

Extracting quantum field theory dynamics from an approximate ground state

Sophie Mutzel
in collaboration with Antoine Tilloy
arXiv:2512.19594

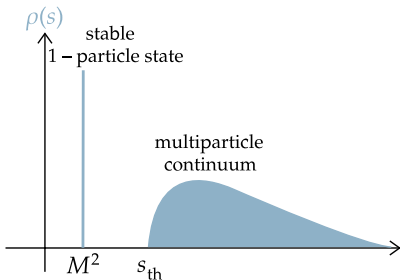
Laboratoire de Physique de l'Ecole Normale Supérieure, Mines Paris - PSL, CNRS,
Inria, PSL Research University, Paris, France

Benasque, February 19, 2026



Introduction

How much dynamical information about a relativistic QFT can we extract from the knowledge of its ground-state $|\Omega\rangle$?



$$H = \int dx \frac{\pi^2}{2} + \frac{(\nabla\phi)^2}{2} + \frac{m^2}{2}\phi^2 + V(\phi)$$

$$|\Omega\rangle \approx \operatorname{argmin}_{|\psi\rangle \in \text{MCH}} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$\rho(s) = Z_\phi \delta(s - M^2) + \rho_{\text{cont}}(s)\theta(s - s_{\text{th}})$$

?

- $\rho(s)$ contains (part of) the spectrum of the theory; in particular, the mass gap $M \equiv E_1 - E_0$

A “simple” relativistic QFT: renormalized ϕ_2^4 theory

$$H = \int dx \underbrace{\frac{:\pi^2:}{2} + \frac{:(\nabla\phi)^2:}{2} + \frac{m^2}{2} : \phi^2 :}_{H_{\text{free}}} + g : \phi^4 :$$

1. Rigorously defined relativistic QFT without cutoff, a **Wightman QFT** (Wightman QFT = Hilbert space + local fields + Lorentz invariance + stable vacuum)
2. Energy density of the interacting vacuum is negative but finite
3. Very hard to solve unless $g \ll m^2$ (perturbation theory)
4. Phase transition around $g_c \simeq 2.7$

Relativistic continuous matrix product states

R

"change in basis"

C

continuum limit

MPS

entanglement limited
ansatz for lattice models

- Use RCMPs to variationally determine ground state of 1+1 dimensional QFTs:

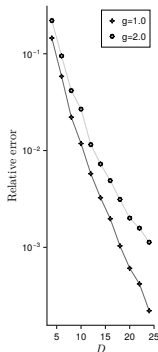
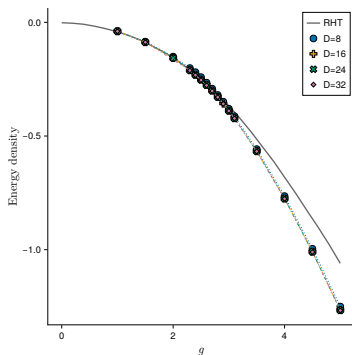
$$H = \int dx \frac{:\pi^2:}{2} + \frac{:(\nabla\phi)^2:}{2} + \frac{m^2}{2} : \phi^2 : + :V(\phi):$$

- parametrized by two complex $D \times D$ dimensional matrices Q, R (variational parameters, analogous to matrices for MPS)
- approximation of ground state of **interacting** theory found by minimizing: $|Q, R\rangle = \operatorname{argmin} \langle Q, R | h | Q, R \rangle \simeq |\Omega\rangle$

Renormalized ϕ_2^4 theory with RCMPS

$$H = \int dx \frac{:\pi^2:}{2} + \frac{:(\nabla\phi)^2:}{2} + \frac{m^2}{2} :\phi^2: + g :\phi^4:$$

Obtain approximation of GS with cost $\propto D^3$, minimum is an **upper bound** of true GS energy density e_0 , $\langle Q, R | h | Q, R \rangle = e_0 + \epsilon$



Tilloy '21, Tiwana '25

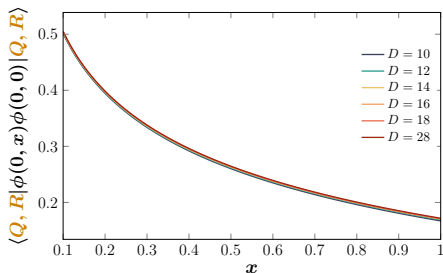
Renormalized ϕ_2^4 theory with RCMPS

$$H = \int dx \frac{:\pi^2:}{2} + \frac{:(\nabla\phi)^2:}{2} + \frac{m^2}{2} : \phi^2 : + g : \phi^4 :$$

Can obtain equal-time **static** n-point correlation functions

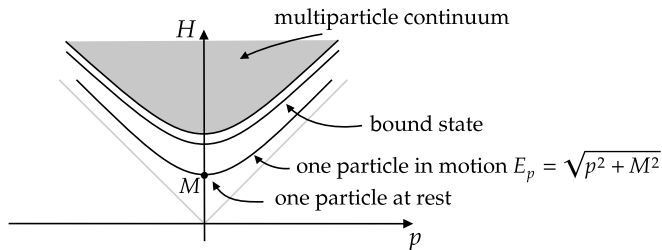
$$\langle Q, R | \phi(0, x_1) \phi(0, x_2) \dots \phi(0, x_n) | Q, R \rangle$$

directly in **continuum and thermodynamic limit** ($\forall x \in \mathbb{R}$) with cost $\propto D^3$

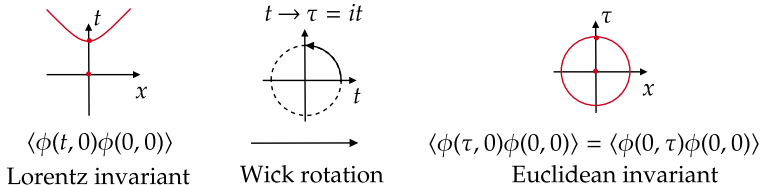


Relativity makes things special

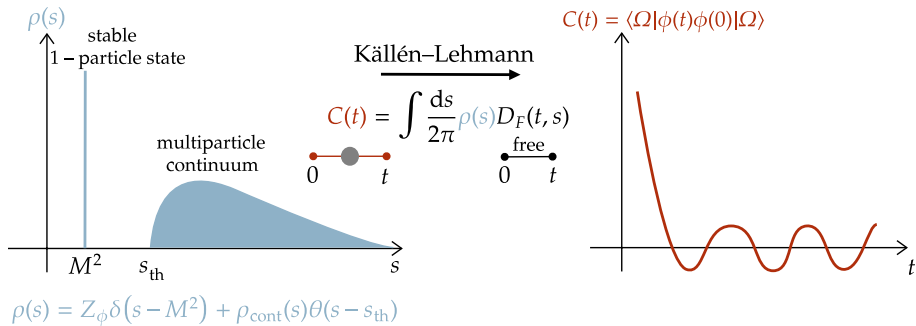
- Once we have the energy spectrum of the eigenstates at rest, we can obtain the full spectrum by applying a boost



- After Wick rotating to imaginary time $t \rightarrow \tau = it$ the two-point functions are equal



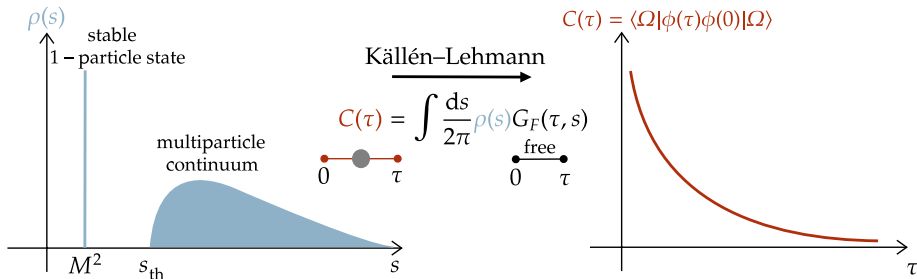
The Källén-Lehmann transform



- $\rho(s)$ contains (part of) the spectrum of the theory; in particular, the mass gap $M \equiv E_1 - E_0$
- Lorentz/Euclidean invariance: $\langle \phi(\tau, 0) \phi(0, 0) \rangle = \langle \phi(0, \tau) \phi(0, 0) \rangle$

The Källén-Lehmann transform

$$t \rightarrow \tau = it$$

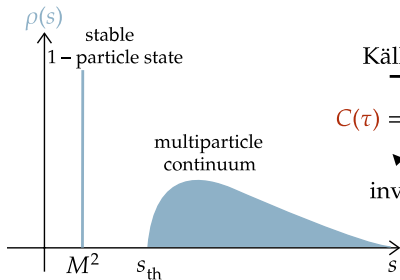


$$\rho(s) = Z_\phi \delta(s - M^2) + \rho_{\text{cont}}(s) \theta(s - s_{\text{th}})$$

- $\rho(s)$ contains (part of) the spectrum of the theory; in particular, the mass gap $M \equiv E_1 - E_0$
- Lorentz/Euclidean invariance: $\langle \phi(\tau, 0) \phi(0, 0) \rangle = \langle \phi(0, \tau) \phi(0, 0) \rangle$

The Källén-Lehmann transform

$$t \rightarrow \tau = it$$



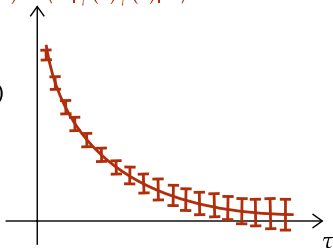
$$\rho(s) = Z_\phi \delta(s - M^2) + \rho_{\text{cont}}(s) \theta(s - s_{\text{th}})$$

Källén-Lehmann

$$C(\tau) = \int \frac{ds}{2\pi} \rho(s) G_F(\tau, s)$$

inverse transform

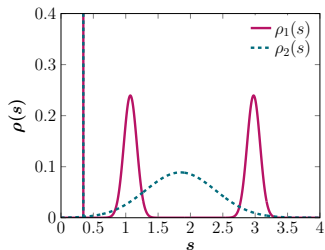
$$C(\tau) = \langle \Omega | \phi(\tau) \phi(0) | \Omega \rangle$$



→ ill – posed problem

- $\rho(s)$ contains (part of) the spectrum of the theory; in particular, the mass gap $M \equiv E_1 - E_0$
- Lorentz/Euclidean invariance: $\langle \phi(\tau, 0) \phi(0, 0) \rangle = \langle \phi(0, \tau) \phi(0, 0) \rangle$

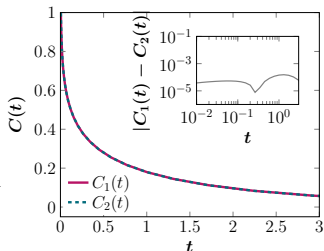
The Källén-Lehmann transform



Källén-Lehmann



inverse transform



→ ill – posed problem

- $\rho(s)$ contains (part of) the spectrum of the theory; in particular, the mass gap $M \equiv E_1 - E_0$
- Lorentz/Euclidean invariance: $\langle \phi(\tau, 0)\phi(0, 0) \rangle = \langle \phi(0, \tau)\phi(0, 0) \rangle$

Formulation as a linear program

- The spectral function is

$$\rho(s) = \sum_{\lambda} (2\pi) \delta(s - s_{\lambda}) \langle \Omega | \phi(0) | \lambda \rangle^2 ,$$

with $|\lambda\rangle$ **physical states**

- $\langle \Omega | \phi(0) | \lambda \rangle^2$ probabilities and therefore

$$\rho(s) \geq 0 \quad \forall s \geq 0$$

→ Space of physically allowed spectral functions is **convex**

Formulation as a linear program

Define smeared spectral function centered at μ^2

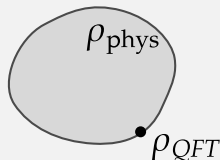
$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

$$\max_{\rho} / \min_{\rho} \quad \rho^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho(s) \geq 0 \quad \forall s \geq 0$$

$$\cdot \int ds \rho(s) = 1$$

$$\cdot C(x)_{Q,R} = \int_{\mathbb{R}^+} ds \rho(s) G_F(s, x) \quad \forall x \in \mathbb{R}$$



Formulation as a linear program

Define smeared spectral function centered at μ^2

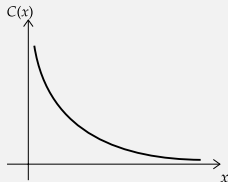
$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

$$\max_{\rho} / \min_{\rho} \quad \rho^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho(s) \geq 0 \quad \forall s \geq 0$$

$$\cdot \int ds \rho(s) = 1$$

$$\cdot C(x)_{\mathbb{Q},\mathbb{R}} = \int_{\mathbb{R}^+} ds \rho(s) G_F(s, x) \quad \forall x \in \mathbb{R}$$



Formulation as a linear program

Define smeared spectral function centered at μ^2

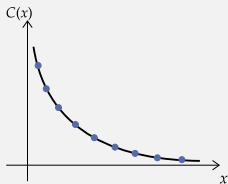
$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

$$\max_{\rho} / \min_{\rho} \quad \rho^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho(s) \geq 0 \quad \forall s \geq 0$$

$$\cdot \int ds \rho(s) = 1$$

$$\cdot C(x_i)_{Q,R} = \int_{\mathbb{R}^+} ds \rho(s) G_F(s, x_i) \quad \text{for } i = 1 \dots N_C$$



- Restricting to **discrete points** is a true **relaxation**

Formulation as a linear program

Define smeared spectral function centered at μ^2

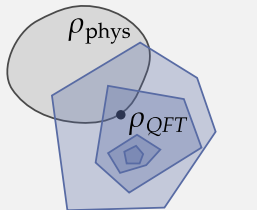
$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

$$\max_{\rho} / \min_{\rho} \quad \rho^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho(s) \geq 0 \quad \forall s \geq 0$$

$$\cdot \int ds \rho(s) = 1$$

$$\cdot C(x_i)_{Q,R} = \int_{\mathbb{R}^+} ds \rho(s) G_F(s, x_i) \quad \text{for } i = 1 \dots N_c$$



- Restricting to **discrete points** is a true **relaxation**

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

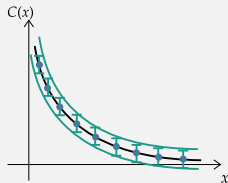
$$\max_{\rho} / \min_{\rho} \quad \rho^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho(s) \geq 0 \quad \forall s \geq 0$$

$$\cdot \int ds \rho(s) = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \int_{\mathbb{R}^+} ds \rho(s) G_F(s, x_i) \leq C(x_i)_{Q,R} + \delta C$$

$$\text{for } i = 1 \dots N_C$$



- Restricting to **discrete points** is a true **relaxation**
- $C_{Q,R}$ has systematic error: need to allow a certain **slack** on correlator

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

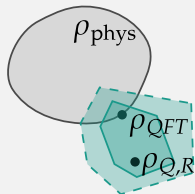
$$\max_{\rho} / \min_{\rho} \quad \rho^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho(s) \geq 0 \quad \forall s \geq 0$$

$$\cdot \int ds \rho(s) = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \int_{\mathbb{R}^+} ds \rho(s) G_F(s, x_i) \leq C(x_i)_{Q,R} + \delta C$$

$$\text{for } i = 1 \dots N_c$$



- Restricting to **discrete points** is a true **relaxation**
- $C_{Q,R}$ has systematic error: need to allow a certain **slack** on correlator

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbf{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

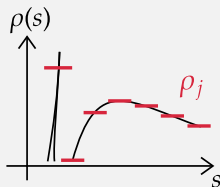
$$\max_{\rho_j} / \min_{\rho_j} \quad \rho_j^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho_j \geq 0 \quad \text{for } j = 1, \dots, N_v$$

$$\cdot \sum_j \rho_j \Delta_j = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \sum_j \Delta_j \rho_j G_F(s_j, x_i) \leq C(x_i)_{Q,R} + \delta C$$

$$\text{for } i = 1 \dots N_c$$



- Restricting to **discrete points** is a true **relaxation**
- $C_{Q,R}$ has systematic error: need to allow a certain **slack** on correlator
- Discretising in s is an **approximation**,

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbf{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

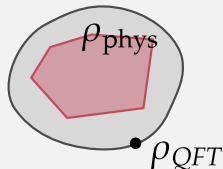
$$\max_{\rho_j} / \min_{\rho_j} \quad \rho_j^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho_j \geq 0 \quad \text{for } j = 1, \dots, N_v$$

$$\cdot \sum_j \rho_j \Delta_j = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \sum_j \Delta_j \rho_j G_F(s_j, x_i) \leq C(x_i)_{Q,R} + \delta C$$

$$\text{for } i = 1 \dots N_c$$



- Restricting to **discrete points** is a true **relaxation**
- $C_{Q,R}$ has systematic error: need to allow a certain **slack** on correlator
- Discretising in s is an **approximation**,

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbf{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

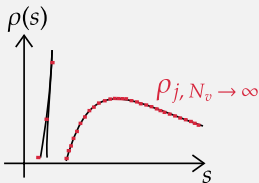
$$\max_{\rho_j} / \min_{\rho_j} \quad \rho_j^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho_j \geq 0 \quad \text{for } j = 1, \dots, N_V$$

$$\cdot \sum_j \rho_j \Delta_j = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \sum_j \Delta_j \rho_j G_F(s_j, x_i) \leq C(x_i)_{Q,R} + \delta C$$

$$\text{for } i = 1 \dots N_C$$



- Restricting to **discrete points** is a true **relaxation**
- $C_{Q,R}$ has systematic error: need to allow a certain **slack** on correlator
- Discretising in s is an **approximation**, choose $N_V \sim 10^4$

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbf{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

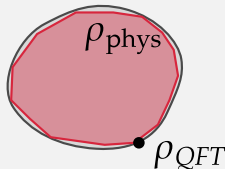
$$\max_{\rho_j} / \min_{\rho_j} \quad \rho_j^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho_j \geq 0 \quad \text{for } j = 1, \dots, N_v$$

$$\cdot \sum_j \rho_j \Delta_j = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \sum_j \Delta_j \rho_j G_F(s_j, x_i) \leq C(x_i)_{Q,R} + \delta C$$

$$\text{for } i = 1 \dots N_c$$



- Restricting to **discrete points** is a true **relaxation**
- $C_{Q,R}$ has systematic error: need to allow a certain **slack** on correlator
- Discretising in s is an **approximation**, choose $N_v \sim 10^4$

Formulation as a linear program

Define smeared spectral function centered at μ^2

$$\rho^\sigma(\mu^2) \equiv \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbf{R}^+} \rho(s) e^{-(\mu^2-s)^2/(2\sigma^2)} ds$$

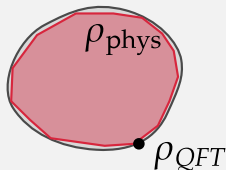
$$\max_{\rho_j} / \min_{\rho_j} \quad \rho_j^\sigma(\mu^2)$$

$$\text{under} \quad \cdot \rho_j \geq 0 \quad \text{for } j = 1, \dots, N_v$$

$$\cdot \sum_j \rho_j \Delta_j = 1$$

$$\cdot C(x_i)_{Q,R} - \delta C \leq \sum_j \Delta_j \rho_j G_F(s_j, x_i) \leq C(x_i)_{Q,R} + \delta C$$

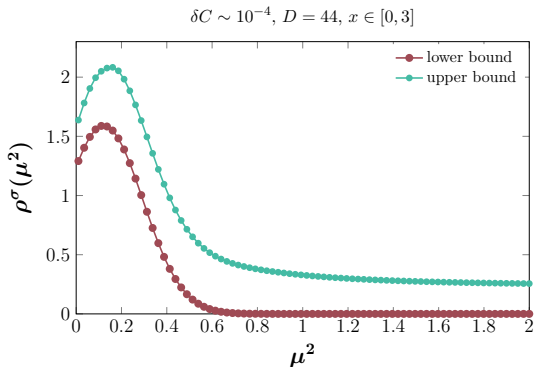
$$\text{for } i = 1 \dots N_c$$



Can use convex optimization tools to obtain rigorous **upper/lower bounds** on spectral density

Lawrence '24

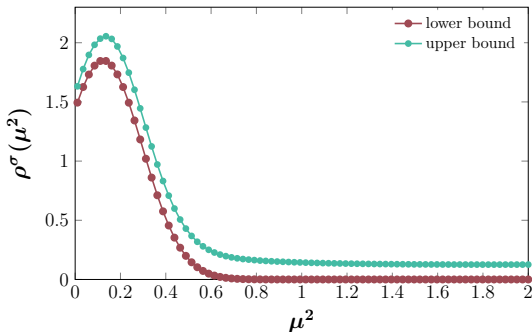
Bootstrapping smeared spectral functions



- Solve linear program for a few μ^2 gives upper/lower bound on smeared spectral density centered at μ^2
- Increase number of points x_i (=number of constraints N_c) until result does not change
- reduce δC until upper and lower bounds coincide

Bootstrapping smeared spectral functions

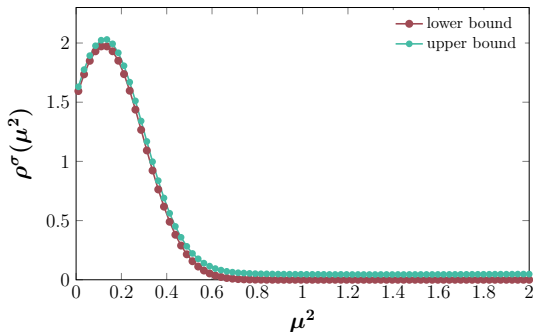
$$\delta C \sim 5 \cdot 10^{-5}, D = 44, x \in [0, 3]$$



- Solve linear program for a few μ^2 gives upper/lower bound on smeared spectral density centered at μ^2
- Increase number of points x_i (=number of constraints N_c) until result does not change
- reduce δC until upper and lower bounds coincide

Bootstrapping smeared spectral functions

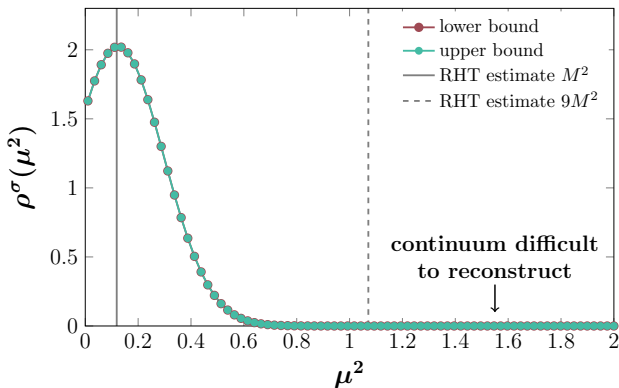
$$\delta C \sim 2 \cdot 10^{-5}, D = 44, x \in [0, 3]$$



- Solve linear program for a few μ^2 gives upper/lower bound on smeared spectral density centered at μ^2
- Increase number of points x_i (=number of constraints N_c) until result does not change
- reduce δC until upper and lower bounds coincide

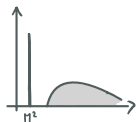
Bootstrapping smeared spectral functions

$$D = 44, x \in [0, 3], \delta C_{\text{out}} \sim 10^{-5}$$



- Perfectly reconstruct the Dirac without any prior theoretical input
- **Output:** *a-posteriori* estimate (lower bound) of systematic error of $C_{Q,R}^D(x)$

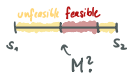
Bootstrapping the mass gap



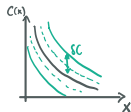
We know from theory:

$$\rho(s) = Z_\phi \delta(s - M^2) + \tilde{\rho}(s) \theta(s - s_{\text{th}})$$

→ optimize over Z_ϕ and $\tilde{\rho}(s)$ instead



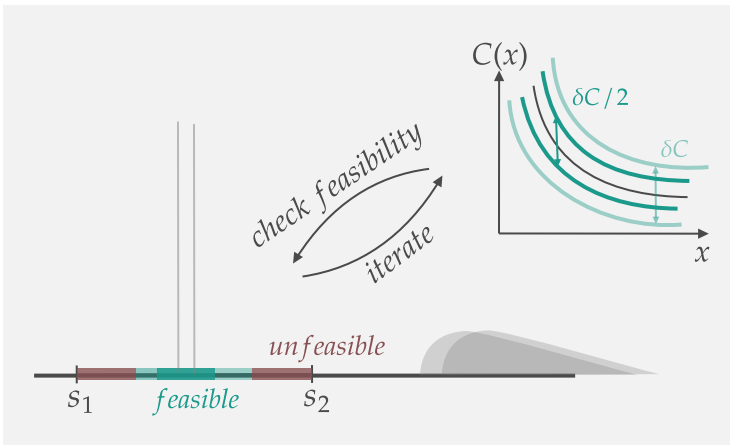
Check for which $M^2 \in [s_1, s_2]$ problem is **feasible**



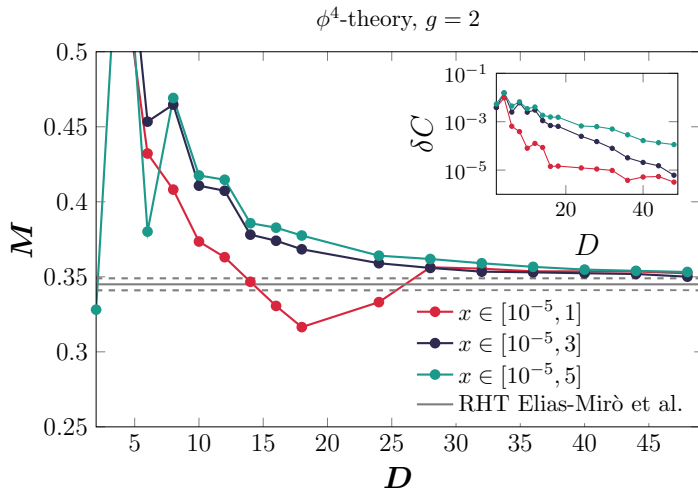
Decrease slack and reiterate

→ find allowed interval via bisection

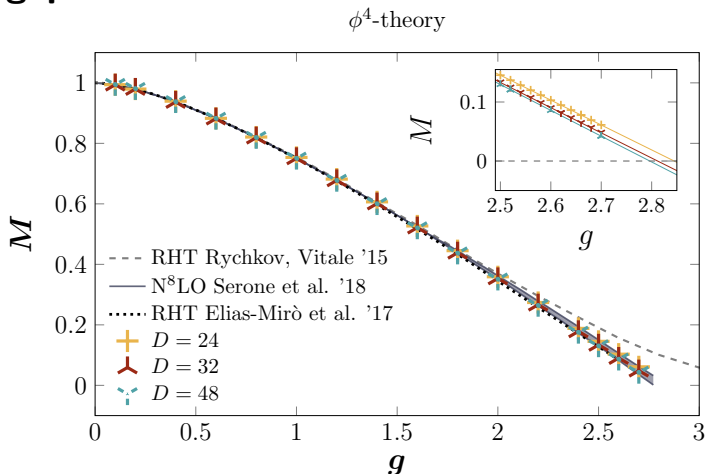
Bootstrapping the mass gap



Bootstrapping the mass gap



Mass gap



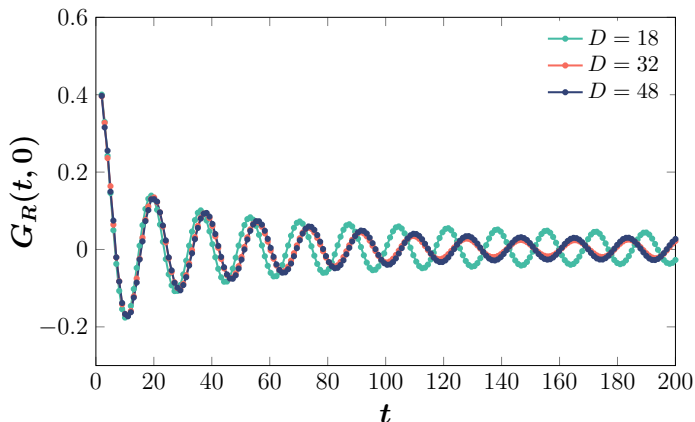
- Good agreement with renormalized HT and perturbation theory
- Extrapolation $M \sim c|g - g_c|$ gives precise estimate of critical point $g_c = 2.796$ for $D = 48$ (at $D = 32$, we get $g_c = 2.812$)

Real-time evolution

Replace objective function by

$$\max_{\rho} / \min_{\rho} \quad G_R(\rho; t, 0) = \frac{1}{2} \int_0^{\infty} ds \rho(s) J_0(\sqrt{s}t)$$

ϕ^4 -theory, $g = 2$



Conclusion

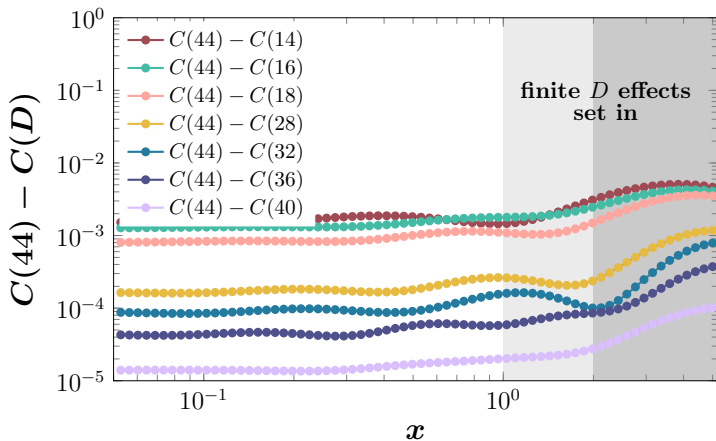
- New method to extract dynamical information from the GS of relativistic 1+1-d QFTs directly in the continuum and thermodynamic limit
- Method consists in turning the inversion problem into a linear program Lawrence '24
- Bootstrap approach gives us in return an a-posteriori estimate of the systematic error on our RCMPS correlation functions
- Estimates for mass-gap in ϕ^4 theory agree well with the ones from renormalized Hamiltonian truncation and perturbation theory

Future work

- Dual formulation directly in the continuum (without discretization of ρ)
 - gives fully rigorous characterization of spectral bounds
 - if unfeasible for $\delta C = 0$: (computer-assisted) proof that RCMPs do not have a local, relativistic parent Hamiltonian?
- Study excitations in \mathbb{Z}_2 symmetry-broken phase
- Finite entanglement scaling for estimate of g_c

Backup Slides

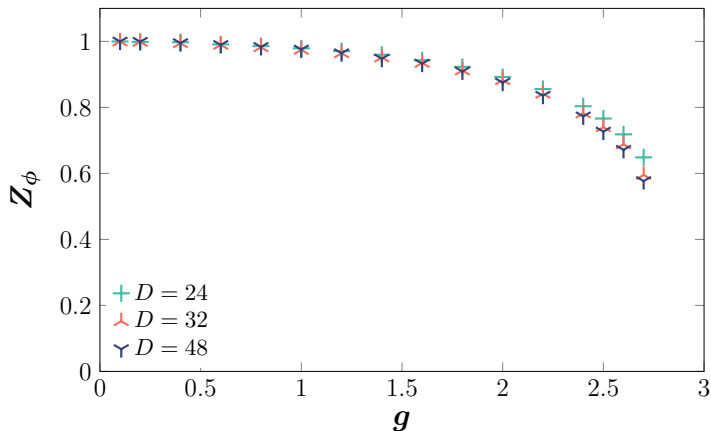
Slack on correlator



- Error of $C_{Q,R}(x)$ is more or less constant as a function of x
- Around $x_{\max} \sim 1 - 2$ error starts increasing
- Choose $x_i \in [0, \text{a few } M^{-1}]$

Field-renormalization constant

ϕ^4 -theory



Get non-perturbative estimate for field-renormalization constant

$$\rho(s) = Z_\phi \delta(s - M^2) + \tilde{\rho}(s) \theta(s - s_{\text{th}})$$