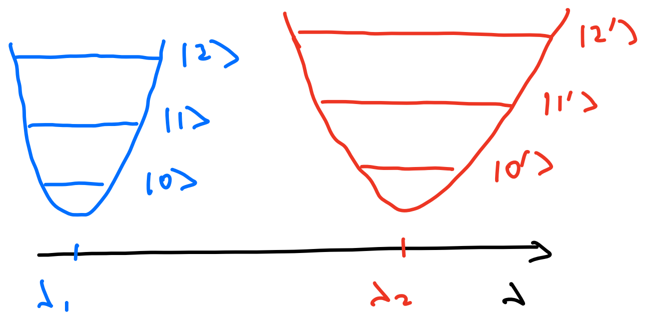
Adiabatic & Counterdiabatic Driving

- consider Hamiltonian w/ changing parameter $\lambda(t)$

  $H_{ctrl} = H_{ctrl}(\lambda(t))$

  e.g. $H_{ctrl}(\lambda(t)) = H_0 + \lambda(t)\, H_1$

- instantaneous e'states : $H_{ctrl}(\lambda)\,|n[\lambda]\rangle = E_n(\lambda)\,|n[\lambda]\rangle$



- time evolved e'states $\neq$ instant. e'states

$$|n(t)\rangle = \mathcal{T} e^{-i\int_0^t ds\, H_{ctrl}(\lambda(s))}\,|n[\lambda(0)]\rangle \neq |n[\lambda(t)]\rangle$$

- Adiabatic theorem

A quantum system remains in its inst. e'state upon changing parameter $\lambda(t)$, iff:

(i) the inst. e'state is gapped at all times

(ii) the change in parameter, $\dot{\lambda}$, remains small compared to the energy gap Δ to nearby levels:

$$\left|\frac{\dot{\lambda}}{\Delta^2(\lambda)}\right| \times |\langle m[\lambda]|\partial_\lambda H_{ctrl}|n[\lambda]\rangle| \ll 1 \quad \forall \lambda$$

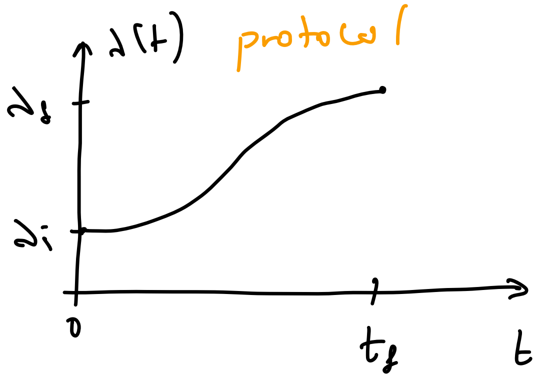
inst. gap : $\Delta(\lambda) = E_m(\lambda) - E_n(\lambda)$

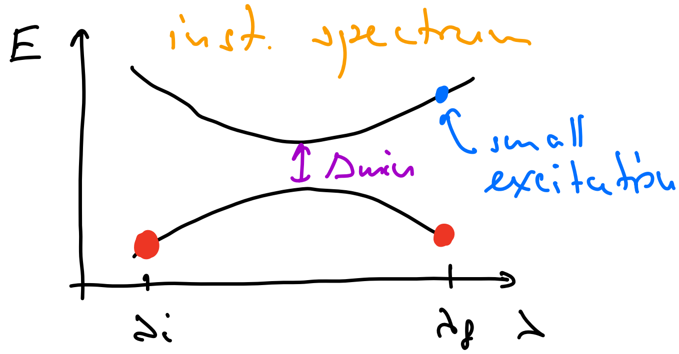
intuitively: "total ramp/evolution time t_f should be longer than inverse gap Δ^{-1}"

(but imprecise!)

→ textbook proof via t-dep. pert. theory (e.g. Sakurai QM book)

– <u>important</u>: statement is perturbative, i.e. for any <u>finite</u> protocol duration t_f there are small excitations, which vanish as $t_f \to \infty$





– consequence of adiabatic theorem:

$$\mathcal{T} e^{-i\int_0^{t_f} H_{ctrl}(t)\, ds} |n[0]\rangle = |n(t_f)\rangle \xrightarrow{\text{ad. thm.}} e^{-i\varphi_n(t_f)} e^{-i\gamma_n(t_f)} |n[\lambda(t_f)]\rangle$$

evolved e'state $\quad i\partial_t |n(t)\rangle = H_{ctrl}(t)|n(t)\rangle$

inst. e'state $\quad H_{ctrl}(\lambda)|n[\lambda]\rangle = E_n|n[\lambda]\rangle$

with:

<u>dynamical "phase"</u>: $\varphi_n(t_f) = \int_0^{t_f} dt\, E_n(\lambda(t))$ $\in \mathbb{R}$ defined on entire real line; not just on $[0, 2\pi)$, since $E_n \in \mathbb{R}$

<u>geometric phase</u>: $\gamma_n(t_f) = \int_0^{t_f} dt\, \dot\lambda(t) \langle n[\lambda(t)] | i\partial_\lambda | n[\lambda(t)]\rangle$

$$\overset{\text{single ctrl param. } \lambda}{=} \int_{\lambda(0)}^{\lambda(t_f)} d\lambda\, \langle n[\lambda] | i\partial_\lambda | n[\lambda]\rangle$$

path-indep.

– adiabatic theorem on the level of operators:

recall:

- static unitary changes the basis
  (observable expectations remain unchanged)
- time-dep. unitary changes reference frame
  (expect. values of obs. may change: e.g. energy, currents, etc.)

$$\begin{tikzcd}
\text{lab frame} \arrow[r, bend right, "V^\dagger(t)"'] & \text{moving frame} \arrow[l, bend right, "V(t)"']
\end{tikzcd}$$

lab frame	moving frame
$H_{lab}(t)$	$H_{mov}(t)$
$U_{lab}(t_2, t_1)$	$U_{mov}(t_2, t_1)$
$\vert\psi_{lab}(t)\rangle$	$\vert\psi_{mov}(t)\rangle$

→ relation b/w states: $|\psi_{mov}(t)\rangle = V^\dagger(t)\,|\psi_{lab}(t)\rangle$

→ – // – evo. operators:

$$U_{lab}(t_2, t_1) = V(t_2)\, U_{mov}(t_2, t_1)\, V^\dagger(t_1)$$

$$U_{mov}(t_2, t_1) = \mathcal{T}\exp\left(-i\int_{t_1}^{t_2} dt\, H_{mov}(t)\right)$$

→ relation b/w Hamiltonians:

lab-frame Schr. eq: $\quad i\partial_t |\psi_{lab}(t)\rangle = H_{lab}(t)\underbrace{|\psi_{lab}(t)\rangle}_{= V(t)|\psi_{mov}(t)\rangle}$

$$\Rightarrow \underbrace{i\partial_t\Big(V(t)\,|\psi_{mov}(t)\rangle\Big)}_{= i\partial_t V(t)\,|\psi_{mov}(t)\rangle + V(t)\, i\partial_t|\psi_{mov}(t)\rangle} = H_{lab}(t)\,V(t)\,|\psi_{mov}(t)\rangle \quad / V^\dagger(t)\cdot$$

$$i\partial_t|\psi_{mov}(t)\rangle = \Big(V^\dagger(t)\, H_{lab}(t)\, V(t) - V^\dagger(t)\, i\partial_t V(t)\Big)\,|\psi_{mov}(t)\rangle$$

$$\Rightarrow H_{mov}(t) = V^\dagger(t)\, H_{lab}(t)\, V(t) - \underbrace{V^\dagger(t)\, i\partial_t V(t)}_{\text{Galilean term}}$$

(e.g. centrifugal force, etc.)

— now consider frame defined by unitary $V(\lambda)$ that diagonalizes

$$V^\dagger(\lambda)\, H_{ctrl}(\lambda)\, V(\lambda) = \mathrm{diag}\big(E_1(\lambda), \dots, E_n(\lambda)\big) =: D(\lambda) = \sum_n E_n(\lambda) \underbrace{|n_{mov}\rangle}_{\text{indep. of } \lambda!} \langle n_{mov}|$$

& recall that $\lambda = \lambda(t)$:

$$\Rightarrow H_{mov}(t) = \underbrace{V^\dagger(\lambda(t))\, H_{ctrl}(\lambda(t))\, V(\lambda(t))}_{\substack{= D(\lambda(t)) \text{ diagonal:} \\ \text{cannot cause excitations!}}} - \dot\lambda\, \underbrace{V^\dagger(\lambda(t))\, i\partial_\lambda V(\lambda(t))}_{\substack{=: A_{mov}(\lambda) \\ \text{adiabatic gauge potential:} \\ \text{creates all non-adia.} \\ \text{excitations}}}$$

→ ad. limit:

• ignore local (in time) transitions due to A_{mov} in moving frame

$$\Rightarrow U_{mov}(t,0) = \mathcal{T} e^{-i\int_0^t ds\, H_{mov}(s)} \xrightarrow{\text{ad. thm}} \mathcal{T} e^{-i\int_0^t ds\, D(\lambda(s))}$$

$$\xrightarrow{\text{D diag.}} = e^{-i\int_0^t ds\, D(\lambda(s))} = e^{-i\Phi_{mov}(t)}$$

w/ dyn. phase op. $\Phi_{mov}(t) = \sum_n \varphi_n(t)\, |n_{mov}\rangle\langle n_{mov}|$

• go back to lab frame:

$$A_{lab}(\lambda) = V(\lambda)\, A_{mov}(\lambda)\, V^\dagger(\lambda) = V V^\dagger i(\partial_\lambda V) V^\dagger = \big(i\partial_\lambda V(\lambda)\big) V^\dagger(\lambda)$$

↪ matrix el.: $\langle n_{lab}|A_{lab}|n_{lab}\rangle = \langle n_{mov}|A_{mov}|n_{mov}\rangle = \langle n_{mov}|V_\lambda^\dagger i\partial_\lambda V_\lambda|n_{mov}\rangle$

$$= \langle n[\lambda]|\, i\partial_\lambda\, |n[\lambda]\rangle$$

$|n_{lab}\rangle = |n[\lambda]\rangle$; n_{mov} indep. of λ

• A_{lab} satisfies Schrödinger eq. $\quad i\partial_\lambda V(\lambda) = A_{lab}(\lambda)\, V(\lambda)$

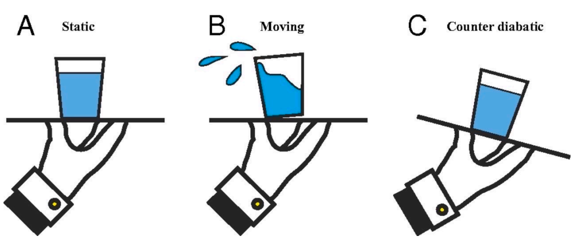
w/ soln: $V(\lambda) = \mathcal{P} e^{-i\int_{\lambda(0)}^{\lambda} d\mu\, A_{lab}(\mu)} \ast V(\lambda(0))$ ← initial cond.

$$U_{lab}(t,0) = V(\lambda(t))\, U_{mov}(t,0)\, V^\dagger(\lambda(0)) \xrightarrow{\text{ad. thm}} \underbrace{\mathcal{P} e^{-i\int_{\lambda(0)}^{\lambda(t)} d\mu\, A_{lab}(\mu)}}_{\substack{\text{geometric phase;} \\ \text{Wilson line operator}}} \underbrace{V(\lambda(0))\, e^{-i\Phi_{mov}(t)}\, V^\dagger(\lambda(0))}_{\substack{= e^{-i\Phi_{lab}(t)} \\ \text{dyn. phase op.}}}$$

$$\Phi_{lab}(t) = \sum_n \varphi_n(t)\, |n[\lambda(0)]\rangle\langle n[\lambda(0)]|$$

<u>Counter-diabatic driving</u>: can we eliminate the non-adia. exc.?

def. CD Hamiltonian: $\quad H_{CD}(\lambda) := H_{ctrl}(\lambda) + \underbrace{\dot\lambda\, A_{lab}(\lambda)}_{\text{counteracts excitations}}$



waiter's co-moving frame

Sels & Polkovnikov, PNAS '17

• in moving frame wrt $V(\lambda)$:

$$H_{CD,mov}(\lambda) = \underbrace{V^\dagger H_{ctrl} V}_{= D(\lambda)} - \dot\lambda \mathcal{A}_{mov}(\lambda) + \dot\lambda \underbrace{V^\dagger \mathcal{A}_{lab} V}_{= \mathcal{A}_{mov}} = D(\lambda) \text{ exact!}$$

=>

$$U^{CD}_{lab}(t,0) = \mathcal{T} e^{-i\int_0^t ds\, \overbrace{H_{CD,lab}(\lambda(s))}^{= H_{ctrl} + \dot\lambda \mathcal{A}_{lab}}} \overset{\text{exact}}{=} \mathcal{P} e^{-i\int_0^{\lambda(t)} d\mu\, \mathcal{A}_{lab}(\mu)}\, e^{-i\Phi_{lab}(t)}$$

→ CD driving makes adiabatic evo. exact

→ many body systems: $\mathcal{A}_{lab}$ is typically nonlocal (→ unphysical)
unless we focus on adia. evo. of the gapped GS of H_{ctrl}
a.k.a. spectral flow thm; Hastings PRB '06
↳ can compute $\mathcal{A}_{lab}$ variationally: Sels & Polkovnikov PNAS '17

Gauge freedom:

– rephase e'states: $|n[\lambda]\rangle \longrightarrow e^{i\chi_n(\lambda)}|n[\lambda]\rangle$

=> AGP not gauge-inv.: $A_n(\lambda) := \langle n|\mathcal{A}|n\rangle \longrightarrow \langle n|\mathcal{A}|n\rangle - \partial_\lambda \chi_n$

$$\Rightarrow \mathcal{A}(\lambda) \longrightarrow \mathcal{A}(\lambda) - \sum_n \partial_\lambda \chi_n(\lambda)\, |n[\lambda]\rangle\langle n[\lambda]|$$

=> H_{CD} not unique! => all H_{CD} achieve transitionless driving
different gauges lead to different overall phases (per e'state!)
↳ Duncan et. al., PRX Q '25 (tutorial)

Q: which gauge corresponds to adia. evolution?

– parallel transport gauge: (Bradlyn & Iraola, SciPost '22)

Kato AGP: $\mathcal{A}_K(\lambda) := \mathcal{A}(\lambda) - \sum_n \langle n[\lambda]|\mathcal{A}(\lambda)|n[\lambda]\rangle \overbrace{|n[\lambda]\rangle\langle n[\lambda]|}^{=: \Pi_n[\lambda] \text{ e'state proj.}}$

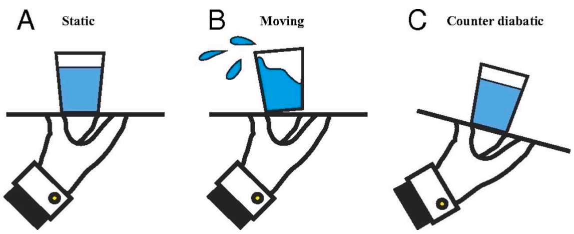
$$= -\frac{1}{2}\sum_n [\Pi_n[\lambda], i\partial_\lambda \Pi_n[\lambda]]$$

→ analogy:

	EM	QM	
derivative	$i\partial_\lambda$	$i\partial_\lambda = \mathcal{A}$	AGP
vector pot.	A_λ	$A_n(\lambda)$	Berry connection
covariant derivative	$i\partial_\lambda - \underbrace{A_\lambda}_{\text{vector pot.}}$	$\mathcal{A}_K = \mathcal{A} - \sum_n A_n \Pi_n$	Kato AGP

key takeaways:

(1) waiter's co-moving frame: mechanism behind CD driving



(2) CD driving: evolved e'state = inst. e'state up to phase

$$\mathcal{T} e^{-i\int_0^{t_f} H_{ctrl}(t)\, ds} |n[0]\rangle = |n(t_f)\rangle \xrightarrow{\text{ad. thm.}} e^{-i\varphi_n(t_f)} e^{-i\gamma_n(t_f)} |n[\lambda(t_f)]\rangle$$

$$\mathcal{T} e^{-i\int_0^{t_f} H_{CD}(t)\, ds} |n[0]\rangle = |n(t_f)\rangle = e^{-i\varphi_n(t_f)} e^{-i\gamma_n(t_f)} |n[\lambda(t_f)]\rangle$$