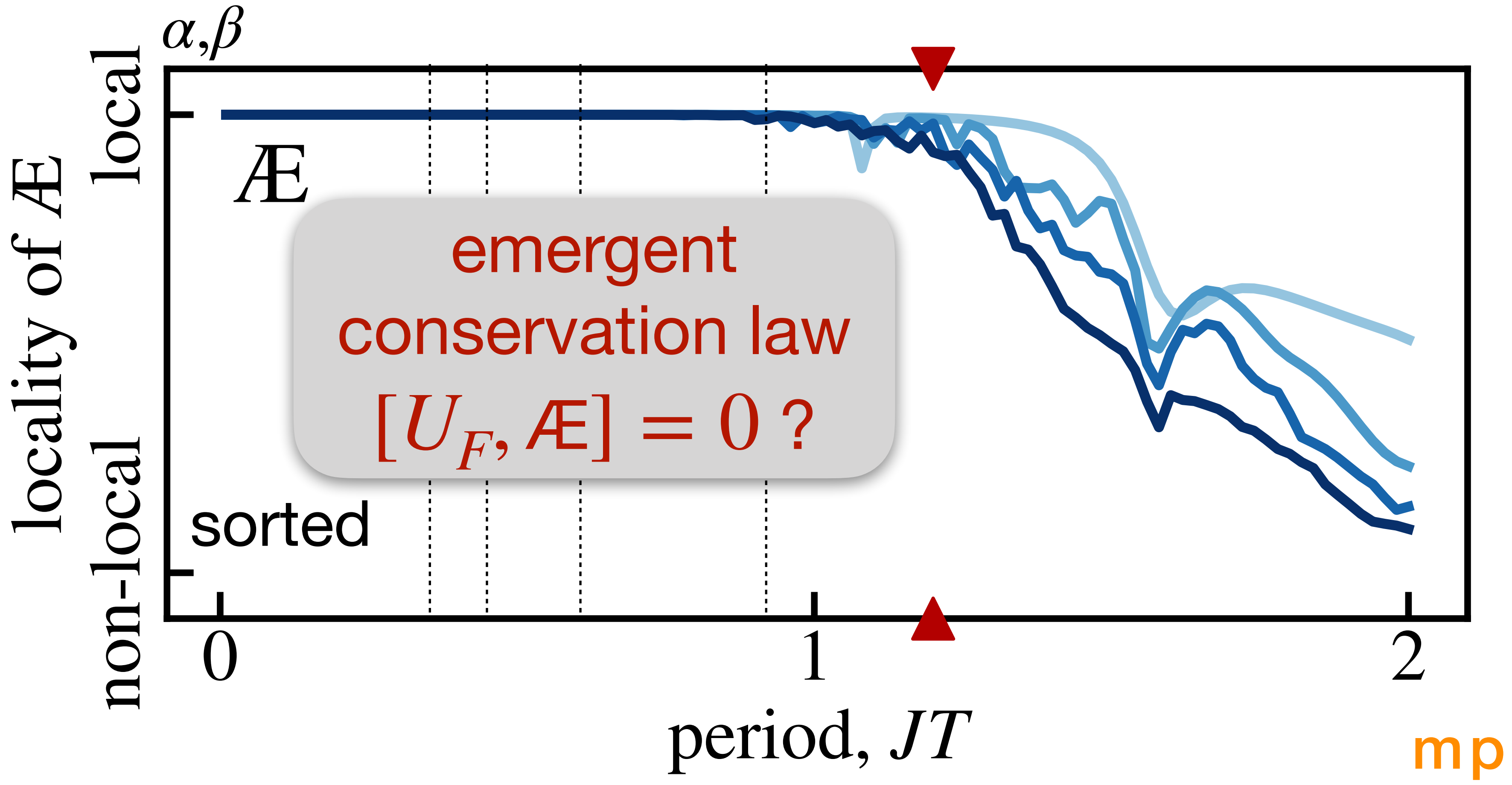




emergent conservation law $[U_F, Æ] = 0$?

Families of periodic drives

- ‘elementary’ families of periodic drives:
  - *Floquet decomposition:* $H(t) = H_F[t] + \mathscr{A}_F(t)$

    (i) equilibrium ‘drives’: $\mathscr{A}_F \equiv 0 \implies H(t) = \text{const}$ **static**

    (ii) pure-micromotion drives: $H_F \equiv 0 \implies U(t) = P(t)$ **no heating**

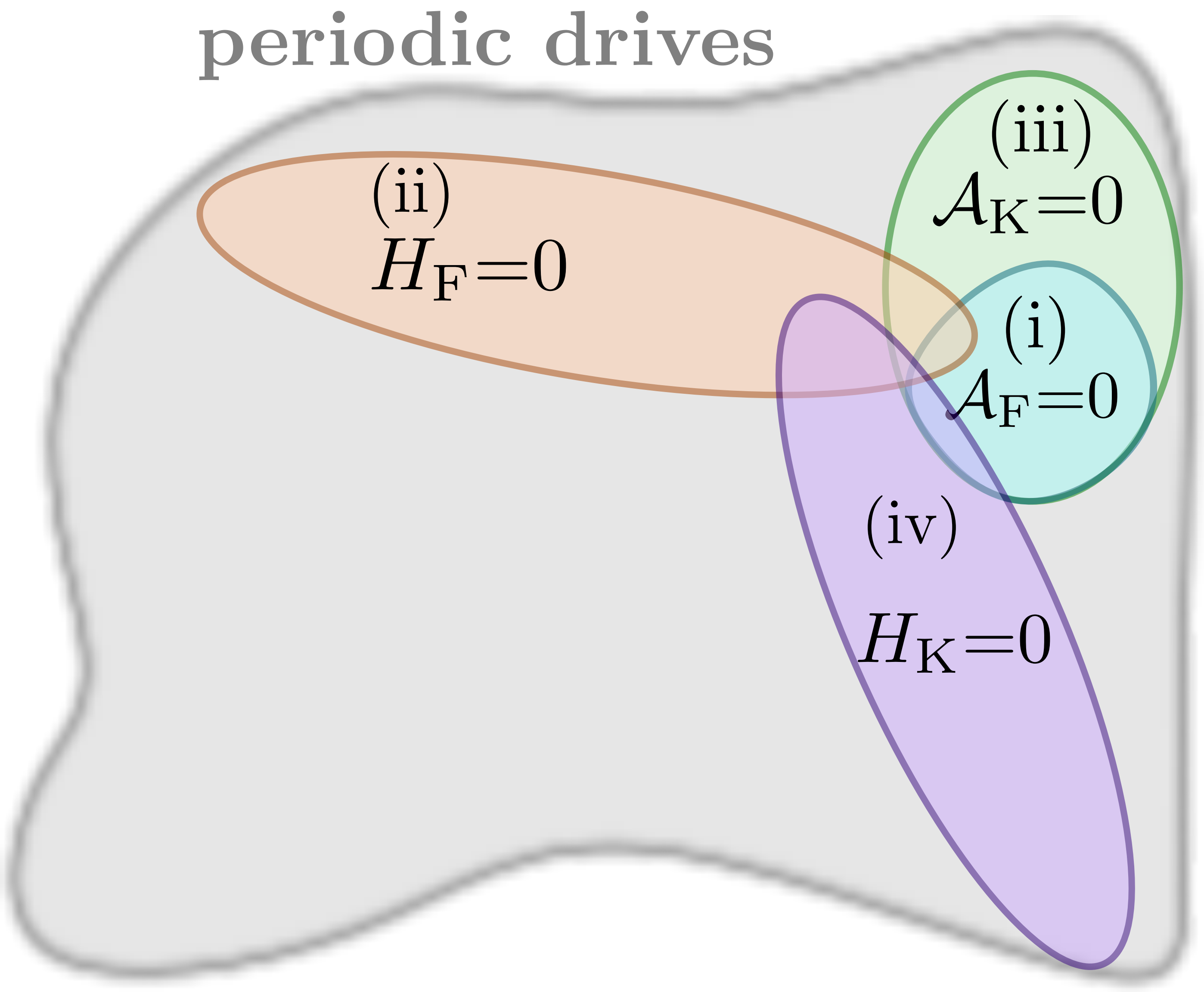
  - *Kato decomposition:* $H(t) = H_K(t) + \mathscr{A}_K(t)$

    (iii) flat drives: Æ

    (iv) pure-geometric drives: $H_K \equiv 0 \implies U(t) = \mathscr{W}(t)$

    Wilson line

    q’energy = geometric phase
    (no Floquet ground state!
    maximally nonequilibrium drives)



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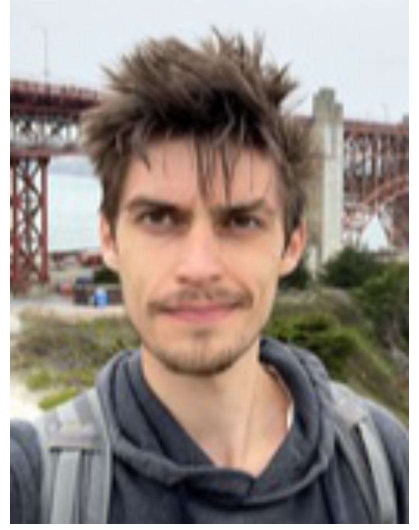

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Summary & Outlook

Schindler & MB, PRX 15, 031037 (2025)

www.pks.mpg.de/nqd

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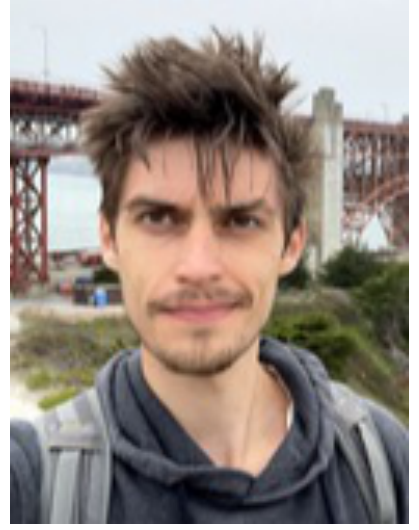
$$H(t) = H_F[t] + \mathscr{A}_F(t)$$



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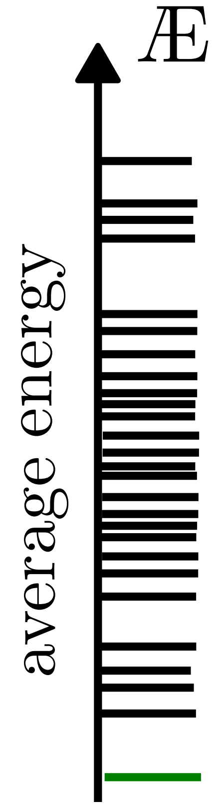

unambiguous ground state

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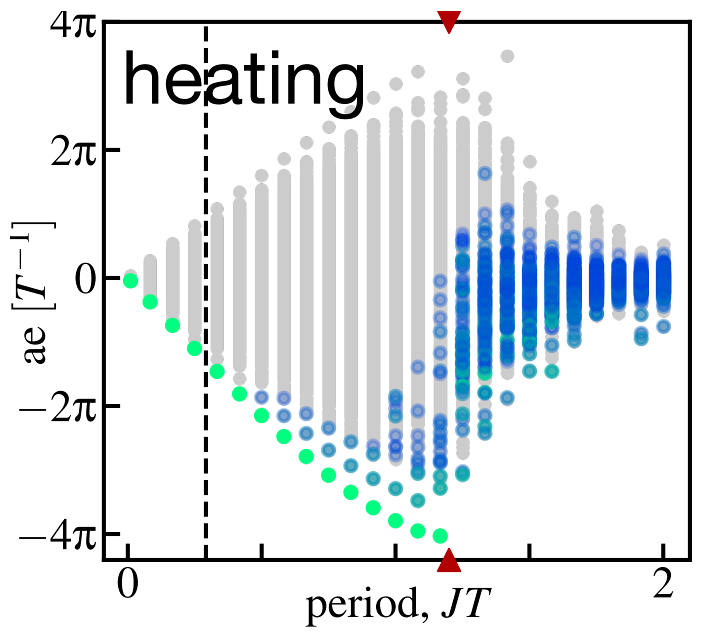

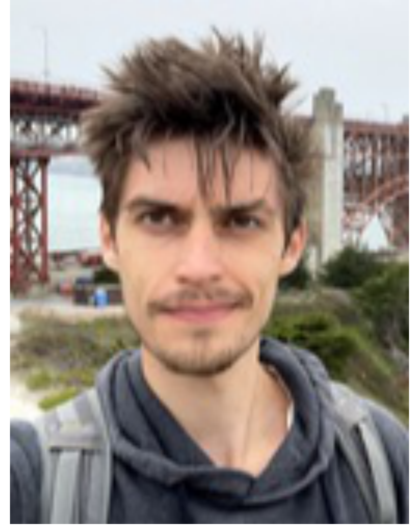
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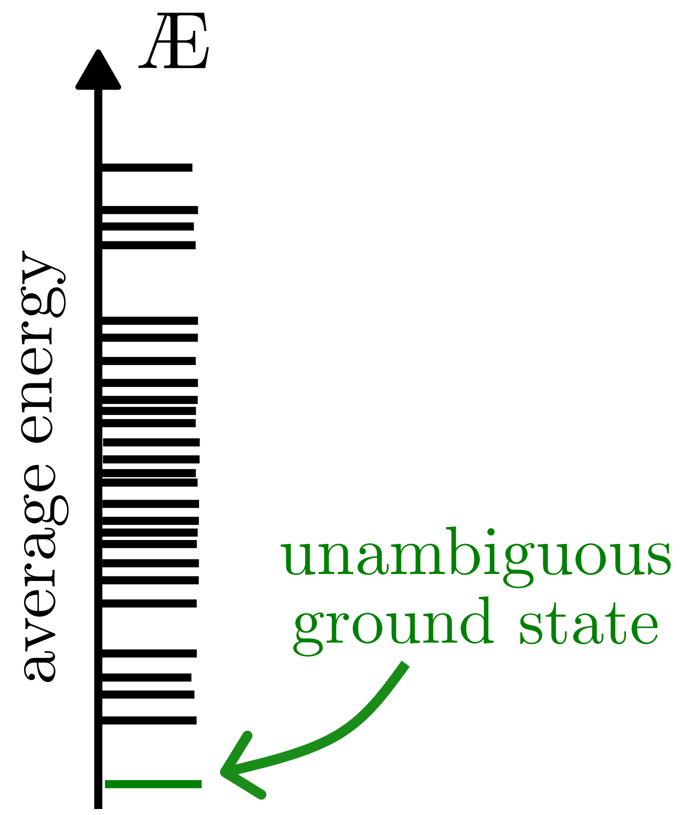

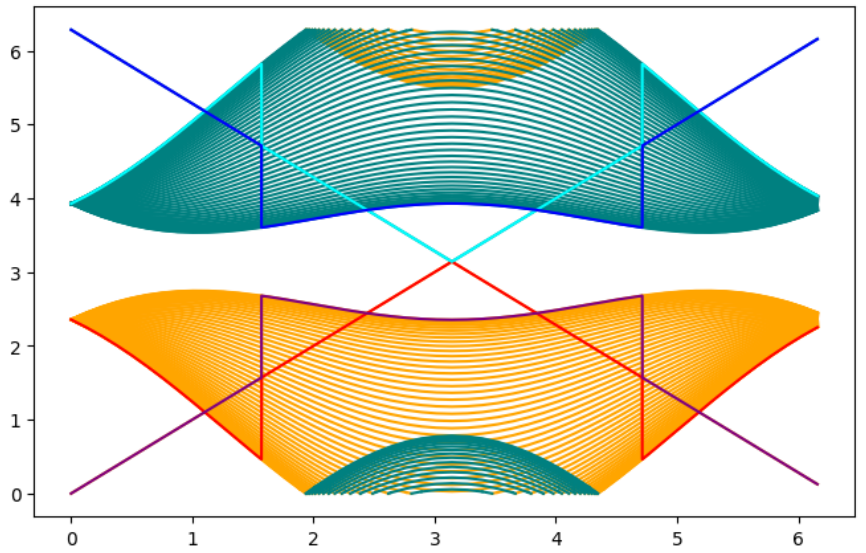

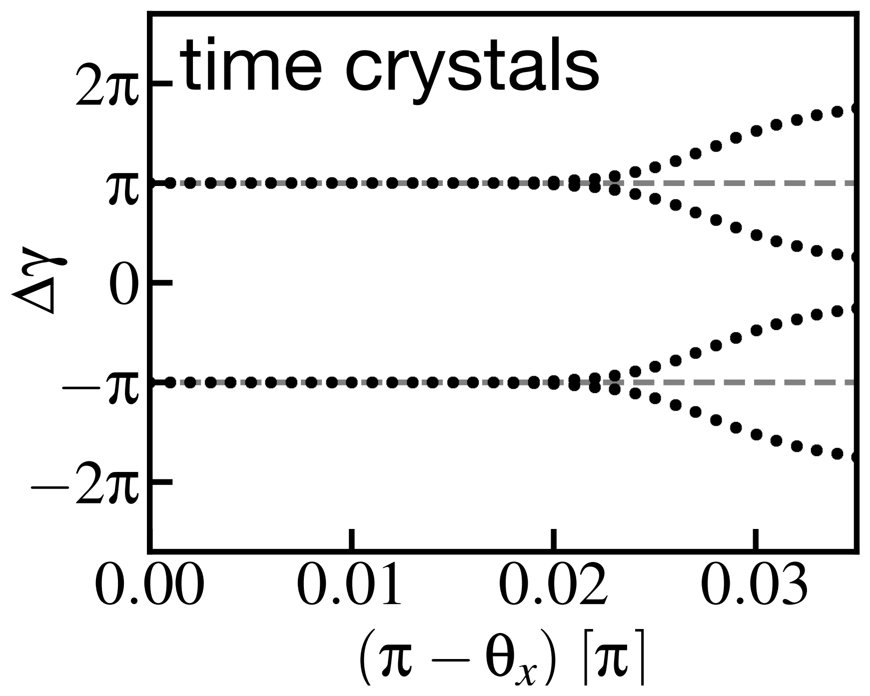

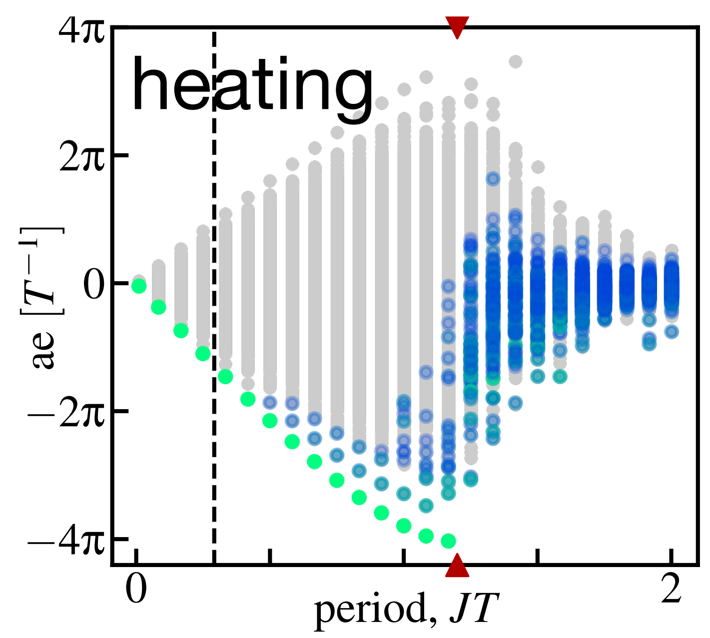

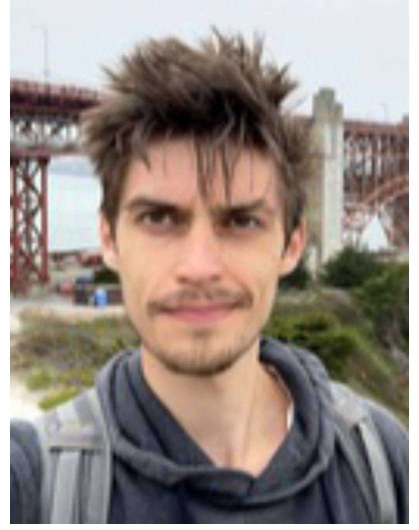
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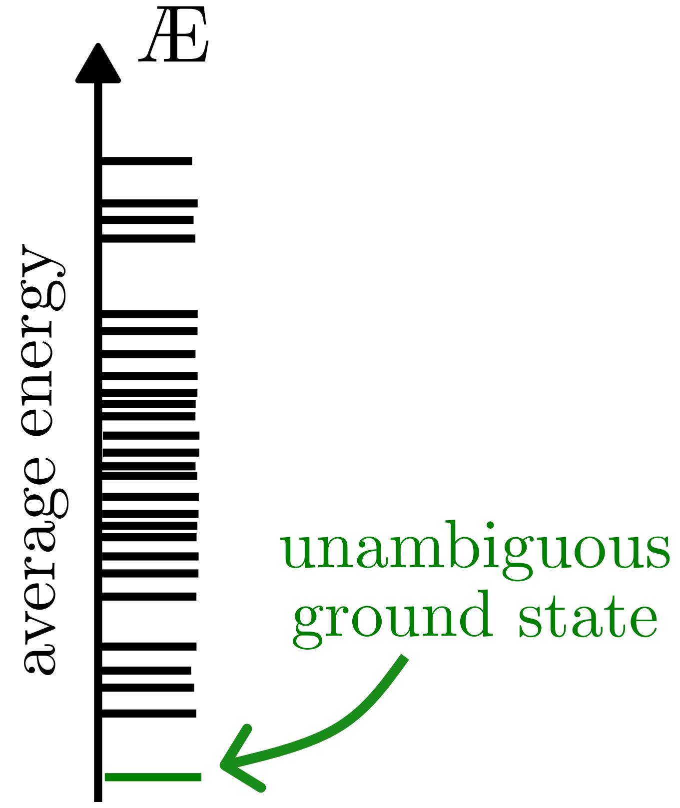

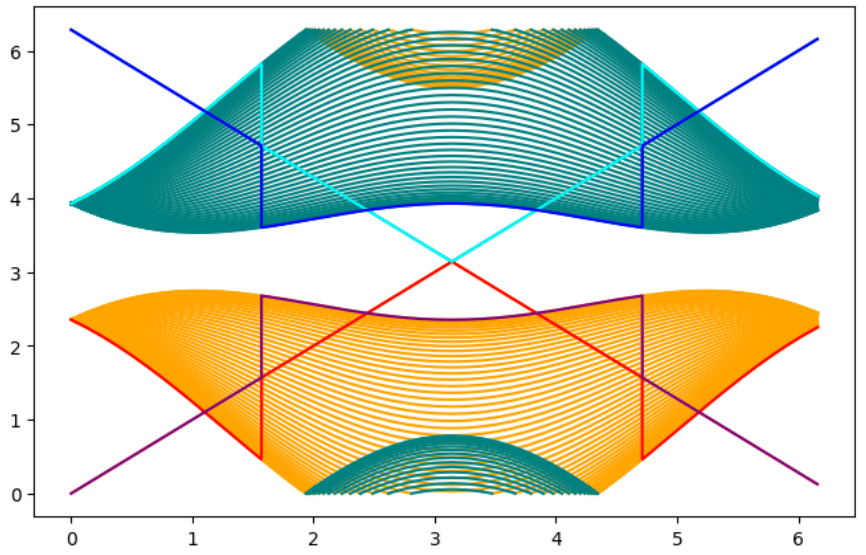

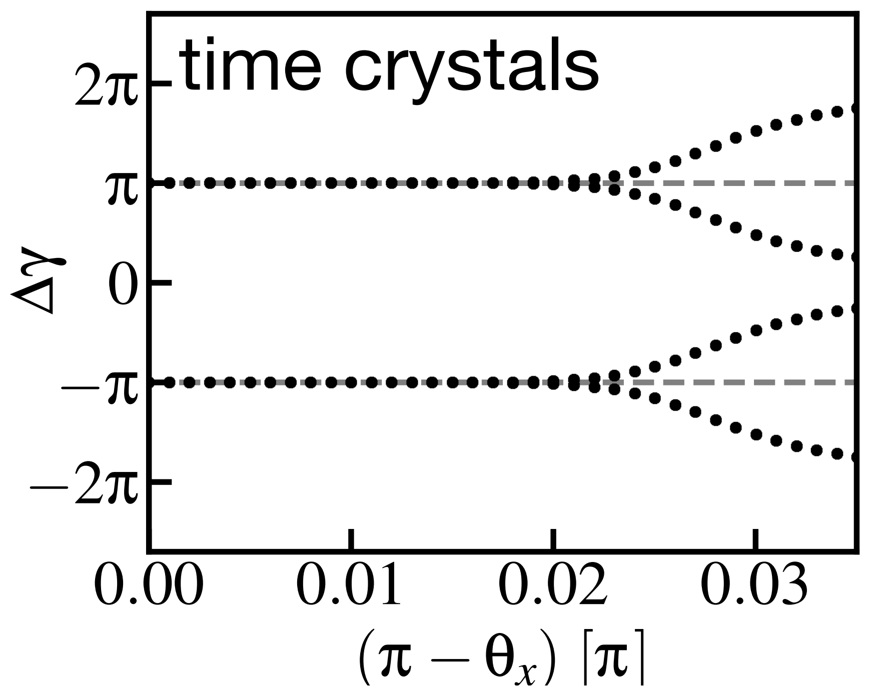

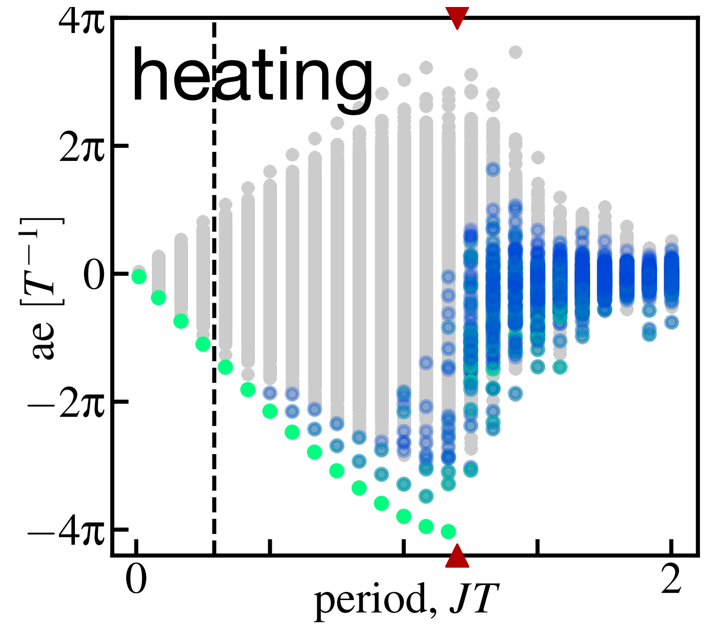

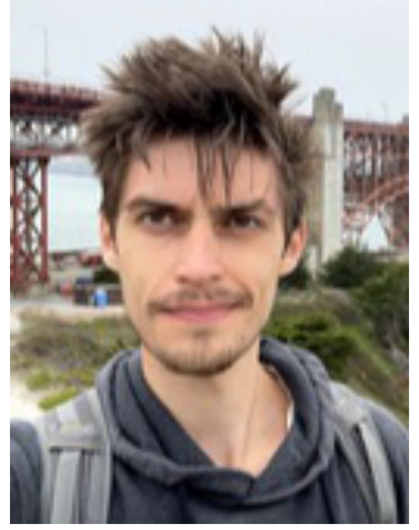
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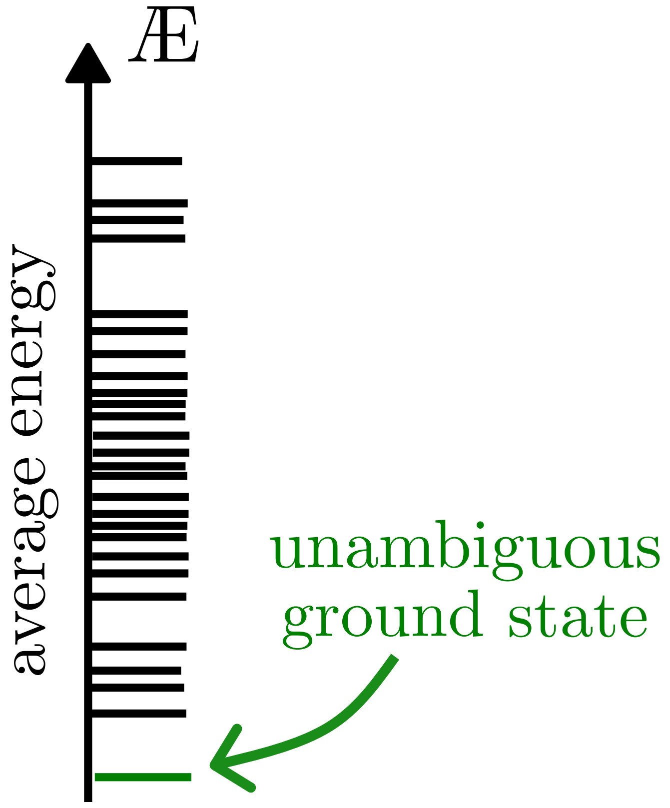
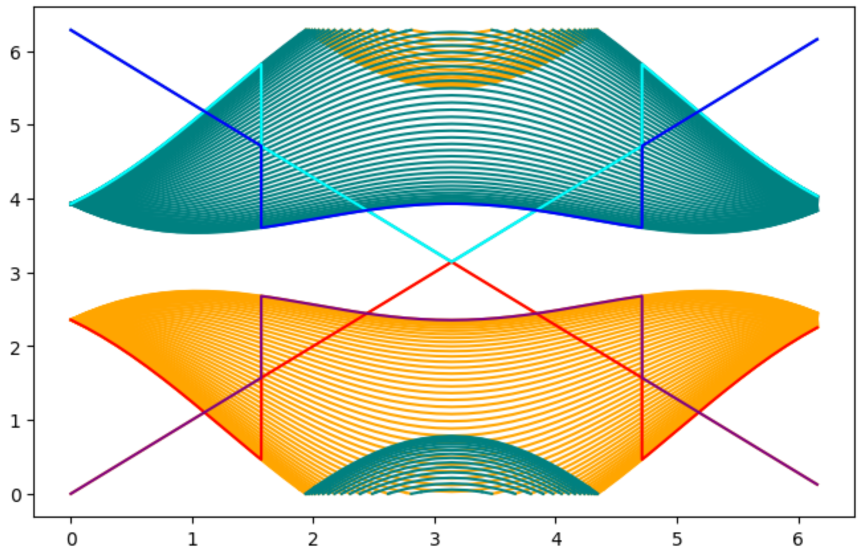
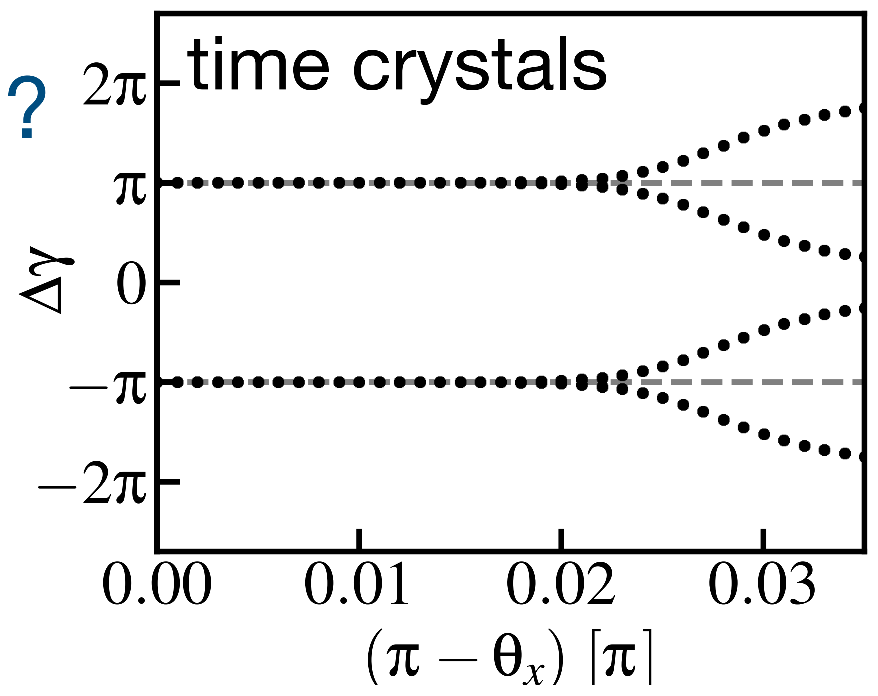
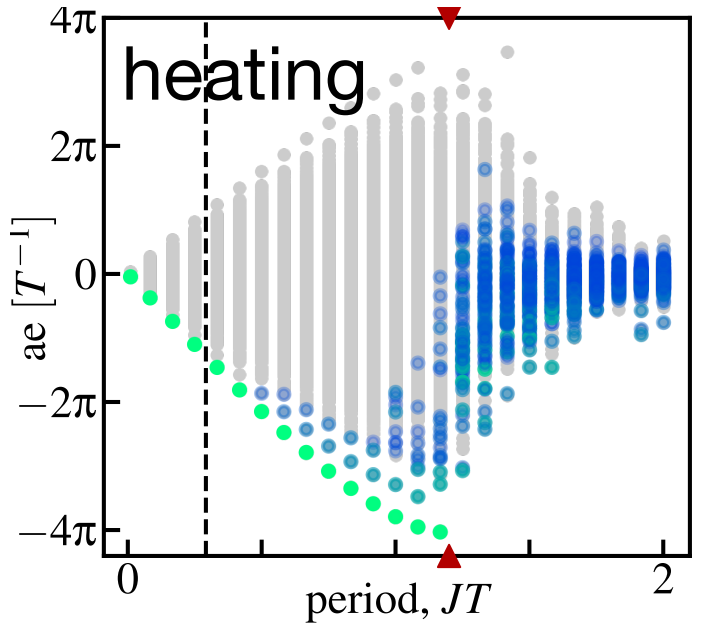
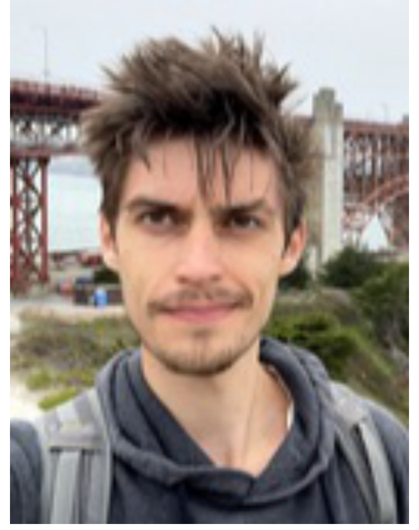
  - variational principle for many-body Floquet ground states: MPS, NQS, etc.
    - gapped?
    - ordered?
    - are certain Floquet states special (nonthermal)?











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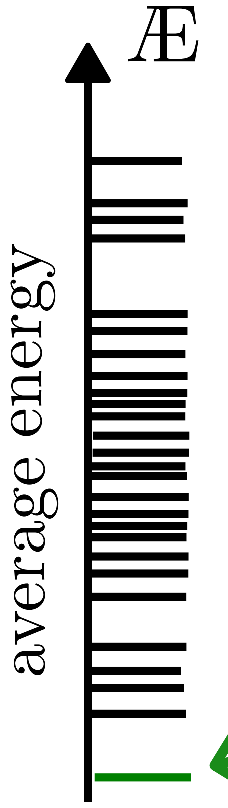

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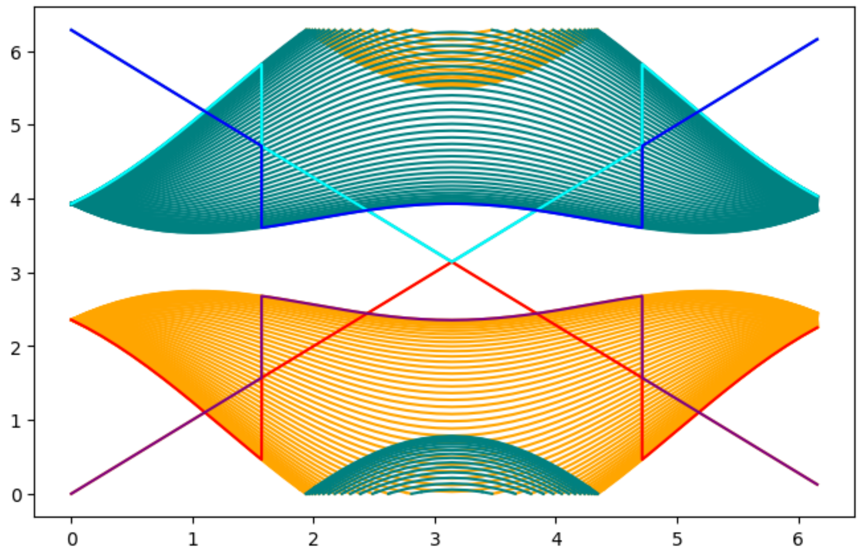

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    - gapped?
    - ordered?
    - are certain Floquet states special (nonthermal)?
  - experiments: control away from equilibrium (Floquet-engineered states)
    - Thouless pumps *away from adiabatic limit*
    - geometric quantum gates (based on Berry phases)

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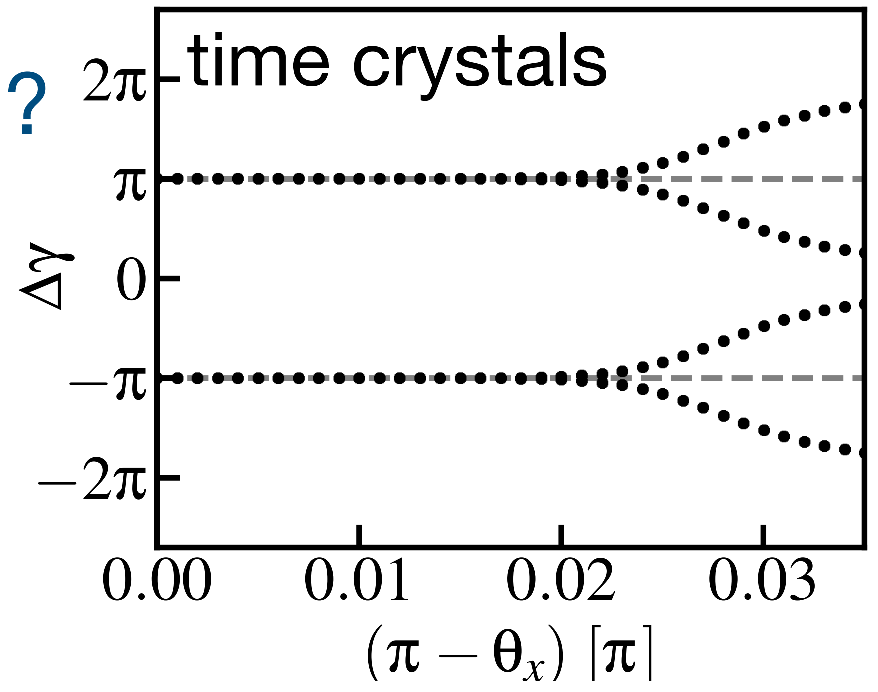
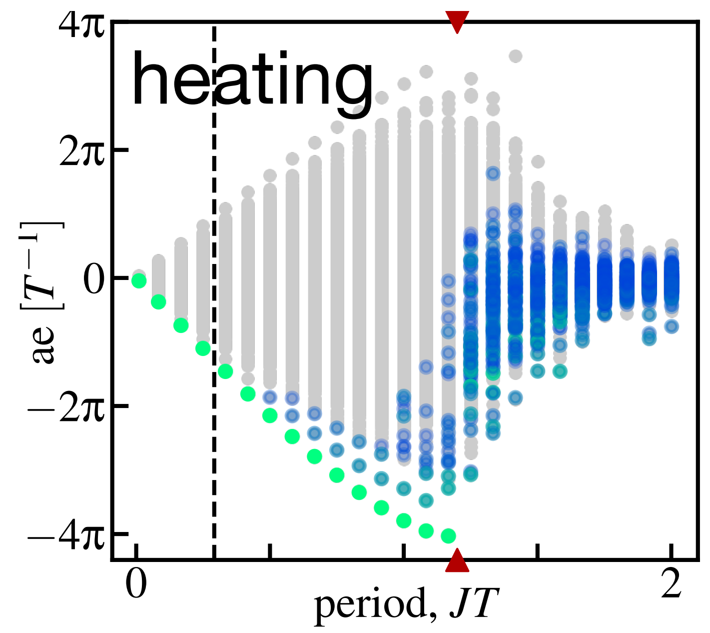
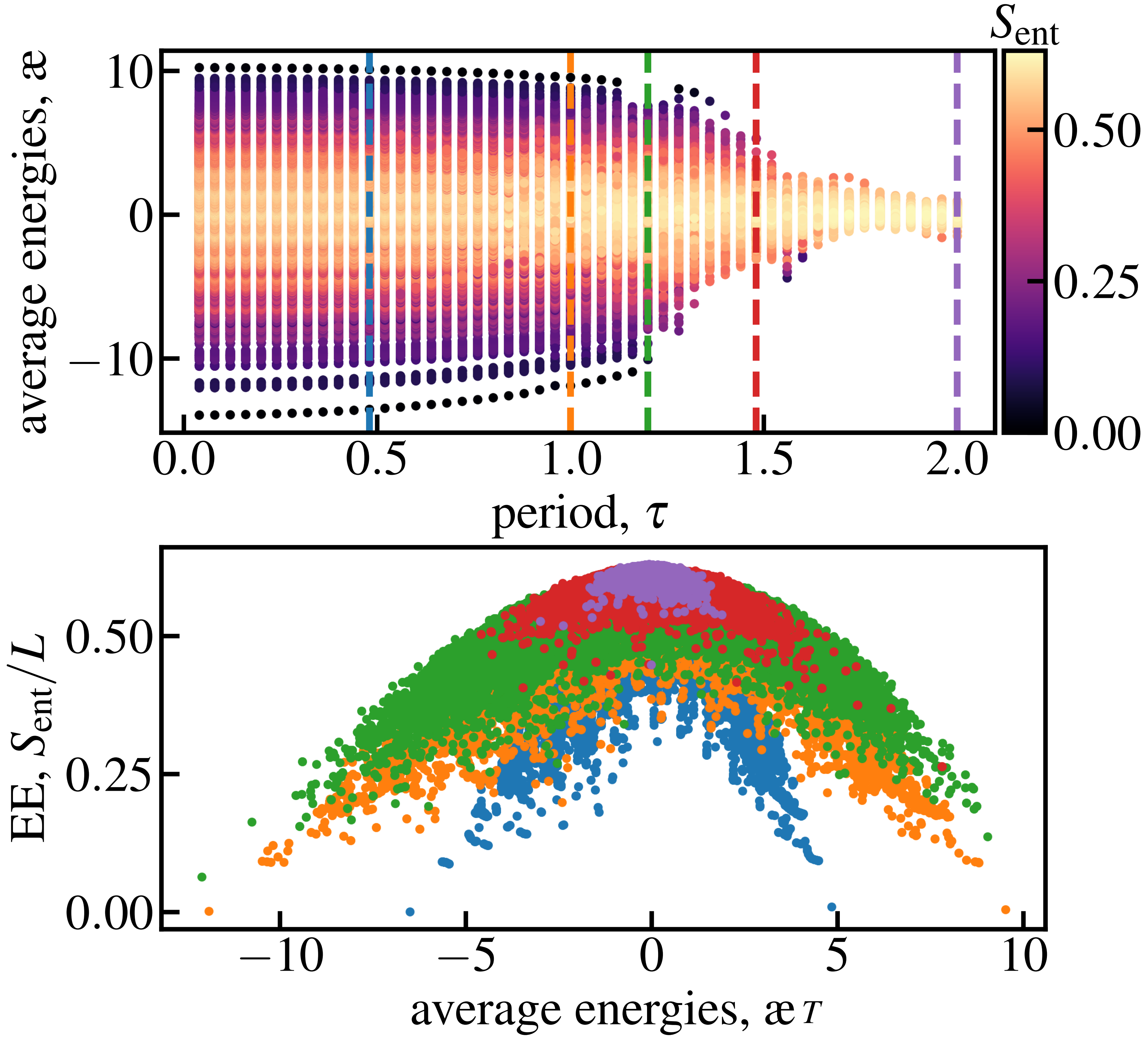
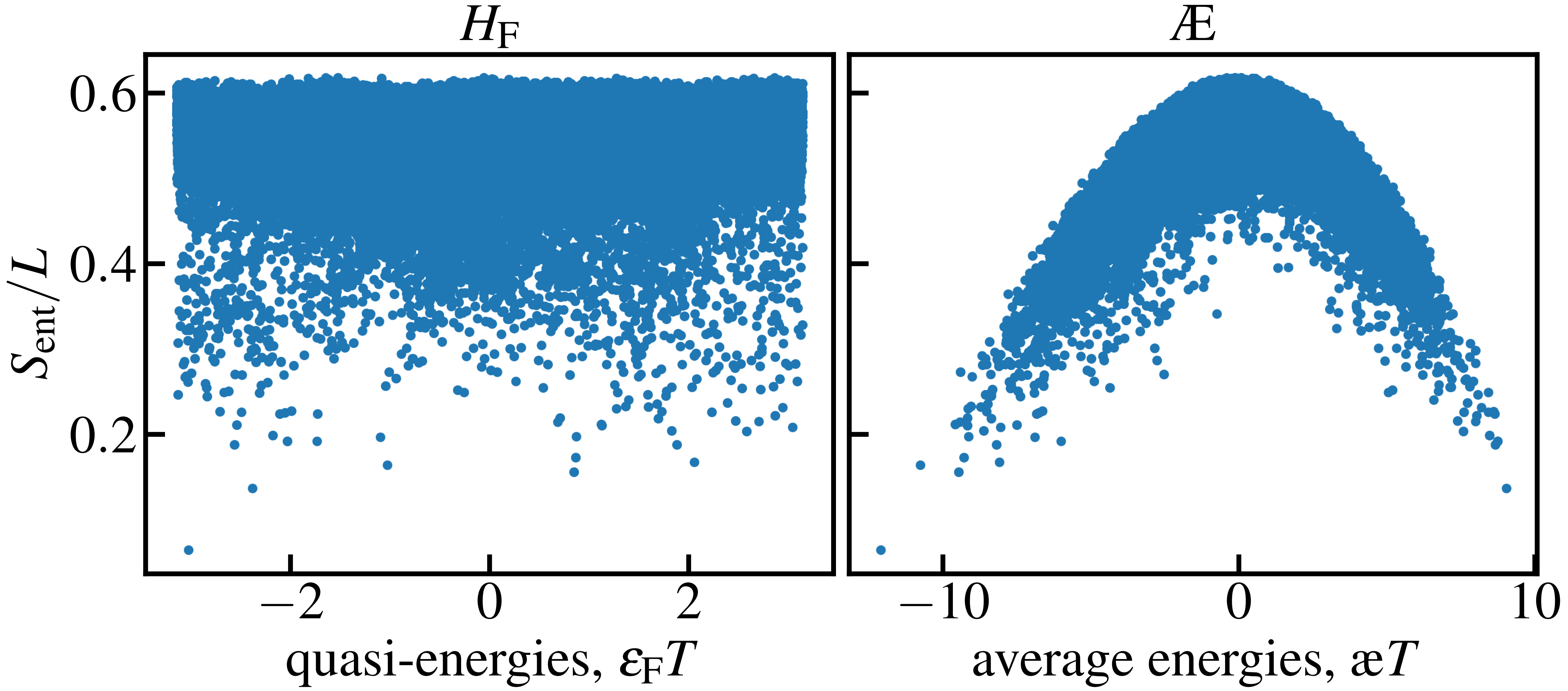





S_{ent}
0.50
0.25
0.00
average energies, æ
10
0
−10
0.0
0.5
1.0
1.5
2.0
period, τ
EE, S_{ent}/L
0.50
0.25
0.00
−10
−5
0
5
10
average energies, æT


H_{F}
Æ
S_{ent}/L
0.6
0.4
0.2
−2
0
2
quasi-energies, $\varepsilon_{\mathrm{F}}T$
−10
0
10
average energies, æT

Floquet resonances

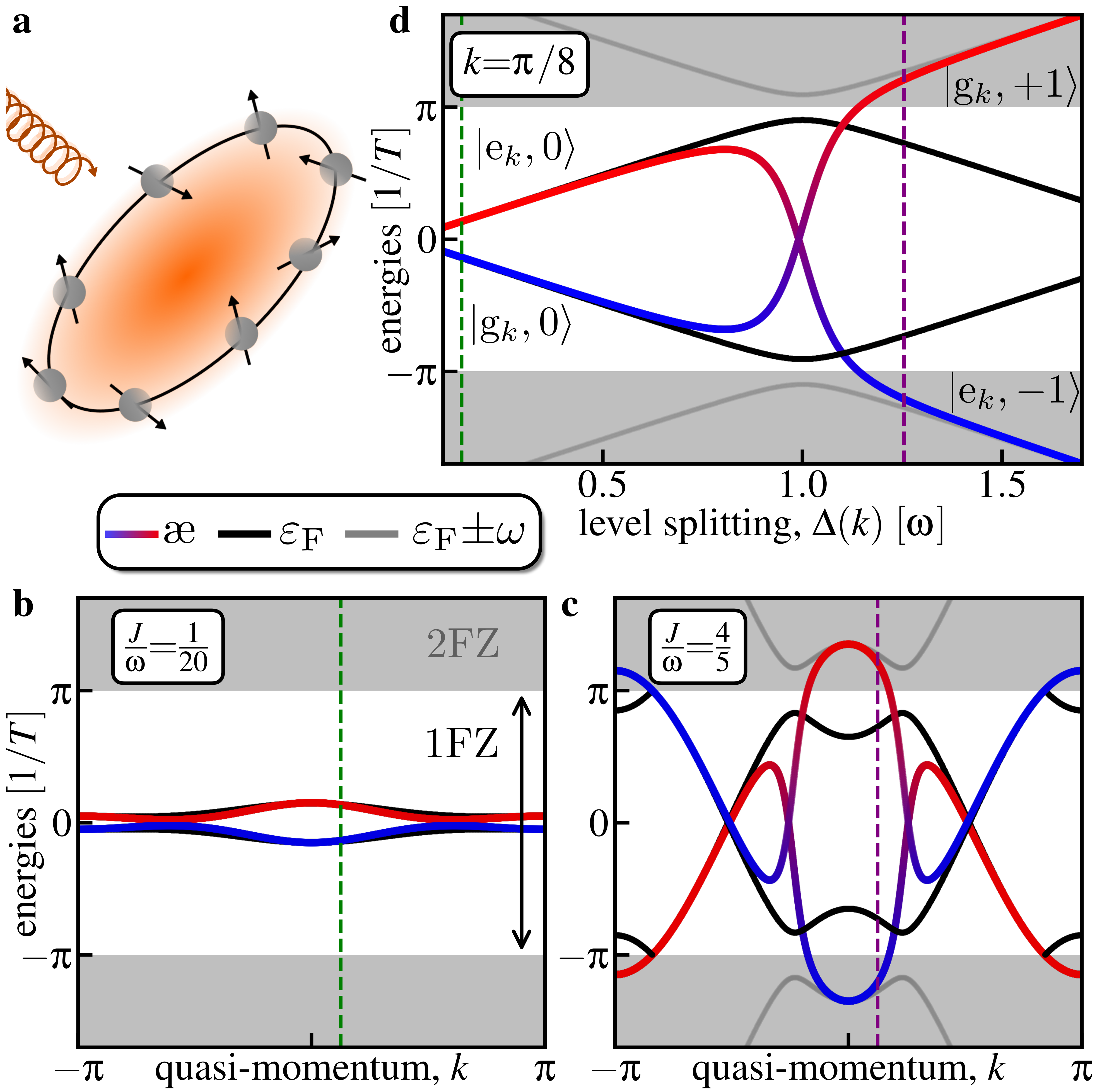
$$H(t) = \frac{1}{2}\sum_{n=1}^{L}\left[\left(J\sigma^{+}_{n+1}\sigma^{-}_{n} + Aie^{-i\omega t}\sigma^{+}_{n+1}\sigma^{+}_{n} + \mathrm{h.c.}\right) + \frac{g}{2}\sigma^{z}_{n}\right]$$

$$H(t) = \sum_{k} \boldsymbol{\psi}^{\dagger}_{k} h(k,t) \boldsymbol{\psi}_{k}$$

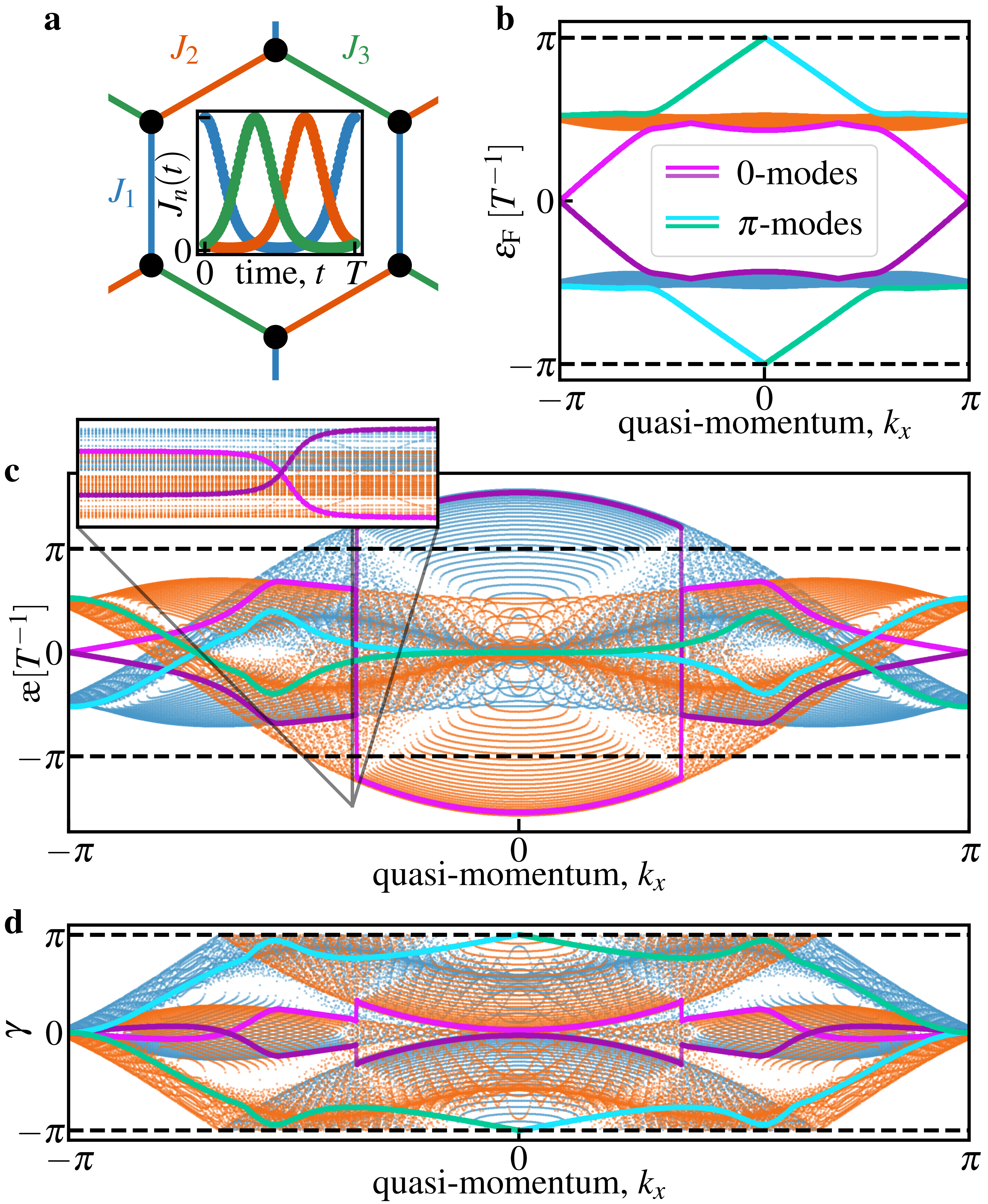
$$h(k,t) = \Delta_k \tau^z + A_k\left[\cos(\omega t)\tau^x + \sin(\omega t)\tau^y\right]$$

$$\Delta_k = g + J\cos(k)$$

$$A_k = A\sin(k)$$



Anomalous Floquet topological insulators



Counterdiabatic driving for periodically driven systems

transitionless driving of Floquet-engineered states $|n_F\rangle$?

- **Floquet toolbox is *incomplete***
  - (i) engineer H_F, (ii) suppress heating, *(iii) prepare state*

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- control Floquet states on top of periodic drives

1. **uncontrolled**, *periodically driven*

$$U_F(t,0) = \mathcal{T}\exp\left(-i\int_0^t H(s)\mathrm{d}s\right) = P(t)\ \exp\left(-iTH_F\right)$$

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1. **uncontrolled**, *periodically driven*

$$U_F(t,0) = \mathcal{T}\exp\left(-i\int_0^t H(s)\mathrm{d}s\right) = P(t)\ \exp\left(-iTH_F\right)$$

2. **adiabatic**, w.r.t. $\lambda(t)$ *(on top of periodic drive)*

$$U_{F,\lambda(t)}(t,0) = \mathcal{T}\exp\left(-i\int_0^t H_{\lambda(t)}(s)\mathrm{d}s\right) \to P_{\lambda(t)}(t)\ \exp\left(-itH_{F,\lambda(t)}\right)$$

PM Schindler & MB, PRL 133, 123402 (2024)

Counterdiabatic driving for periodically driven systems

transitionless driving of Floquet-engineered states $|n_F\rangle$?

- **Floquet toolbox is *incomplete***
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3. **counterdiabatic** *(on top of control & periodic drive)*

$$U_{\mathrm{FCD},\lambda(\mathrm{t})}(t,0) = \mathcal{T}\exp\left(-i\int_0^t \overbrace{H_{\lambda(s)}(s)+\dot{\lambda}(s)\mathscr{A}_{\lambda(s)}(s)}^{=H_{\mathrm{CD},\lambda(\mathrm{s})}(s)}\,\mathrm{d}s\right)$$

$\neq$ *different!*

$$= P_{\lambda(t)}(t)\ \exp\left(-it\left[H_{F,\lambda(t)}+\dot{\lambda}(t)\mathscr{A}_{F,\lambda(s)}(t)\right]\right)$$

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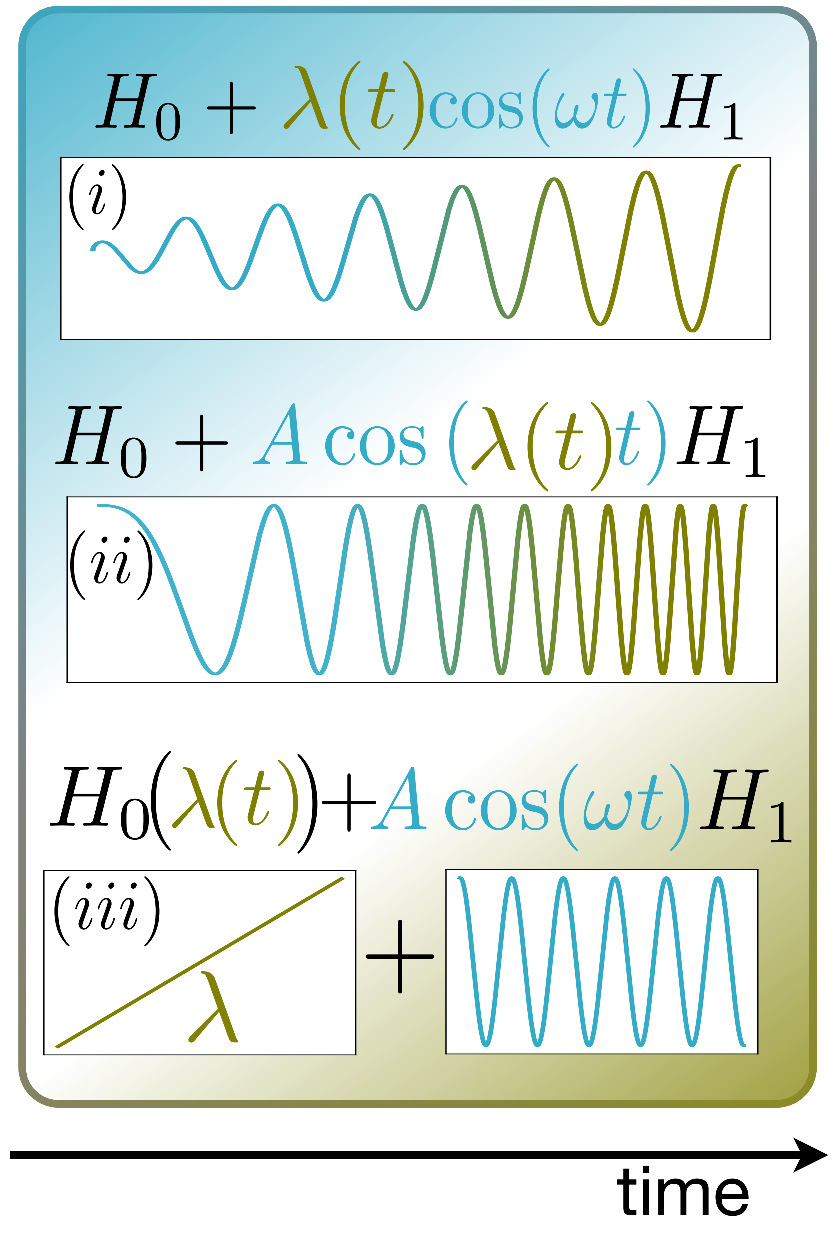
  - (i) engineer H_F, (ii) suppress heating, *(iii) prepare state*
- **control Floquet states on top of periodic drives**
  - so far: initial time/phase of the drive
    - what about other control parameters?

✦ amplitude ramps
$\lambda(t) = A(t)$

✦ **frequency chirps**
$\lambda(t) = \omega(t)$

▸ external ramps
$\lambda(t)$

▸ phase of drive
$\varphi(t)$



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Nonperturbative variational ansatz for Floquet Hamiltonian

$$H(t) = H_F[t] + \mathscr{A}_F(t)$$

- ◉ **Issues:**
  - ▸ **perturbative approximations to H_F (e.g., Magnus expansion) fail to capture resonances**
  - ▸ ‘no Floquet ground state’ (quasi-energy not ordered)

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- given drive $H(t)$,
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can use CD driving techniques to study Floquet systems!

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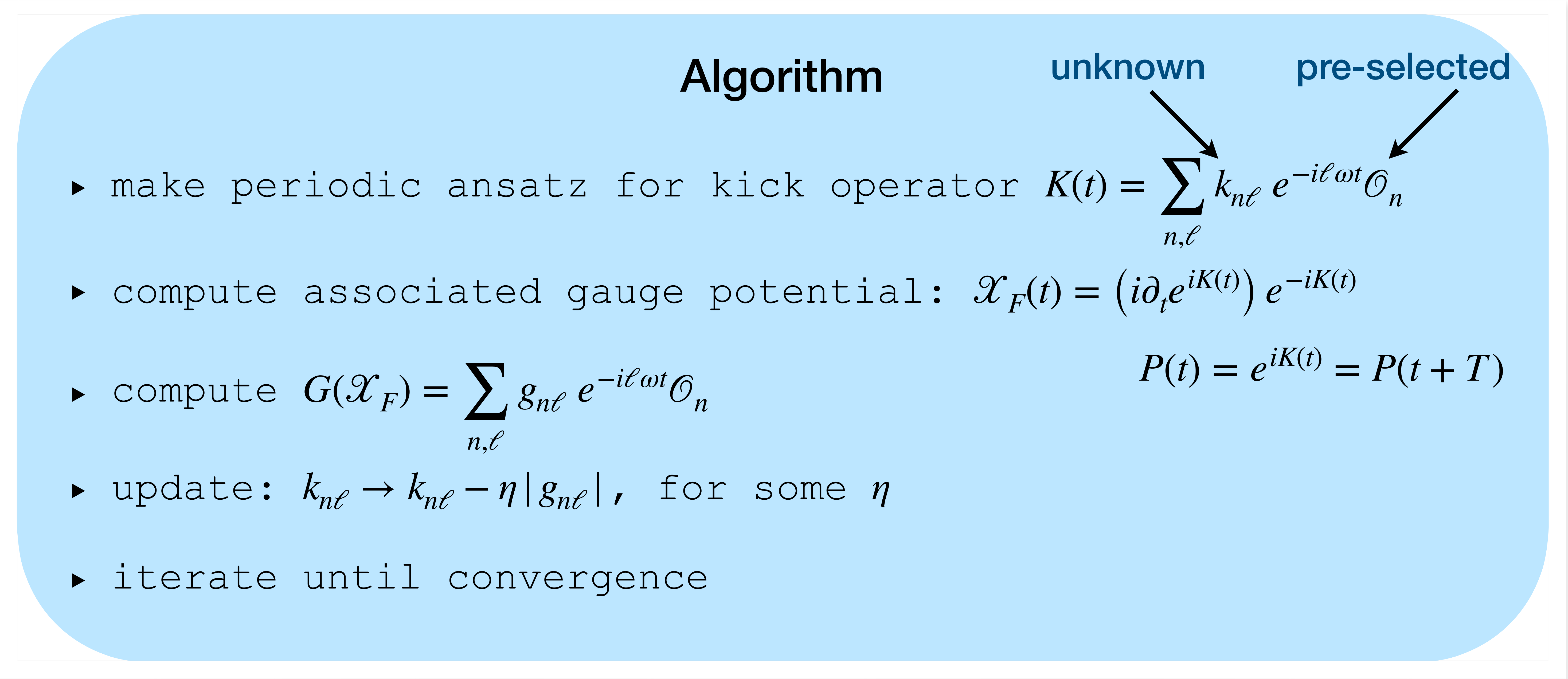

Algorithm

```
▸ make periodic ansatz for kick operator $K(t) = \sum_{n,\ell} \overset{\text{unknown}}{k_{n\ell}}\, e^{-i\ell\omega t} \overset{\text{pre-selected}}{\mathcal{O}_n}$
▸ compute associated gauge potential: $\mathscr{X}_F(t) = \left(i\partial_t e^{iK(t)}\right) e^{-iK(t)}$
▸ compute $G(\mathscr{X}_F) = \sum_{n,\ell} g_{n\ell}\, e^{-i\ell\omega t} \mathcal{O}_n$
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▸ iterate until convergence
```

$$P(t) = e^{iK(t)} = P(t+T)$$

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Algorithm

unknown → $k_{n\ell}$; pre-selected → $\mathcal{O}_n$

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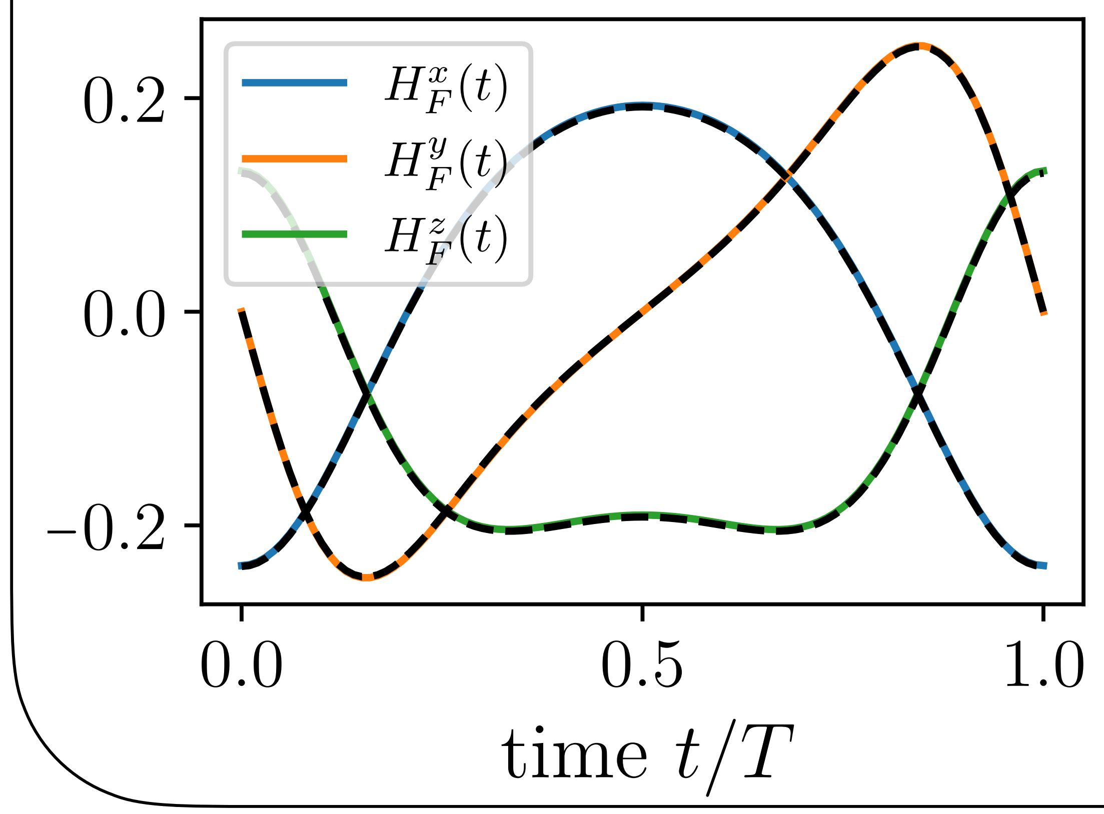
two-level system: $H(t) = \frac{\Delta}{2}Z + \frac{g}{2}(1 + 2\cos\omega t)X$



X, Y, Z: Pauli matrices

resonant drive

$\omega/g = 2$

$\Delta/g = 2$

dashed: *numerics*
solid: *variational*

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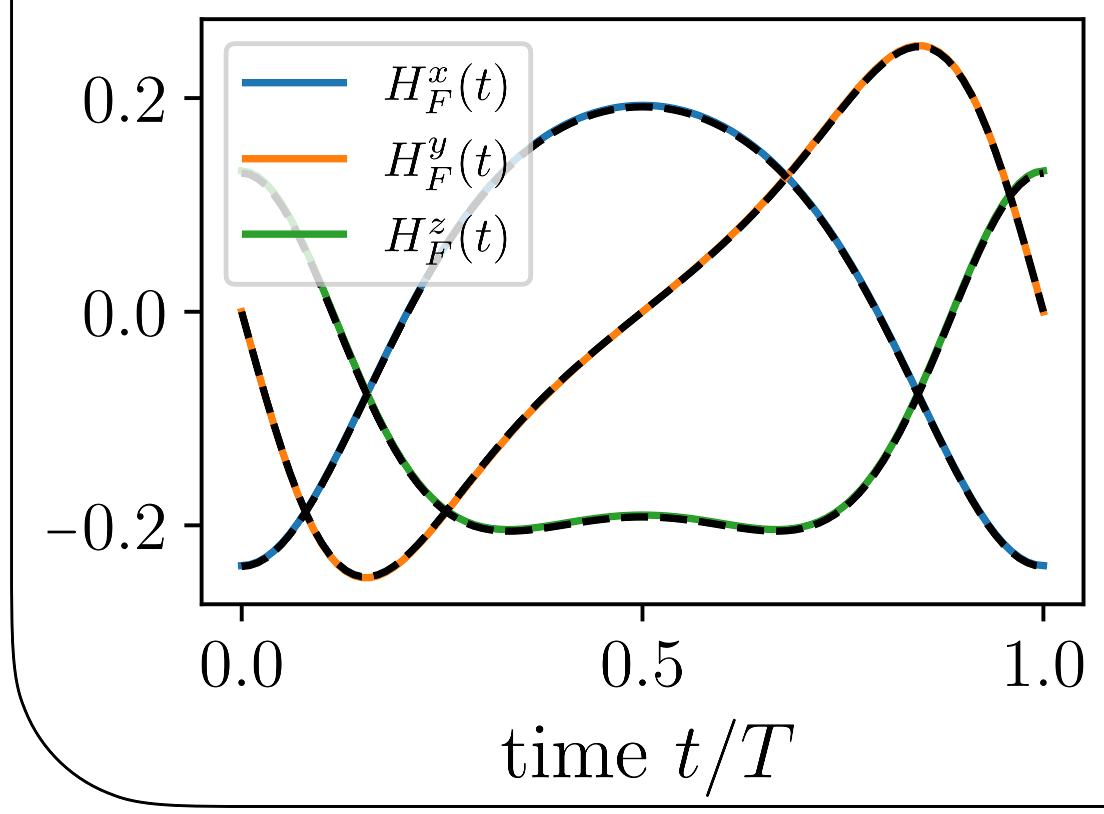

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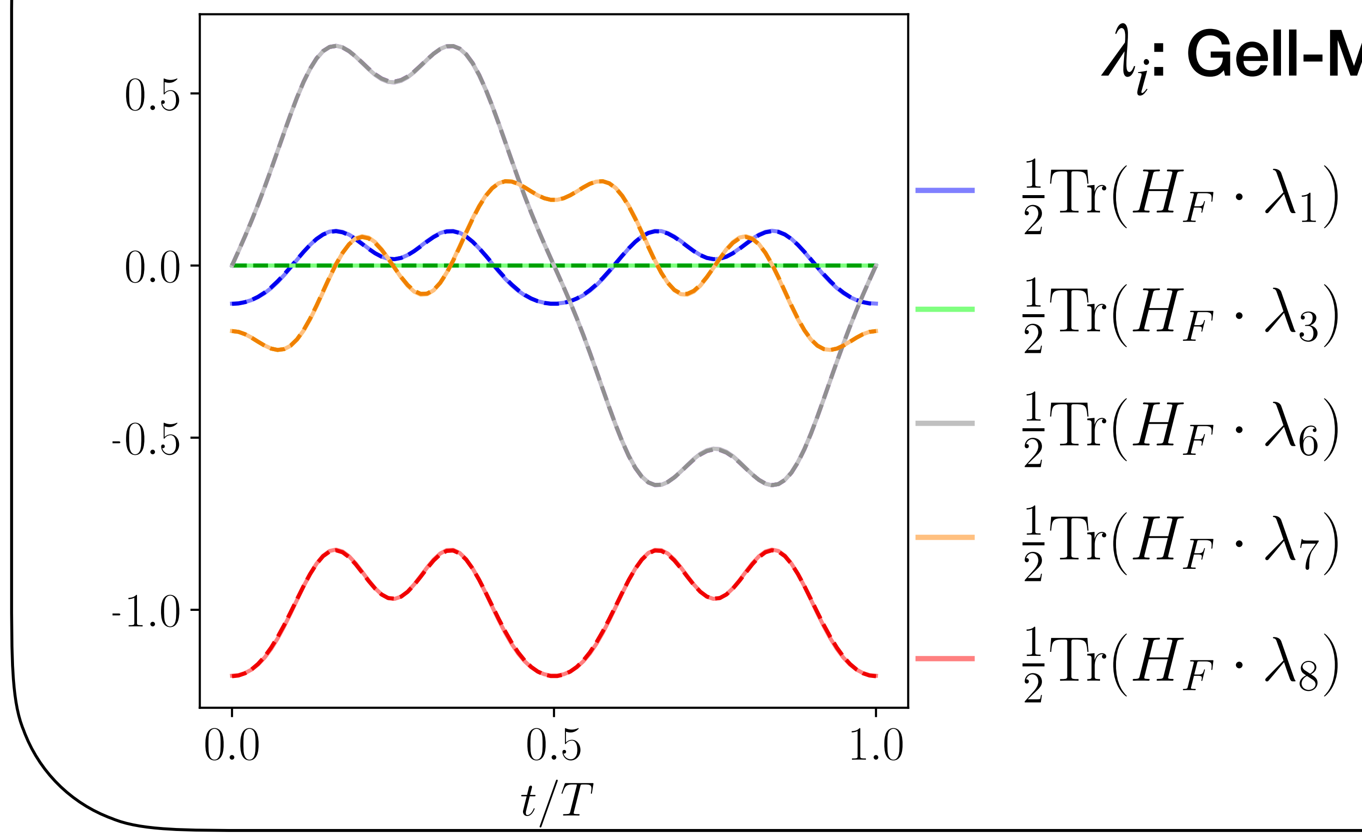
Λ-system: $H(t) = -\Delta\lambda_8 + g \sin\omega t\,(\lambda_4 + \lambda_6)$



λ_i: Gell-Mann matrices

resonant drive

$\omega/g = 0.5$

$\Delta/g = 1$

G. Aleksandrov, MSc thesis (2025)

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✓ no diagonalization

✓ no time-ordered exponentials

✓ no high-frequency regime

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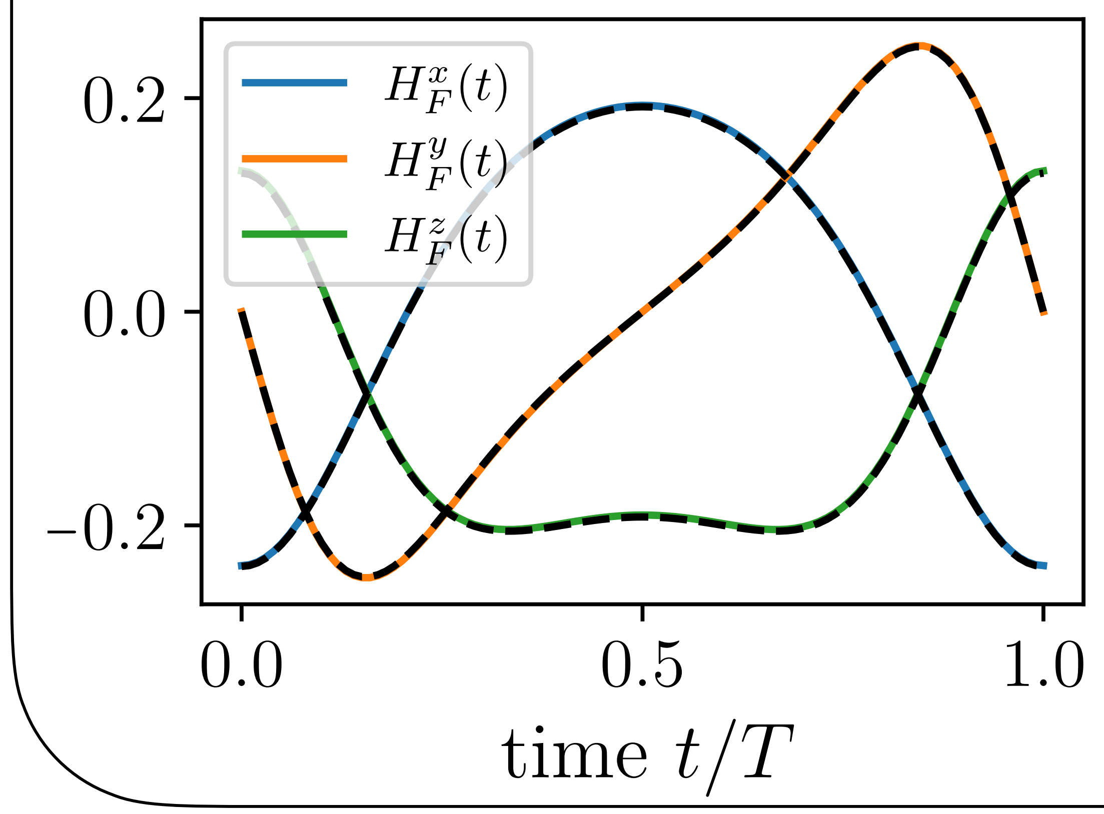

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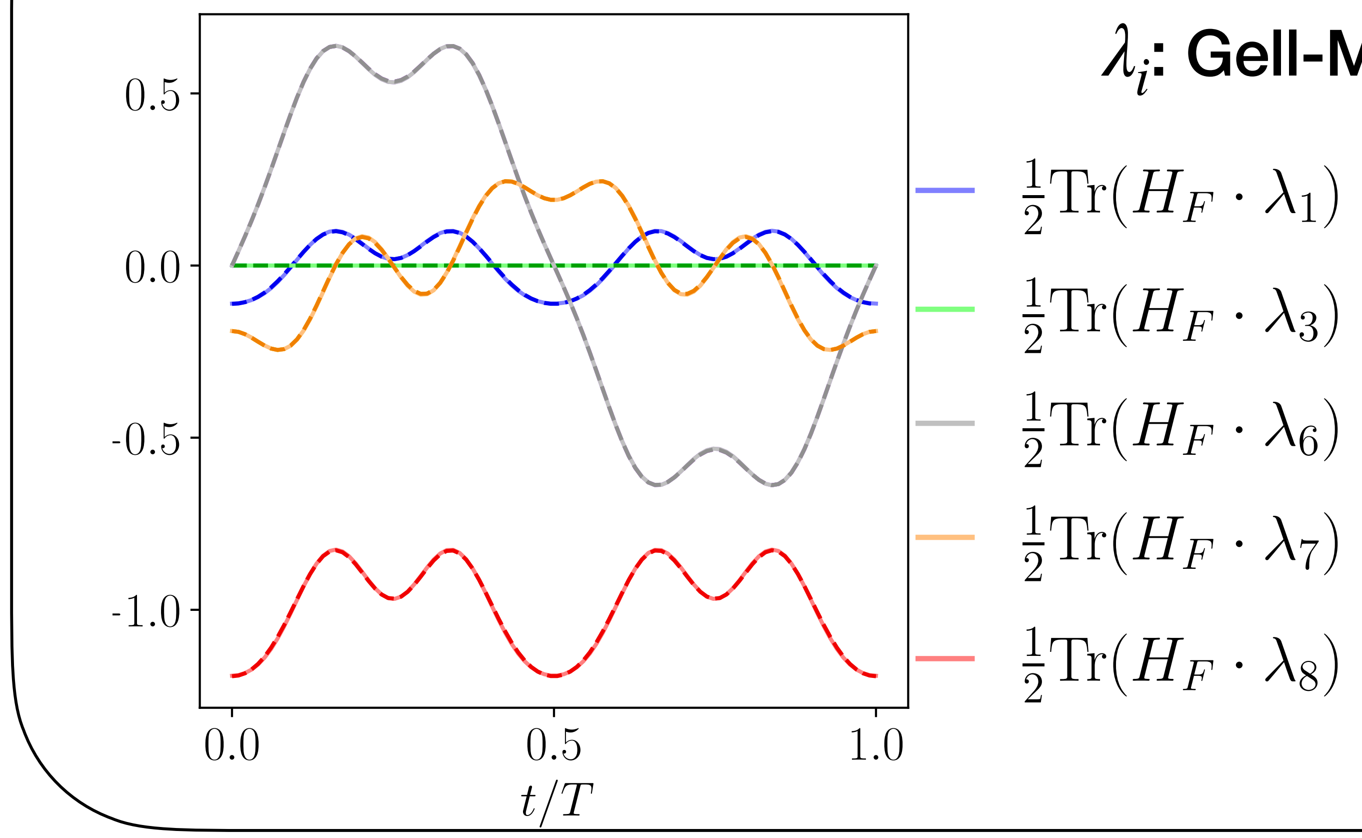

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Variational approximation of H_F

- **nonintegrable Ising chain:** $H(t) = \sum_j J Z_{j+1} Z_j + h_z Z_j + h_x \sin \omega t X_j$

$$K \in \left\{ \sum_j X_j, \sum_j Y_j, \sum_j Z_j, \sum_j X_j X_{j+1}, \sum_j Y_j Y_{j+1}, \sum_j Z_j Z_{j+1}, \sum_j X_j Y_{j+1} + Y_j X_{j+1}, \sum_j Y_j Z_{j+1} + Z_j Y_{j+1}, \sum_j Z_j X_{j+1} + X_j Z_{j+1} \right\}$$

$$\|A - B\|^2 = 1 - \frac{1}{\dim(H)} \mathrm{Re}\ \mathrm{tr}(A^\dagger B)$$

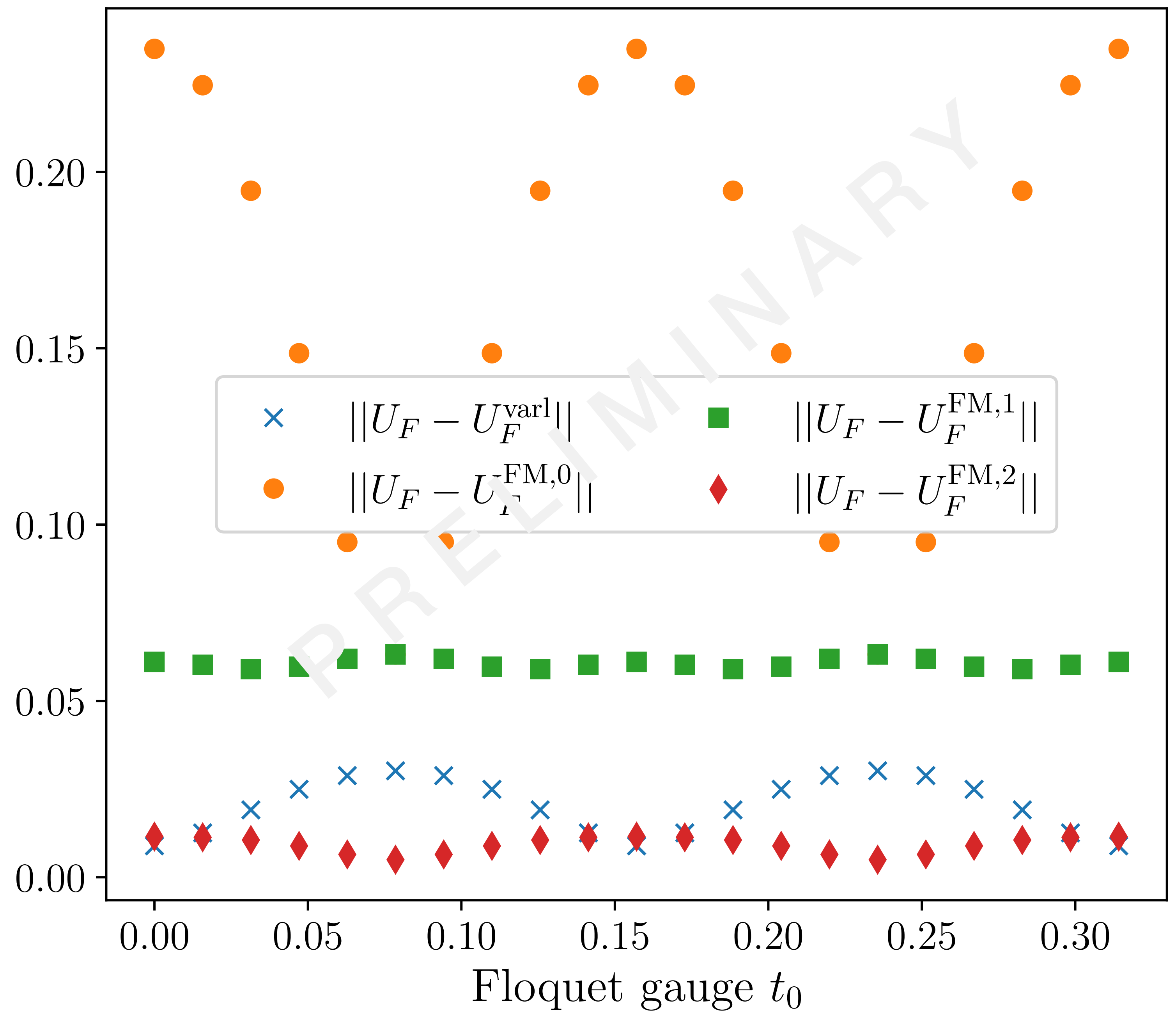
$$\|A - B\|^2 \in [0,2]$$

$$U_F = e^{-iTH_F}$$

$$\|e^{-iTH_F} - e^{-iT\mathscr{H}_F}\|$$

$$\|e^{-iTH_F} - e^{-iTH^{(n)}_{\mathrm{FM}}}\|$$

high frequency regime



$\omega/J = 20, \quad h_x/J = \sqrt{3}, \quad h_z/J = 0.9$

- 2nd order FM: $[ZZ, [X, ZZ]] \sim ZXZ$, etc.

12 **spins,** 21 **harmonics**

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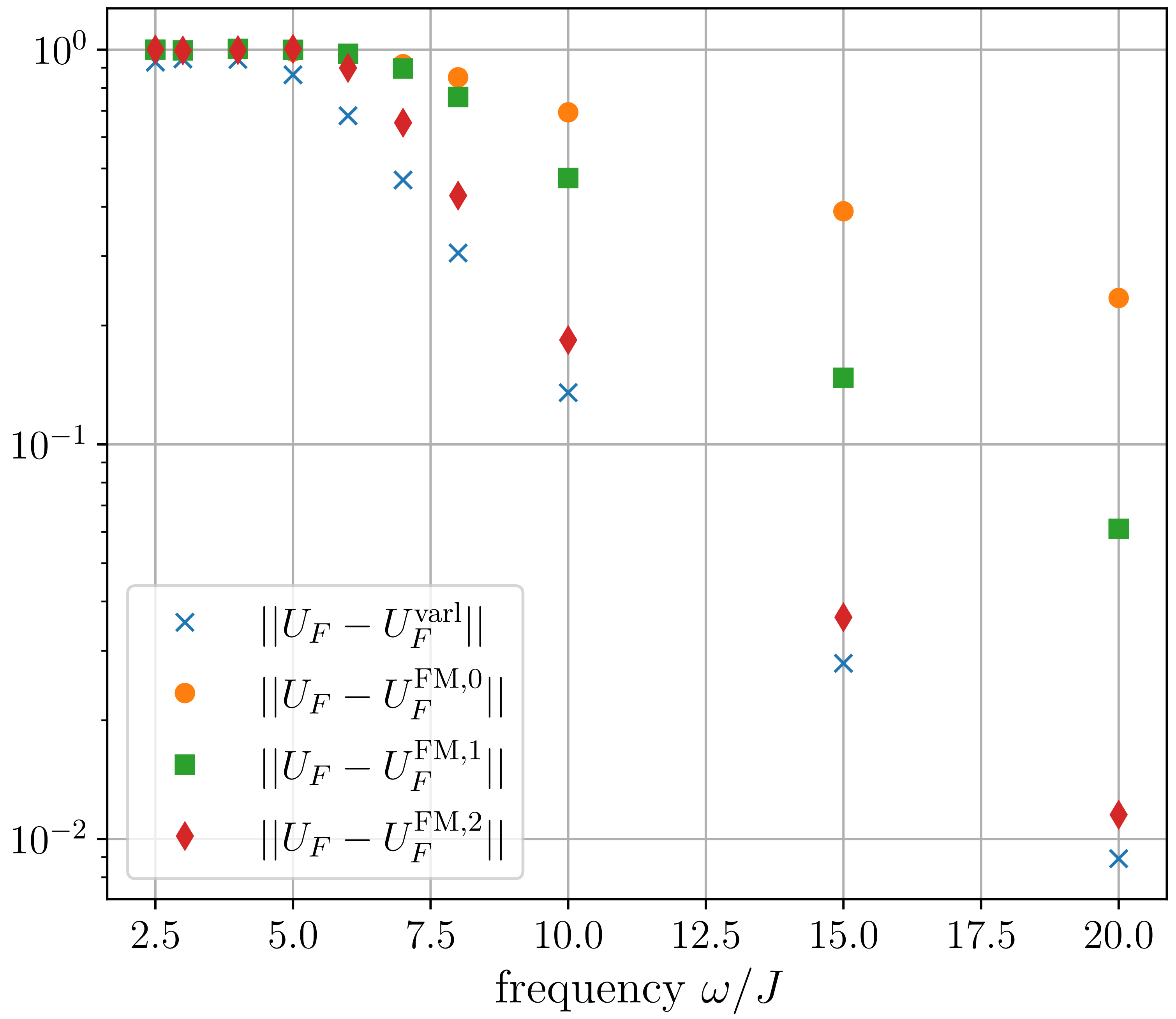
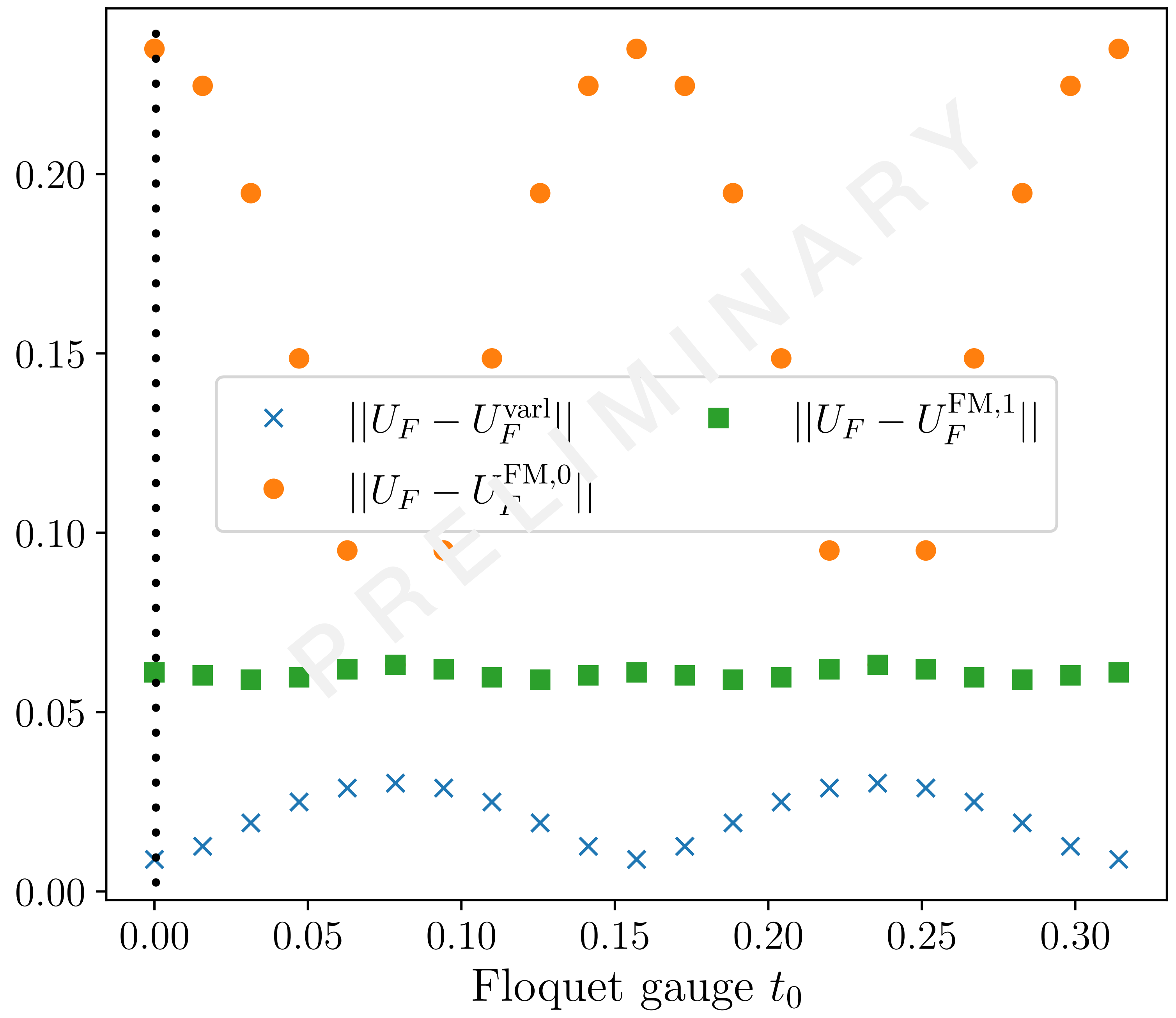
frequency scan



$t_0 = 0$

high frequency regime



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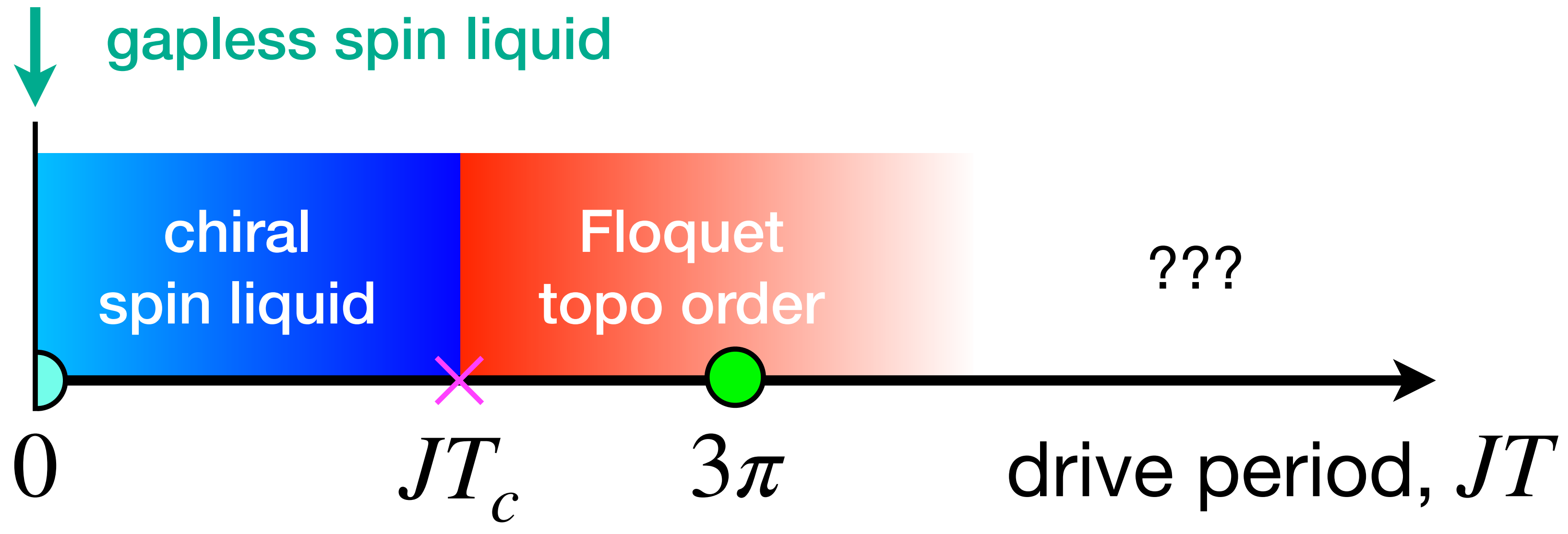

gapless spin liquid

chiral spin liquid

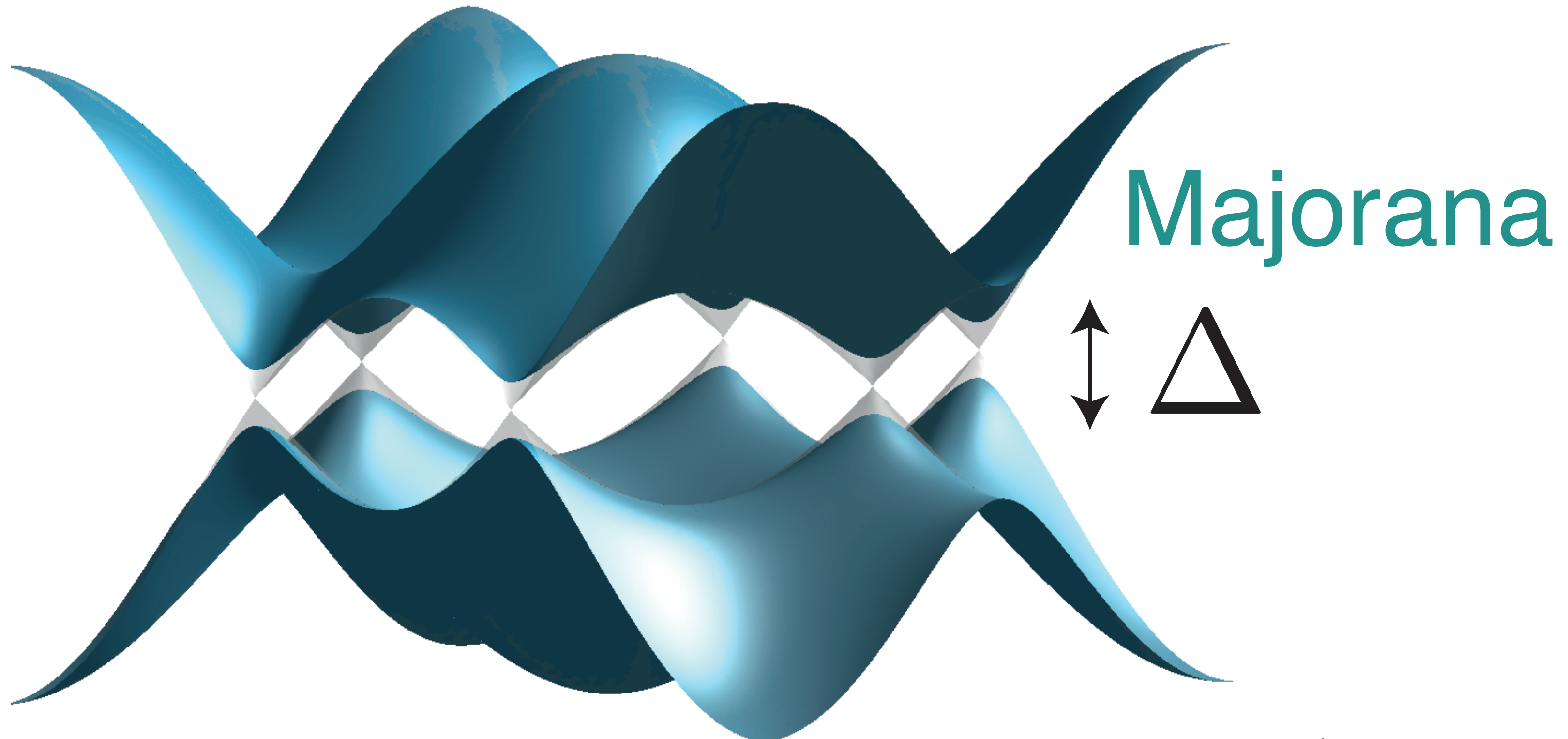
Floquet topo order

???

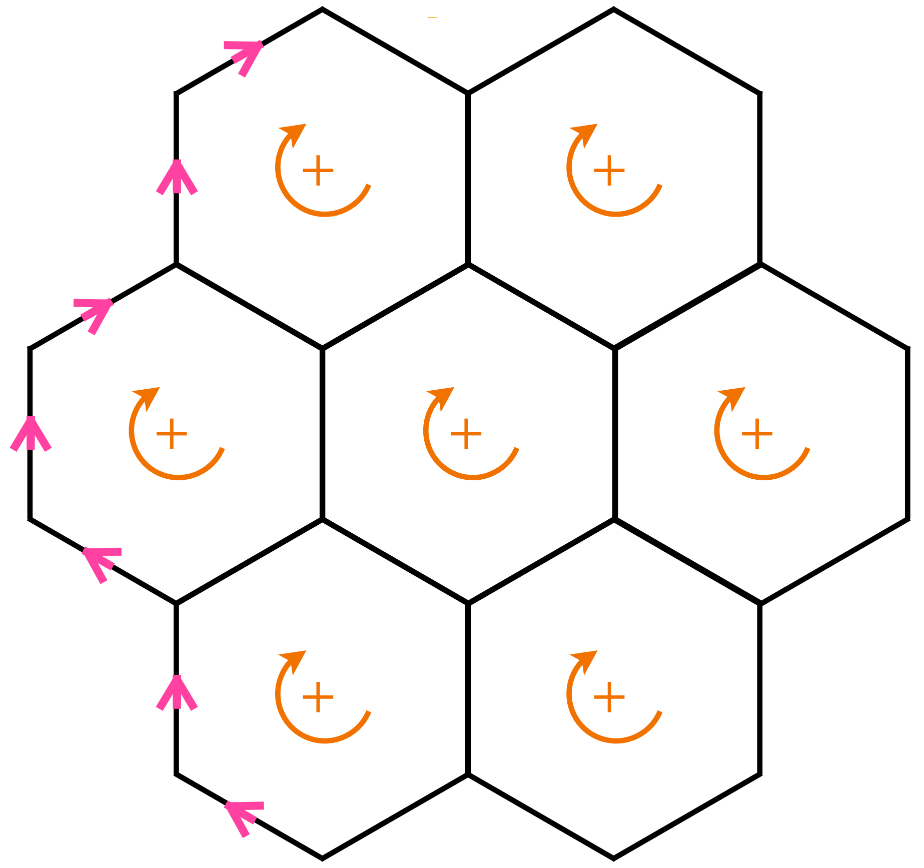
0 JT_c 3π drive period, JT

Thermal Hall effect

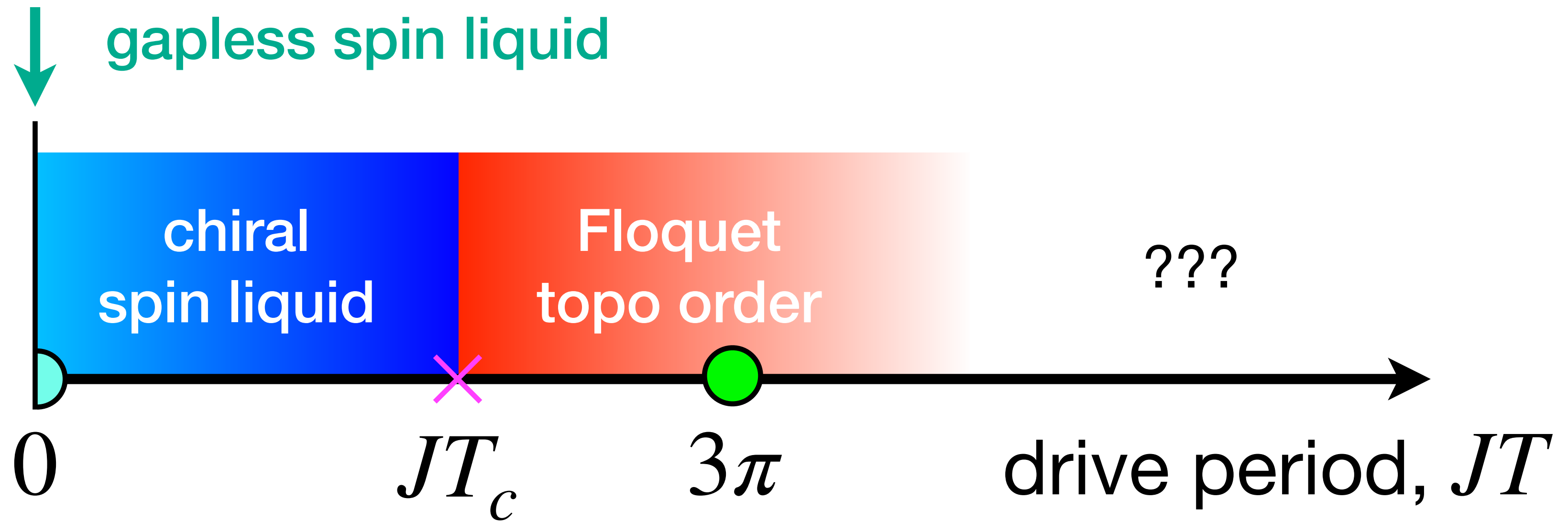
- **chiral spin liquid phase**
  - Majoranas do not conserve particle number
    - no particle edge current, *but:*
    - half quantized **chiral *energy density* edge currents**



chiral edge
energy density current



Kitaev, Annals of Physics 321, 2 (2006)



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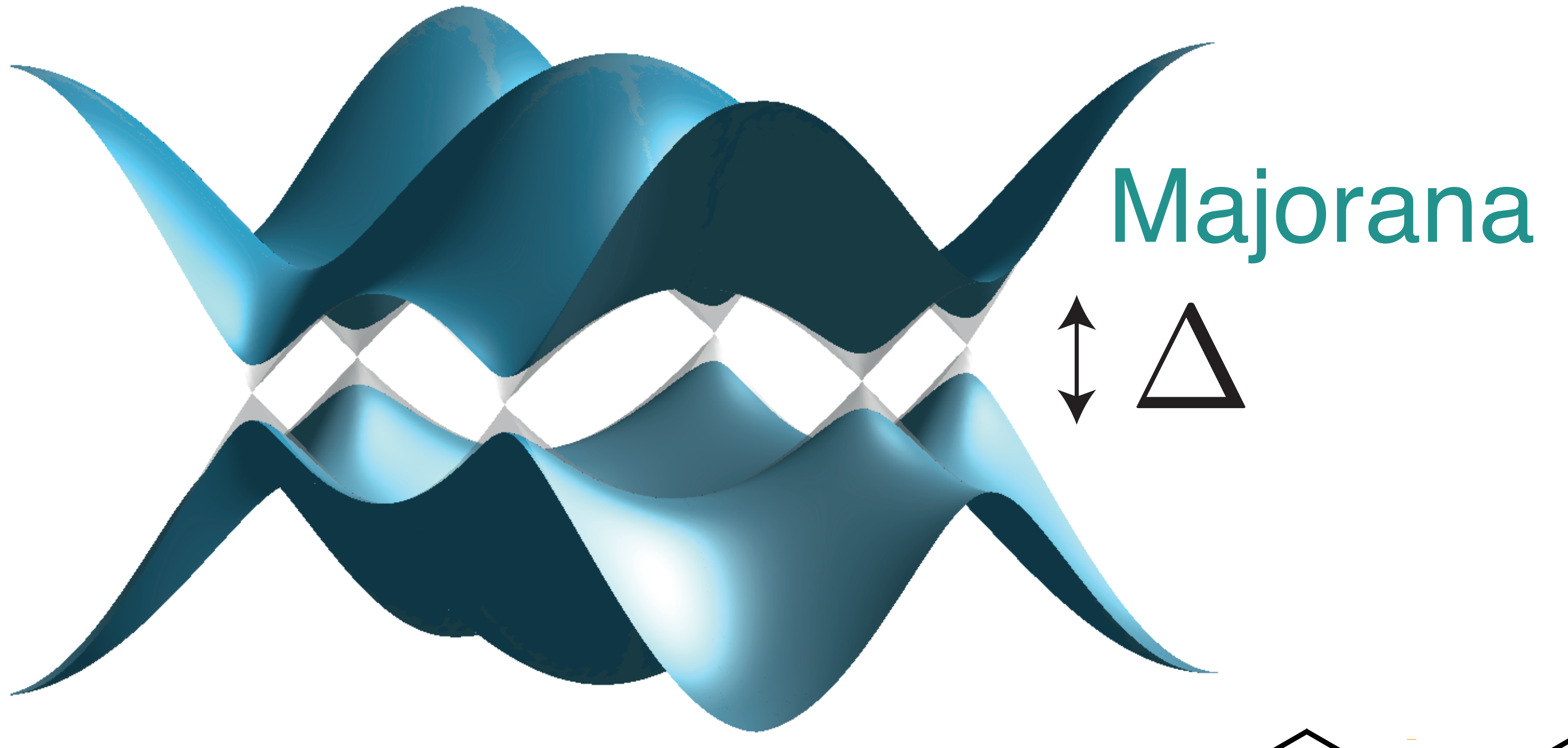
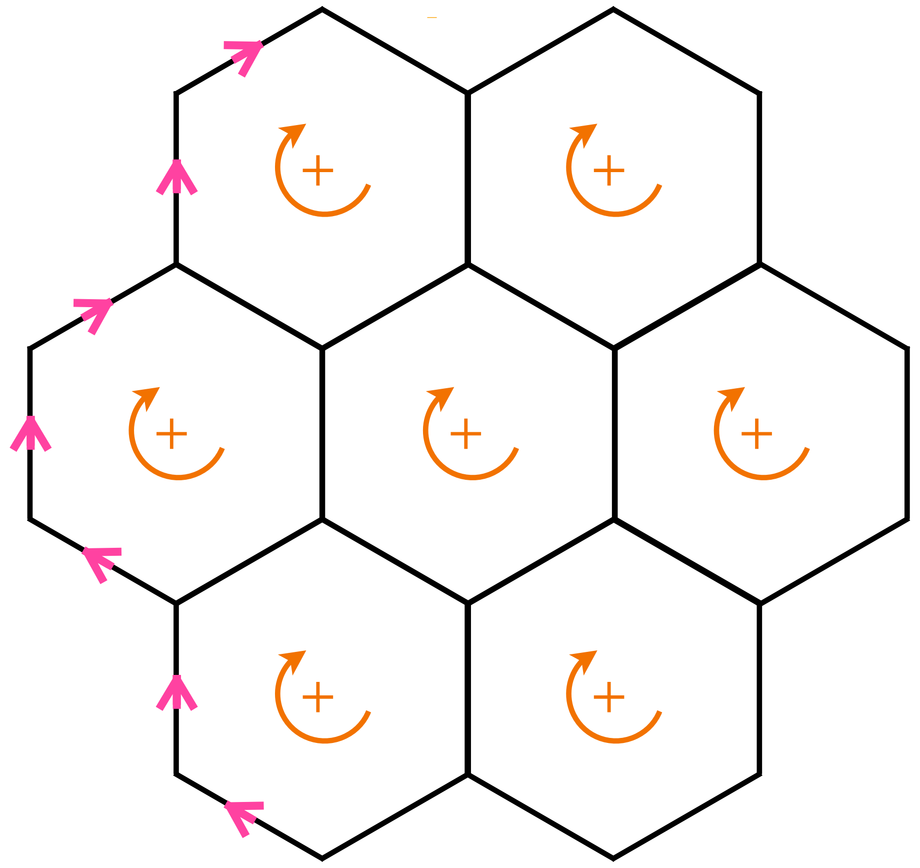


◉ mechanism & topological origin

- nonzero Chern number ν: quantized edge transport
  - complex fermions: $\beta\kappa = \nu \times \dfrac{\pi}{6}$
  - **Majorana fermions:** $\beta\kappa = \dfrac{\nu}{2} \times \dfrac{\pi}{6}$ (half-quantized)



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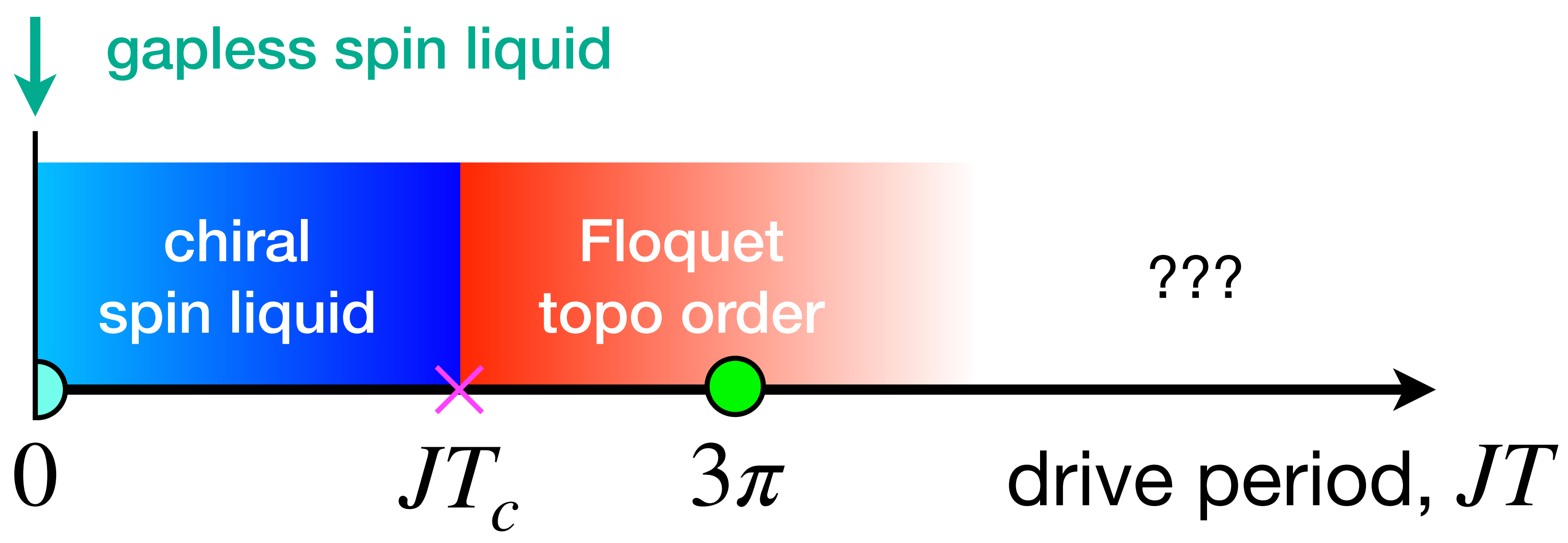

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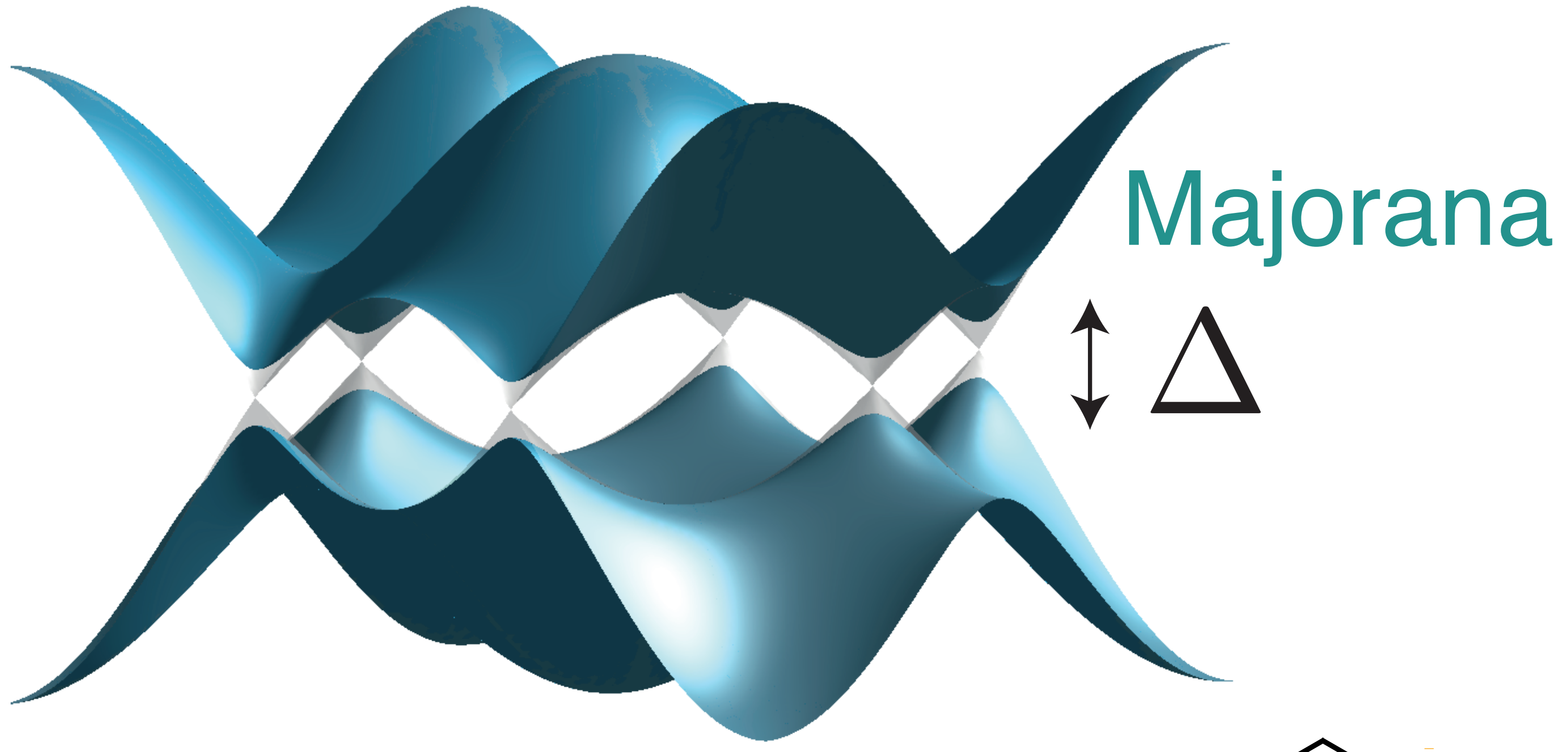

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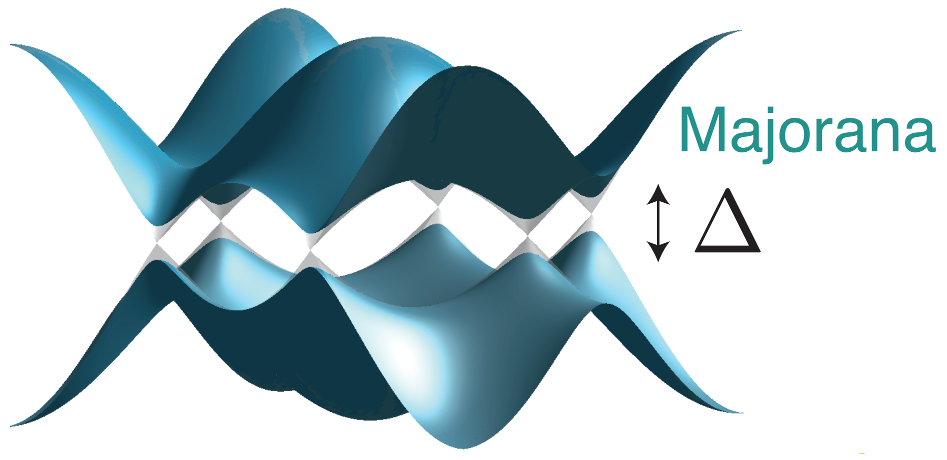
    - **Majorana fermions:** $\beta\kappa = \dfrac{\nu}{2} \times \dfrac{\pi}{6}$ (half-quantized)
  - promising candidate $\alpha-\mathrm{RuCl}_3$: phonon scattering, material imperfections, etc.

Q: can we observe quantized Thermal Hall effect in intermediate-scale quantum simulators?

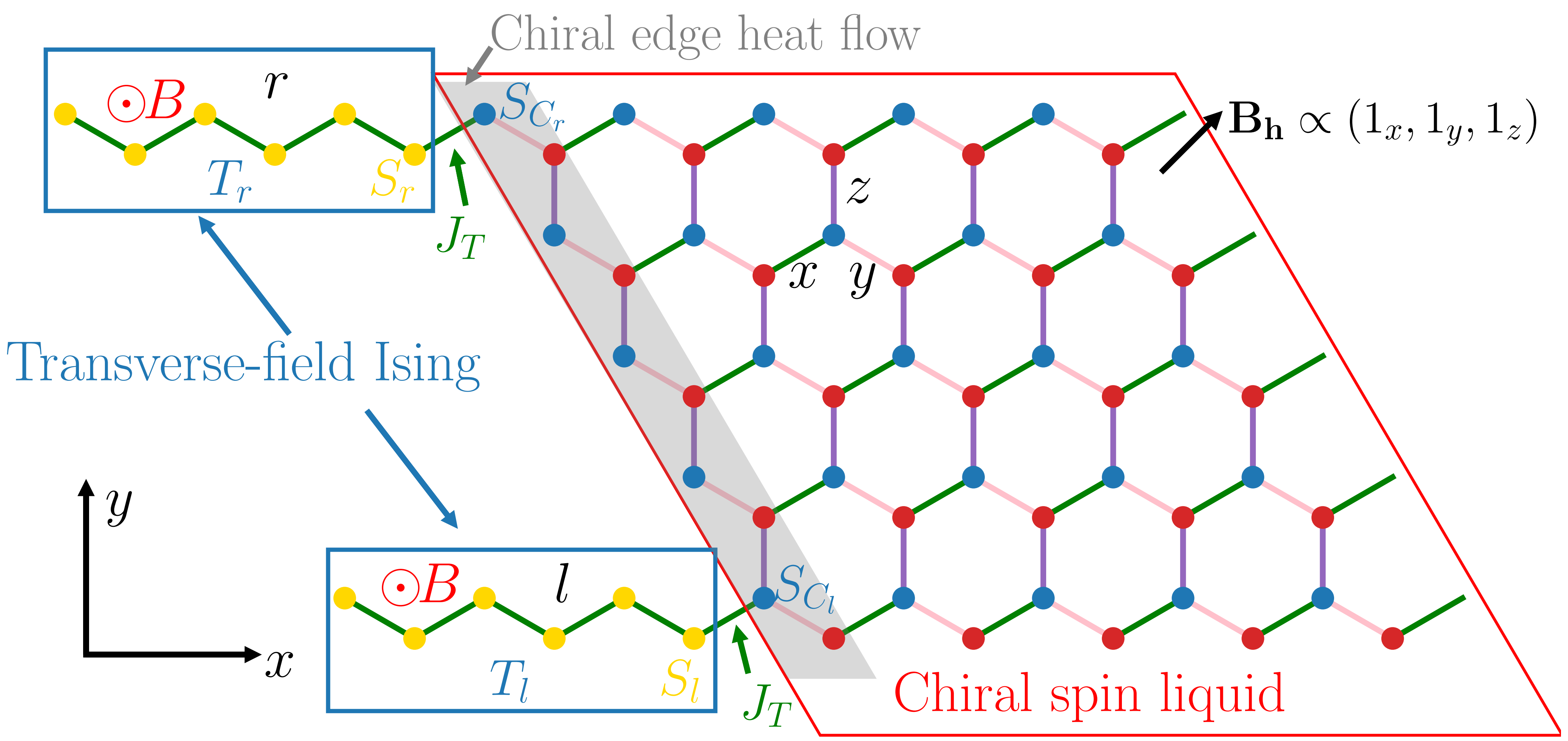
Kitaev, Annals of Physics 321, 2 (2006)



Thermal Hall effect in quantum simulators

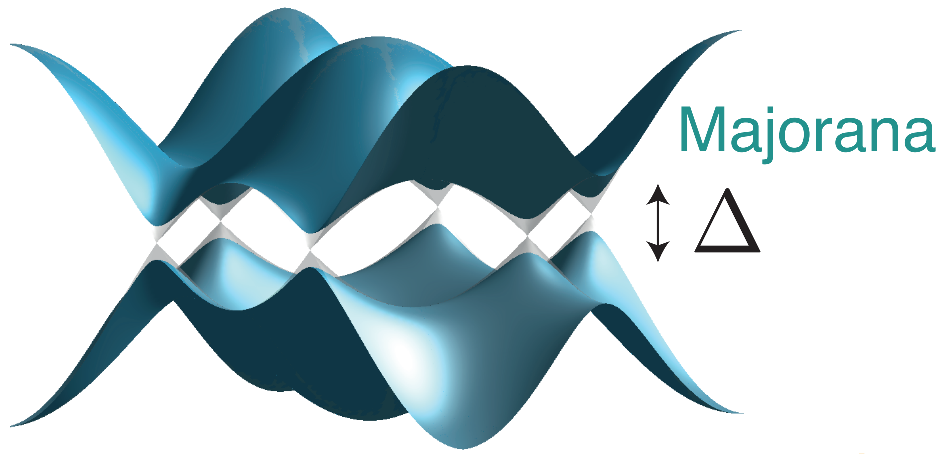
$$\kappa/T = \frac{\nu}{2} \times \frac{\pi}{6}$$

- two-terminal heat transport setup
  - finite-size Kitaev chiral spin-liquid device, H_{Kitaev}



$$H = H_{\text{Kitaev}} +$$

Sun et al, arXiv:2509.03355



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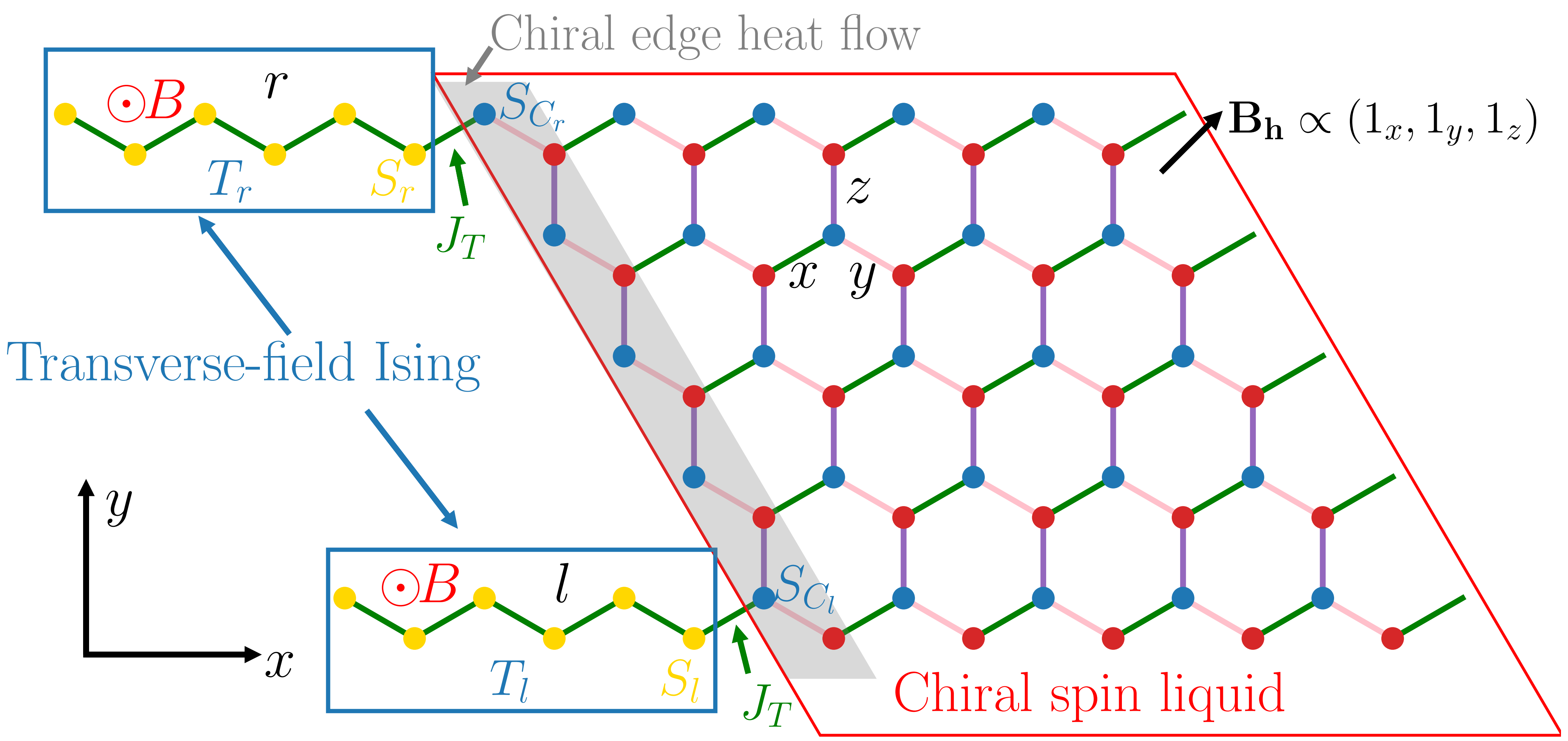

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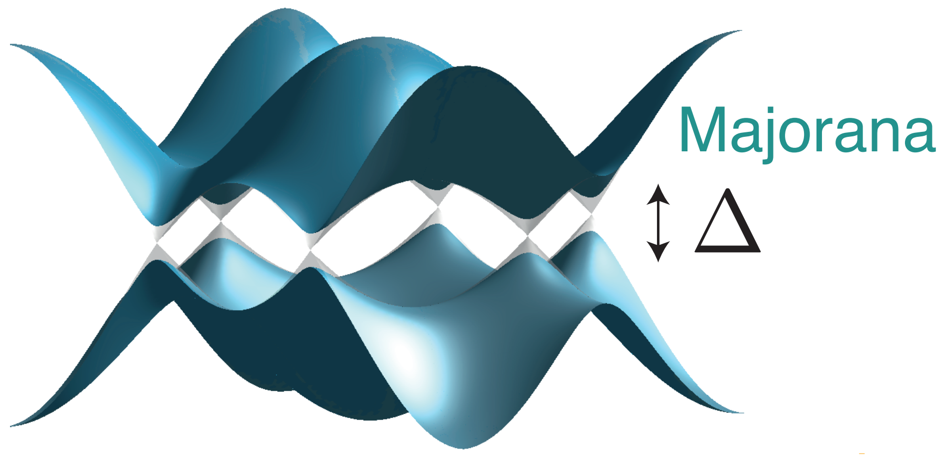

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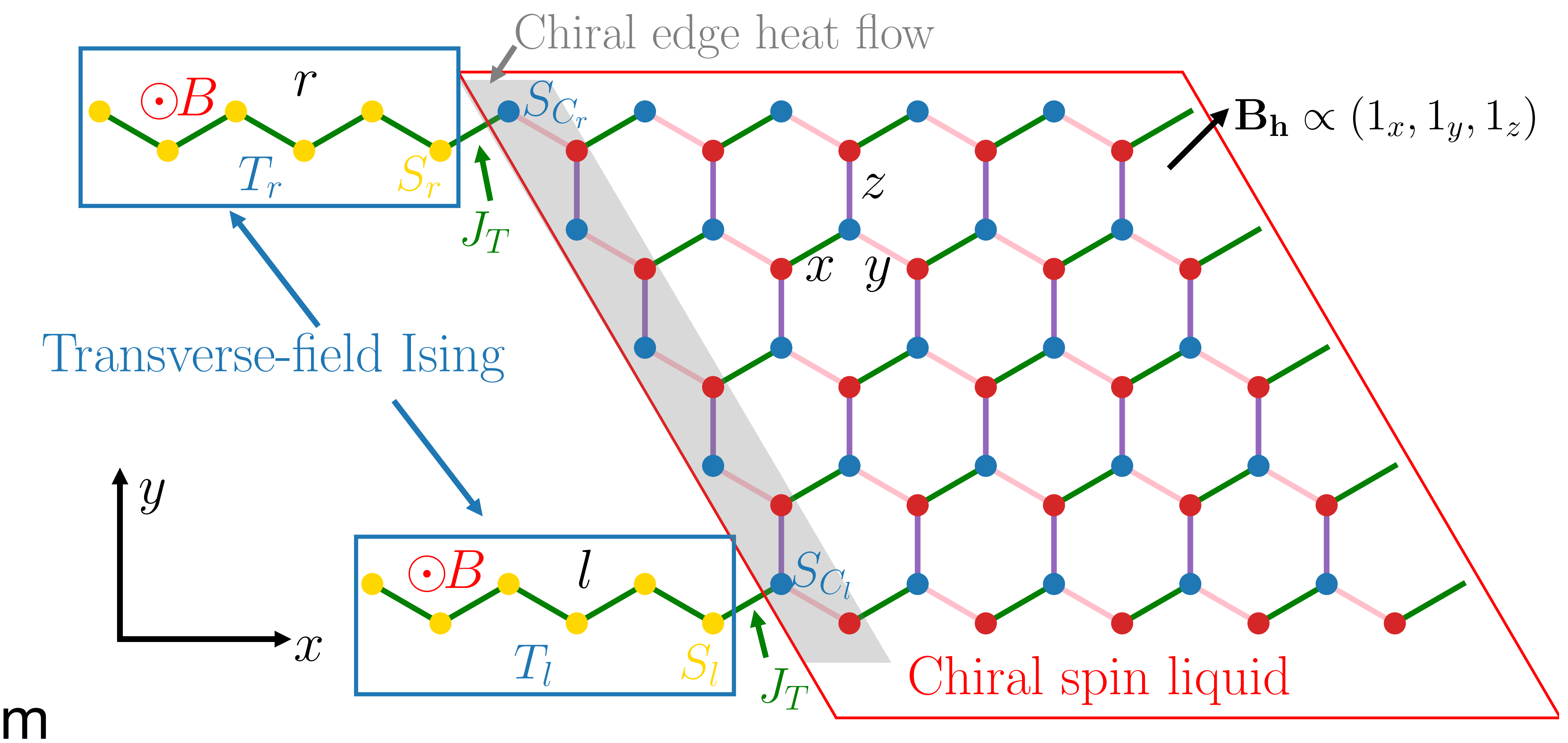
    - turned on adiabatically, until steady-state heat currents form

$$J_E^{l/r}(t) = -J_T B \langle S_{l/r}^y(t) S_{C_{l/r}}^x(t) \rangle \qquad \kappa^{l/r}(t)/T = 2J_E^{l/r}(t)/|T_l^2 - T_r^2|$$

$$H = H_{\text{Kitaev}} + H_{\text{res}}^l + H_{\text{res}}^r + H_{\text{coupling}}$$



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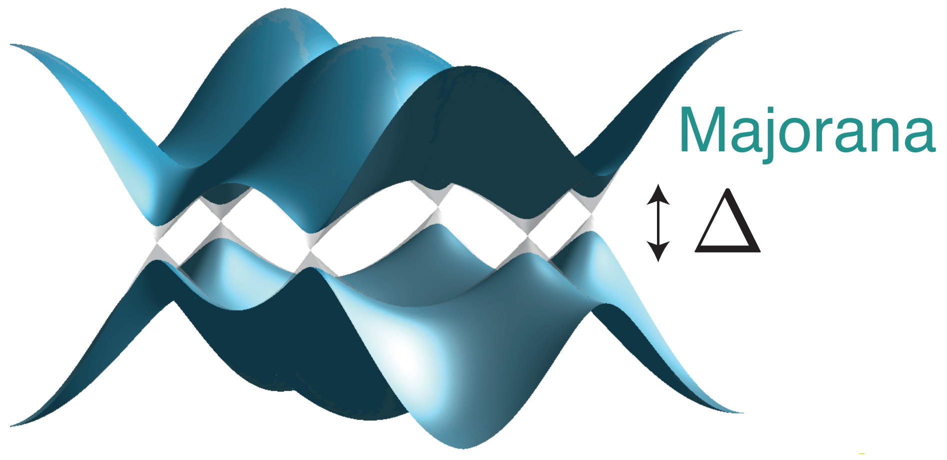

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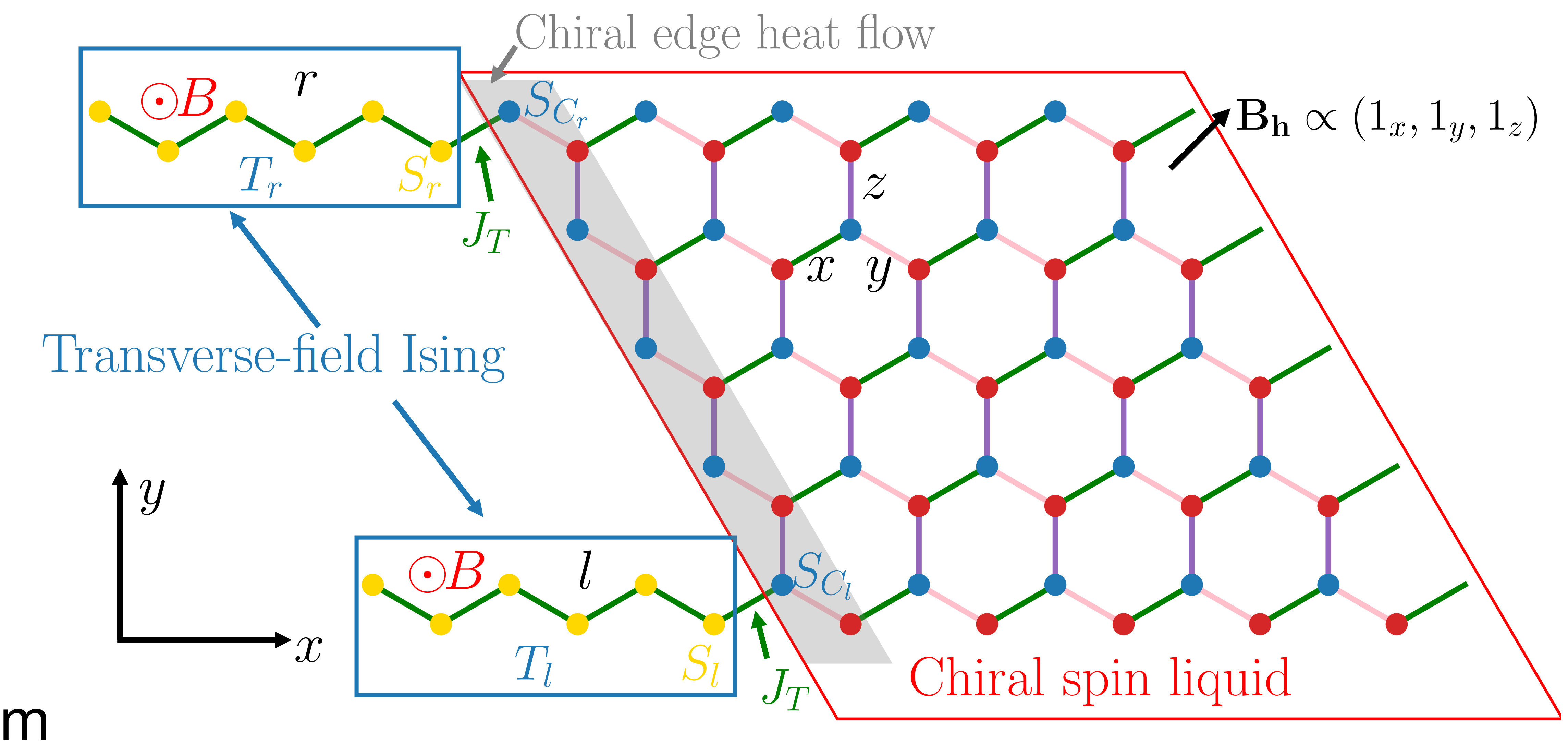
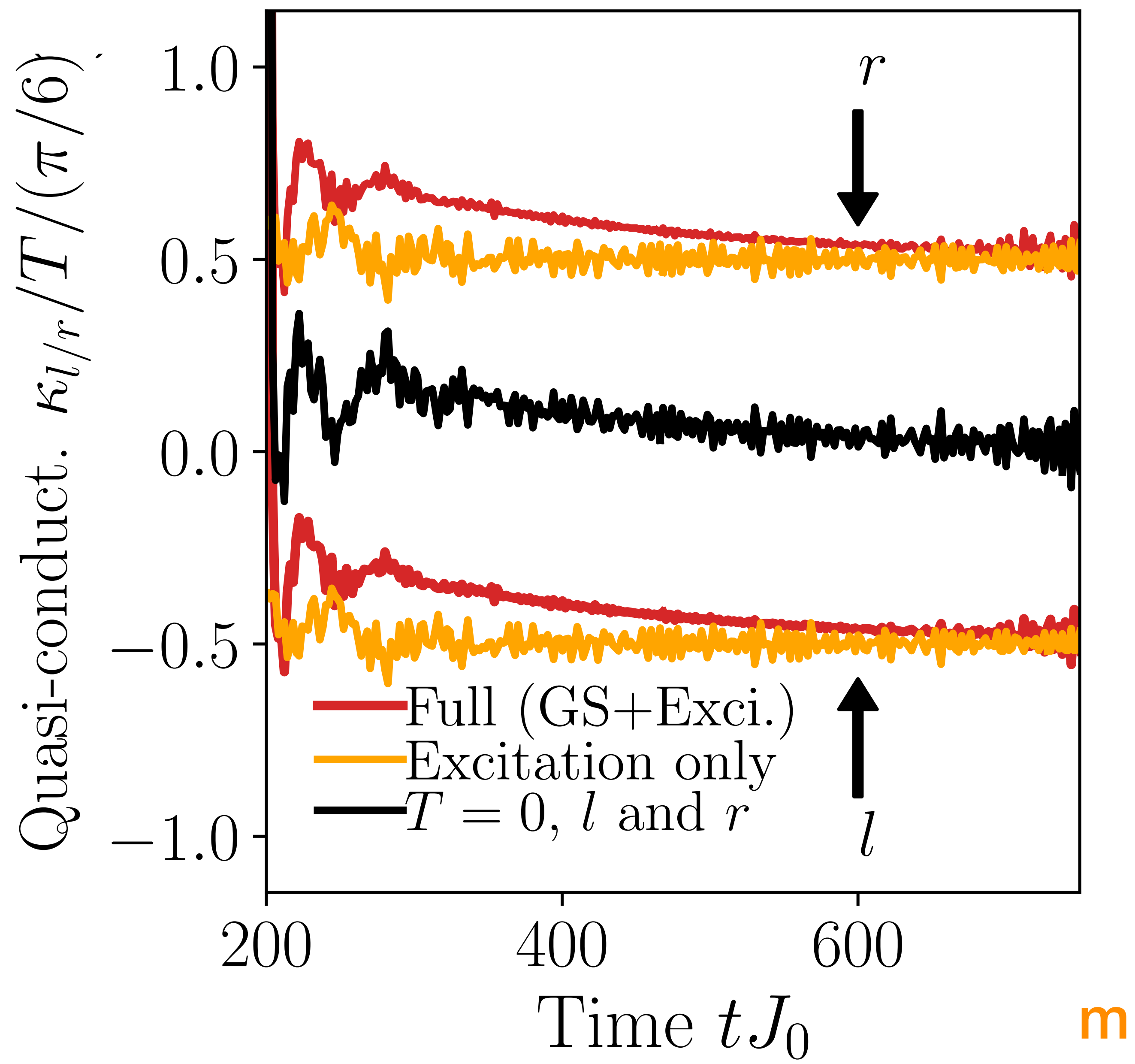
  - coupling term: $H_{\text{coupling}} = -J_T S^x_l S^x_{C_l} - J_T S^x_r S^x_{C_r}$
    - turned on adiabatically, until steady-state heat currents form

$$J^{l/r}_E(t) = -J_T B \langle S^y_{l/r}(t) S^x_{C_{l/r}}(t) \rangle \qquad \kappa^{l/r}(t)/T = 2J^{l/r}_E(t)/|T^2_l - T^2_r|$$

$$H = H_{\text{Kitaev}} + H^l_{\text{res}} + H^r_{\text{res}} + H_{\text{coupling}}$$

- **results in a nutshell**
  - stable heat flow region, even for finite reservoirs (70-200 sites)
  - half-quantized conductance measurements require very low reservoir temperatures $T/J_{\text{Kitaev}} \sim 10^{-3}$
    - optimal coupling strength $J_T \approx 0.8 J_{\text{Kitaev}}$





Sun et al, arXiv:2509.03355