Periodically Driven (Floquet) Systems

- models using time-dep. Hamiltonian:

$$H(t) = H_0 + f(t) H_1 , \quad H_1 = H_1^\dagger$$

$$H(t) = H_0 + g(t) V + g^*(t) V^\dagger$$

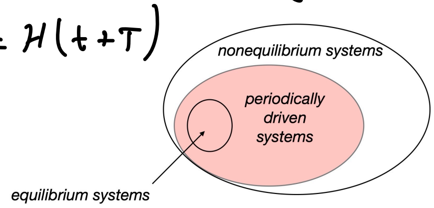
- time evo: solve Schrödinger's eq.

$$i\partial_t |\psi(t)\rangle = H(t) |\psi(t)\rangle \quad ; \quad [H(t_1), H(t_2)] \neq 0$$

↳ soln: time-ordered exponential

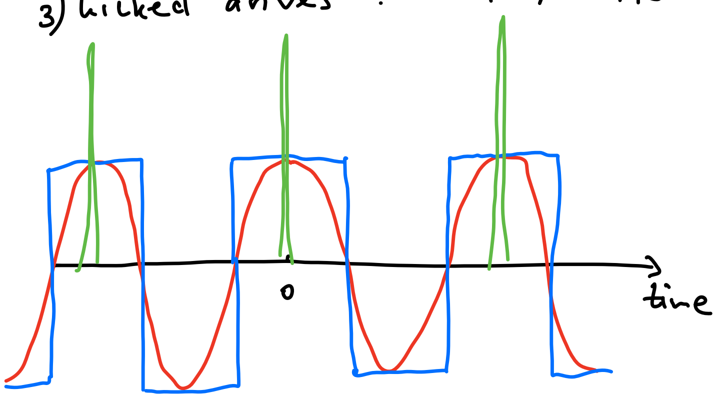
$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle \qquad U(t, t_0) = \mathcal{T} \exp\left(-i \int_{t_0}^{t} ds \, H(s)\right)$$

- periodic time dependence: $H(t) = H(t+T)$
  → relevant params.:
  - amplitude A
  - frequency $\omega = 2\pi/T$
  - phase of drive φ / starting time t_0



cos vs sin

- examples of periodic drives:
  1) continuous drives: $H(t) = H_0 + A \cos\omega t \, H_1$
  2) step/square drives: $H(t) = H_0 + A \, \mathrm{sign}(\cos\omega t) \, H_1$
  3) kicked drives: $H(t) = H_0 + A \sum_{n=-\infty}^{\infty} \delta(t - nT) \, H_1$



$A \cos\omega t$

$A \, \mathrm{sign}(\cos\omega t)$

$A \sum \delta(t - nT)$

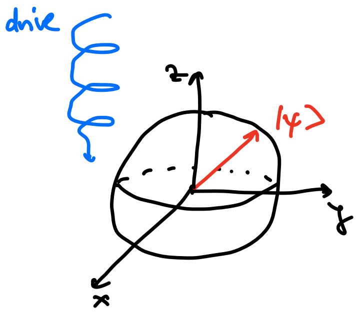
- integer multiples of drive period, ℓT, $\ell \in \mathbb{N}$

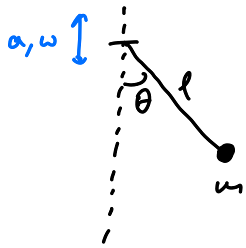
stroboscopic times

- examples of periodically driven systems

1) <u>2LS</u> in circularly polarized "light"

$$H(t) = B_z \sigma^z + B_{\parallel} (\sigma^x \cos\omega t + \sigma^y \sin\omega t)$$





$\downarrow g \qquad \omega_0 = \sqrt{g/\ell}$

$x(t) = \ell \sin\theta(t)$

$y(t) = \ell \cos\theta(t) + a\cos\omega t$

2) <u>Kapitza pendulum</u>: $H(t) = \frac{1}{2m\ell^2} p_\theta^2 - m\ell^2\left(\omega_0^2 + \frac{a\omega}{\ell}\omega\cos\omega t\right)\cos\theta$

re-define: $m\ell^2 \to m$

$\frac{a\omega}{\ell} \to A$

Hamiltonian for Kapitza pendulum:

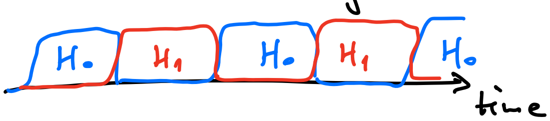
$$H(t) = \frac{p_\theta^2}{2m} - m\left(\omega_0^2 + A\omega\cos\omega t\right)\cos\theta$$

- recall video

$\to$ stabilization of inverted equilibrium position at high enough freq. ω; how high?

3) periodically kicked spin chain:

$$H(t) = \begin{cases} H_0 = \sum_j J\sigma^z_{j+1}\sigma^z_j + h_z\sigma^z_j, & t \in [0, T/2) \bmod T \\ H_1 = \sum_j h_x \sigma^x_j, & t \in [T/2, T) \bmod T \end{cases}$$



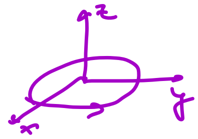
$$U(T,0) = \mathcal{T} e^{-i\int_0^T dt\, H(t)} \underset{\text{comp. def.}}{=} e^{-i\frac{T}{2}H_1} e^{-i\frac{T}{2}H_0}$$

- if $J = J_{ij}$ random numbers / disordered ($\to$ MBL)

$\to$ Floquet time crystal ($\to$ later)

Example: solution to 2LS: $H(t) = B_z \sigma^z + B_{\parallel} (\cos\omega t\, \sigma^x + \sin\omega t\, \sigma^y)$

want to compute dynamics under $H(t)$, i.e.



$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$

trick: let's transform to co-rotating frame

use: $V(t) = e^{-i\omega t \frac{\sigma^z}{2}}$

$\Rightarrow H_{rot} = B_z \sigma^z + B_{\parallel} \sigma^x - \frac{\omega}{2} \sigma^z$ time-indep!

$\Rightarrow U_{rot}(t_2, t_1) = \exp\left(-i(t_2 - t_1) H_{rot}\right)$ easy in rot. frame!

- rotating frame for a generic problem is difficult/impossible to identify

Q: can we exploit the time-periodic structure of $H(t)$?

Thm (Gaston Floquet, 1883) (theory ODE's)

let $H: \begin{array}{l} \mathbb{R} \to \mathbb{C}^{n\times n} \\ t \mapsto H(t) \end{array}$ be continuous, matrix-valued fn w/ period T: $H(t+T) = H(t)$

(Hamiltonian)

let $U(t)$ be the fundamental matrix (time-evo op.) to the first-order linear ODE

$$i\partial_t \psi(t) = H(t)\psi(t) \quad ; \quad i\partial_t U(t) = H(t)\, U(t), \quad U(0) = \mathbb{1}$$

then:

1) $U(t+T)$ is also a fundamental matrix

2) there exists a non-singular, continuously diff'ble matrix valued fn:

$P: \begin{array}{l} \mathbb{R} \to \mathbb{C}^{n\times n} \\ t \mapsto P(t) \end{array}$ with period T: $P(t+T) = P(t)$

and a time-indep. matrix $H_F \in \mathbb{C}^{n\times n}$, s.t.

$$\boxed{U(t) = P(t)\, e^{-i t H_F}}$$

Corollary: stroboscopically, i.e. at $t = \ell T$

$$U(\ell T) = e^{-i \ell T H_F}$$

Proof:

1) $i\partial_t \underline{U(t+T)} = i\dot{U}(t+T) \overset{\text{def. of } U(t)}{=} H(t+T)\,U(t+T) \overset{H(t)=H(t+T)}{=} \underline{H(t)\,U(t+T)}$ ✓

2) $U(t)$ & $U(t+T)$ are both fundamental matrices

$\Rightarrow$ there is a static linear transformation that relates them

$$\Rightarrow \quad U(t+T) = U(t)\,\underline{U_F} \qquad (**)$$

$$U_F : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}$$

by the existence of matrix log, define H_F via:

$$U_F = e^{-i T H_F}$$

set: $P(t) := U(t)\, e^{+i t H_F}$ $\quad (*)$

check periodicity: $P(t+T) \overset{(*)}{=} U(t+T)\, e^{+i(t+T)H_F}$

$$\overset{(**)}{=} U(t) \underbrace{U_F\, e^{+i t H_F}\, e^{+i T H_F}}_{} \quad = \mathbb{1}$$

$$= U(t)\, e^{+i t H_F} = P(t) \; ✓$$

invert $(*)$: $U(t) = P(t)\, e^{-i t H_F}$ ✓ ∎

Rmks:

1) note: this requires a *linear* ODE

2) proof is not constructive

- Floquet thm: $H(t+T) = H(t)$

$$\Rightarrow \quad U_{lab}(t,0) = P(t)\,\underbrace{e^{-itH_F}}_{=U_{rot}(t,0)}\,\underbrace{P^\dagger(0)}_{=\mathbb{1}} \;,\; b/c \;\; U(0,0)=\mathbb{1}$$

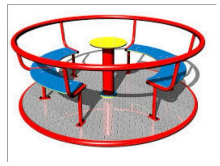
→ looks like transf. law for evo. op's b/w lab & rot frames!

$P(t)$: rot ⟶ lab

⇒ Hamiltonian in rot frame:

$$H_{rot}(t) = P^\dagger(t)\,H_{lab}(t)\,P(t) - i\,P^\dagger(t)\,\partial_t P(t) \overset{\text{Floquet thm}}{=} H_F$$

time-indep



- Floquet's thm: statement about existence of a reference frame in which the dynamics is governed by the time indep. Hamiltonian H_F; moreover this reference frame is a rotating frame: $P(t+T) = P(t)$

note: thm doesn't tell us how to compute H_F (nor how to find that special rot. frame)!

- notation:

• effective / Floquet Hamiltonian : H_F (H_{eff})

→ this is NOT the same as H_0, $H(t=0)$

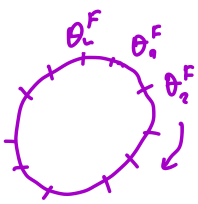
• Floquet unitary: $U_F = U(T,0) = e^{-iTH_F}$

$$\Rightarrow U(\ell T,0) = e^{-i\ell T H_F} = U_F^\ell$$

→ evo. op. over one drive cycle/period T

→ eigendecomposition: $U_F = \sum_n e^{i\theta_n^F} |n_F\rangle\langle n_F|$

Floquet phases   Floquet states



spectrum of U_F

$\{\theta_n^F\}$ lives on unit circle

$$\Rightarrow H_F = \sum_n \varepsilon_n^F |n_F\rangle\langle n_F| \;, \quad T\varepsilon_n^F = \theta_n^F$$

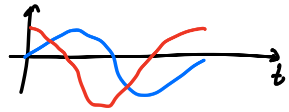
quasi-energies

q'energies defined mod ω : $e^{iT(\varepsilon_n^F + \ell\omega)} = e^{iT\varepsilon_n^F} \quad \forall \ell \in \mathbb{Z}$

⇒ q'energies cannot be unambiguously ordered
there is no Floquet ground state (↪ issue for q'simulation)

- Floquet Hamiltonian H_F is <u>not</u> unique (depends on the choice for branch cut of log) but U_F is unique

– <u>Floquet "gauge"</u>: dependence on the initial time t_0 (or phase of drive)



$$U(t,t_0) = P(t,t_0)\, e^{-i(t-t_0)H_F[t_0]}$$

Floquet gauge

→ $H_F[t_0]$ matrix depends on t_0

<u>but</u> ε_n^F are <u>indep.</u> of t_0 (→ hence t_0 is a gauge)

⇒ Floquet states carry dependence on t_0: $|u_F[t_0]\rangle$

$$H_F[t_0]\,|u_F[t_0]\rangle = \varepsilon_n^F\,|u_F[t_0]\rangle$$

<u>note</u>: Floquet "gauge" is physical; it is set by initial time/condition

<u>claim</u>: $P(t,t_0)\,H_F[t_0]\,P^\dagger(t,t_0) = H_F[t]$

<u>pf</u>: $$e^{-iTH_F[t]} = U(T+t,t) = U(T+t,t_0)\,\overbrace{U(t_0,t)}^{=U^\dagger(t,t_0)}$$

$$= \underbrace{P(T+t,t_0)}_{=P(t,t_0)}\, e^{-i(T+t-t_0)H_F[t_0]}\, e^{i(t-t_0)H_F[t_0]}\, P^\dagger(t,t_0)$$

$$= P(t,t_0)\, e^{-iTH_F[t_0]}\, P^\dagger(t,t_0)$$

$$\Rightarrow H_F[t] = P(t,t_0)\,H_F[t_0]\,P^\dagger(t,t_0) \checkmark$$

<u>claim</u>: $P(t,t_0)\,|u_F[t_0]\rangle = |u_F[t]\rangle$ inst. e'state of $H_F[t]$

→ micromotion op. moves b/w inst. Floquet states

<u>pf</u>: $$H_F[t]\,\underline{P(t,t_0)\,|u_F[t_0]\rangle} = P(t,t_0)\,\underbrace{P^\dagger(t,t_0)\,H_F[t]\,P(t,t_0)}_{=H_F[t_0]}\,|u_F[t_0]\rangle$$

$$= \varepsilon_n^F\, \underline{P(t,t_0)\,|u_F[t_0]\rangle} \checkmark$$

Q: How do we compute H_F & P in practice?

→ inverse-frequency expansions: $\omega \gg \omega_0$, $\omega \to \infty$ (or $T \to 0$)

(1) Floquet-Magnus expansion

(2) van Vleck expansion

(3) Brillouin-Wigner expansion

review: Bukov et al., Adv. in Physics '15