Projecting the MPS to be “exact”

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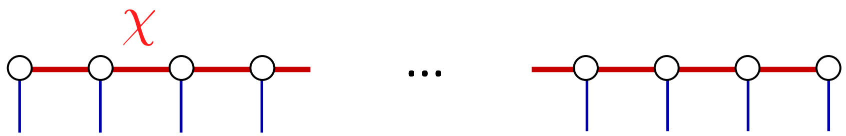
Kitaev-honeycomb paper: Just submitted (2026)

Phys. Rev. Res.7 , 013086 (2025)

Phys. Rev. Lett. 132, 166701 (2024)

MPS/TN as variational ansatz

- What was nice ?

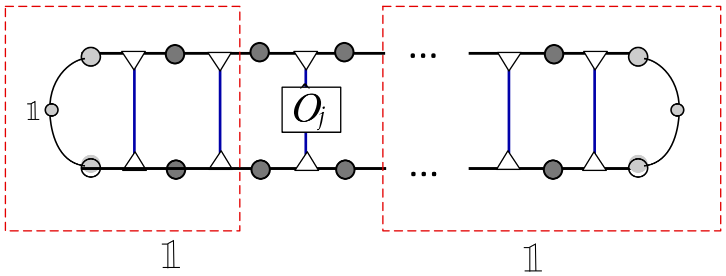


Canonical form:

Ability to compress the info efficiently. "minimally entangled state"

MPO's are evaluated locally due to its isometric nature.

$\langle \Psi | \hat{O}_j | \Psi \rangle =$



Translational invariance is realized: iTEBD, iPEPS.

- What was the bound ?

MPS has constant (Area law) bound.

$$S_A = -\sum_l \lambda_l^2 \ln \lambda_l^2 \leq \ln \chi_{max} = \text{const}$$

Go beyond Area law: thermal pure state

Iwaki-Shimizu-Hotta, PRR **3**,L022015(2021)

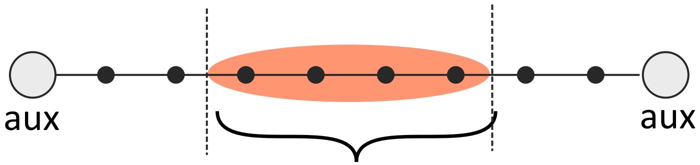
Iwaki-Hotta, PRB **106**, 094409 (2022)

TPQ-MPS

finite temperature pure quantum state

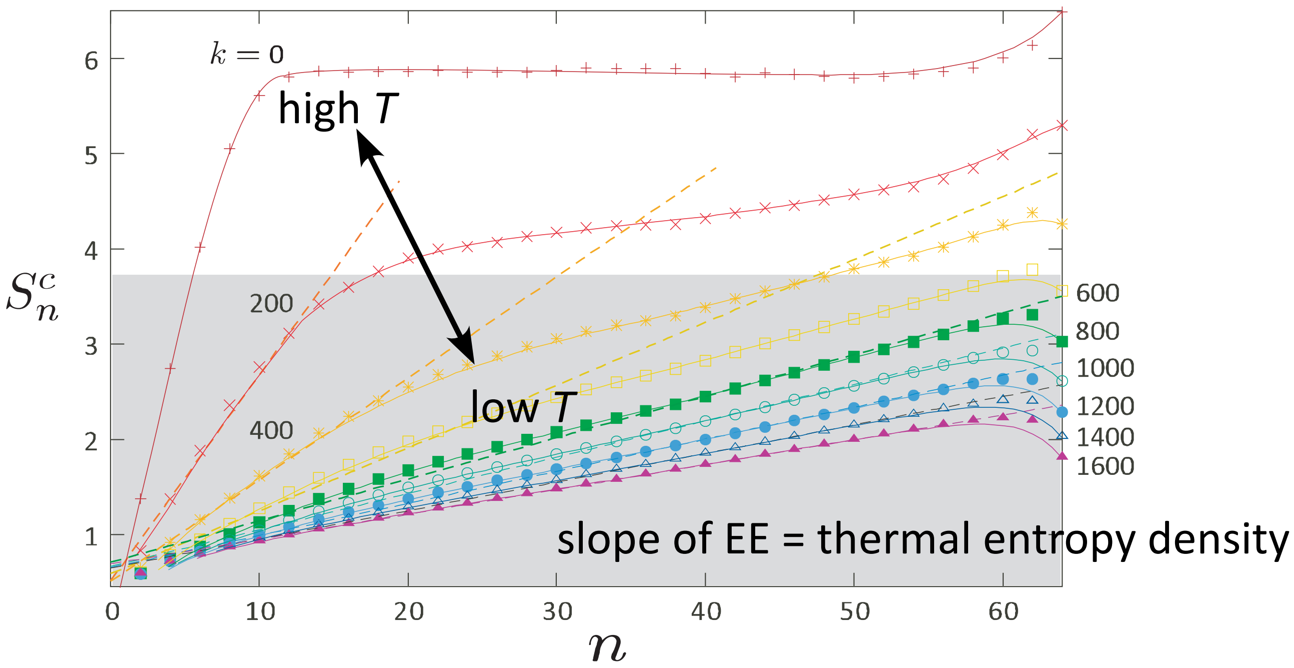
MPS + two auxiliaries on both edges.

$$\left|\psi_\beta\right\rangle = e^{-\beta\hat{H}/2}\left|\psi_0\right\rangle$$



take n sites at the center

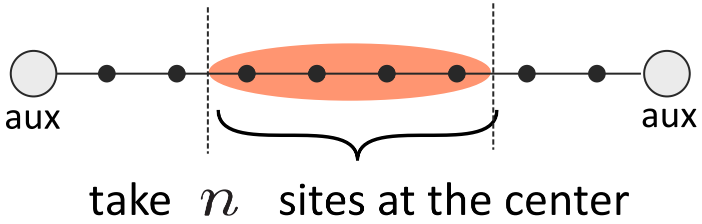
Uniform bipartite EE (prev slide)
will guarantee an entanglement volume law.

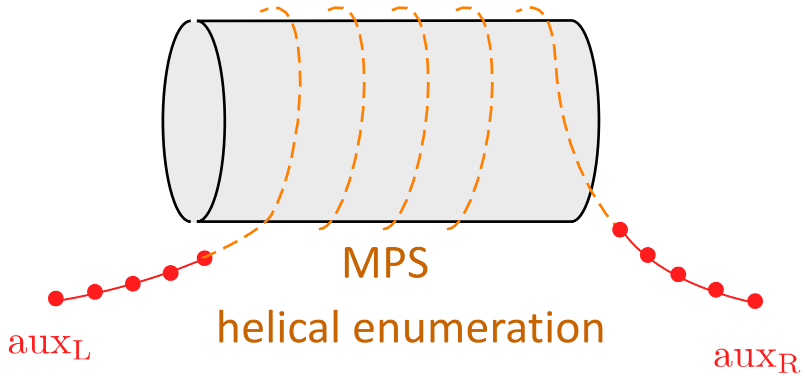


Go beyond Area law: thermal pure state

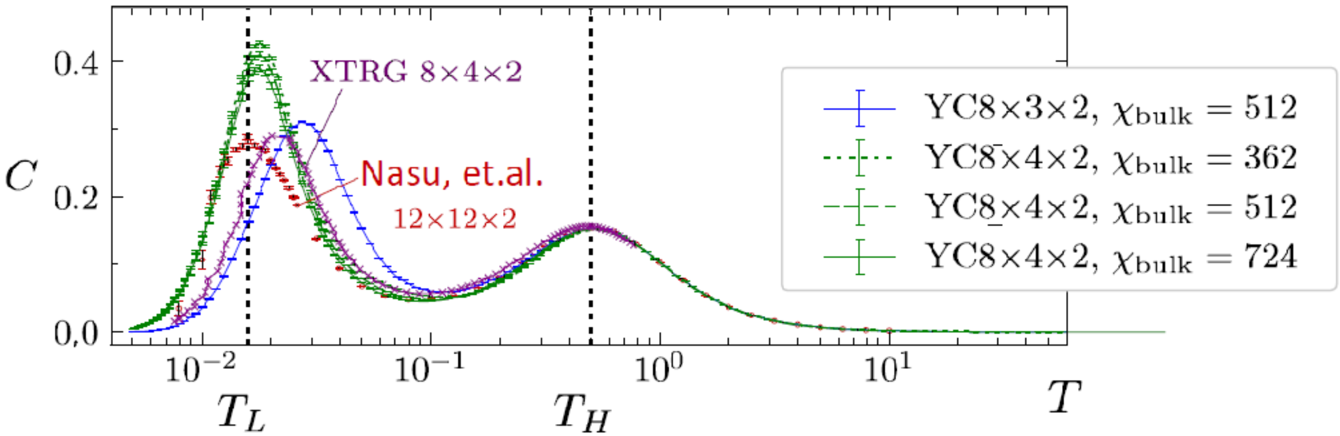
TPQ-MPS finite temperature pure quantum state

MPS + two auxiliaries on both edges.



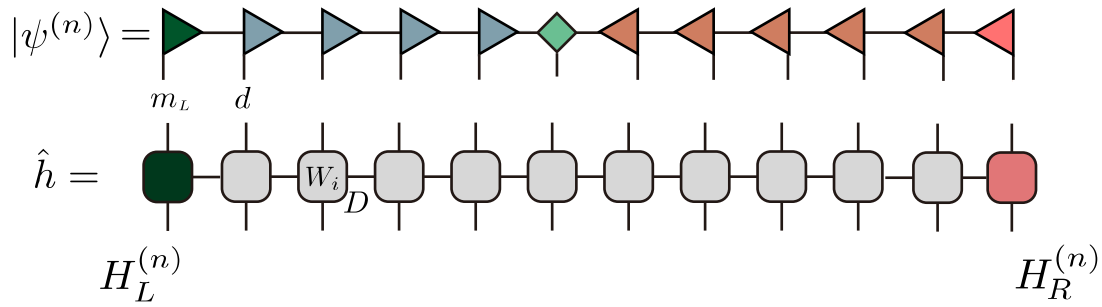


Kitaev honeycomb spin liquid:



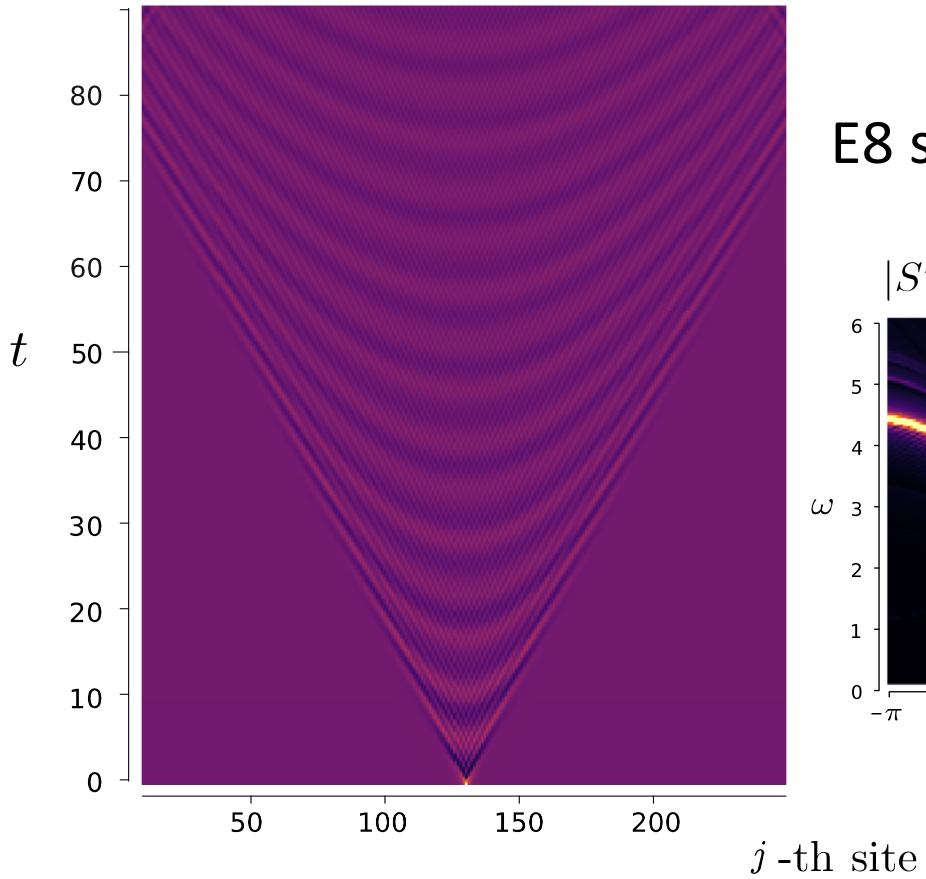
Gohlke, Iwaki, Hotta, SciPost Physics 15, 206 (2023)

Go beyond Area law: long-time dynamics

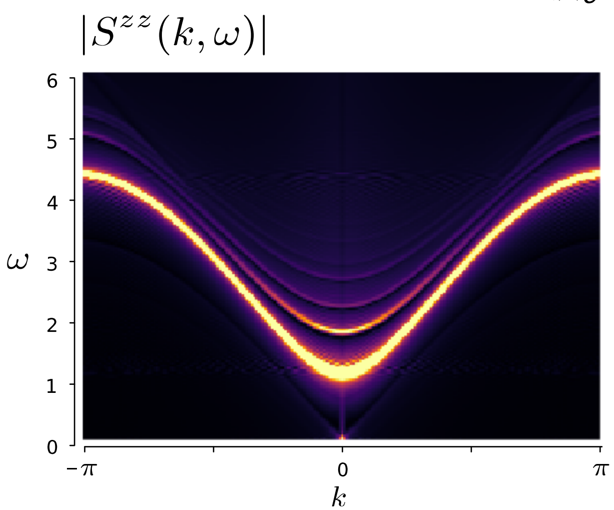


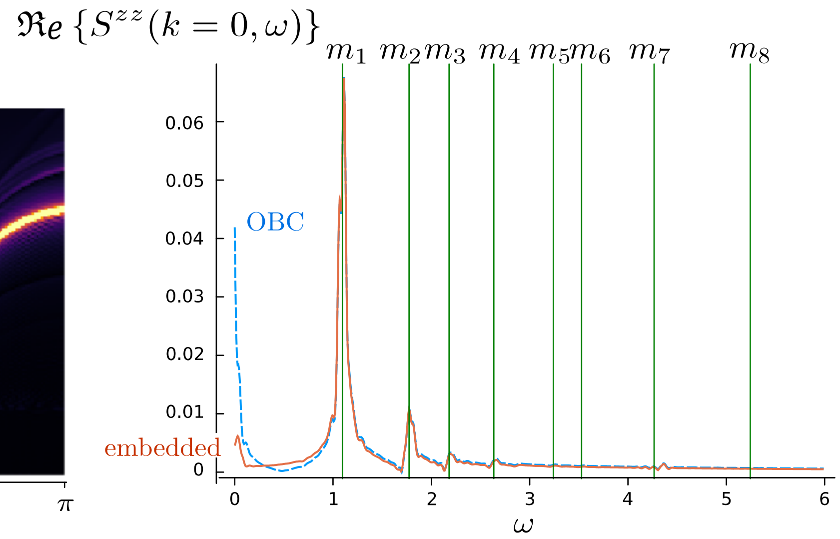
Shimozono-Hotta, arXiv:2512.07923

Environment MPO + MPS



E8 spectrum of transverse+longitudinal field Ising





Success in MPS

- Handy building blocks of DMRG. S. White (1992)

Take lowest energy wave function , do SVD, and keep only large Schmidt states.

- Capability of representing infinite-system using translational invariance (TI).

e.g. extract symmetries,
prove that for all MPS there is a parent Hamiltonian having exact sols.

- Time evolutions, TEBD, iTEBD, TDVP. G. Vidal (2007-)

Haegeman, .. Verstraete(2011)

We are promoting

- Finite T & dynamical state (volume law) : TPQ-MPS with edge auxiliaries
- Exact solutions using MPS with local projectors (new algorithm).

Exact MPS(TN) solutions for

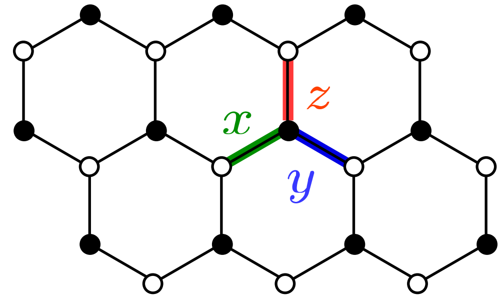
topic 1

Kitaev-honeycomb QSL ground state & all excited states

topic 2

general (all possible) frustration-free models

Kitaev honeycomb model



$$\mathcal{H} = -\sum_{\langle jk\rangle^\gamma} J_\gamma \hat{\sigma}_j^\gamma \hat{\sigma}_k^\gamma,$$

Physical Hibert space

quantum manybody $|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle$

up/down spins

$|\uparrow\rangle, |\downarrow\rangle$

Gutzwiller projection

$D_i = 1$

$|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle$

Four Majorana fermions $\{\ \hat{b}_i^x, \hat{b}_i^y, \hat{b}_i^z, \hat{c}_i\ \}$

$$\hat{\sigma}_i^\alpha = i\hat{b}_i^\alpha \hat{c}_i$$

extended Hibert space

$$\hat{u}_{jk} = i\hat{b}_j^\gamma \hat{b}_k^\gamma$$

bond gauge (link) operator

$$\hat{D}_i = \hat{b}_i^x \hat{b}_i^y \hat{b}_i^z \hat{c}_i = \pm 1$$

$$\mathcal{H}_M = i\sum_{\langle jk\rangle^\gamma} J_\gamma \hat{u}_{jk} \hat{c}_j \hat{c}_k = \sum_{\ell=1}^{N/2} \epsilon_\ell (\hat{a}_\ell^\dagger \hat{a}_\ell - 1/2)$$

$\hat{n}_\ell$

$|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$ free Majorana eigenstate

Quantum numbers that specify the solution

$$\{u_{jk}\} = \pm 1$$

$$\{\ n_\ell\ \} = 0 \text{ or } 1$$

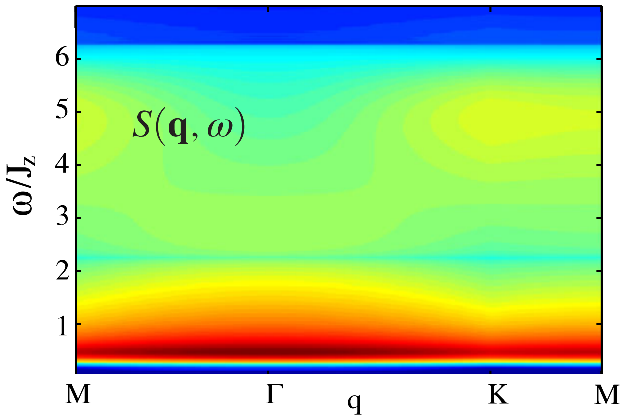
$\{u_{jk}\} = \pm 1$ diagonalize

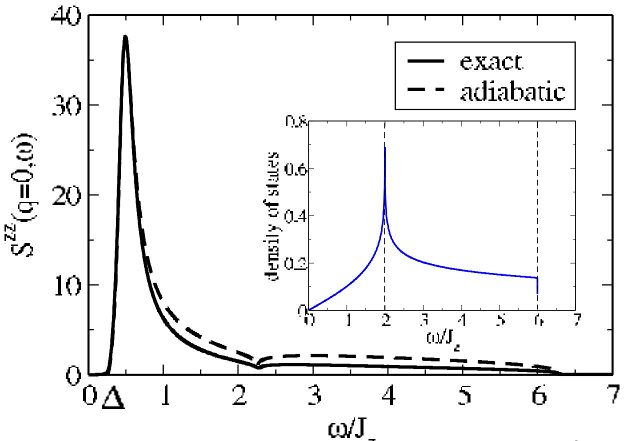
$$\hat{a}_\ell = \frac{1}{\sqrt{2}} \sum_{j=1}^{N} Q_{j\ell}\ \hat{c}_j$$

Kitaev, Ann. Phys. **321**,2 (2006)

2026/2/18 Benasque Spain

Kitaev honeycomb model

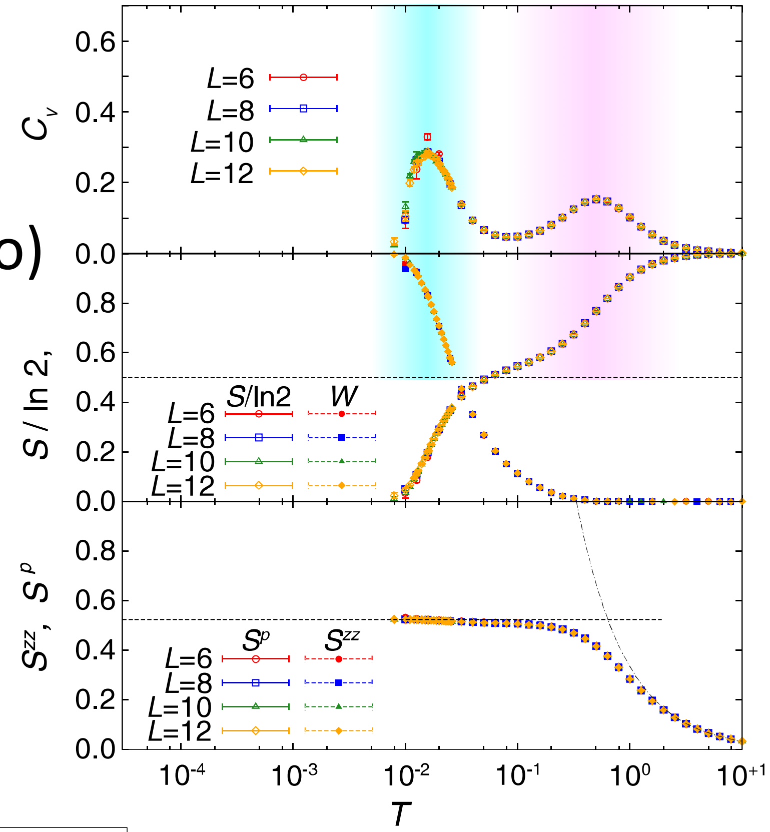




Knolle, et.al. PRL **112**, 207203 (2014)

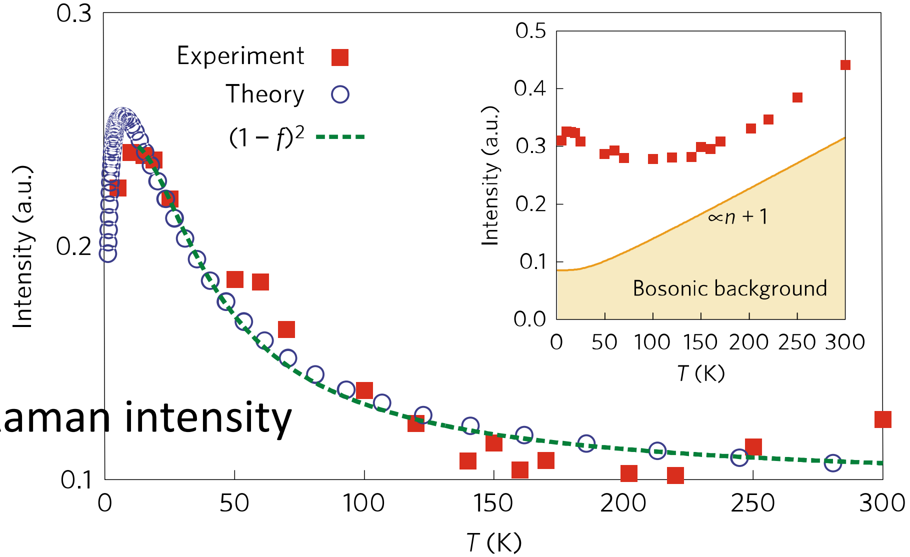
Z_2 gauge (classical Monte Carlo)
+ free Majorana

Noninteracting framework
was enough for
thermodynamic quantities.



Nasu, Udagawa, Motome
PRB **92**, 115122 (2015)

α -$RuCl_3$



integrated Raman intensity

Nasu, Knolle, *et al.*
N. Phys. 12,912 (2016)

Kitaev honeycomb model

It is solved... In what sense?

If the thermodynamic quantities are available, and if we could get a nice (but rough) cartoon picture on the degrees of freedom.

How can it form a QSL ?

Q. Do you recognize that the QSL appears ONLY AFTER the Gutzwiller projection?

$$|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle \propto \mathcal{P}_D|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

Physical state Z2+Majorana

Q. Did you know how the entanglement entropy can achieve topological term ?

H. Yao, S.-C. Zhang, S. A. Kivelson, PRL **102**, 217202 (2009)
H. Yao, X.-L. Qi, PRL **105**, 080501 (2010)
F. L. Pedrocchi, S. Chesi, D. Loss, PRB **84**, 165414(2011)

Exact quantum many-body solution can give an answer.

$$|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle \propto \mathcal{P}_D|\tilde{\Psi}^{M}_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

Physical state

Z_2+Majorana

Project MPS to form the exact solution

Projections to have exact QSL solution

extended Hibert space 4^N

represent
full Majorana Hilbert space

$$|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle = \mathcal{P}_a\mathcal{P}_u|\tilde{\Psi}_{\mathrm{random}}\rangle$$

Projection onto Quantum number sector

$$\{u_{jk}\} = \pm 1$$

$$\{\, n_\ell \,\} = 0 \text{ or } 1$$

$$\mathcal{P}_u = \prod_{\langle jk\rangle^\gamma} \frac{1 \pm \hat{u}_{jk}}{2}$$ Z_2 bond gauge

fermionic #

$$\mathcal{P}_a = \prod_{\ell=1}^{N/2} \hat{q}_\ell, \qquad \hat{q}_\ell = \begin{cases} \hat{a}^\dagger_\ell \hat{a}_\ell & (n_\ell = 1) \\ \hat{a}_\ell \hat{a}^\dagger_\ell & (n_\ell = 0) \end{cases}$$

Gutzwiller projection

$$|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle \propto \mathcal{P}_D|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

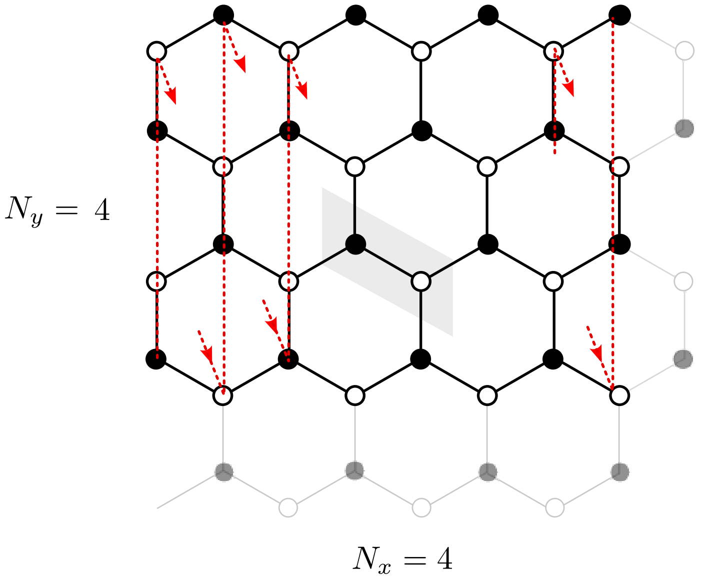
$$\mathcal{P}_D = \prod_{i=1} \frac{1 + \hat{D}_i}{2}$$

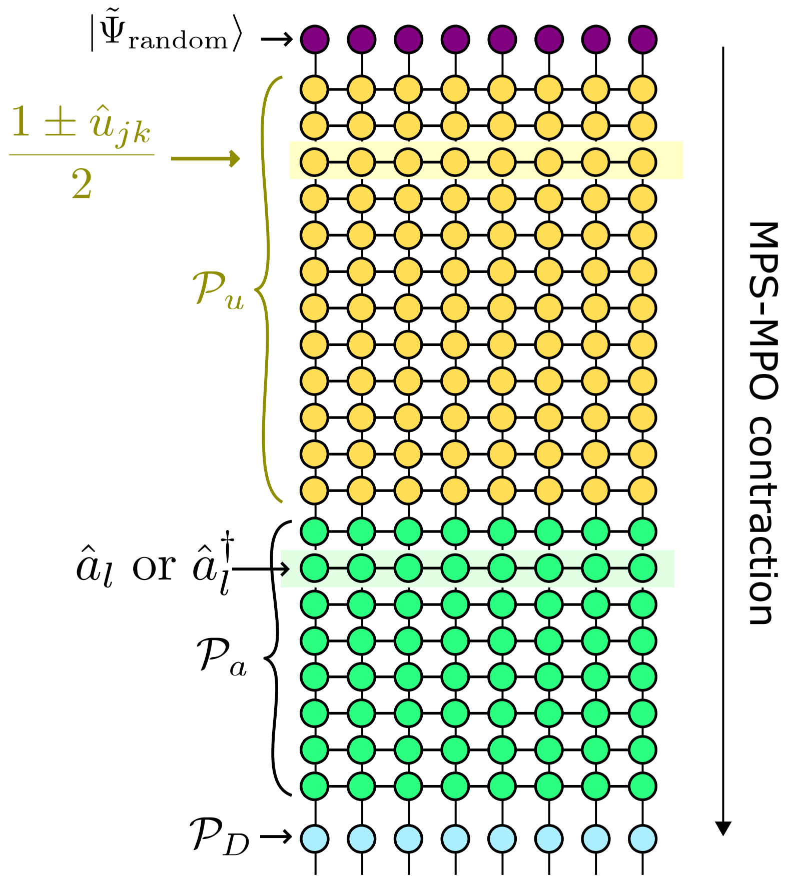
We want $D_i = 1$

$$\hat{D}_i = \hat{b}^x_i \hat{b}^y_i \hat{b}^z_i \hat{c}_i = \pm 1$$

physical Hilbert space 2^N

Exact 2D tensor network solution





Key :

Ground state & excited states, same cost.

Initial random product state does a good job.

Projectors are factorized into D=2 local MPOs.

Previous TN solutions

exact 3D TN

Schmoll & Orus

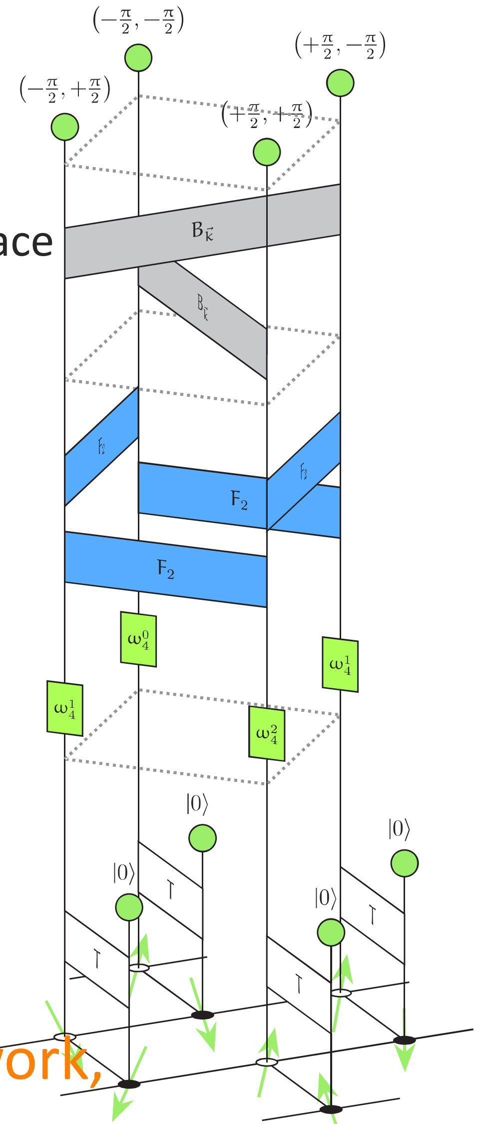
PRB **95**, 045112 (2017)

k space & Bogoliubov space

Bogoliubov tr

Fourier tr

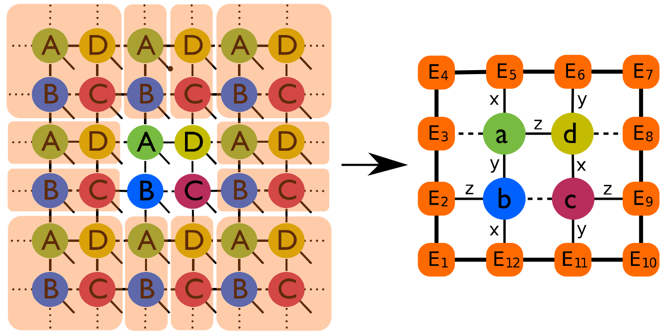
Majorana braiding



Complex 3D network, nonlocal operations

2D approximate TN

Iregui, Corboz, Troyer, PRB **90**, 195102 (2014)

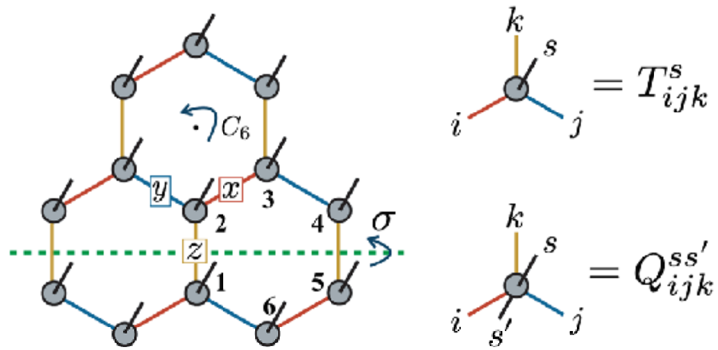


iPEPS

$(E - E_{exact})/N = 10^{-3}$

Lee, Kaneko, Okubo, Kawashima, PRL **123**, 087203 (2019)

Prepare a symmetric tensor and optimize



$(E - E_{exact})/N = 10^{-4}$

Jin, Tu, Zhou, PRB **104**, L020409 (2021) DMRG+projector

Variational, only for ground states

What we can do with our exact TN ?

“numerical experiment"
giving us a clue to understand
quantum many-body physics

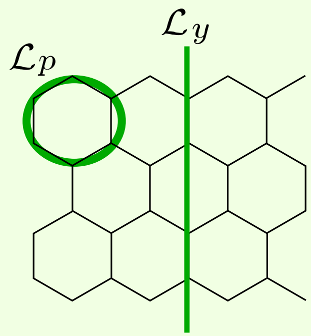
What we can do with our TN: braiding

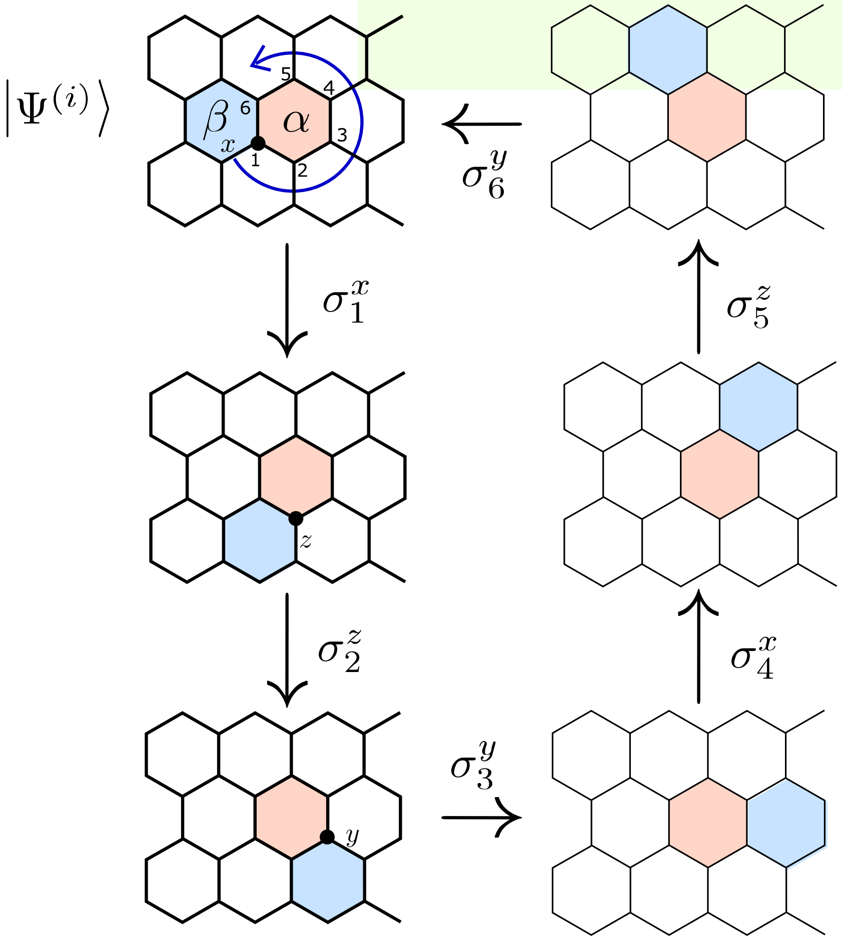
$$\{w_p, w_x, w_y\} = \{1, 1, 1\}$$

$$w_\alpha = w_\beta = -1$$

loop operators

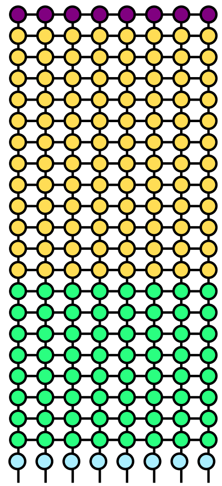
$$\hat{w}_\gamma = \prod_{j \in \mathcal{L}_\alpha} \sigma_j^{\alpha_j}$$

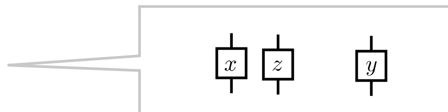




$$|\Psi^{(f)}\rangle = \sigma_6^y \sigma_5^z \sigma_4^x \sigma_3^y \sigma_2^z \sigma_1^x |\Psi^{(i)}[\alpha, \beta]\rangle$$

$$\langle \Psi^{(i)} | \Psi^{(f)} \rangle = -1$$



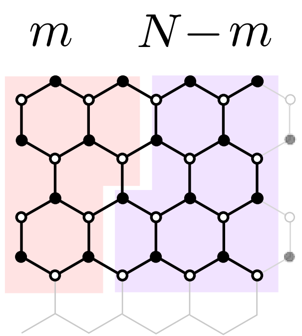


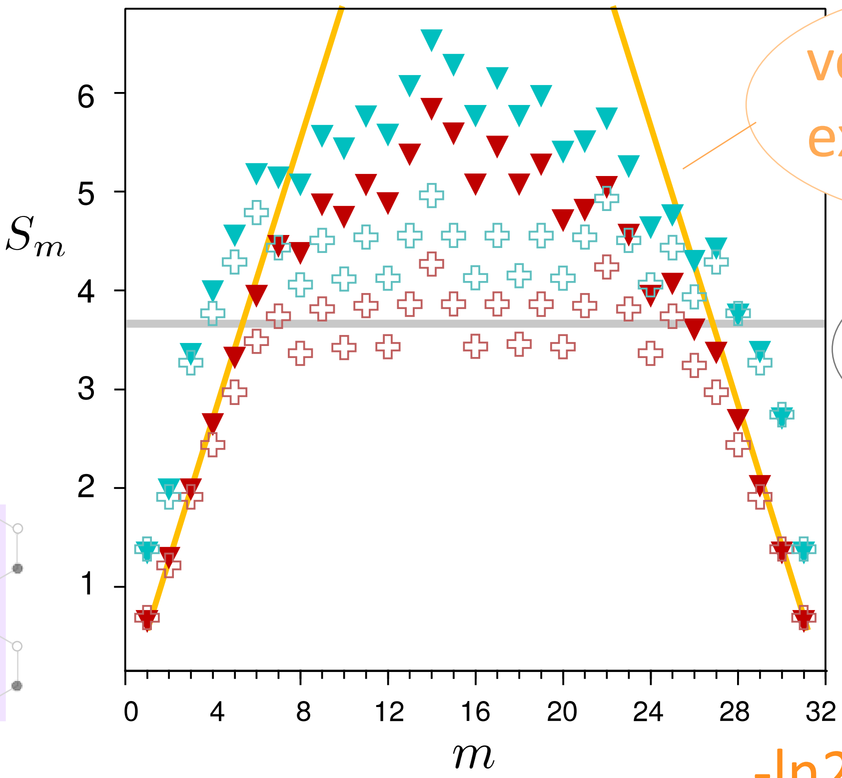
We only need to multiply local (1site) MPOs.

What we can do with our TN: Entanglement

$$|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle \propto \mathcal{P}_D |\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

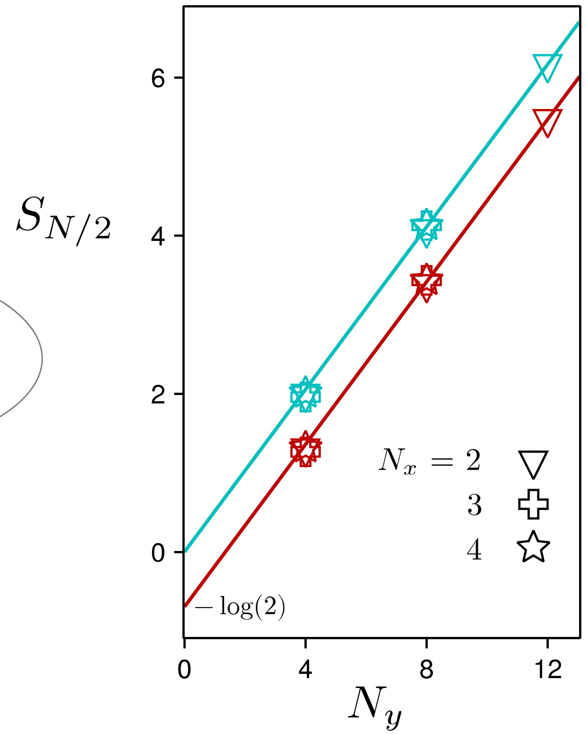
physical state   Fermionic state





volume law excited states

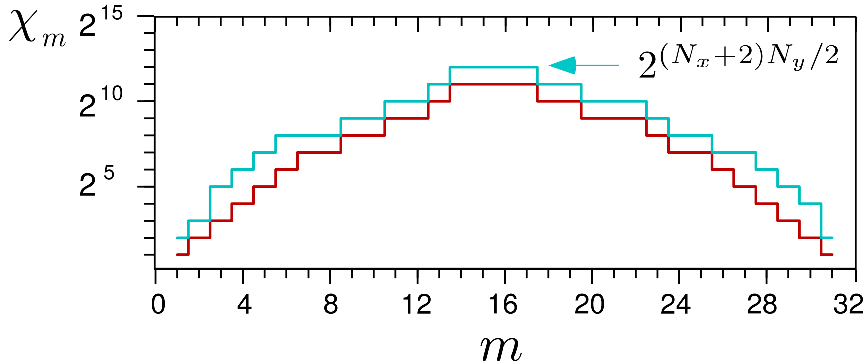
Area-law ground state



-ln2 appears only after Gutzwiller projection

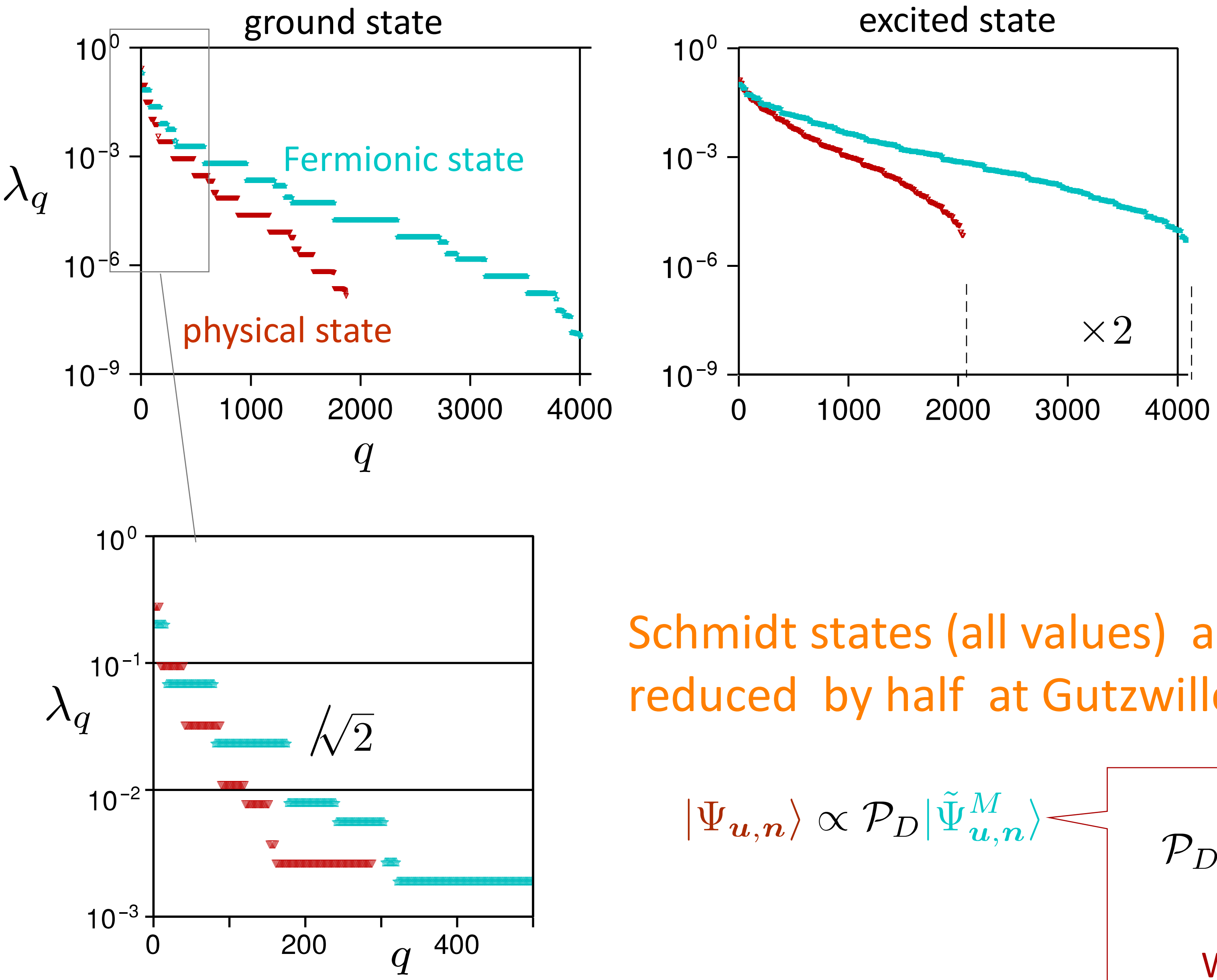
$$S = (\alpha + \ln 2)L - \ln 2$$

c.f. H. Yao, X.-L. Qi, PRL 105, 080501 (2010)



bond dimension the SAME for ground state & excited state

What we can do with our TN: Schmidt states



Schmidt states (all values) are exactly reduced by half at Gutzwiller projection.

$$|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle \propto \mathcal{P}_D|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

$$\mathcal{P}_D = \prod_{i=1} \frac{1+\hat{D}_i}{2}$$

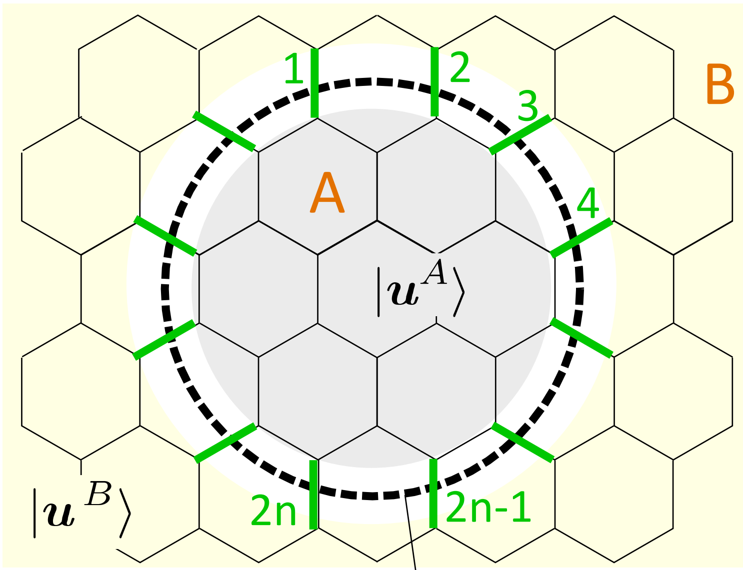
We want $D_i = 1$

What's going on ?

$$|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{v}}\rangle \;=\; |M_{\boldsymbol{u}}\rangle \;\; |\boldsymbol{u}\rangle \qquad = \left(\sum_{\ell=1}^{\chi} \lambda_\ell |M^A_{\boldsymbol{u}\ell}\rangle |M^B_{\boldsymbol{u}\ell}\rangle \right) \;\; |\boldsymbol{u}^A\rangle|\boldsymbol{u}^B\rangle \;\Big|\, \text{boundary gauge} \Big\rangle$$

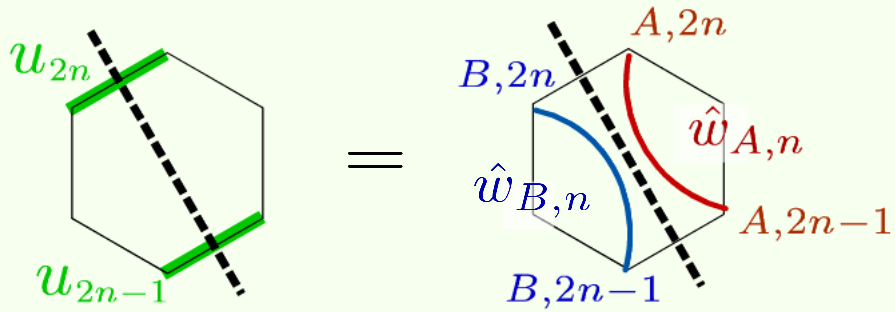
Majorana × bond gauge

$$\{u_{jk}\} \;=\; \pm 1$$



Boundary cuts $2L$ bond gauges

$$\hat{u}_{jk} \;=\; i\hat{b}^\gamma_j \hat{b}^\gamma_k \qquad \hat{w}_{\substack{A,n\\B}} = i\hat{b}^{\alpha}_{\substack{A,2n-1\\B}} \hat{b}^{\beta}_{\substack{A,2n\\B}}$$



$(u_{2n-1},\ u_{2n})$	$\longleftrightarrow$	$(w_{A,n}, w_{B,n})$	
$(\ \ 1,\ \ \ 1)$		$(1,-1)-(-1,1)$	
$(-1,-1)$		$(1,-1)+(-1,1)$	$/\sqrt{2}$
$(\ \ 1,-1)$		$(1,1)+(-1,-1)$	
$(-1,\ \ \ 1)$		$(1,1)-(-1,-1)$	

H. Yao, X.-L. Qi, PRL 105, 080501 (2010)

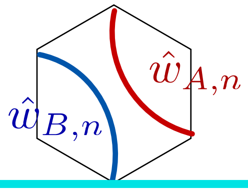
generate 2^L free boundary gauge degrees of freedom

What's going on ?

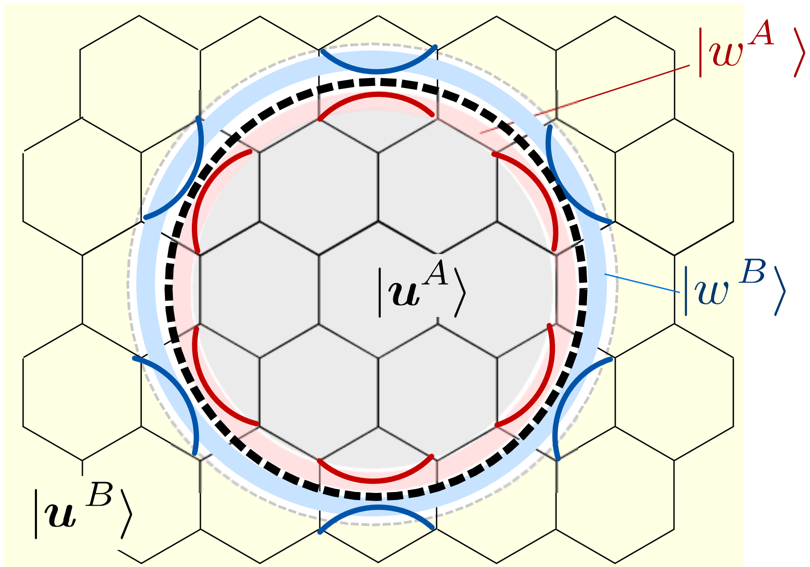
$$|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{v}}\rangle = |M_{\boldsymbol{u}}\rangle \; |\boldsymbol{u}\rangle = \left(\sum_{\ell=1}^{\chi} \lambda_\ell |M^A_{\boldsymbol{u}\ell}\rangle |M^B_{\boldsymbol{u}\ell}\rangle \right) |\boldsymbol{u}^A\rangle|\boldsymbol{u}^B\rangle \sum_{\{w_n=\pm1\}} \prod_{n=1}^{L} |w_{A,n}, w_{B,n}\rangle$$

Majorana × bond gauge

$$= \sum_{\ell=1}^{\chi} \sum_{\{w_n=\pm1\}} \frac{\lambda_\ell \, (-)^w}{\sqrt{2^L}} \left(|M^A_{\boldsymbol{u}\ell}\rangle |\boldsymbol{u}^A\rangle |w^A\rangle \right) \left(|M^B_{\boldsymbol{u}\ell}\rangle |\boldsymbol{u}^B\rangle |w^B\rangle \right)$$



2^L degenerate sets of Schmidt states



Fermionic state is the $D = 1$ eigenstate of $\hat{D} = \prod_j \hat{D}_j$

$$\hat{D}_i = \hat{b}^x_i \hat{b}^y_i \hat{b}^z_i \hat{c}_i = \pm 1$$

$$|\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle \propto \mathcal{P}_D |\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

$$\mathcal{P}_D = \prod_{j=1} \frac{1+\hat{D}_j}{2}$$

Gutzwiller wants

$D_j = 1 \;\; \forall j$

$$\hat{D} = \hat{D}_A \cdot \hat{D}_B = 1$$

$(\;1, \;\; 1\;)$

~~$(-1, -1)$~~

half are discarded.

→ $-\ln 2$

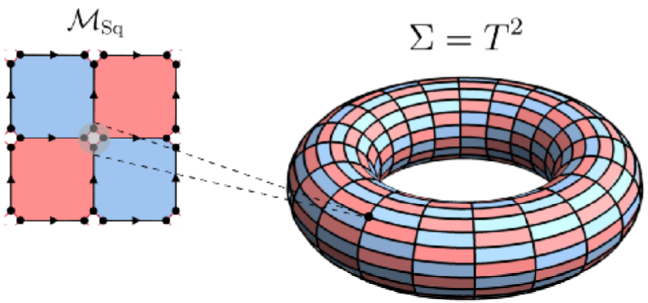
What was known previously

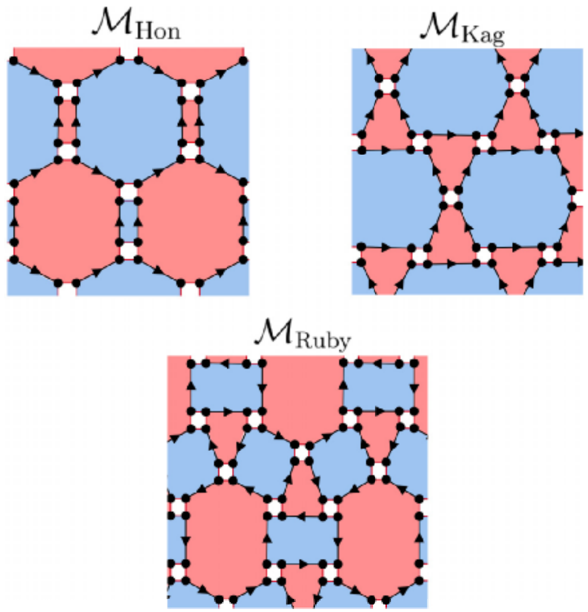
PHYSICAL REVIEW B **112**, 125101 (2025)

Partons from stabilizer codes

Rafael A. Macêdo,[1,2] Carlo C. Bellinati,[3] Weslei B. Fontana,[4] Eric C. Andrade,[3] and Rodrigo G. Pereira[1,2]

The Gutzwiller projection of fermionic wave functions is a well-established method for generating variational wave functions describing exotic states of matter, such as quantum spin liquids. We investigate the conditions under which a projected wave function constructed from fermionic partons can be rigorously shown to possess topological order. We demonstrate that these conditions can be precisely determined in the case of projected Majorana stabilizer codes. We then use matrix product states to study states that interpolate between two distinct Majorana fermion codes, one yielding a $\mathbb{Z}_2$ spin liquid and the other a trivial polarized state upon projection. While the free-fermion states are adiabatically connected, we find that the projected states undergo a phase transition detected by the topological entanglement entropy. Our work underscores the profound impact of the Gutzwiller projection and cautions against inferring properties of quantum spin liquids solely from their unprojected counterparts.





Partons -> physical state

Gutzwiller projection

What was known previously

Let's formally expand the Gutzwiller projector

$$\mathcal{P}_D = \prod_{j=1} \frac{1+\hat{D}_j}{2} = \frac{1}{2^N}\sum_g \hat{D}_g \qquad \hat{D}_g = \prod_{j\in g} \hat{D}_j$$

$$g \subset \mathcal{L} = \{1:N\}$$

H. Yao, S.-C. Zhang, S. A. Kivelson, PRL **102**, 217202 (2009)

H. Yao, X.-L. Qi, PRL **105**, 080501 (2010)

F. L. Pedrocchi, S. Chesi, D. Loss, PRB **84**, 165414(2011)

$$\hat{D} = \prod_j \hat{D}_j = 1$$

$$\hat{D}\,|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle = |\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle \longrightarrow D_g|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle = D_{\mathcal{L}-g}|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

Among 2^N different terms in $\sum_g$ only half are independent.

$$\longrightarrow \quad |\Psi_{\boldsymbol{u},\boldsymbol{n}}\rangle = \frac{1}{\sqrt{2^{N-1}}}\,\frac{1}{2}\sum_g \hat{D}_g|\tilde{\Psi}^M_{\boldsymbol{u},\boldsymbol{n}}\rangle$$

double counting

Equal-weight superpositions of 2^{N-1} different gauge choice.

$\mathbb{Z}_2$ Topological index

$$\hat{D} = \prod_j \hat{D}_j = 1$$

Projections give constraint on the underlying Z2 gauges and make it topological.

$$D = (-)^{N_x + N/2}\, \mathrm{sgn}\big(\mathrm{Pf}(A)\big) \prod_{\ell=1}^{N/2} (1 - 2n_\ell) \prod_j u_{j,k}$$

$$\mathcal{H}_M = i \boxed{\sum_{\langle jk \rangle^\gamma} J_\gamma u_{jk}}\; \hat{c}_j \hat{c}_k$$

$$\{\, A \,\}_{j,k=1}^{N}$$

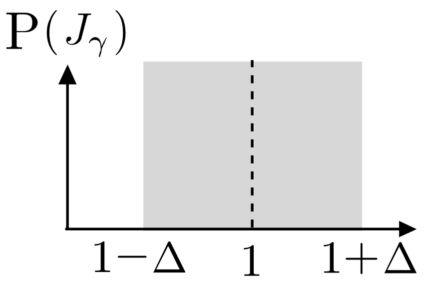
c.f. Kane, Mele, PRL **95**, 146802 (2005)

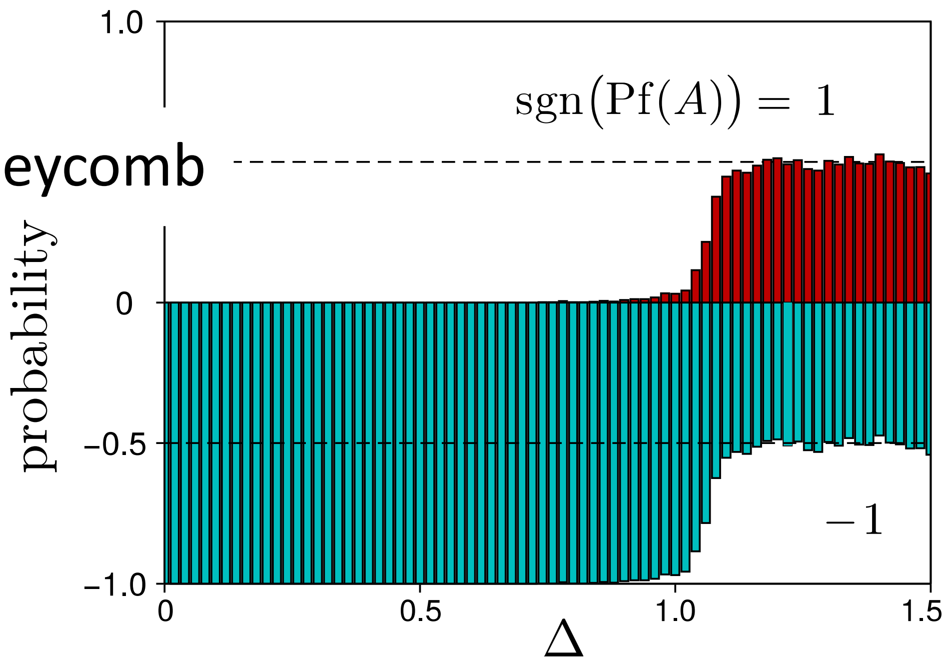
$$I = \frac{1}{2\pi i} \oint_C d\mathbf{k} \cdot \nabla_{\mathbf{k}} \log \mathrm{Pf}\left[\langle u_i(\mathbf{k}) | \Theta | u_j(\mathbf{k}) \rangle\right]$$

Z2 topological index

Quantum Spin Hall Effect

Random-bond Kitaev-honeycomb





Exact MPS(TN) solutions for

topic 1

Kitaev-honeycomb QSL ground state & all excited states

topic 2

general (all possible) frustration-free models

What is “Frustration-free” ?

A local projector Hamiltonian : $\hat{h}_l$, $\langle \hat{h}_l \rangle \geq 0$

Global Hamiltonian of system size *N*, given as a sum of local projectors.

$$\mathcal{H}_N = \sum_{l=1}^{N_c} \hat{h}_l$$

If we find a zero-energy ground state $\langle \mathcal{H}_N \rangle = 0$, all the local $\langle \hat{h}_l \rangle = 0$.

“Frustration-free” = the ground state is the simultaneous ground state of all local projector Hamiltonian.

Given the form of $\mathcal{H}_N$, judging whether it is frustration free or not

is a quantum k-QSAT problem (quantum satisfaction problem).

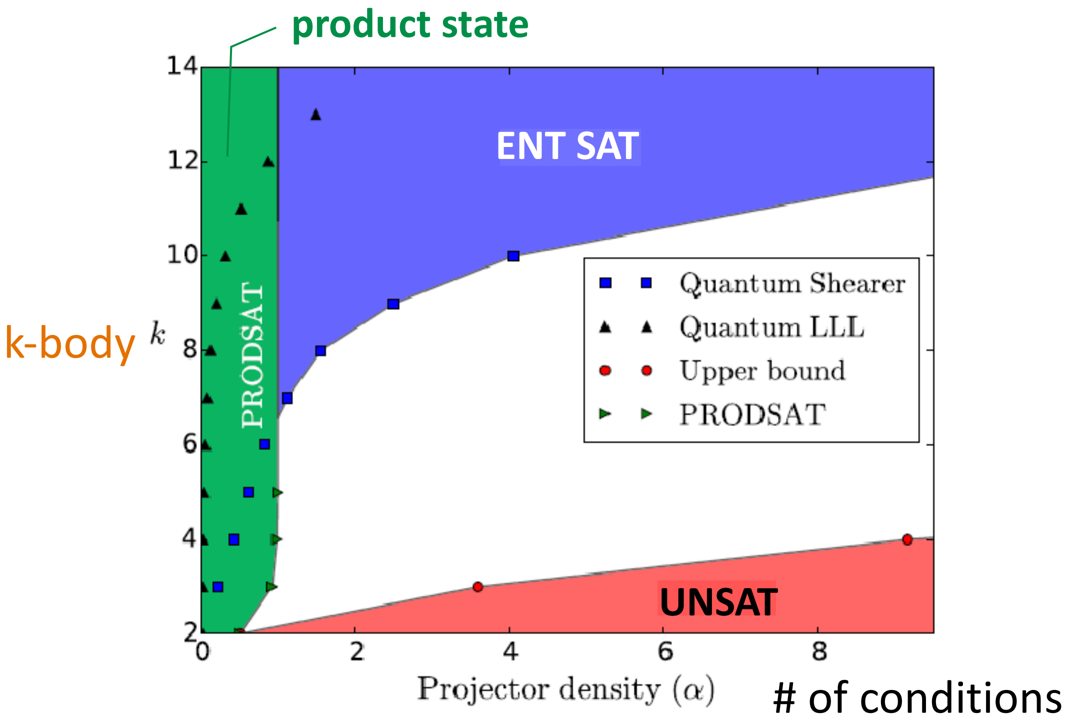
quantum analogue of NP-complete (Non-deterministic polynomial time)

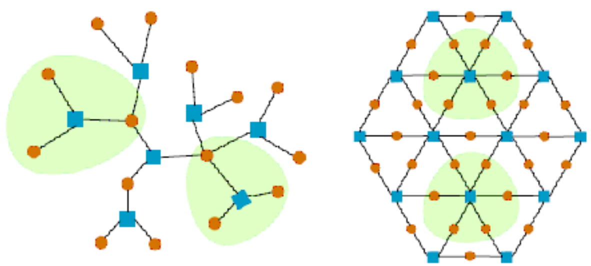
They are widely believed to be intractable.

How to choose this summation,

$$\mathcal{H}_N = \sum_l \hat{h}_l$$

How to solve it exactly.





Sattath, et.al. PNAS | 2016 | vol. 113 | 6433

Frustration-free models

model		exact solutions		
AKLT chain	MPS	SPT	gapped	Affleck, et al. (1987), Den Nijs, et al. (1989)
Majumdar-Ghosh chain		Singlet product state	gapped	Majumdar, et al. (1969)
Motzkin chain	MERA	Motzkin walk	gapless	Bravyi, et al. (2012), Alexander, et al.(2021)
Fredkin chain	MPS	Dyck walk	gapless	Salberger, et al. (2017), Udagawa, et al. (2017)
PXP-like chain	MPS	Quantum scar	gapped	Lesanovsky(2011-2012), Mark, et al. (2020)
Zigzag XYZ chain	product states	Anyon condensation	gapless	Batista (2009,2012)
Three-Coloring kagome model		Three-coloring product	gapless	Changlani, et al. (2018), Palle, et al. (2021)
Kitaev toric code		Quantum spin liquid	gapped	Kitaev (2003)
Rokhsar-Kivelson point		Quantum spin liquid	gapless	Rokhsar, et al. (1988)

Zoo of interesting physics.
But solved in a model-specific language, so far.

Also, it is believed that the system with long range entanglement (QSL) or gapless states CANNOT be exactly solved using MPS.

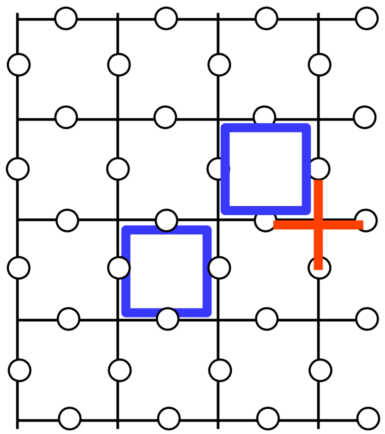
Quantum "Frustration-free"

trivial example: when $[\,\hat{h}_i,\,\hat{h}_j\,] = 0$ for all i, j .

Toric code long-range entangled topologocal order

$$\mathcal{H}_{\text{toric}} = -\sum_v A_v - \sum_p B_p,$$

$$A_v = \prod_{i\in v} \sigma_i^x, \quad B_p = \prod_{i\in p} \sigma_i^z.$$



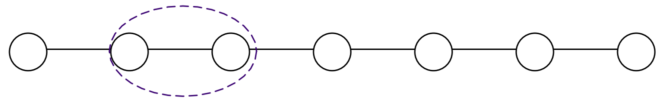
All A_v and B_p commute and have eigenvalues ± 1,

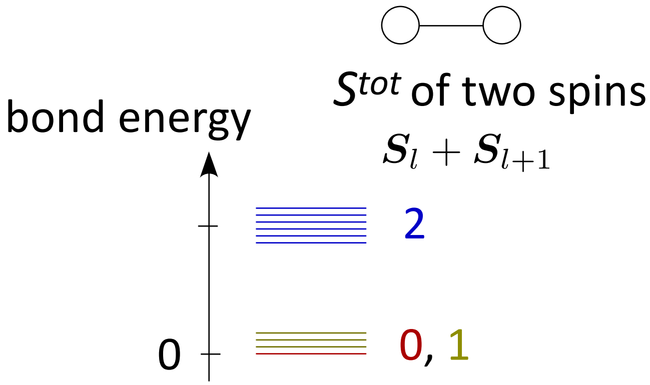
so that the ground state is their simultaneous $+1$ eigenstate.

Quantum "Frustration-free"

AKLT (Affleck-Kennedy-Lieb-Tasaki) model

$S = 1$



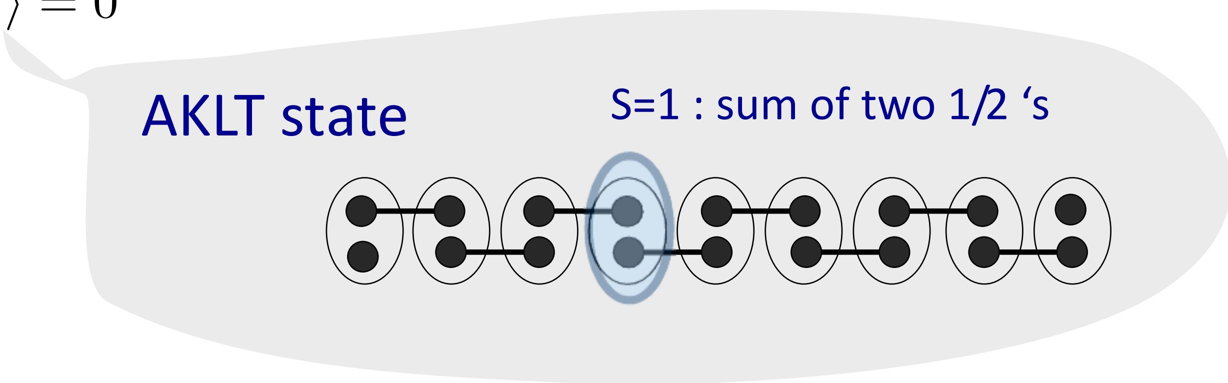


$$\mathcal{H}_N^{\mathrm{AKLT}} = \sum_{l=1}^{N} \boxed{\frac{1}{2}(\boldsymbol{S}_l \cdot \boldsymbol{S}_{l+1}) + \frac{1}{6}(\boldsymbol{S}_l \cdot \boldsymbol{S}_{l+1})^2 + \frac{1}{3}}$$

$$\hat{h}_l = \mathcal{P}_2(\boldsymbol{S}_l + \boldsymbol{S}_{l+1})$$

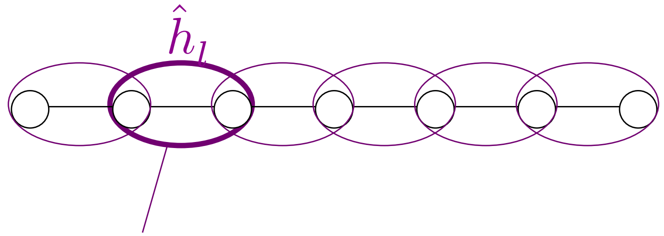
project out $S^{tot} = (S_l + S_{l+1}) = 2$ for all dimers

$$\mathcal{H}_N^{\mathrm{AKLT}} |\Psi_N^{\mathrm{gs}}\rangle = 0$$



Revisit AKLT

$S = 1$



total $S=2$ 5 excluded

$S=1$ 3
$S=0$ 1

select four of them as ground states

For each bond (two S=1's), among 3^2=9 states project out S=2 for all bonds.

$$\hat{h}_l = \sum_{m=1}^{5} |\xi^m\rangle\langle\xi^m| = \mathcal{P}_2(\boldsymbol{S}_l + \boldsymbol{S}_{l+1})$$

This operation is represented by

$$Q_l = \begin{pmatrix} 1 & 0 & \cdots & & & & & & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \cdots & & & & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & \cdots & & & & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \cdots & & & & & & 0 & 1 \end{pmatrix}.$$

constraints

|11> |10> |-1-1>

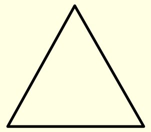
local dim. = 9

Clebsch-Gordan

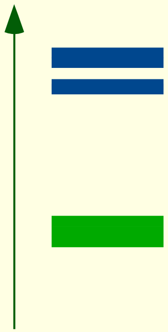
$$Q|\Psi_N^{\rm gs}\rangle = 0$$

Ground state is the zero-eigenstate of all Q_l 's.

Step1 : Prepare a unit cluster.



e.g. spin 1/2 triangle
dimension 2^3=8



excited states $\left\{ |\xi^m\rangle \right\}_{m=1}^{M}$

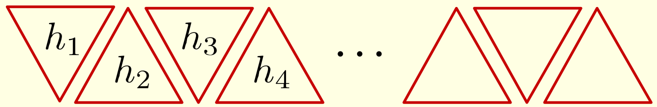
zero energy states $\left\{ |\psi^{m'}\rangle \right\}_{m'=1}^{D_g}$

$\longleftrightarrow$

$$\hat{h}_l = \sum_{m=1}^{M} \epsilon_m |\xi^m\rangle\langle\xi^m|$$

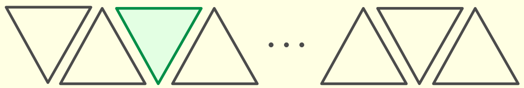
Penalty on excited states $\epsilon_m > 0$

Step2 : Candidate of frustration-free Hamiltonian as a sum of $\hat{h}_l$.



$$\mathcal{H}_N = \sum_{l=1}^{N_c} \hat{h}_l$$

Step3 : Judge whether have $\mathcal{H}_N|\Psi_N^{\rm gs}\rangle = 0$, and if yes, obtain $|\Psi_N^{\rm gs}\rangle$.

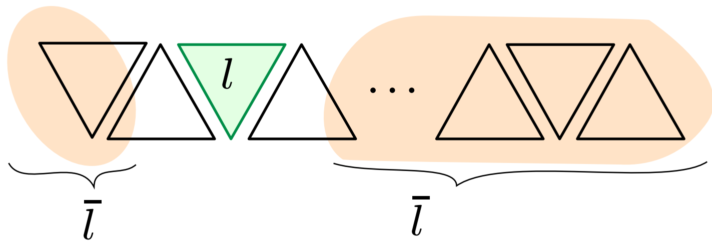


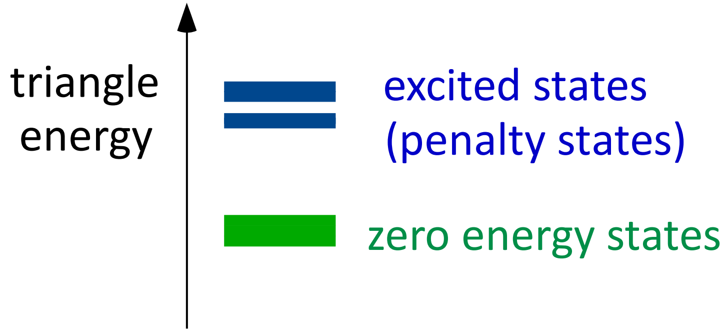
$$|\Psi_N^{\rm gs}\rangle = \sum_{m=1}^{D_g} \lambda_m |\psi_l^m\rangle |\Phi_{\bar{l}}^m\rangle$$

Nonzero Schmidt states on a triangle are the zero-energy states .

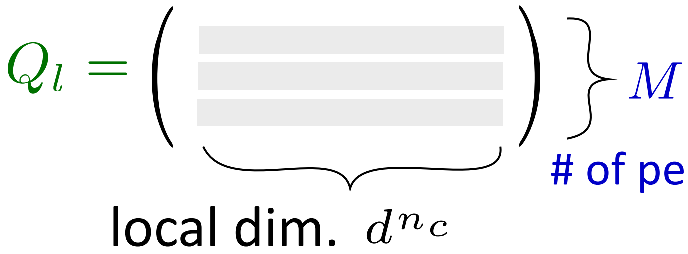
Projector onto zero-energy cluster states

cluster size n_c =3 d : local deg. of freedom





Local projector Q_l that acts locally on l th cluster to zero-out the penalty state.



global projector

$$Q = \sum_{l=1}^{N_c} I^{d^{n_l}} \otimes Q_l \otimes I^{d^{N-n_c-n_l}}$$

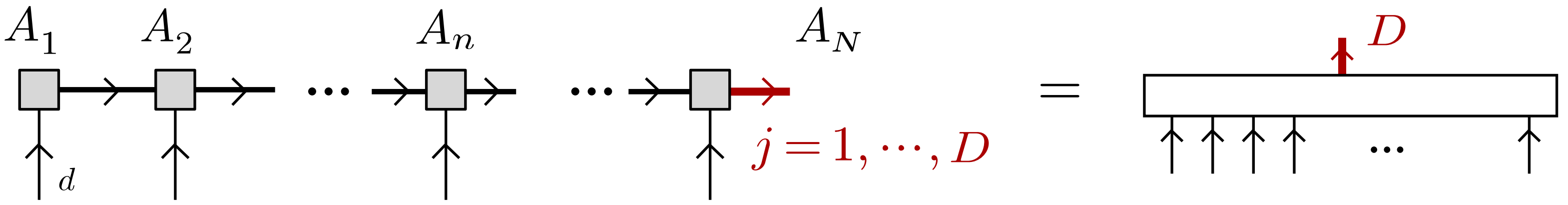
$$Q|\Psi_N^{\mathrm{gs}}\rangle = 0 \qquad D_N = d^N - \mathrm{rank} Q$$

ground state degeneracy

If $D_N = 0$ the model is NOT frustration-free, we can immediately judge.

Cluster projected MPS

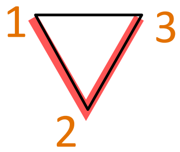
$$\left\{ |\Psi_j^{(N)}\rangle \right\}_{j=1}^{D} =$$



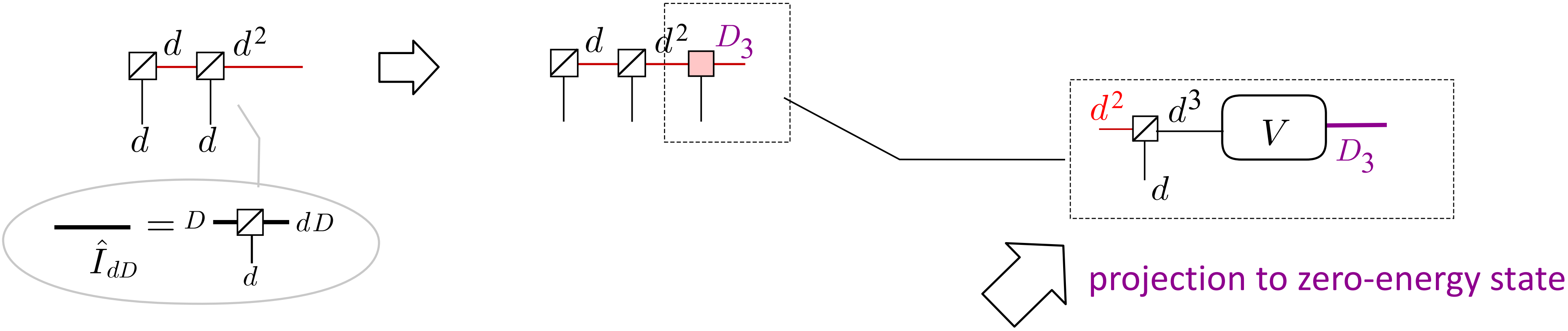
Representing *D* different states as isometry.

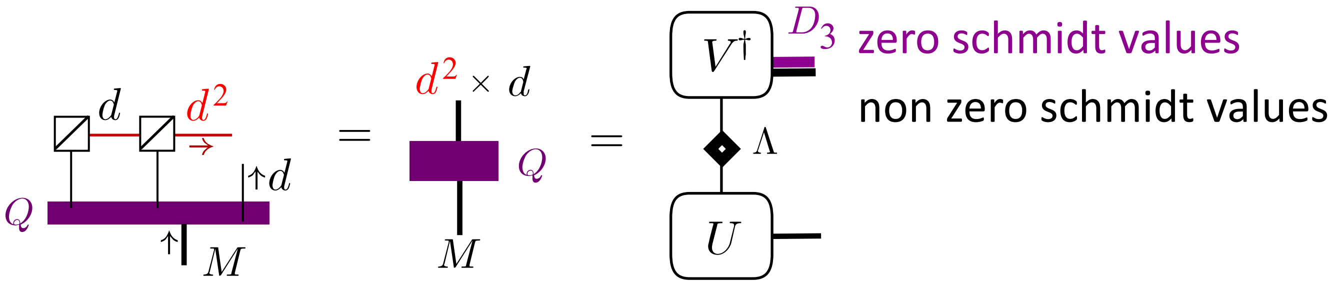
Bond dimension grows from left to right.

Cluster projected MPS

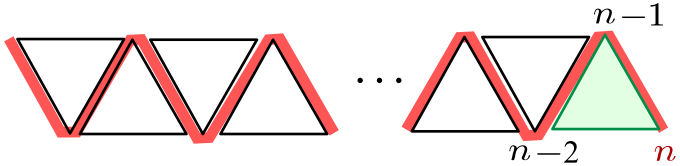


Apply a projector in a unit triangle.

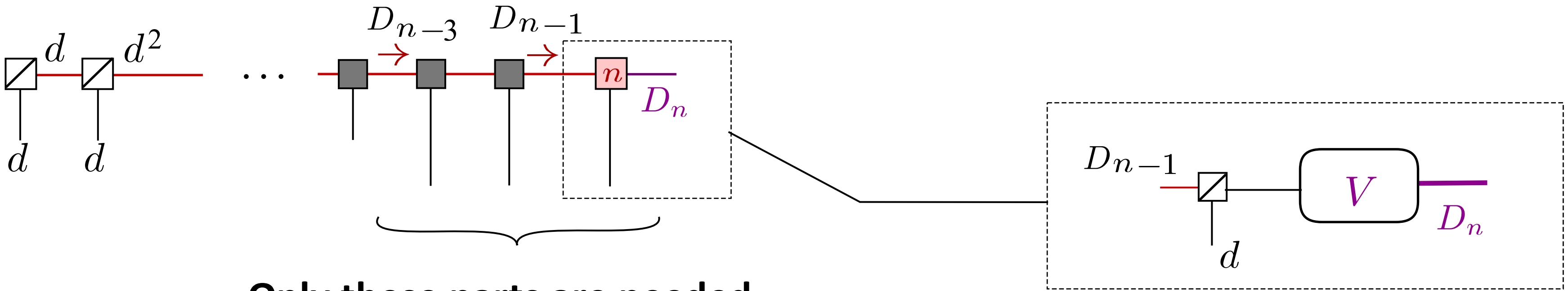




Cluster projected MPS

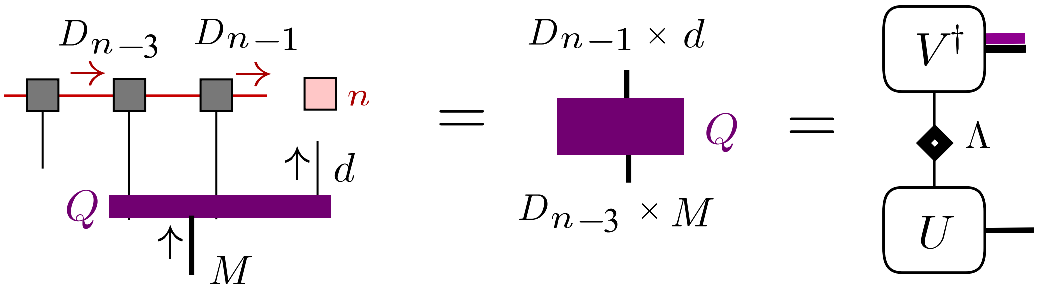


Given a [1:n-1] tensor,
apply a projector on the right edge triangle.



Only these parts are needed.
Previous sites do not matter.

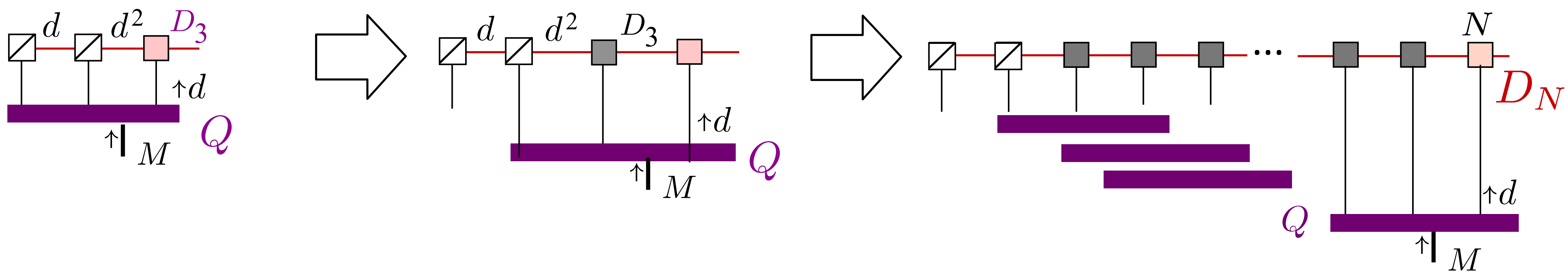
projection to zero-energy state



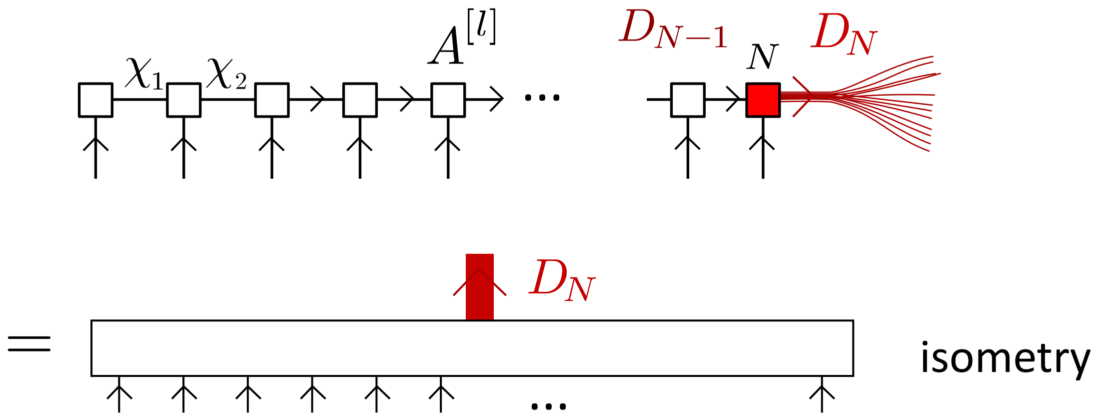
D_n zero schmidt values
non zero schmidt values ➡ discard

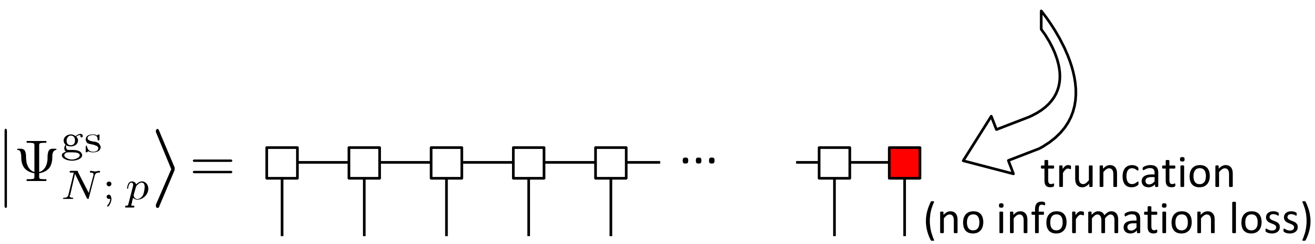
Make full use of the locality.
Left normalized / isometric form.

Cluster projected MPS



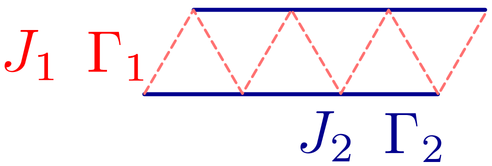
- Increase the matrix one-by-one
- We end up with orthogonal set of D_N degenerate MPS





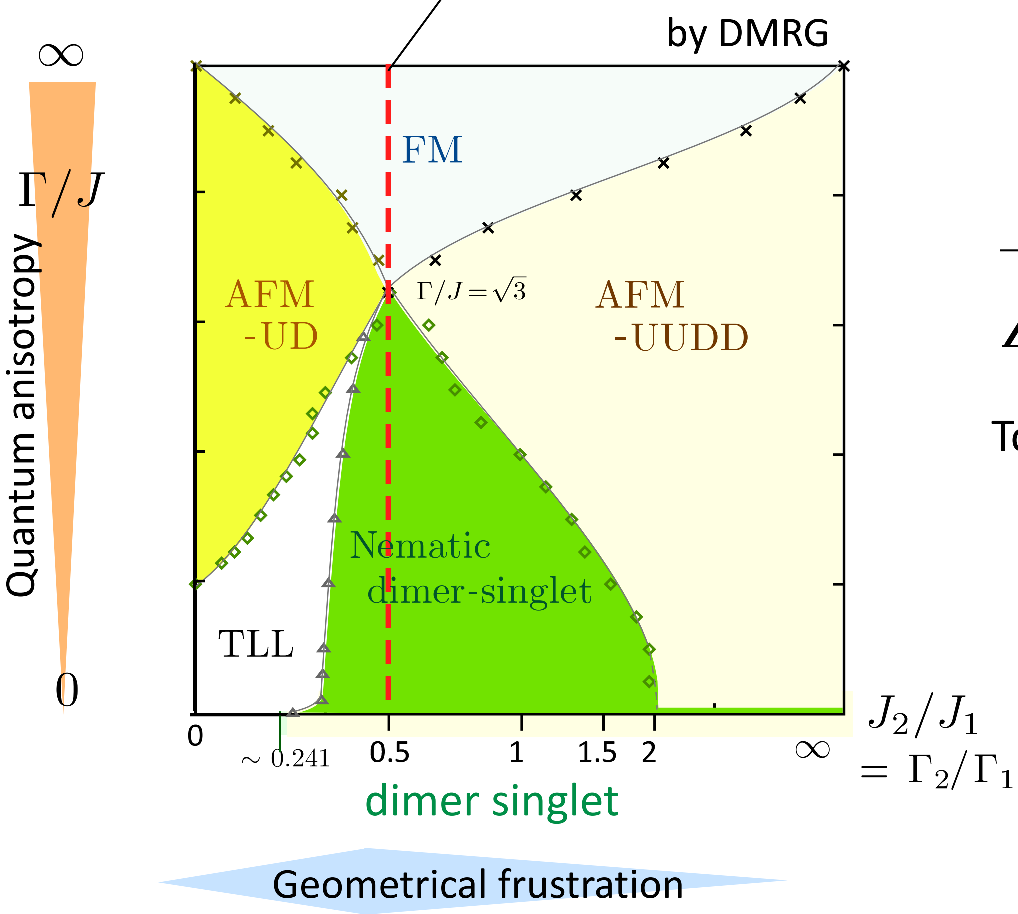
Examples of
obtaining frustration-free solutions

Zigzag chain



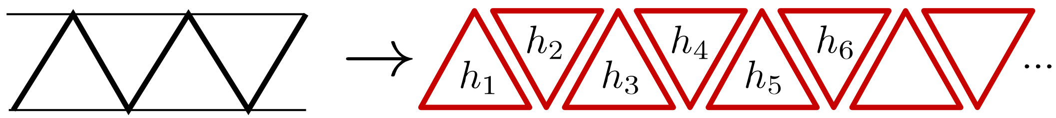
$$\mathcal{H} = \sum_j \sum_{\eta=1,2} J_\eta \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+\eta} + \Gamma_\eta (S_j^x S_{j+\eta}^y + S_j^y S_{j+\eta}^x)$$

exactly solvable line



$$J_1 = 2J_2 \equiv J$$

$$\Gamma_1 = 2\Gamma_2 \equiv \Gamma$$



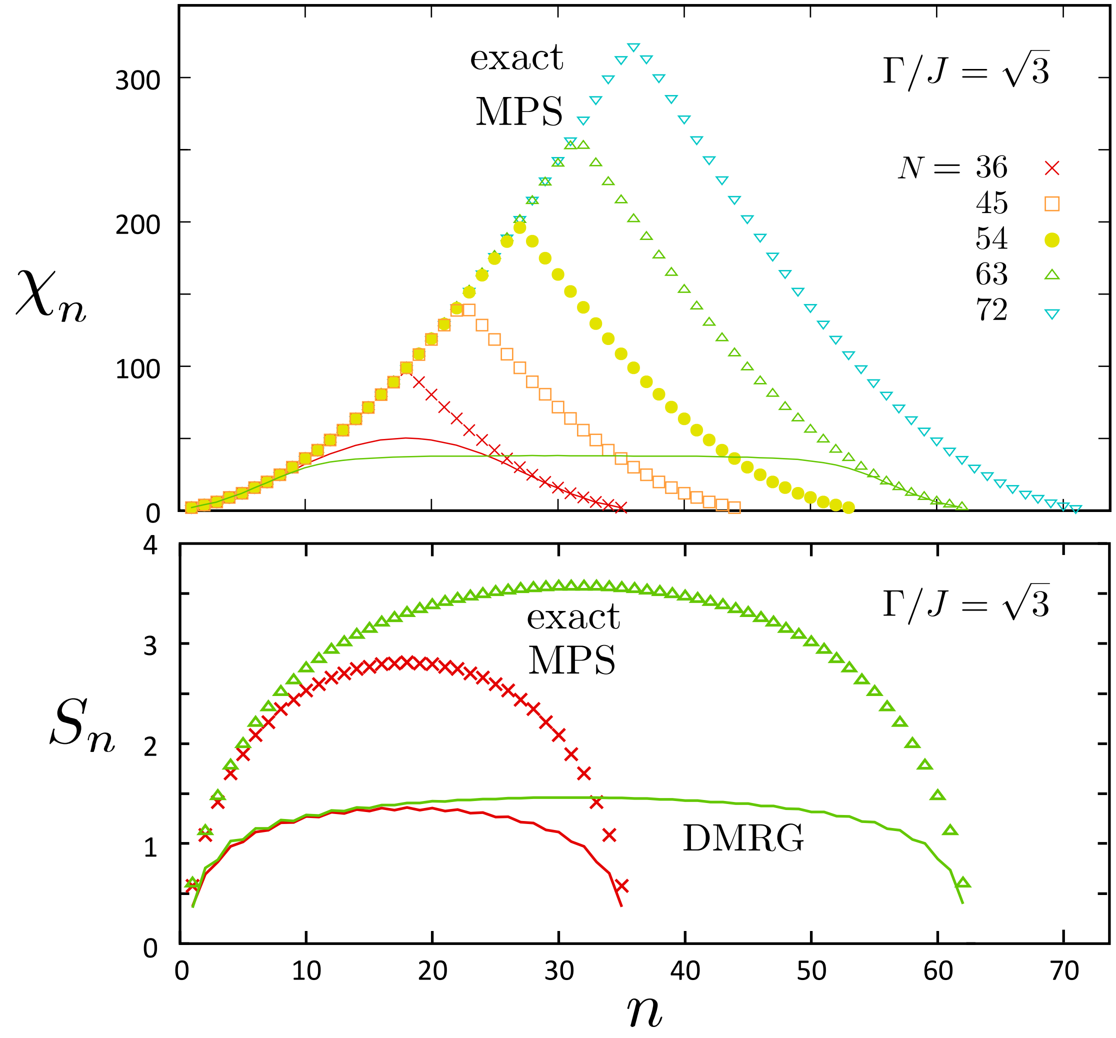
Total ground state energy = Ne_l

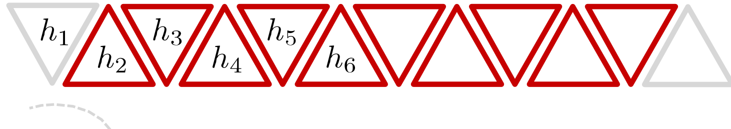
$$e_l = -3J/4$$

lowest energy of the triangle

⇨ “Frustration free”

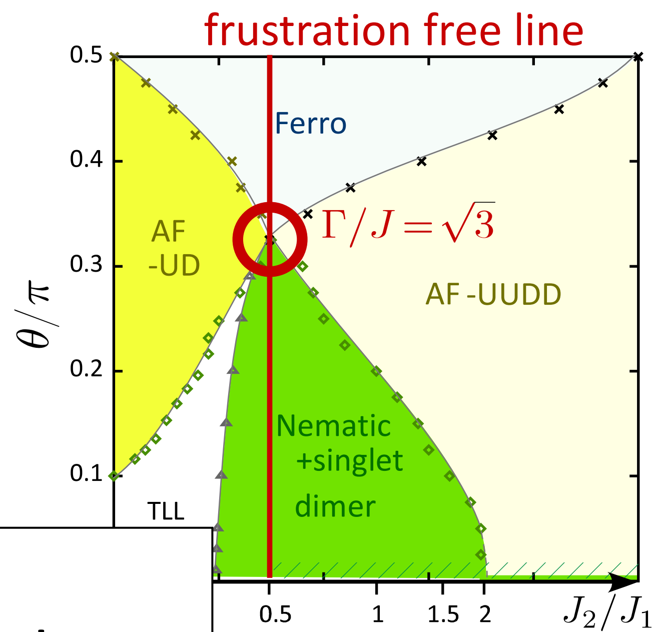
Zigzag chain multicritical point





Ground state has huge degeneracy.

$$D_N^{\mathrm{TOBC}} = \begin{cases} (N+2)^2/4 & (\text{even } N) \\ (N+1)(N+3)/4 & (\text{odd } N). \end{cases}$$



- We can obtain complete set of degenerate solutions.
  This was not possible by DMRG... as they favor minimally entangled states.

Previously known frust-free models are super-easy to recalculate by our cluster-projected MPS.

Lesanovsky model

Lesanovsky (2011-2012)

spin-1/2 PXP model + working term

$$H = \Omega \sum_{k}^{L} P_{k-1}\sigma_x^k P_{k+1} + \Delta \sum_{k}^{L} n_k + V \sum_{m>k+1} \frac{n_m n_k}{|k-m|^\gamma}$$

$$|\langle \mathrm{gs_{PXP}} | \mathrm{gs_{Lesanovsky}} \rangle| = 0.977$$

vicinity of quantum scar

$$\mathcal{H} = \sum_j h_j,$$

$$h_j = P_{j-1}(X_j + z^{-1} n_j + z P_j) P_{j+1} = z^{-1} |\Phi\rangle\langle\Phi|_{j,j+1,j+2}$$

$$n_j = |0\rangle\langle 0|_j, \; P_j = |1\rangle\langle 1|_j$$

$$X_j = |0\rangle\langle 1|_j + |1\rangle\langle 0|_j$$

$$|\Phi\rangle = |101\rangle + z|111\rangle$$

We iteratiecly project out

$$|\Phi\rangle_{j,j+1,j+2}, |00\rangle_{k,k+1}$$

$$|000\rangle, |100\rangle$$

Supplementary of PRL **132**, 166701 (2024)

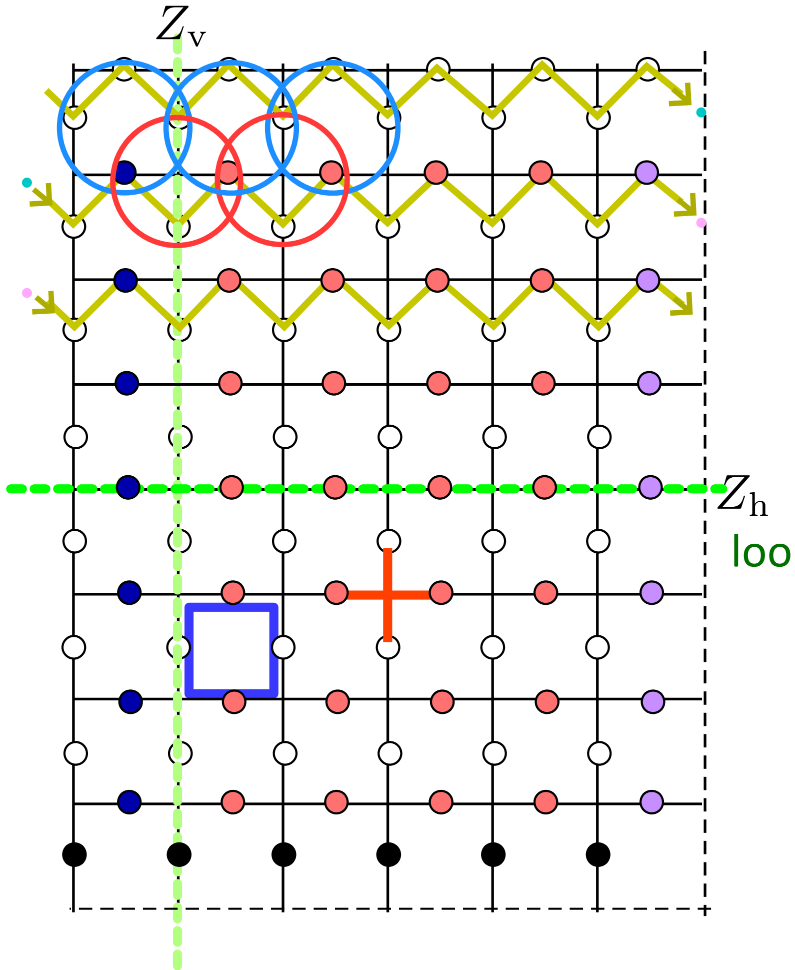
What else can we do ?

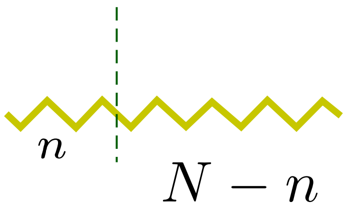
Obtain topologically ordered ground states explicitly.

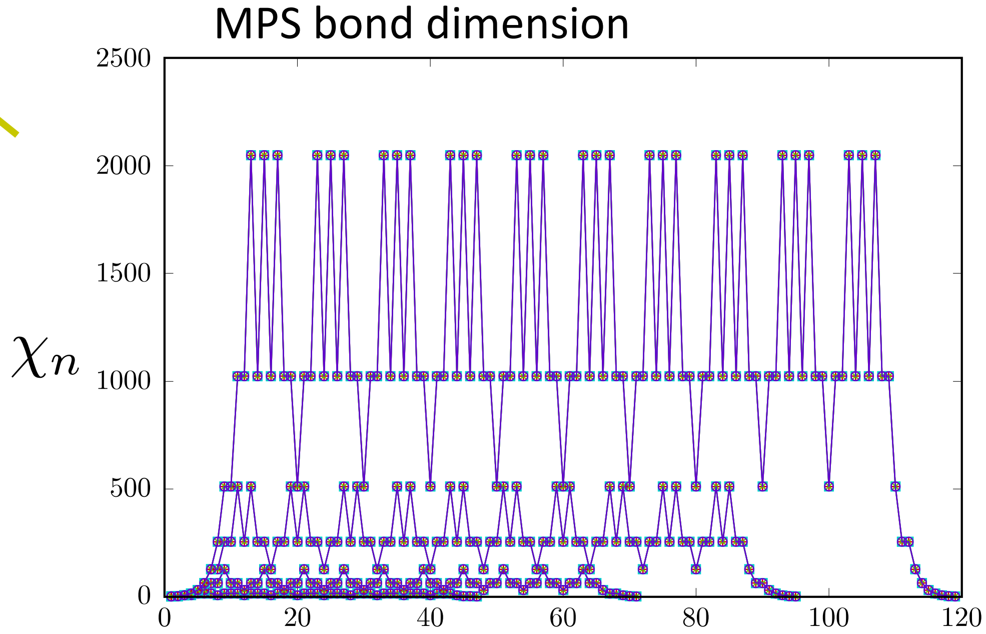
Calculate 2D models with a same cost as 1D.

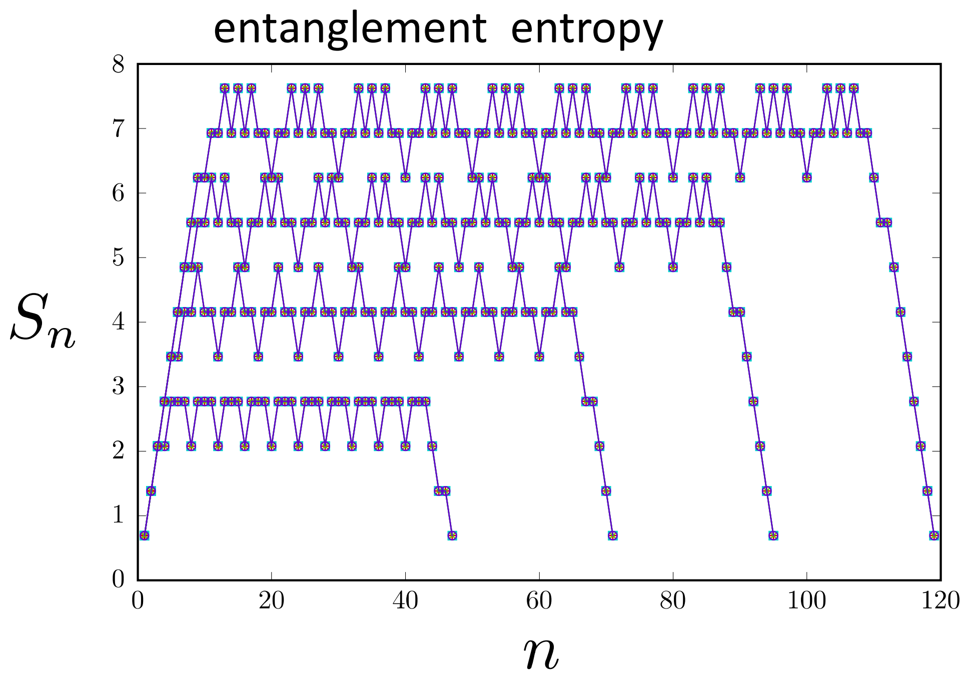
Design frustration-free across different models.

Toric code









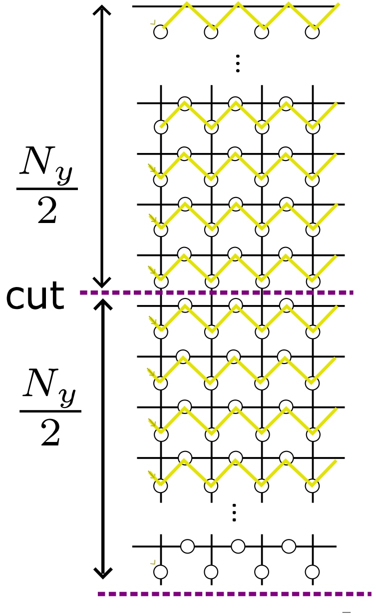
$$\mathcal{H} = -\sum_v A_v - \sum_p B_p,$$

$$A_v = \prod_{i\in v} \sigma_i^x \qquad B_p = \prod_{i \in p} \sigma_i^z$$

= penalty terms of clusters

Ground state is the A=B=+1

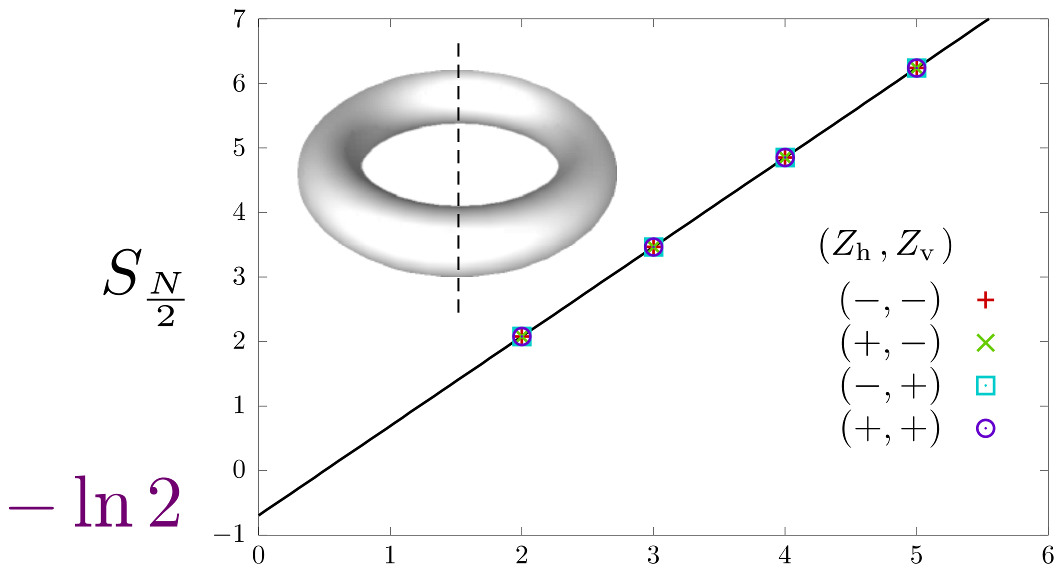
Toric code



- **topological order (long range entangled) can be described**
- **topological sectors are technically hard to detect numerically in DMRG. We do it easily.**

Jiang, et.al.(2013)

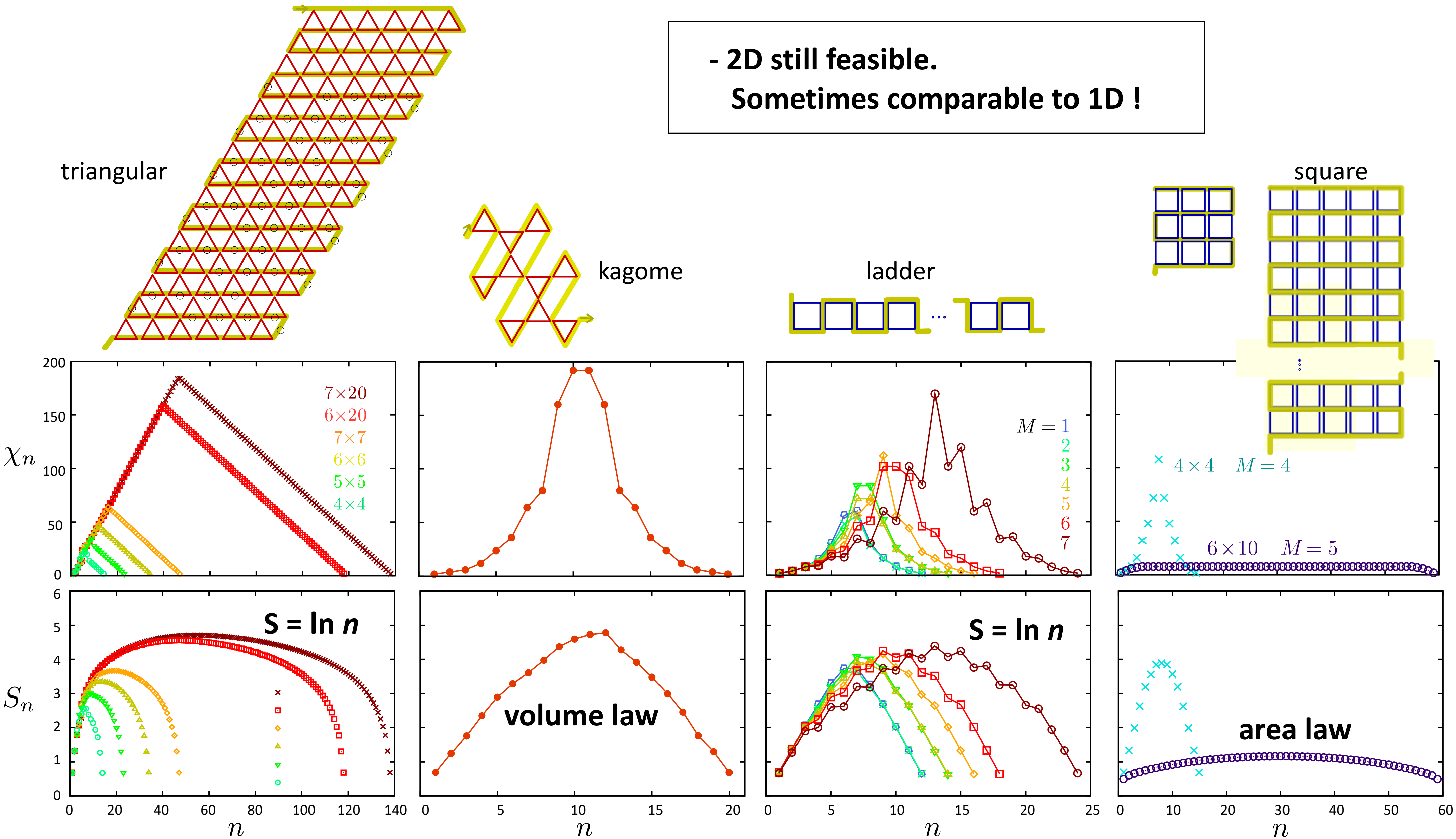
- **Entanglement entropy evaluation, all excited states exactly obtained by popping up A, B.**



Topological EE

2D lattices S=1/2

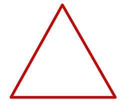
- 2D still feasible.
  Sometimes comparable to 1D !



Design "frustration-free" across various forms of models

$\hat{h}_l$ Choose the cluster shape, degree of freedom.

Choose the states to zero-out.

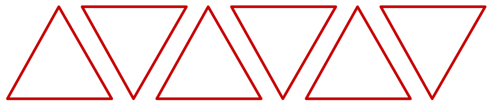


$$|\xi_\Uparrow\rangle = \cos\alpha|000\rangle + i\sin\alpha\left[\cos\beta|101\rangle + \sin\beta\left(|110\rangle + |011\rangle\right)/\sqrt{2}\right]$$
$$|\xi_\Downarrow\rangle = \cos\alpha|111\rangle - i\sin\alpha\left[\cos\beta|010\rangle + \sin\beta\left(|001\rangle + |100\rangle\right)/\sqrt{2}\right]$$

We can parameterize the choice.

$$\mathcal{H}_N = \sum_{l=1}^{N_c} \hat{h}_l$$

Combine clusters and form a lattice.



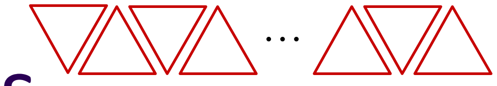
$$\mathcal{H}_N = \sum |\xi_\Uparrow\rangle\langle\xi_\Uparrow| + |\xi_\Downarrow\rangle\langle\xi_\Downarrow| = \sum_{\eta=1,2}\Big(J_\eta^\perp(S_i^xS_j^x + S_i^yS_j^y) + J_\eta^zS_i^zS_j^z + \Gamma_\eta(S_i^xS_j^y + S_i^yS_j^x)\Big)$$

Check whether we have solution by CPMPS.

We can do it immediately!

If not... vary the parameter/lattice and try again.

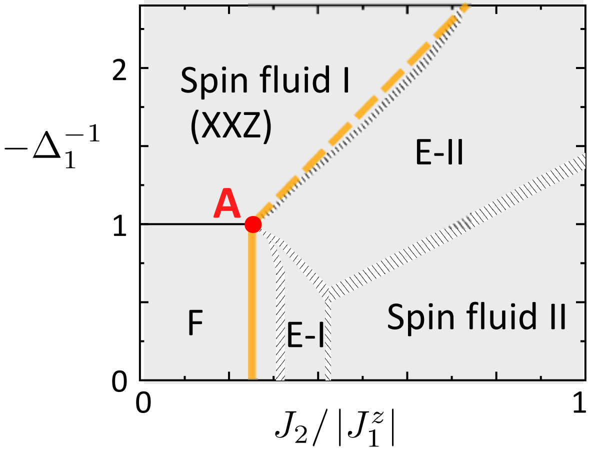
Revisiting various XXZ-anisotropic models

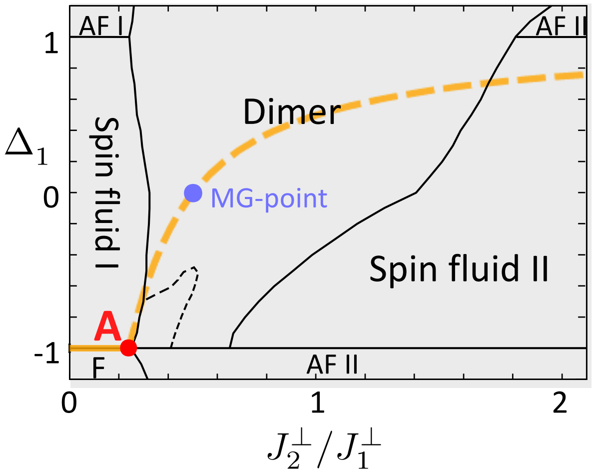


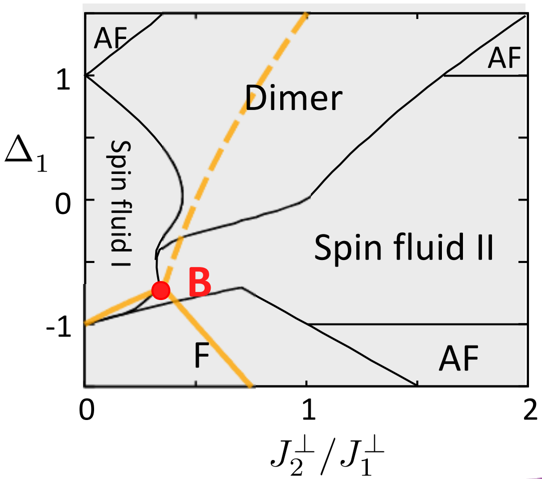
$$\sum_{\eta=1,2} \left(J_\eta^\perp (S_i^x S_j^x + S_i^y S_j^y) + J_\eta^z S_i^z S_j^z + \Gamma_\eta (S_i^x S_j^y + S_i^y S_j^x) \right)$$

A, B, C, D, E, F are frustration-free exact solution points : multicritical

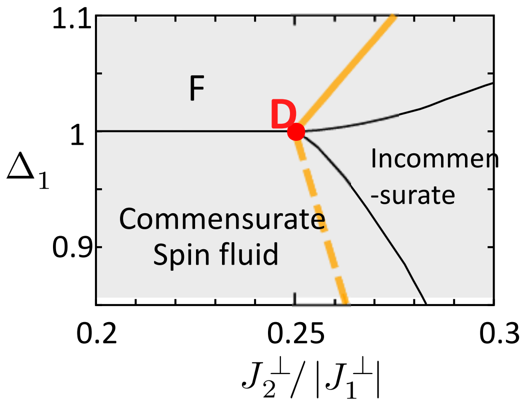
Solid lines are frustration-free product state exact lines.

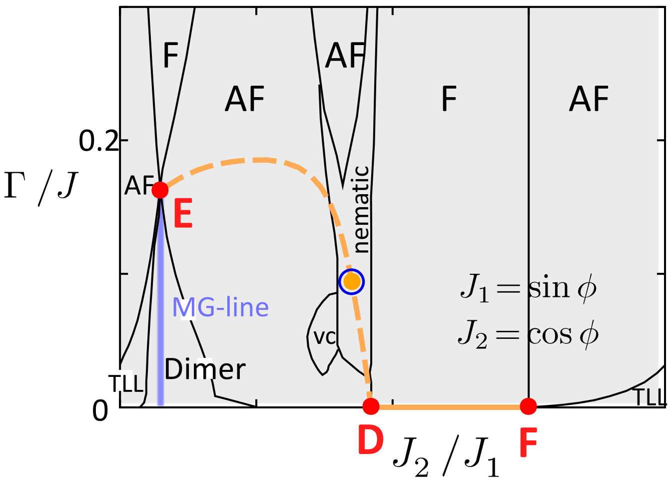


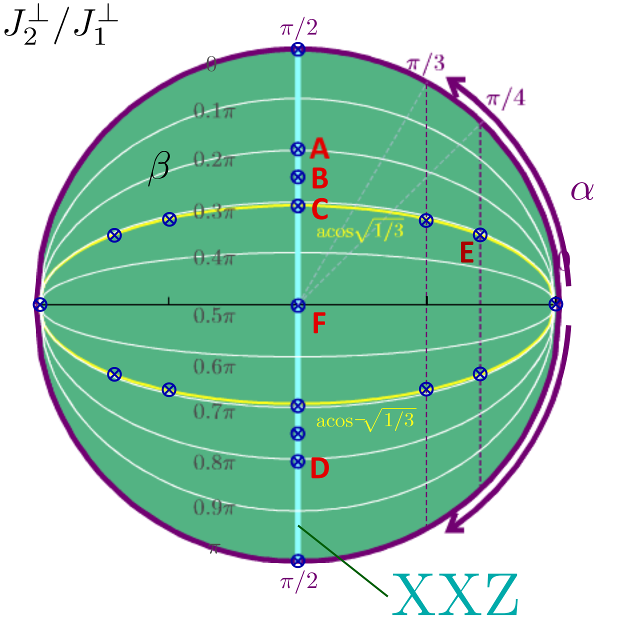




Somma,et.al.(2002)
Jafari ,et.al.(2007)
Hirata ,et.al.(2000)
Gerhardt ,et.al.(1998)
Dmitriev ,et.al.(2008)







Summary

If we know the types of projectors that characterize the eigenstate of the model,
we can project MPS to form an exact solution.

Kitaev honeycomb: quantum number +Gutzwiller projectors

Frustration-free models: sum of local projector Hamiltonian

Algorithm is simple, numerically cheep, and practical.

- 1D and 2D has not much difference in the cost.
- design/solve frustration-free problem.
- Area-law bound is not a serious problem.