Chiral States in Quantum and Classical Frustrated Magnets

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Entanglement and topology in interacting quantum matterICTP,
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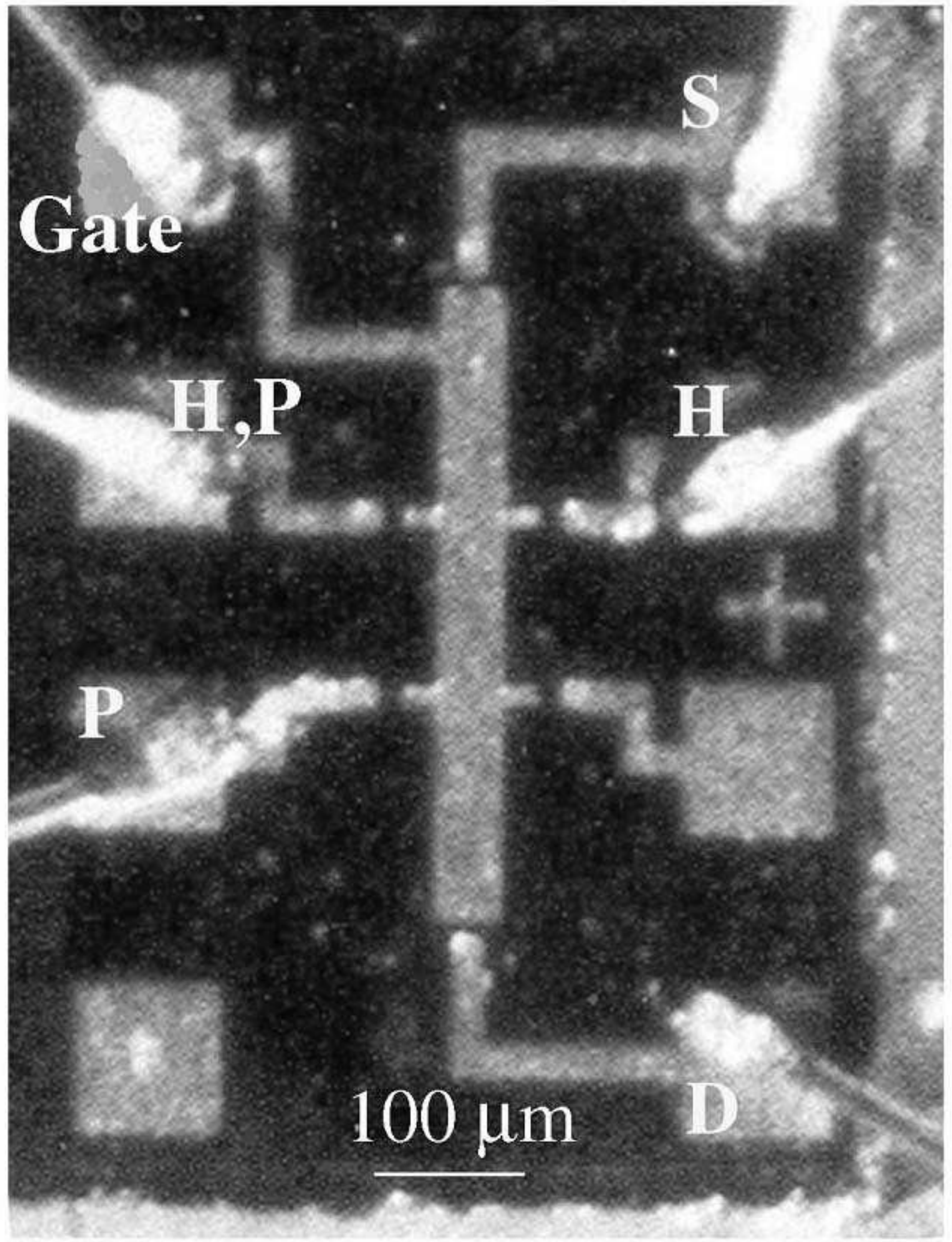
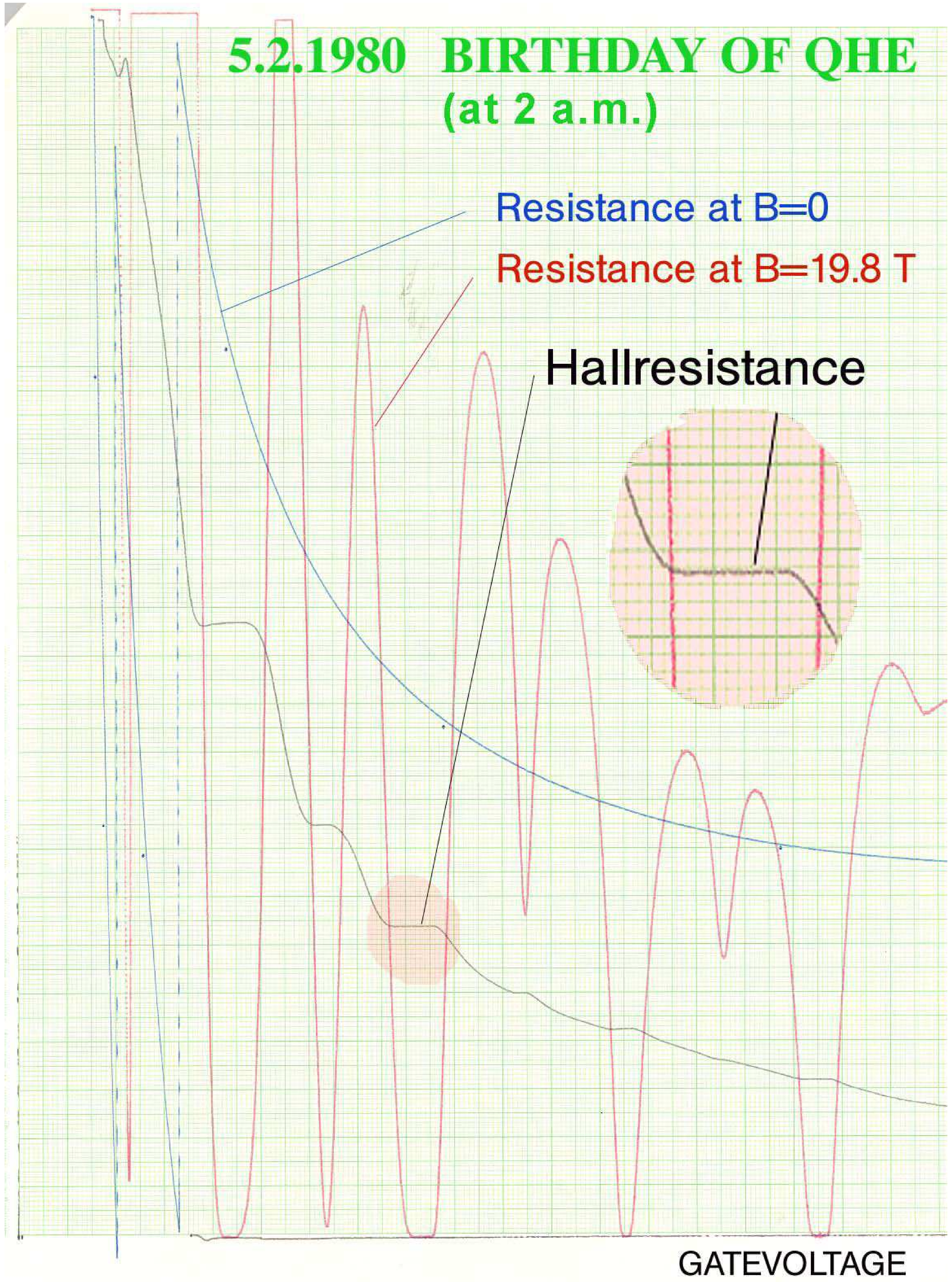
The Integer Quantum Hall effect



K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980)

http://www.bourbaphy.fr/klitzing.pdf



From Kubo formula to Chern number

Linear response:
Ando, Matsumoto, Uemura
1975

$$\sigma_H \propto \sum \frac{\langle\alpha|j_x|\beta\rangle\langle\beta|j_y|\alpha\rangle - \langle\alpha|j_y|\beta\rangle\langle\beta|j_x|\alpha\rangle}{(\epsilon_\alpha - \epsilon_\beta)^2}$$

since $\mathbf{j} \propto \mathbf{v} \propto \frac{\partial \mathcal{H}}{\partial \mathbf{k}}$

$$\sigma_H \propto \sum_{\alpha,\beta} \frac{\langle\alpha|\frac{\partial\mathcal{H}}{\partial k_x}|\beta\rangle\langle\beta|\frac{\partial\mathcal{H}}{\partial k_y}|\alpha\rangle - \langle\alpha|\frac{\partial\mathcal{H}}{\partial k_y}|\beta\rangle\langle\beta|\frac{\partial\mathcal{H}}{\partial k_x}|\alpha\rangle}{(\epsilon_\alpha - \epsilon_\beta)^2}$$

same form as Berry curvature (1984).

Can be recast into a topological invariant, the Chern number (integer):
Avron, Seiler, Simon (1983)
Thouless, Kohmoto, Nightingale, den Nijs (1982)

$$\sigma_{xy} = \frac{e^2}{h} C \qquad C_n = \frac{1}{2\pi i} \int_{\mathrm{BZ}} dk_x dk_y F_n^{xy}$$

Berry curvature of the *n*(**k**) wave function with momentum **k** in the 2D Brillouin zone.

$$F_n^{xy}(\mathbf{k}) = \langle \partial_x n(\mathbf{k}) | \partial_y n(\mathbf{k}) \rangle - \langle \partial_y n(\mathbf{k}) | \partial_x n(\mathbf{k}) \rangle$$

Topological invariants — geometry

Gauss-Bonnet formula for closed surfaces provides a link between
local geometric properties (local curvature K)
and
global topological properties.

total curvature = Euler characteristic

$$\frac{1}{2\pi}\int_S K \mathrm{d}A = 2(1-g)$$

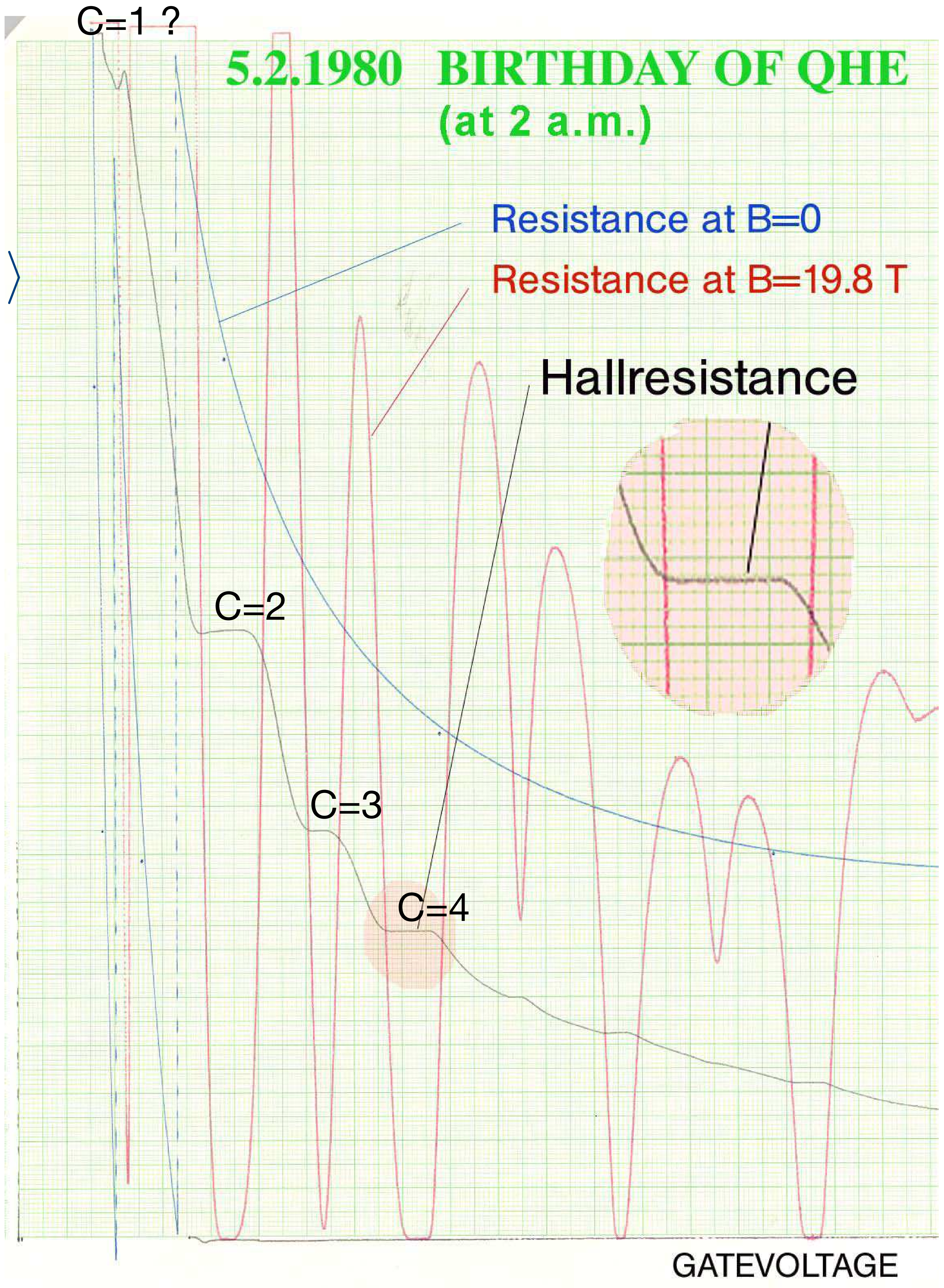
g is the number of handles
(e.g. torus g=1, pretzel g=3)

The Integer Quantum Hall effect

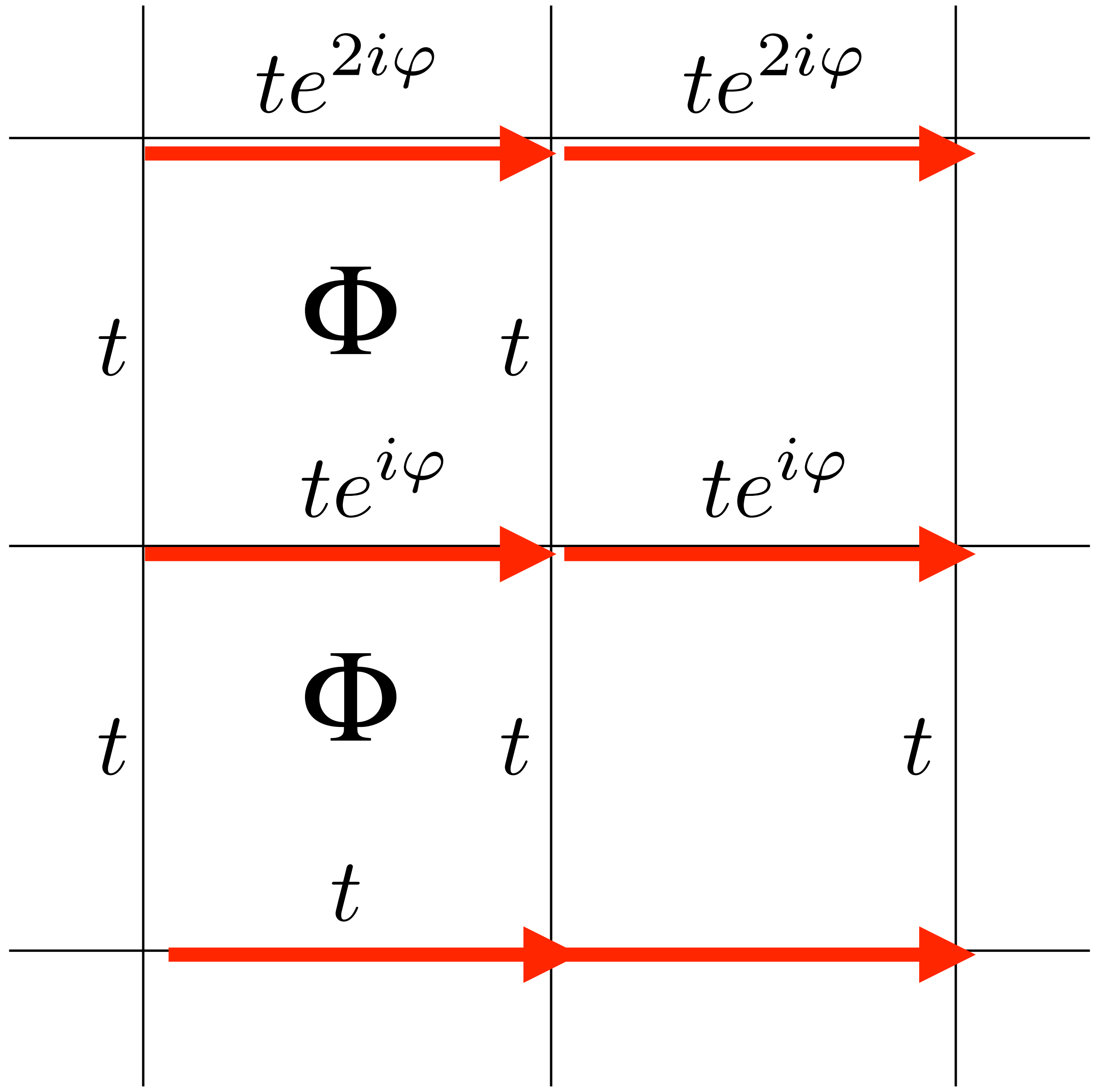
$$F_n^{xy}(\mathbf{k}) = \langle \partial_x n(\mathbf{k}) | \partial_y n(\mathbf{k}) \rangle - \langle \partial_y n(\mathbf{k}) | \partial_x n(\mathbf{k}) \rangle$$

$$C_n = \frac{1}{2\pi i} \int_{\mathrm{BZ}} dk_x dk_y F_n^{xy}$$

$$\sigma_{xy} = \frac{e^2}{h} C$$



The analogy between the electrons and magnons

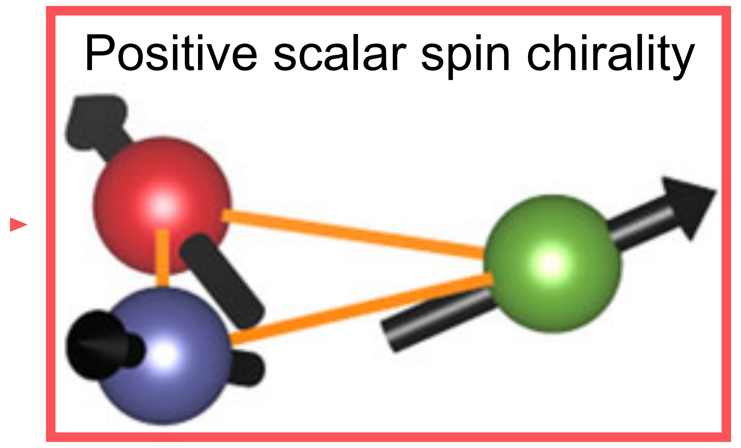


electrons in magnetic fields:

- the hopping amplitudes acquire a finite phase
- phase around the plaquette = Φ/Φ_0
- the complex hopping will lead to Berry curvature and Chern numbers
- Berry curvature is the manifestation of the Lorentz force

magnons in non-coplanar orders

hopping magnon



Katsura, Nagaosa & Lee,. PRL **104**, 066403 (2010).
Shindou, Matsumoto, Murakami, & Ohe, PRB **87**, 174427 (2013).
Matsumoto, Murakami, PRL **106**, 197202 (2011).

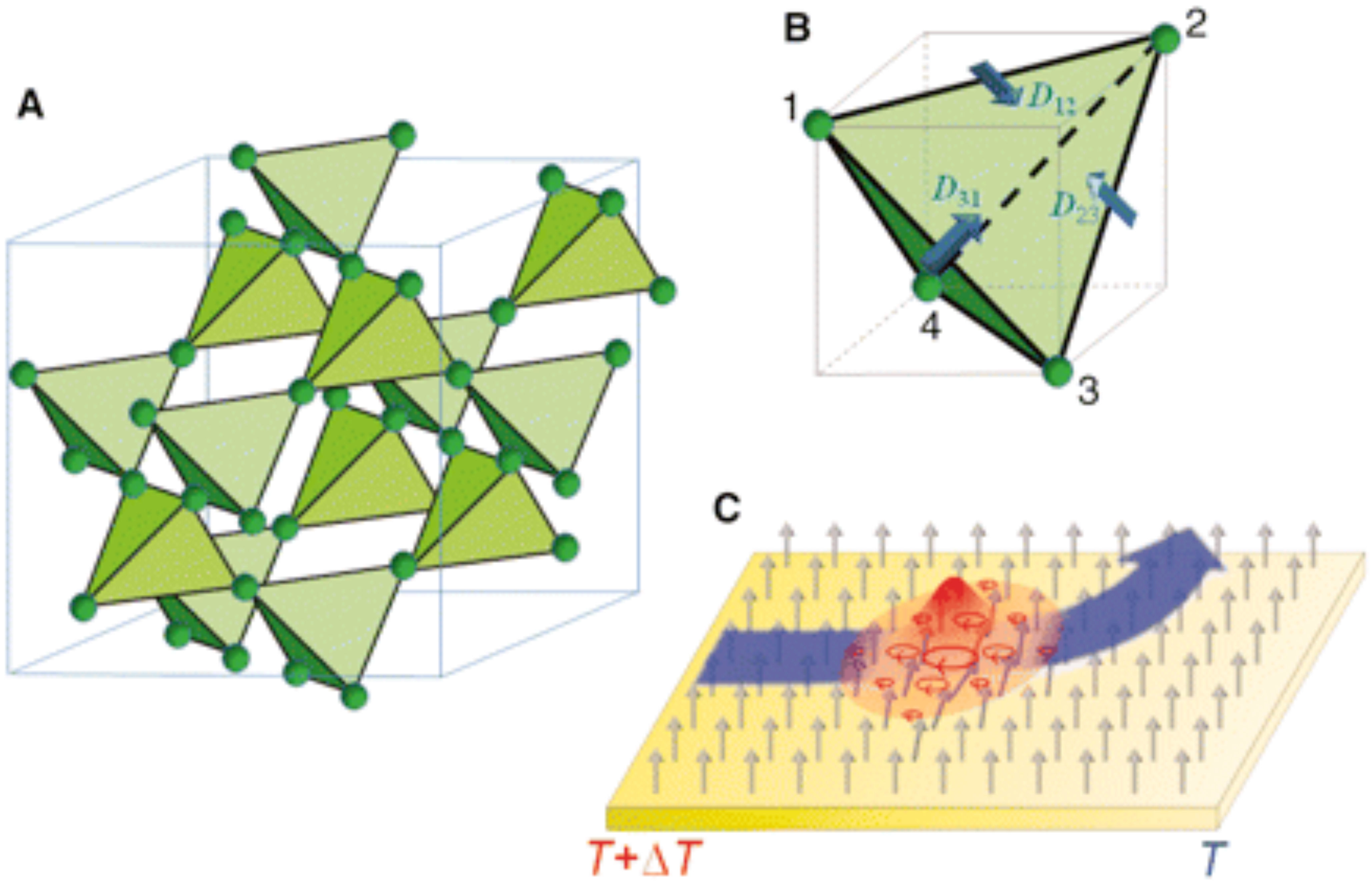
$$\mathbf{S}_i \cdot \mathbf{S}_j \to e^{i\varphi}\tilde{S}_i^{-}\tilde{S}_j^{+} \sim e^{i\varphi}a_i^{\dagger}a_j$$

- during the hopping the magnons picks up a phase due to chirality or anisotropy (cf. anomalous Hall effect)
- the complex hopping will lead to Berry curvature and Chern number
- Berry curvature is the signature of a deflecting force acting on a magnon

Observation of the Magnon Hall Effect

SCIENCE VOL 329 16 JULY 2010

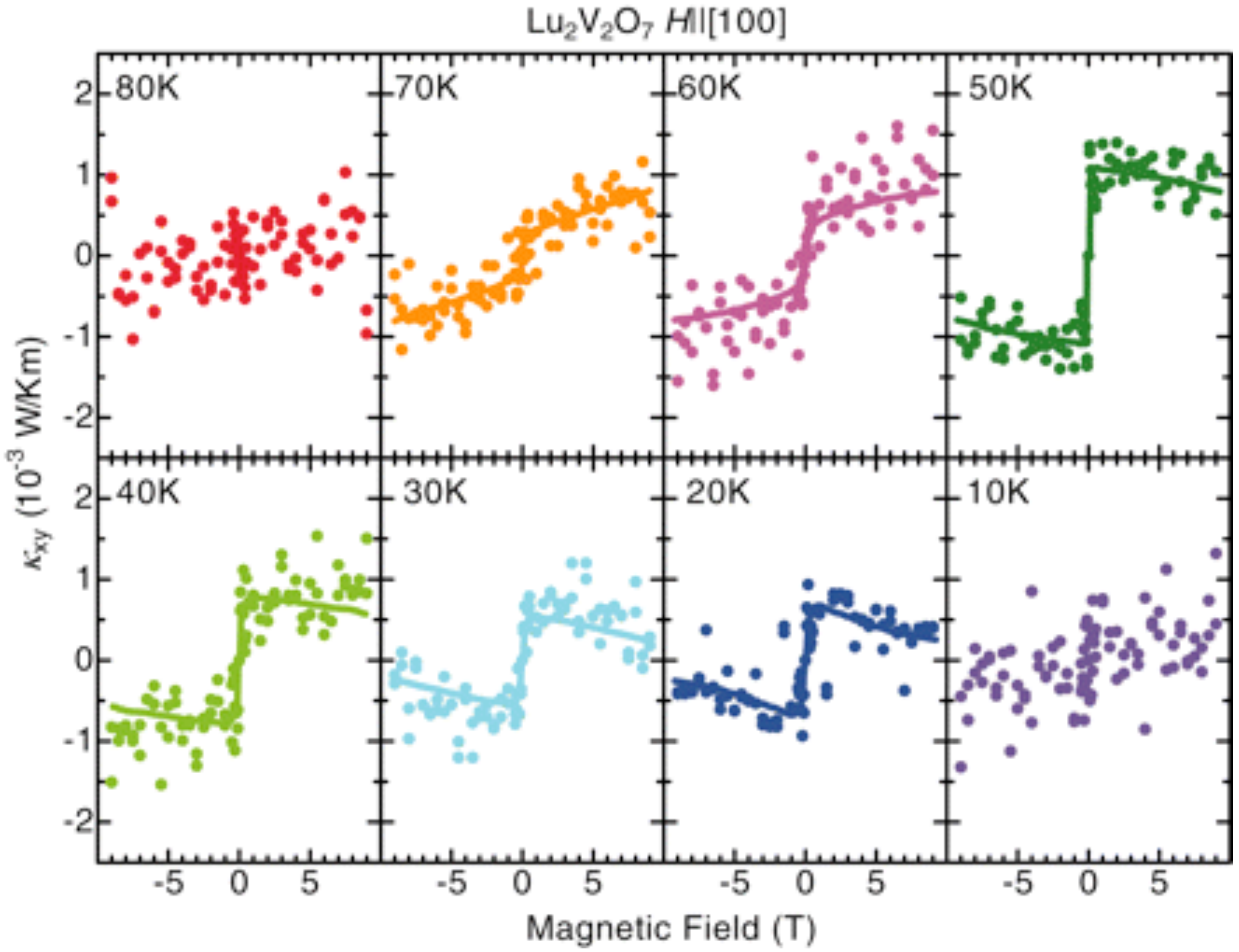
Y. Onose,[1,2]* T. Ideue,[1] H. Katsura,[3] Y. Shiomi,[1,4] N. Nagaosa,[1,4] Y. Tokura[1,2,4]



The crystal structure of $Lu_2V_2O_7$ and the magnon Hall effect. (**A**) The *V* sublattice of $Lu_2V_2O_7$, which is composed of corner-sharing tetrahedra. (**B**) The direction of the Dzyaloshinskii-Moriya vector on each bond of the tetrahedron. The Dzyaloshinskii-Moriya interaction acts between the *i* and *j* sites. (**C**) The magnon Hall effect. **A wave packet of magnon (a quantum of spin precession) moving from the hot to the cold side is deflected by the Dzyaloshinskii-Moriya interaction playing the role of a vector potential.**

$Lu_2V_2O_7$ is a FM insulator

Magnetic field variation of the thermal Hall conductivity of $Lu_2V_2O_7$ at various temperatures. The magnetic field is applied along the [100] direction.



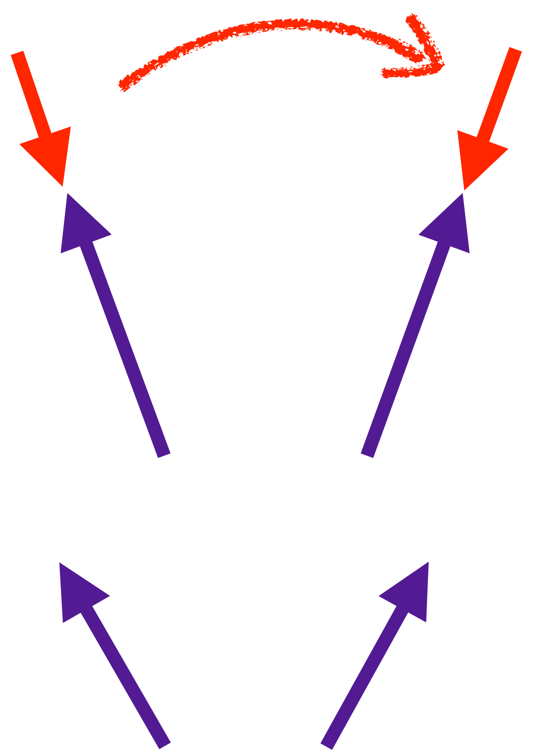
No-go theorem for Bravais lattices

PRL **104,** 066403 (2010) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2010

Theory of the Thermal Hall Effect in Quantum Magnets

Hosho Katsura,[1] Naoto Nagaosa,[1,2] and Patrick A. Lee[3]

hopping magnon



For a finite Hall response, time-reversal symmetry must be broken due to the magnetic field and/or magnetic ordering. The Hall effect in itinerant magnets, where the spin structure and conduction electron motion are coupled, has been studied extensively. In this case, in addition to the usual Lorentz force, the scalar (spin) chirality defined for three spins as $\vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)$ plays an important role [6–8]. The scalar chirality acts as a fictitious magnetic flux for the conduction electrons and gives rise to a nontrivial topology of the Bloch wave functions, leading to the Hall effect. It is natural to expect that a similar effect occurs even in the localized spin systems for, e.g., the spin current [9].

No-go theorem for Bravais lattices

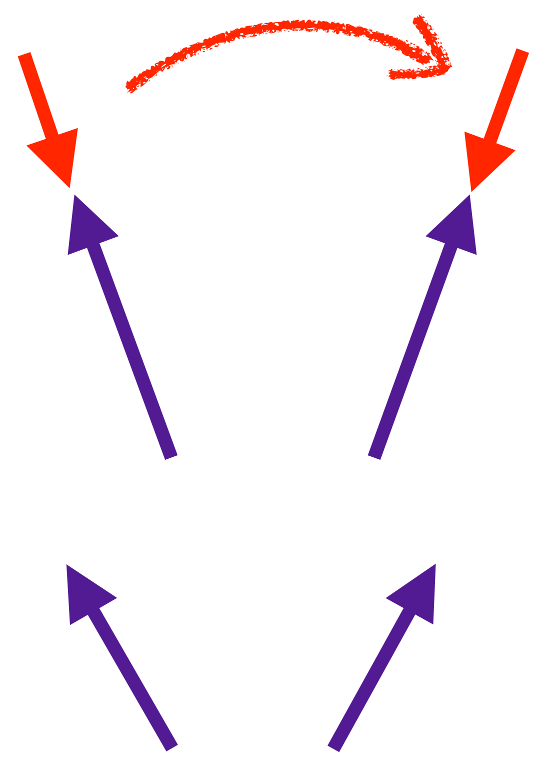
PRL **104,** 066403 (2010) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2010

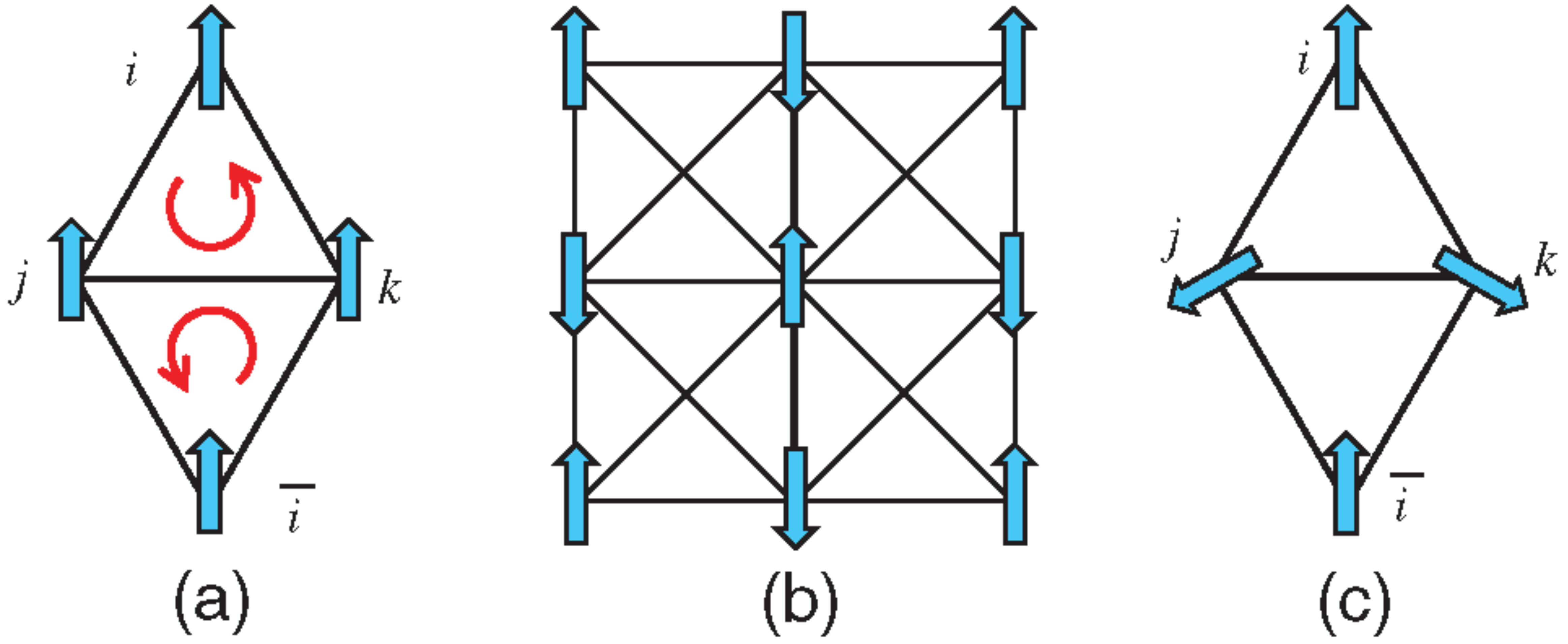
Theory of the Thermal Hall Effect in Quantum Magnets

Hosho Katsura,[1] Naoto Nagaosa,[1,2] and Patrick A. Lee[3]

No-go theorem for the coupling to magnetic flux.—Let us first consider the spin-wave expansion of Eq. (1) to find a system in which the intrinsic thermal Hall effect occurs.

hopping magnon



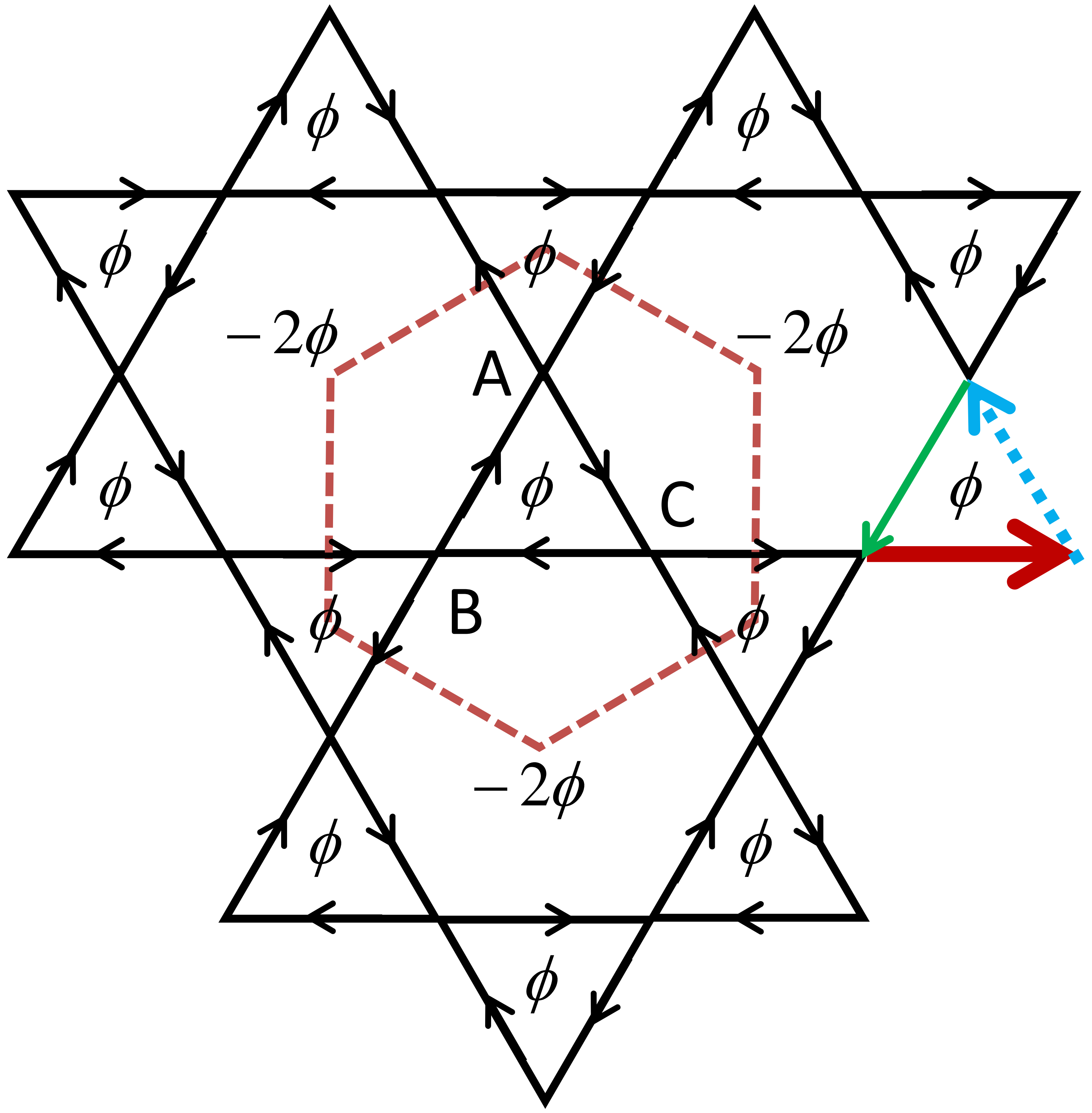


e.g., DM cancels out

PRL **104,** 066403 (2010) PHYSICAL REVIEW LETTERS week ending 12 FEBRUARY 2010

Theory of the Thermal Hall Effect in Quantum Magnets

Hosho Katsura,[1] Naoto Nagaosa,[1,2] and Patrick A. Lee[3]



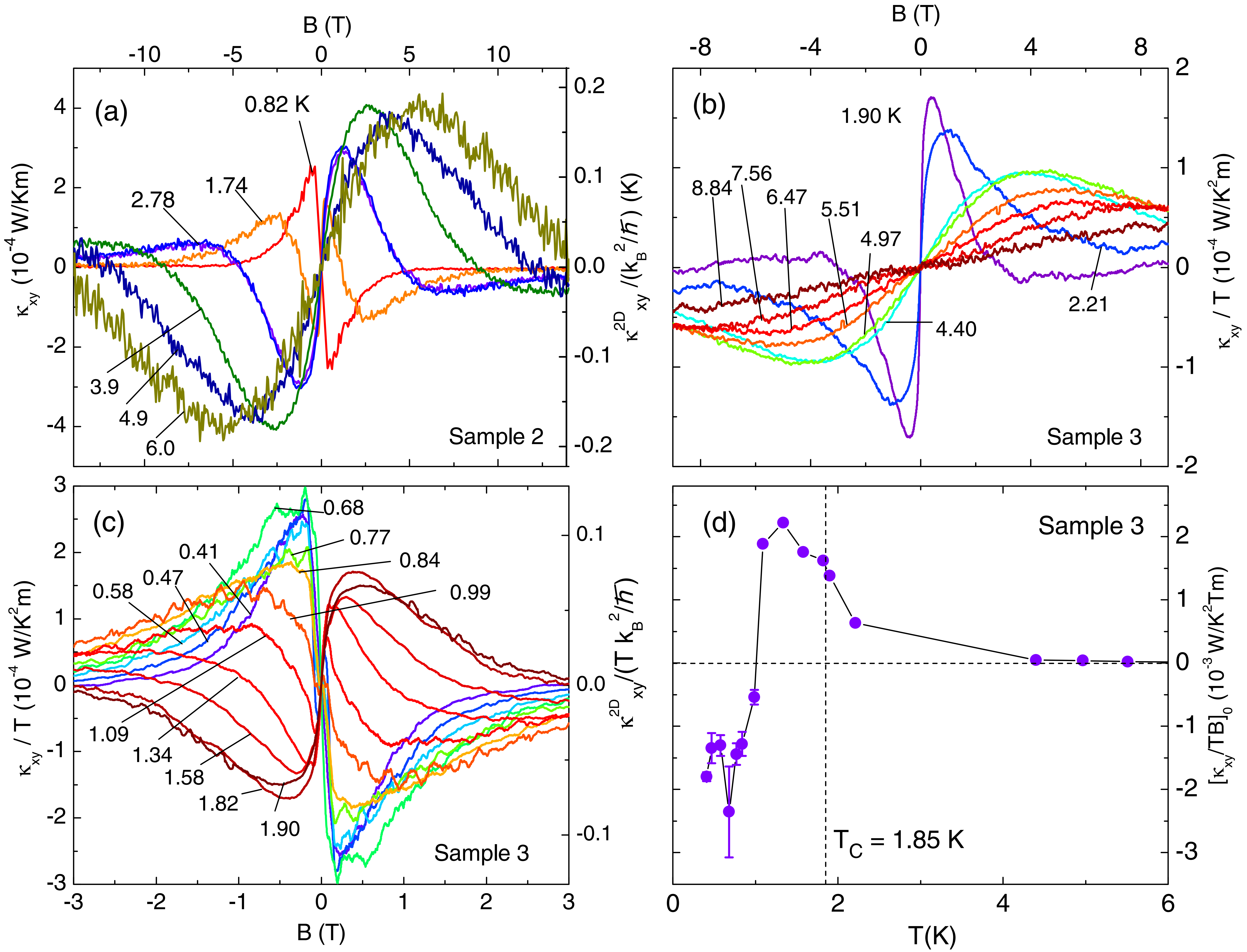
(a)

Intrinsic thermal Hall effect in the spin-wave approximation.—Once we understand the principles of the cancellation, it is rather easy to find an example where it does not occur, namely, when the link is shared by inequivalent cells. An example is the ferromagnetic model on the kagome lattice. In this case, the spin-wave Hamiltonian is

No-go theorem for Bravais lattices (under some conditions)

Thermal Hall Effect of Spin Excitations in a Kagome Magnet

Max Hirschberger, Robin Chisnell, Young S. Lee and N. P. Ong PRL **115**, 106603 (2015)



planar kagome magnet
Cu(1,3-benzenedicarboxylate)

Article https://doi.org/10.1038/s41467-023-44448-9 nature communications

Thermal Hall effects due to topological spin fluctuations in $YMnO_3$

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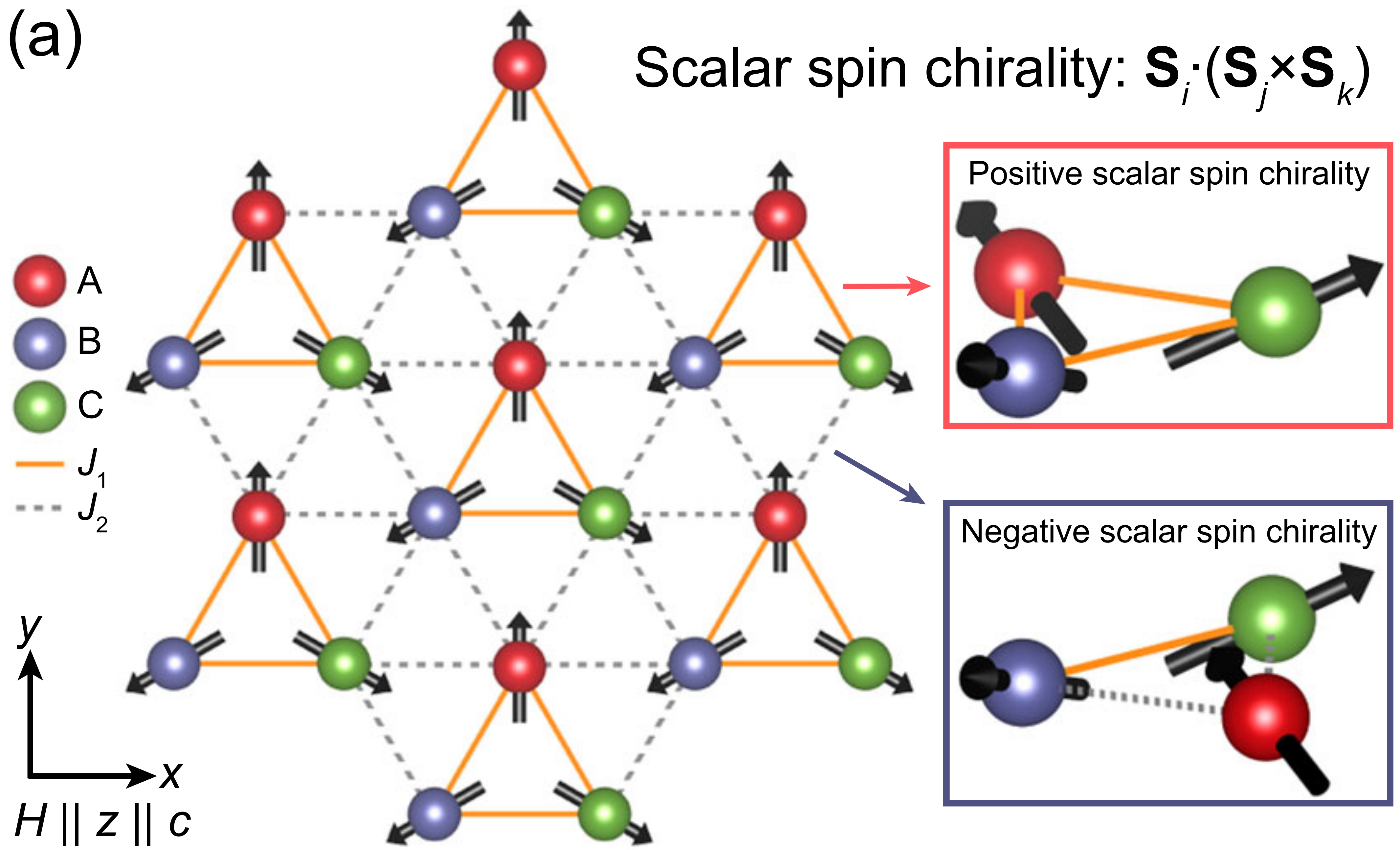
Ha-Leem Kim[1,2,9], Takuma Saito[3,9], Heejun Yang[1,2,9], Hiroaki Ishizuka[4], Matthew John Coak[1,2,5], Jun Han Lee[6], Hasung Sim[1,2], Yoon Seok Oh[6], Naoto Nagaosa[7] & Je-Geun Park[1,2,8]

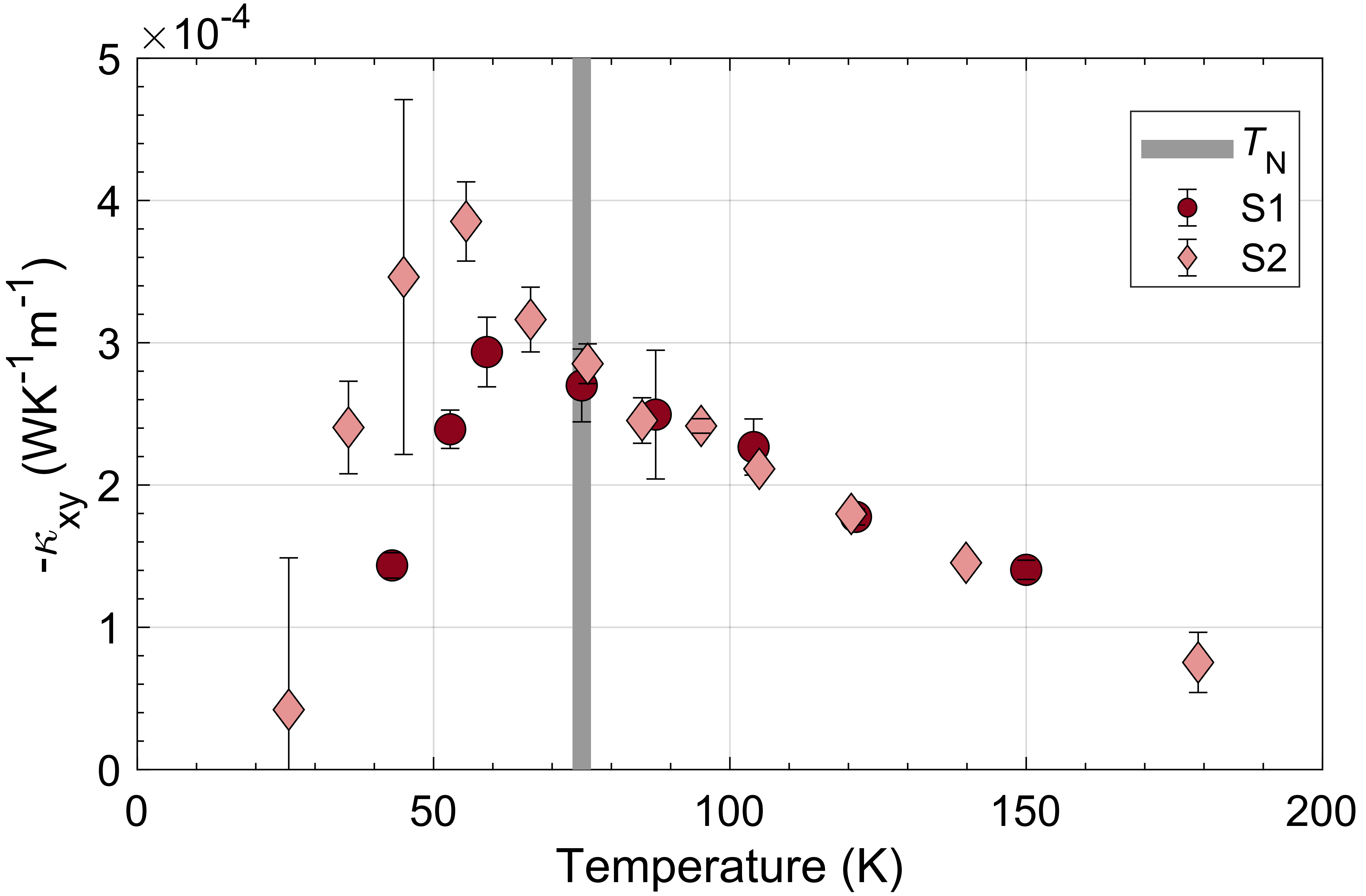
$$H = \sum_{\langle i,j \rangle} \left[J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \mathbf{D}_{ij} \cdot \left(\mathbf{S}_i \times \mathbf{S}_j \right) \right] - \sum_i \mathbf{h} \cdot \mathbf{S}_i + \sum_i \mathbf{S}_i^{\mathrm{T}} \Delta \mathbf{S}_i,$$

$$\kappa_{xy}^{\mathrm{Kubo}} = \frac{1}{k_{\mathrm{B}} T^2} \int_0^{\infty} \mathrm{d}t \left\langle J^{\mathrm{E},x}(t); J^{\mathrm{E},y}(0) \right\rangle,$$

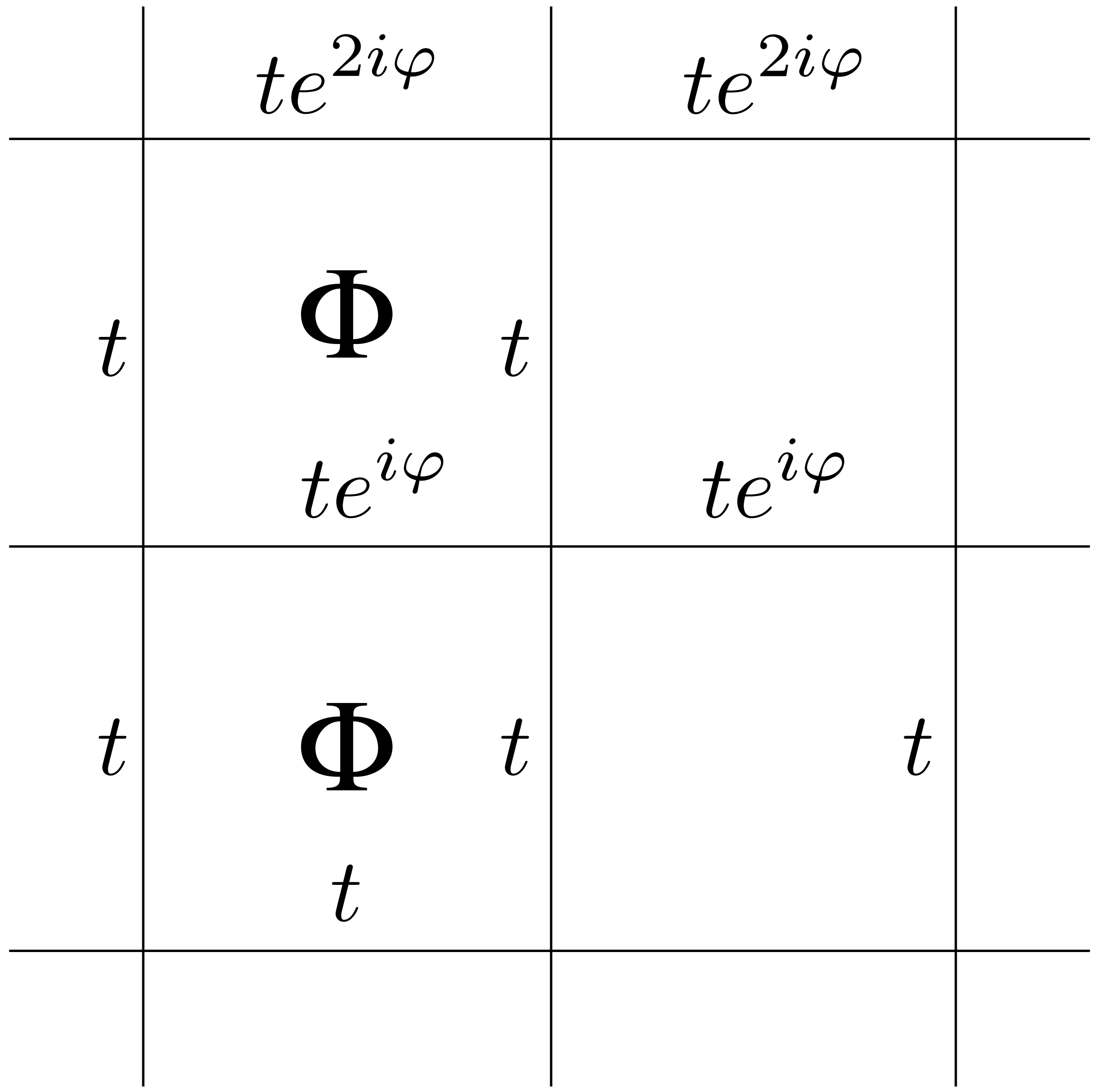
Landau–Lifshitz–Gilbert dynamics



trimerization



Field or no field ?



electrons in magnetic fields:

- the hopping amplitudes acquire a finite phase
- phase around the plaquette = Φ/Φ_0
- the complex hopping will lead to Berry curvature and Chern numbers
- Berry curvature is the manifestation of the Lorentz force

We can do without field (cf anomalous Hall effect)!

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane

Department of Physics, University of California, San Diego, La Jolla, California 92093

(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

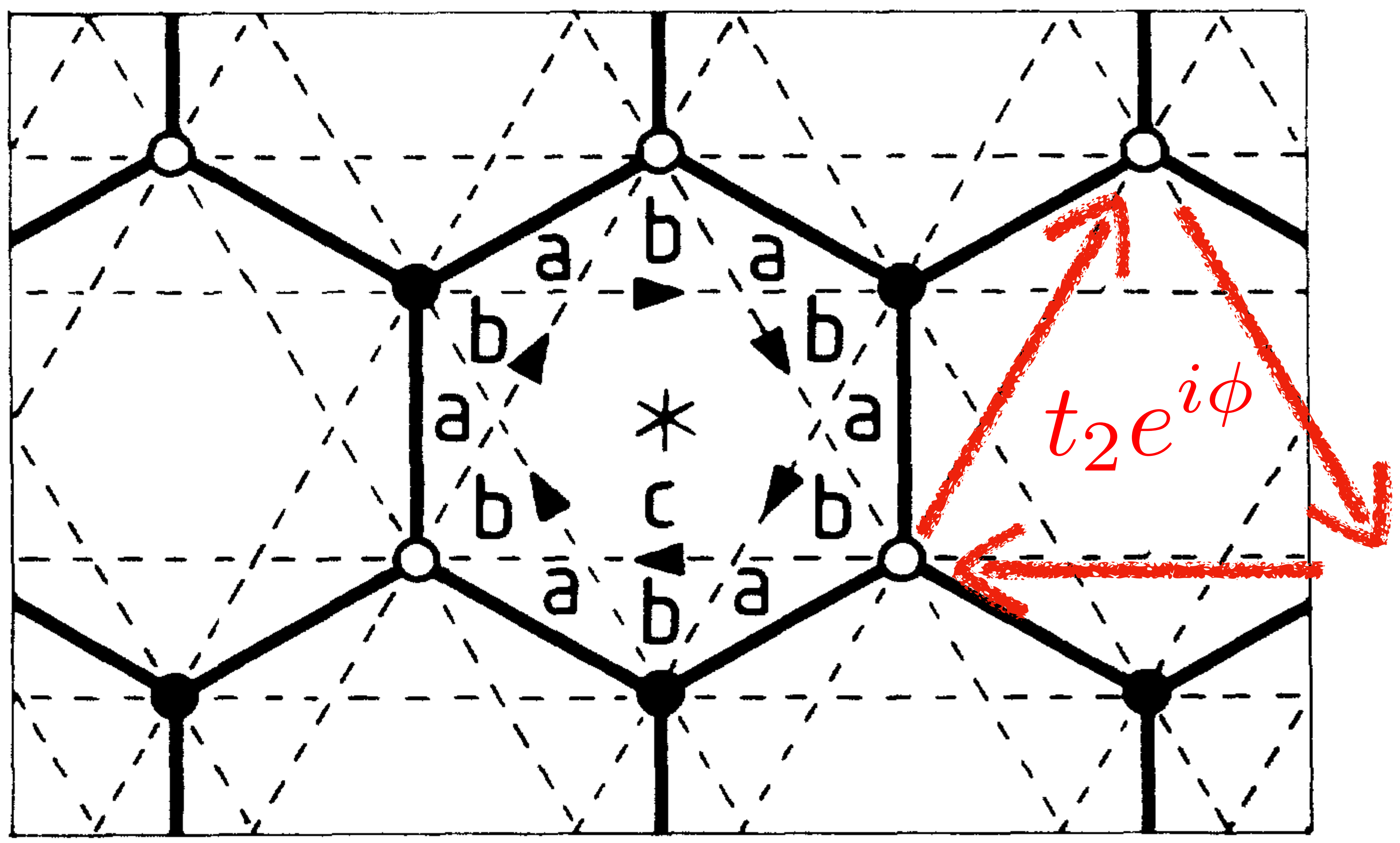
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane

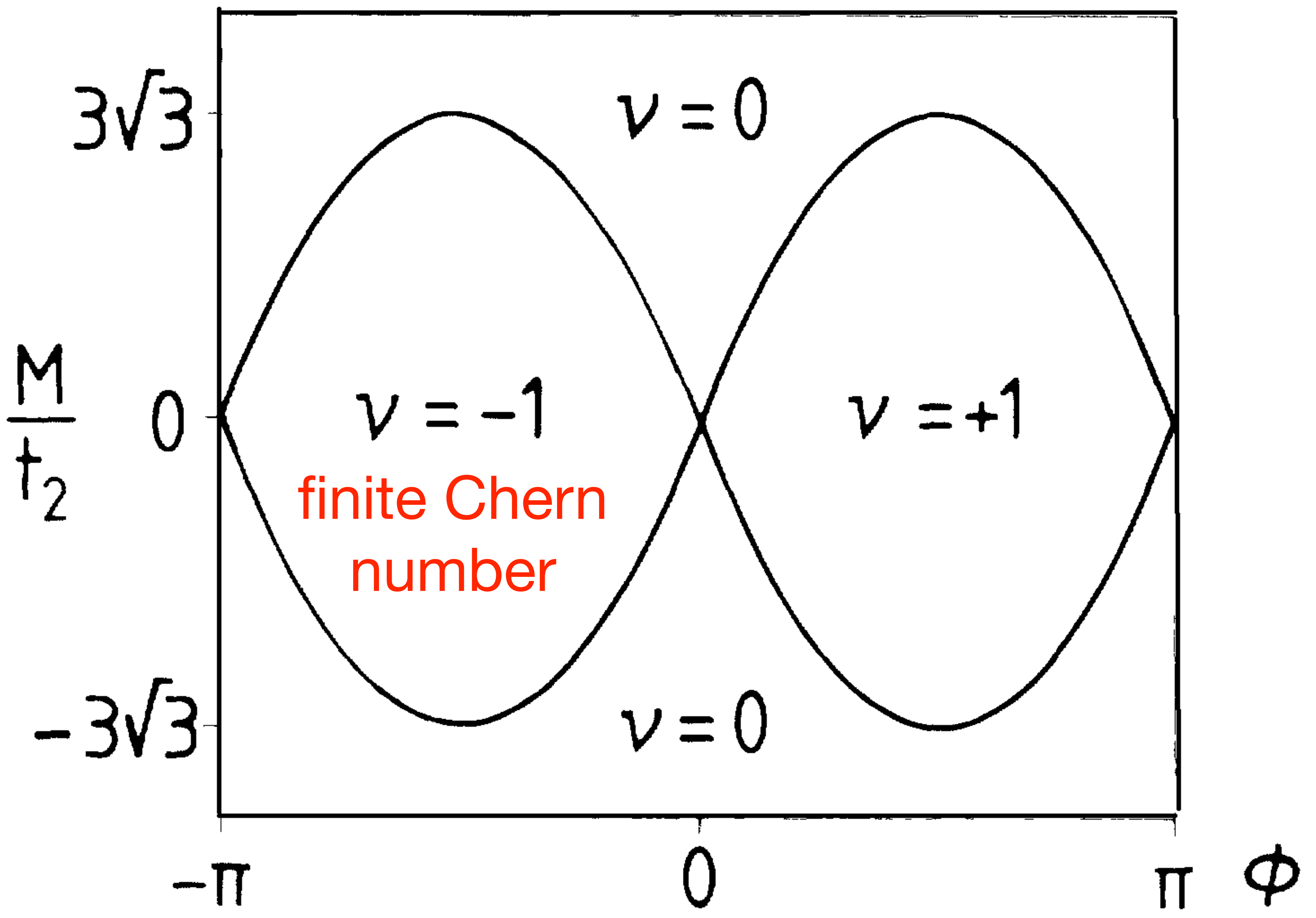
The H(k) is a 2×2 matrix (fermions on the filled and empty circles):

$$\mathbf{H}(\mathbf{k}) = 2t_2\cos\phi\left(\sum_i \cos(\mathbf{k}\cdot\mathbf{b}_i)\right)\mathbf{I} + t_1\left(\sum_i [\cos(\mathbf{k}\cdot\mathbf{a}_i)\sigma^1 + \sin(\mathbf{k}\cdot\mathbf{a}_i)\sigma^2]\right) + \left[M - 2t_2\sin\phi\left(\sum_i \sin(\mathbf{k}\cdot\mathbf{b}_i)\right)\right]\sigma^3$$

The model with finite internal fluxes which sum up to 0 for a hexagon



Phase diagram



How about Bravais lattices?

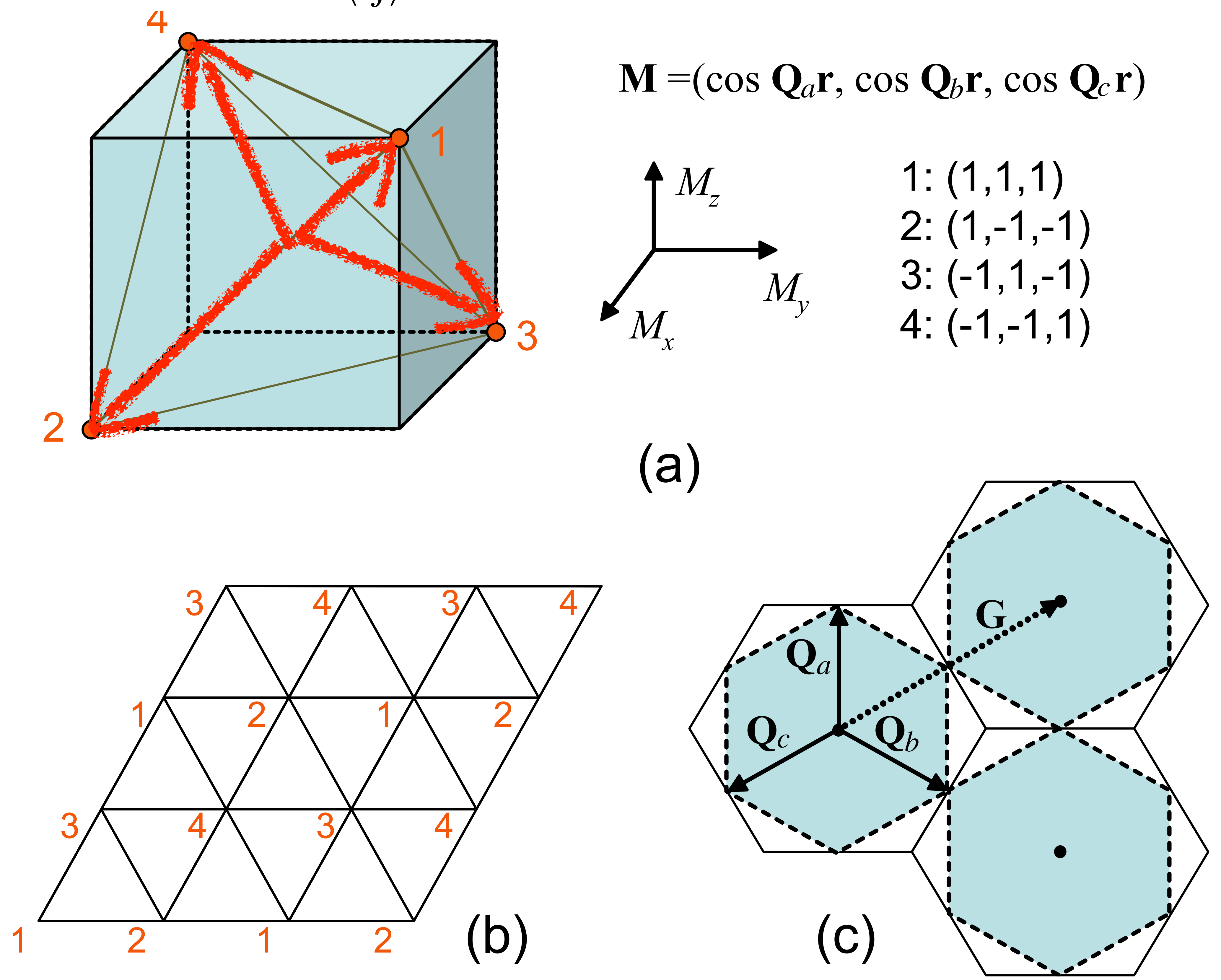
PRL **101**, 156402 (2008) PHYSICAL REVIEW LETTERS week ending 10 OCTOBER 2008

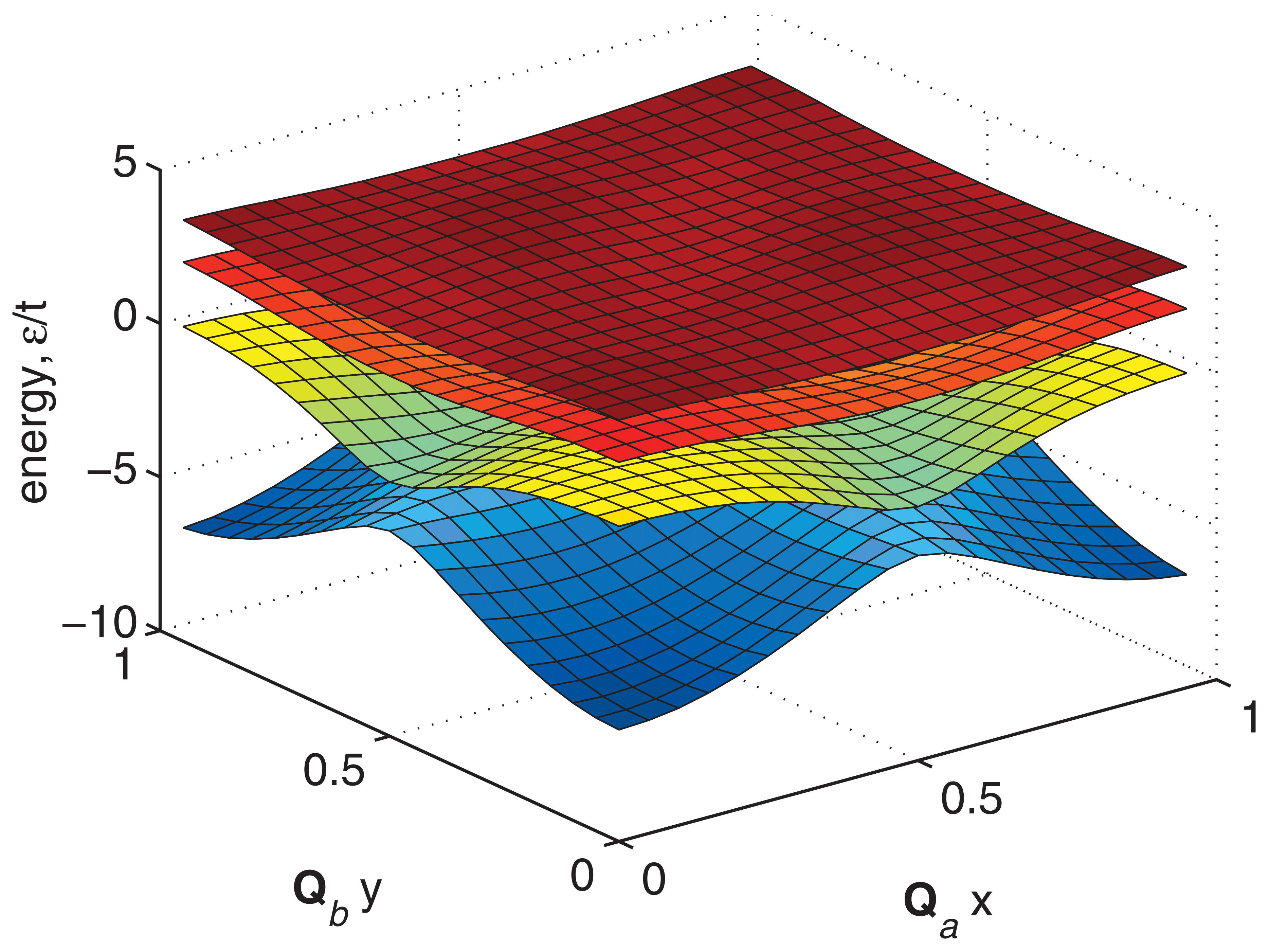
Itinerant Electron-Driven Chiral Magnetic Ordering and Spontaneous Quantum Hall Effect in Triangular Lattice Models

Ivar Martin and C. D. Batista

$$H = -t\sum_{\langle ij\rangle} c_{i\alpha}^{\dagger} c_{j\alpha} - J\sum_{i} \mathbf{S}_i \cdot c_{i\alpha}^{\dagger} \boldsymbol{\sigma}_{\alpha\beta} c_{i\beta},$$

Kondo Lattice Hamiltonian





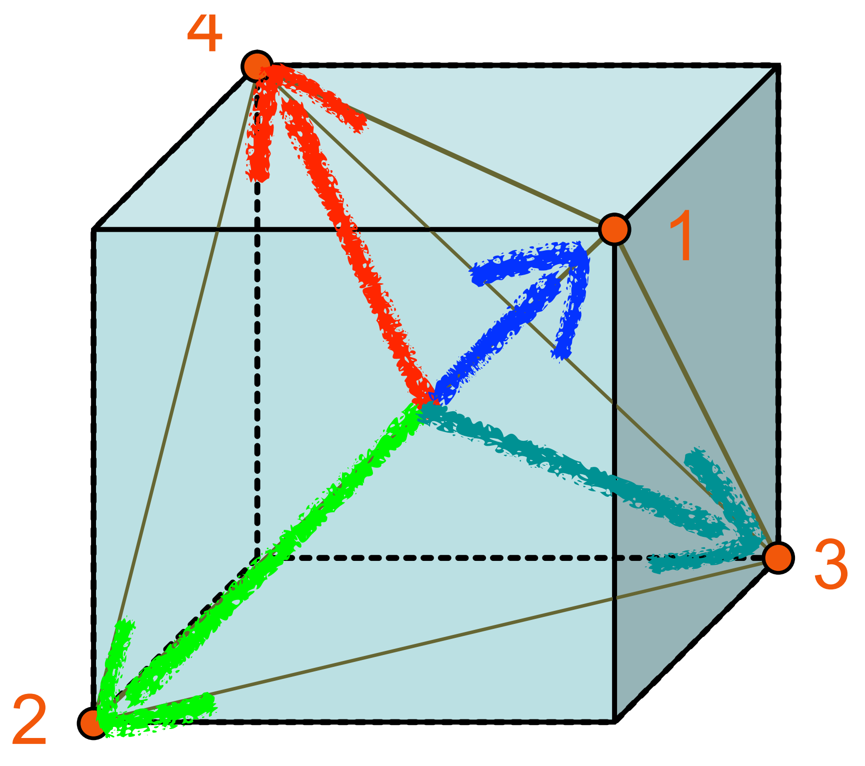
A chiral configuration mimicking external flux

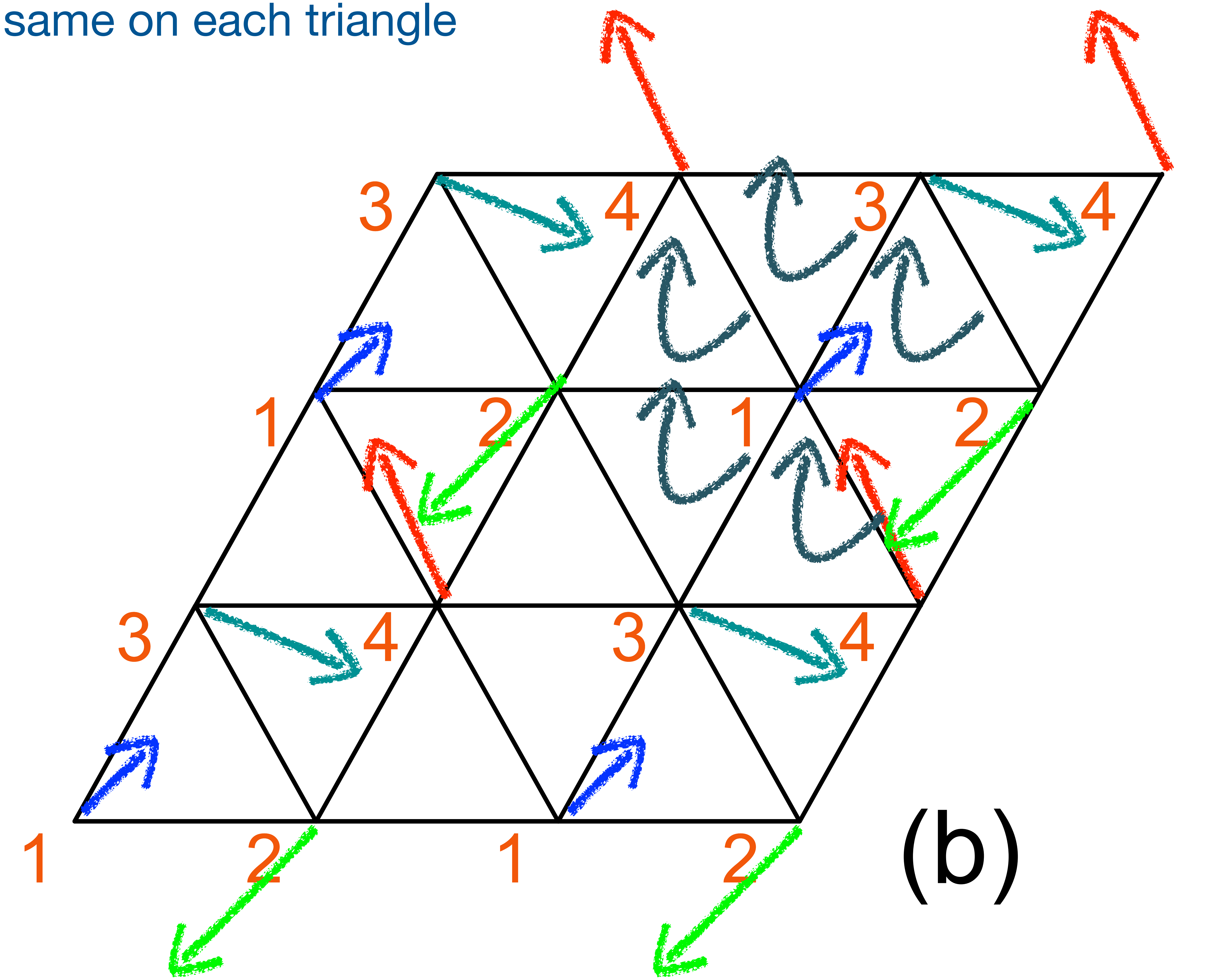
-a triple-**q** structure

-the scalar chirality $\chi_{ijk} = \mathbf{S}_i \cdot \left(\mathbf{S}_j \times \mathbf{S}_k\right)$ is the same on each triangle

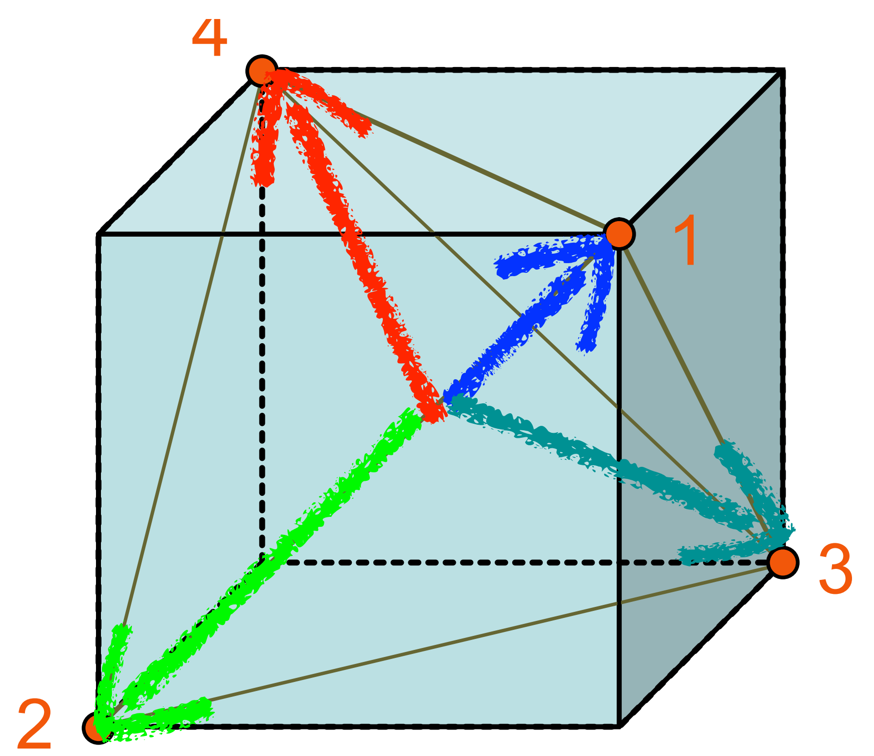
-spherical angle $\Omega = \frac{1}{4}(4\pi) = \pi$

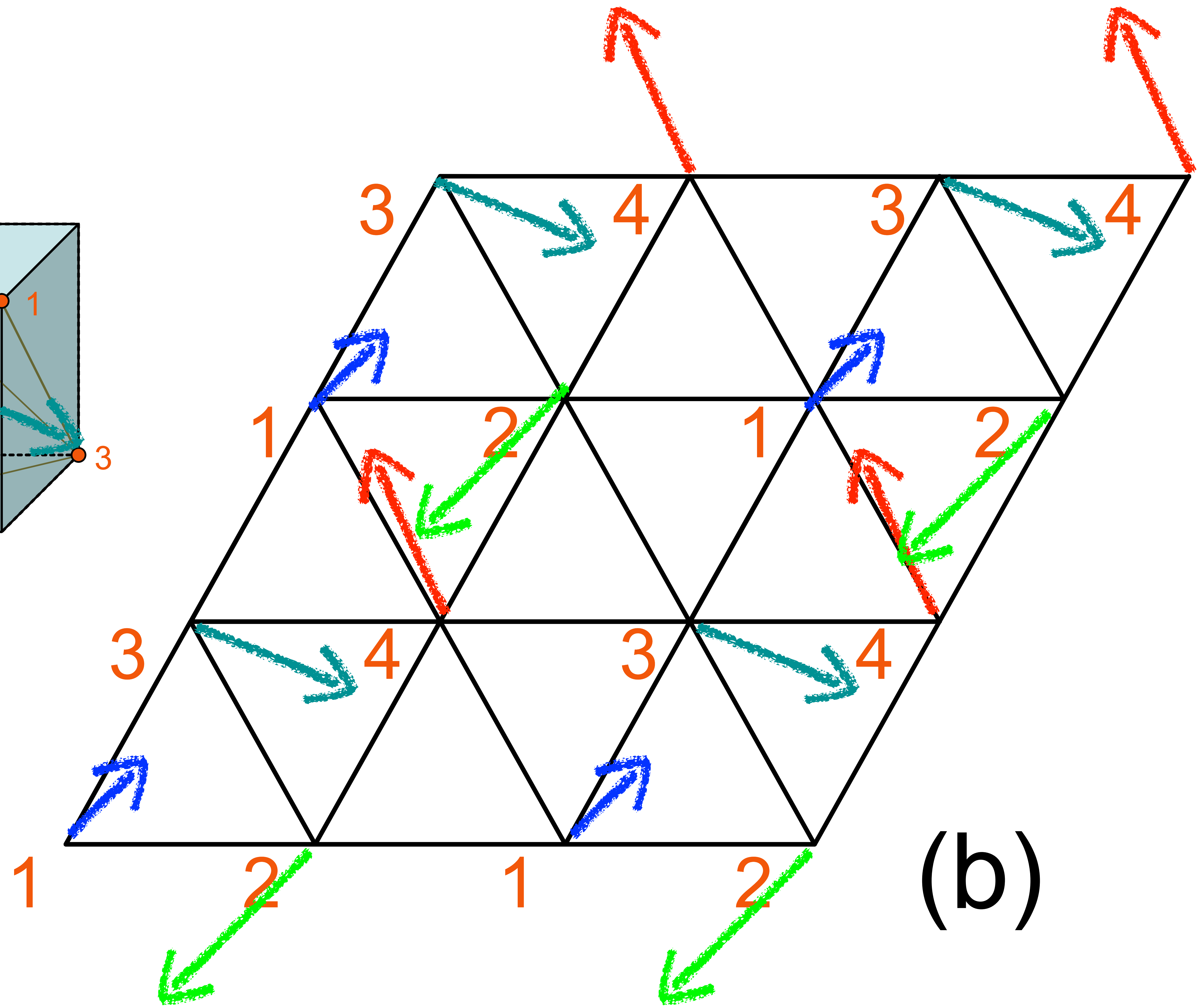
-flux in each triangle is $\Phi = S\Omega = \frac{\pi}{2}$





Where do we find such a configuration?





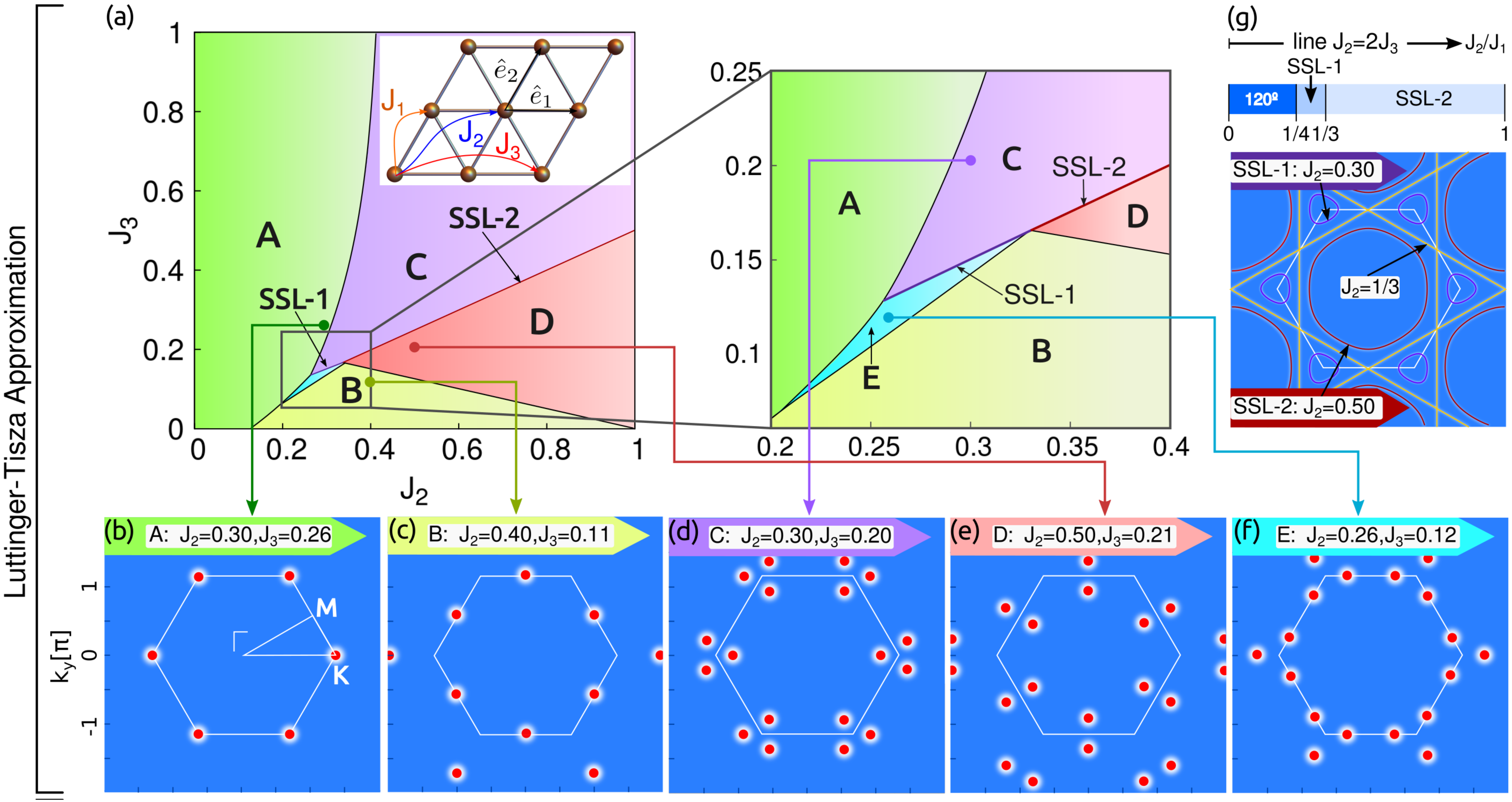
(b)

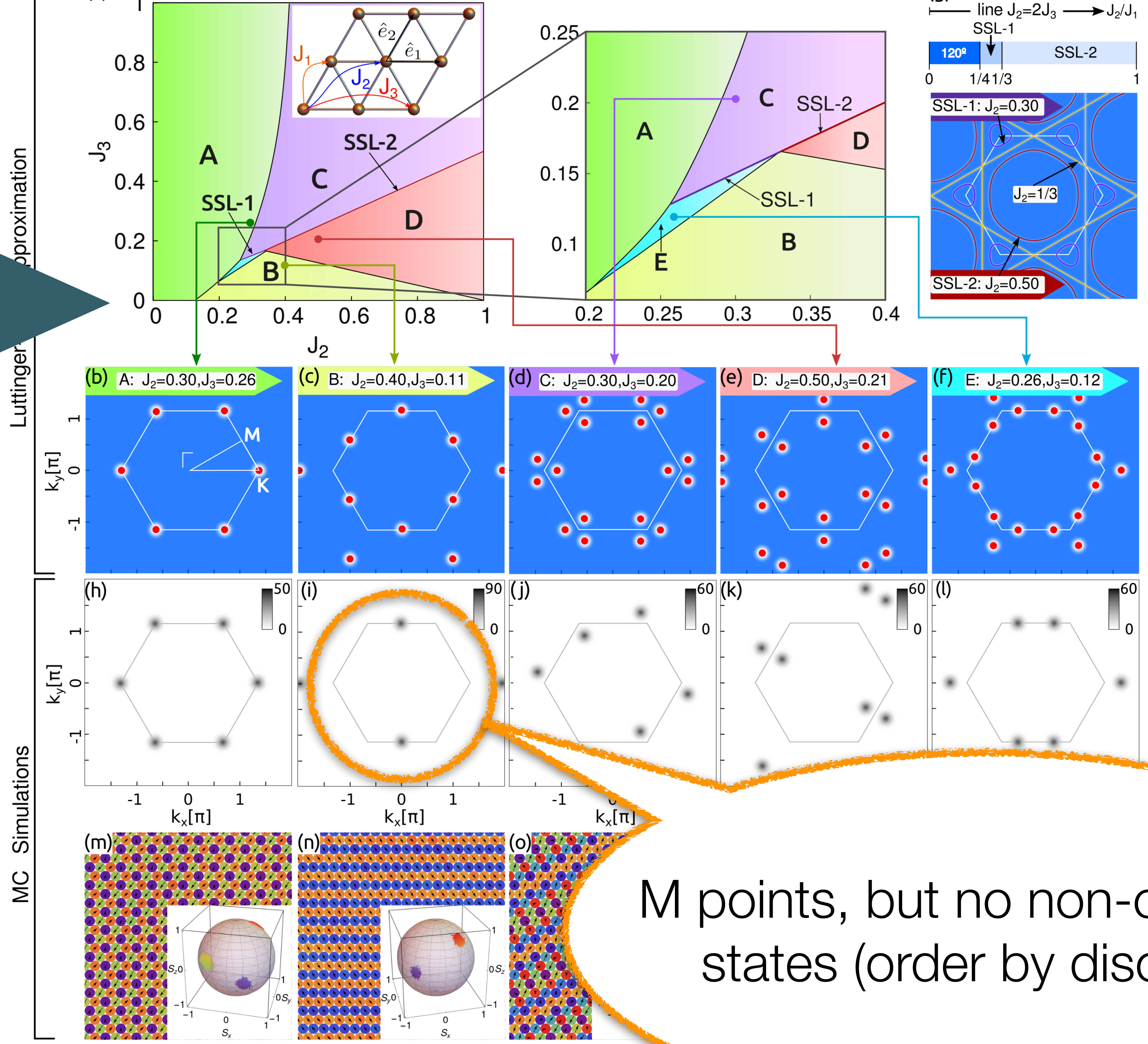
S=1/2 triangular lattice,

semiclassical approach

Spontaneous antiferromagnetic skyrmion/antiskyrmion lattice and spiral spin-liquid states in the frustrated triangular lattice

M. Mohylna,[1] F. A. Gómez Albarracín,[2,3,4] M. Žukovič,[1] and H. D. Rosales[2,3,4,*]





M points, but no non-coplanar states (order by disorder)

Where do we find such a configuration?

How do we stabilise a non-coplanar (chiral) state?

We must fight the tendency towards collinearity in the multi-q states resulting from quantum and thermal fluctuations.

The 4-site ring exchange may work:

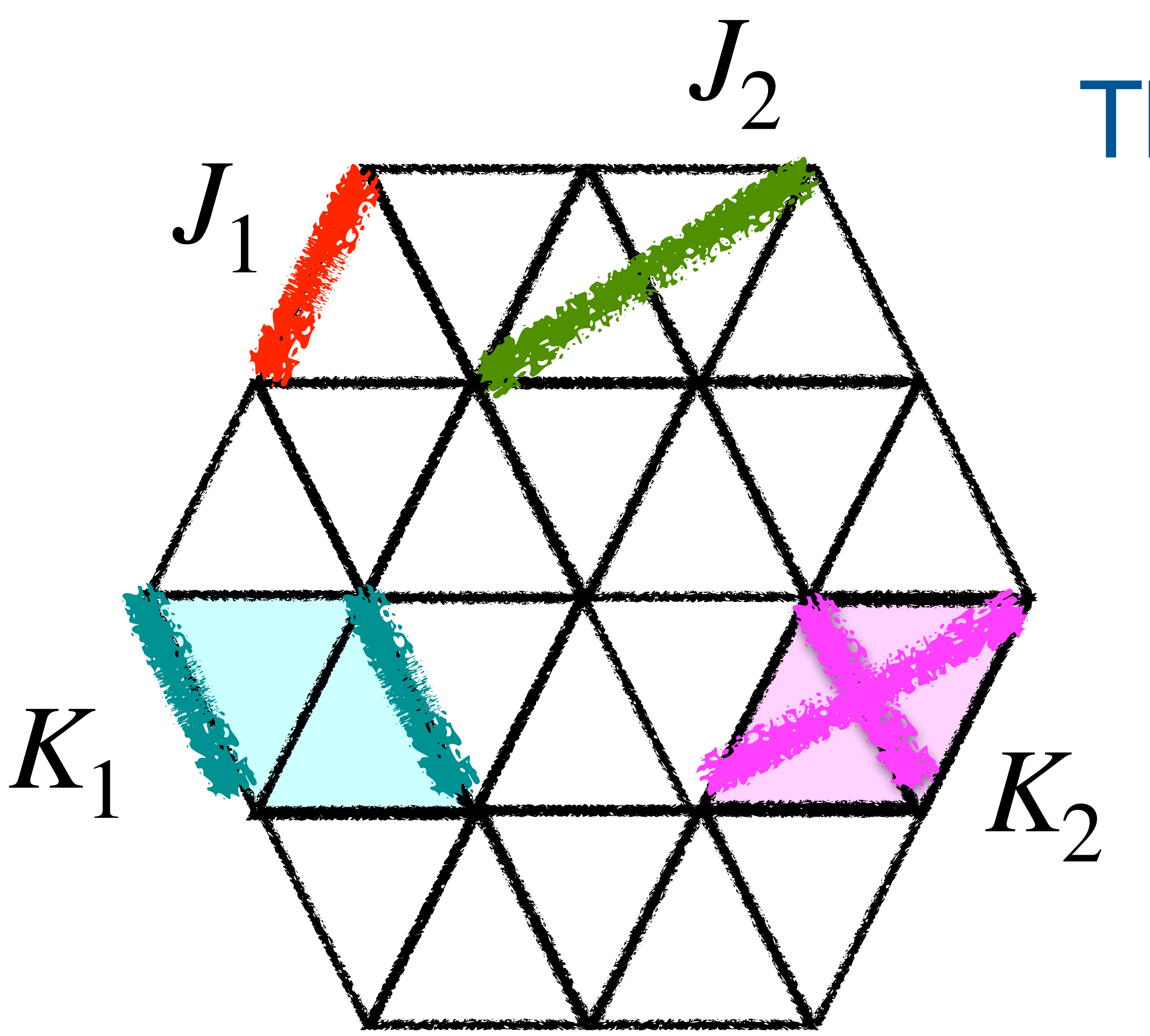
$$\mathcal{P}_{1,2,3,4} + \mathcal{P}_{1,4,3,2} = \frac{1}{4} + \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

$$+ 4 \sum_{\langle i,j,k,l \rangle_{\text{tet}}} \left[\left(\mathbf{S}_1 \cdot \mathbf{S}_2 \right) \left(\mathbf{S}_3 \cdot \mathbf{S}_4 \right) + \left(\mathbf{S}_1 \cdot \mathbf{S}_4 \right) \left(\mathbf{S}_2 \cdot \mathbf{S}_3 \right) - \left(\mathbf{S}_1 \cdot \mathbf{S}_3 \right) \left(\mathbf{S}_2 \cdot \mathbf{S}_4 \right) \right]$$

How do we stabilise a non-coplanar (chiral) state?

4-site ring exchange may work:

$$\mathcal{P}_{1,2,3,4} + \mathcal{P}_{1,4,3,2} = \frac{1}{4} + \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + 4 \sum_{\langle i,j,k,l \rangle_{\text{tet}}} \left[(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_3 \cdot \mathbf{S}_4) + (\mathbf{S}_1 \cdot \mathbf{S}_4)(\mathbf{S}_2 \cdot \mathbf{S}_3) - (\mathbf{S}_1 \cdot \mathbf{S}_3)(\mathbf{S}_2 \cdot \mathbf{S}_4) \right]$$

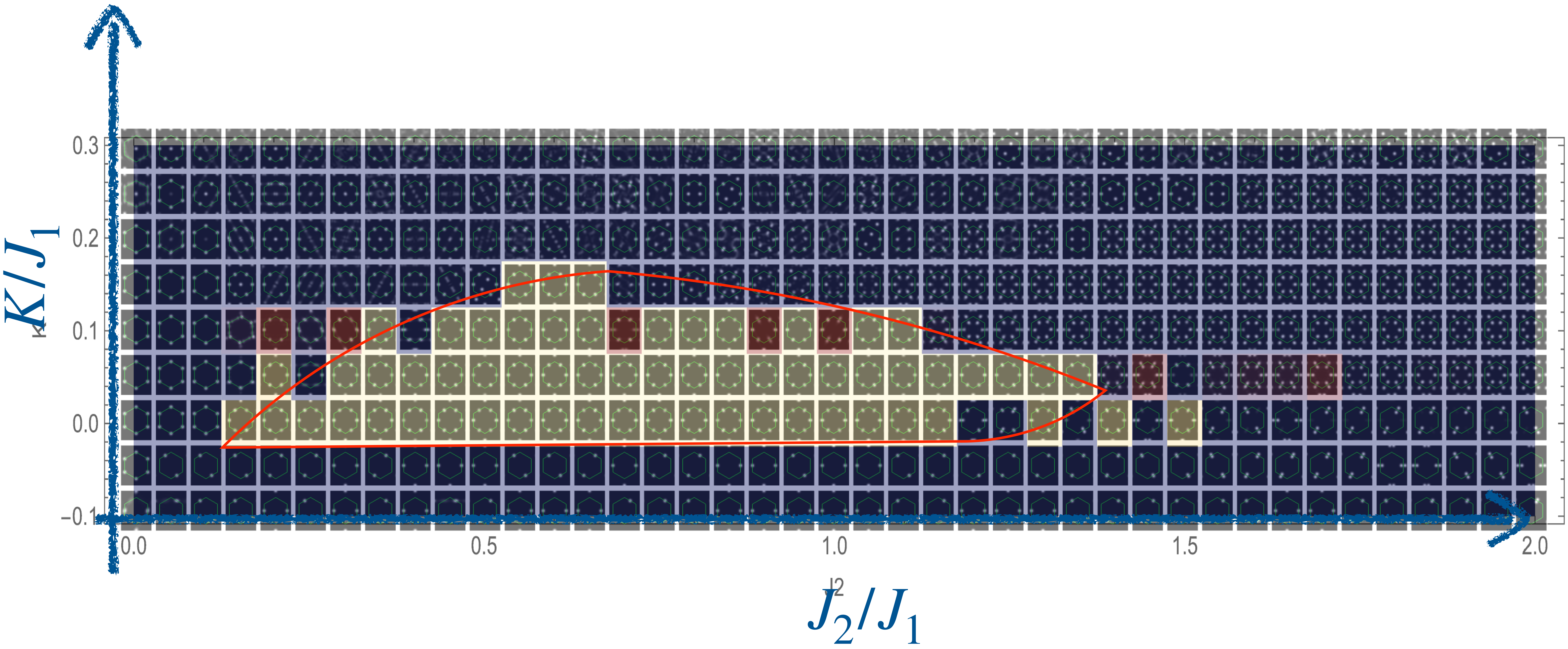


The 4-site Hamiltonian we consider:

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle_{\text{1st}}} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle i,j \rangle_{\text{2nd}}} \mathbf{S}_i \cdot \mathbf{S}_j + K_1 \sum_{\langle i,j,k,l \rangle_{\text{tet}}} \left[(\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l) + (\mathbf{S}_i \cdot \mathbf{S}_k)(\mathbf{S}_j \cdot \mathbf{S}_l) \right] + K_2 \sum_{\langle i,j,k,l \rangle_{\text{tet}}} (\mathbf{S}_i \cdot \mathbf{S}_l)(\mathbf{S}_j \cdot \mathbf{S}_k)$$

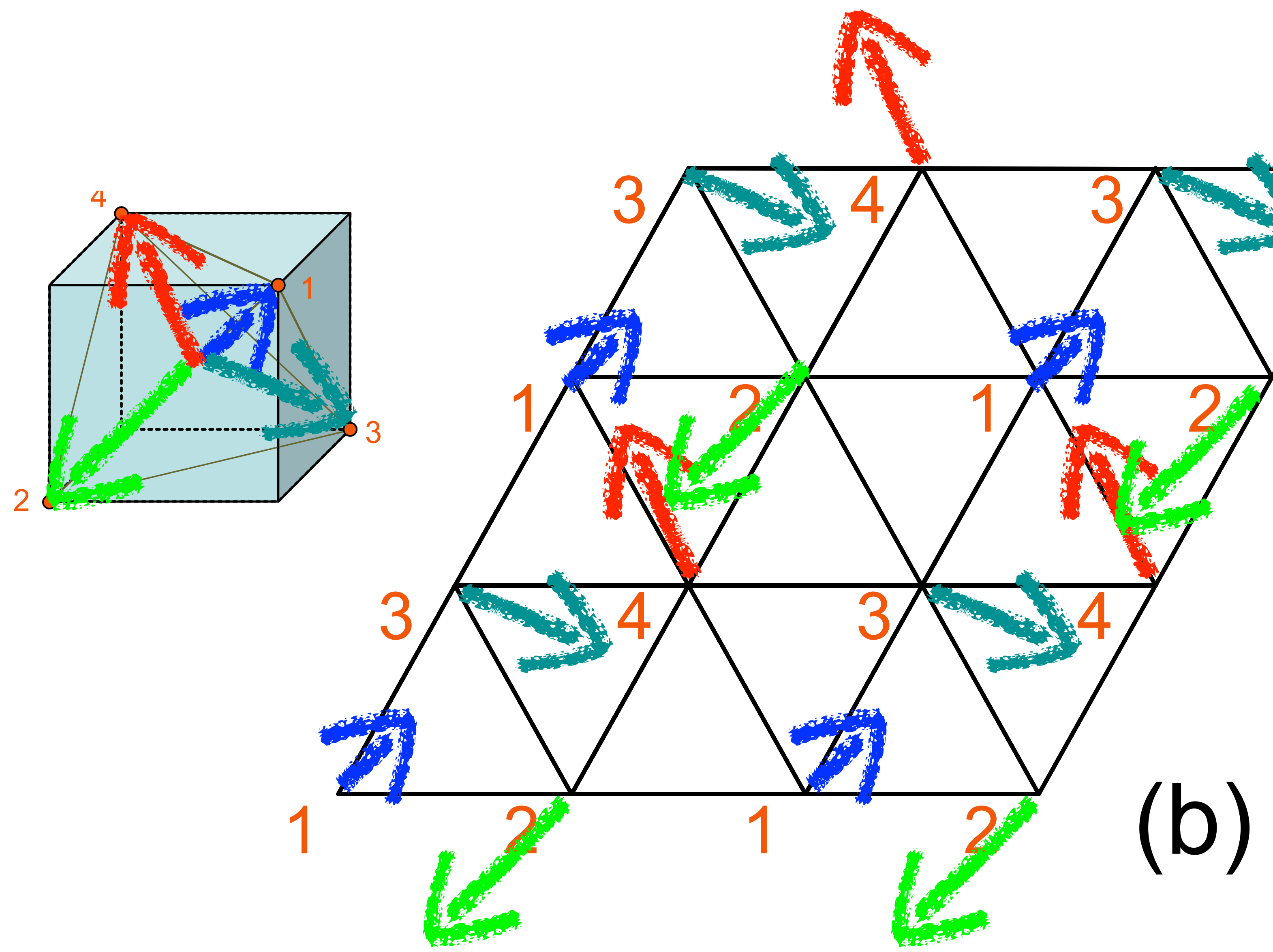
Does it work?

Classical minimisations

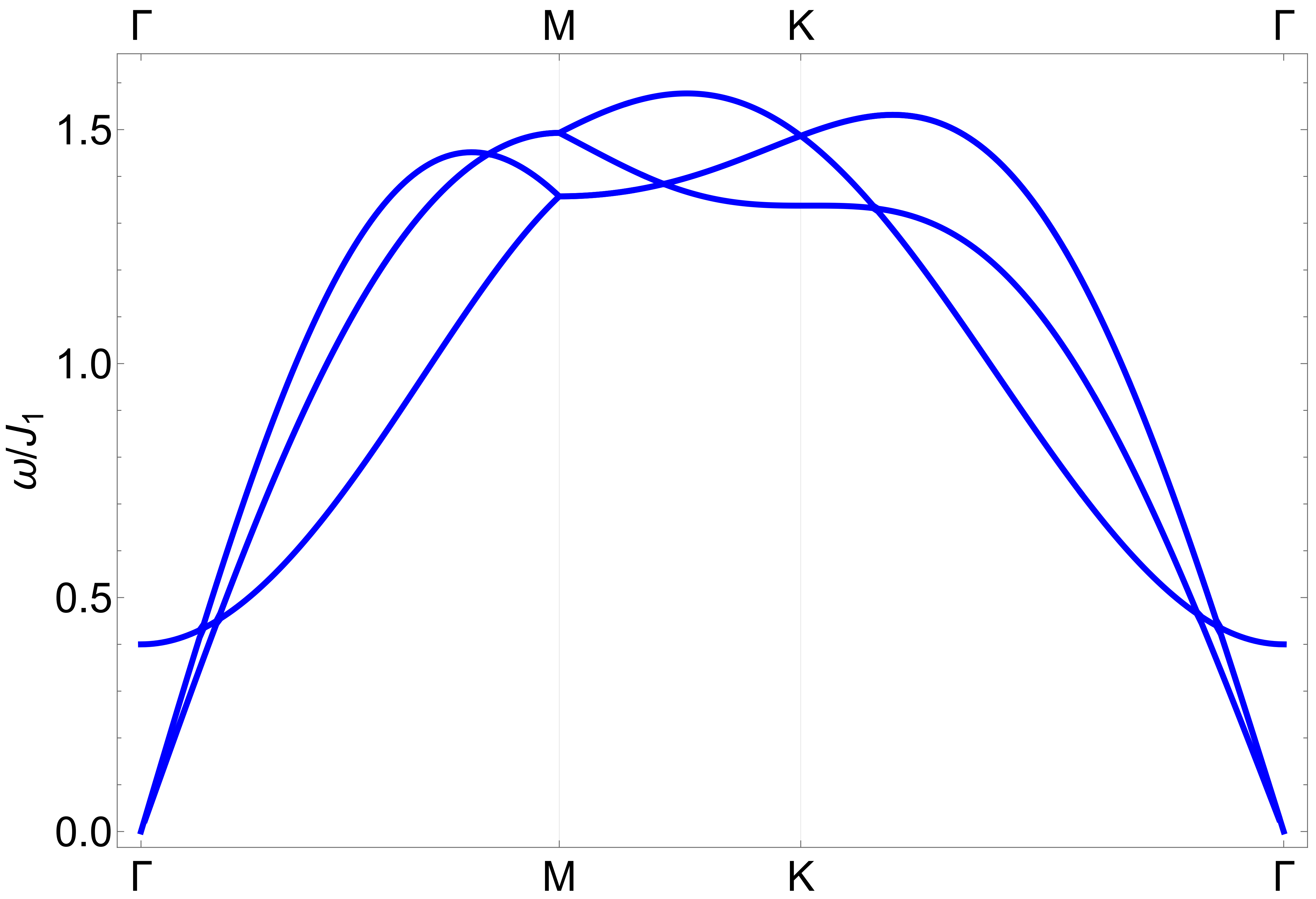


Classical minimisations

Spin waves in tetrahedral/tetratic state



Spin wave dispersions:



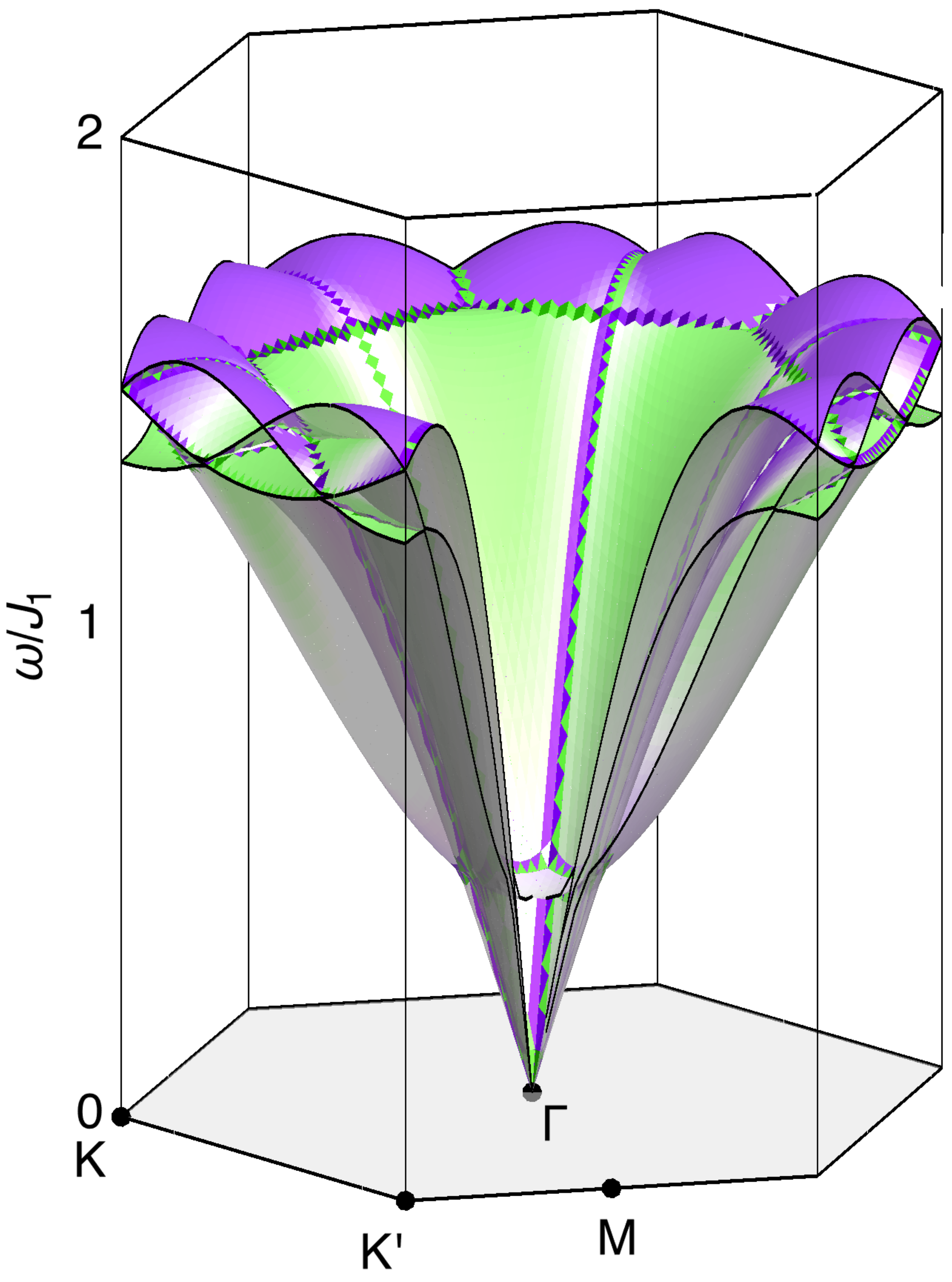
(i) Three Goldstone modes.

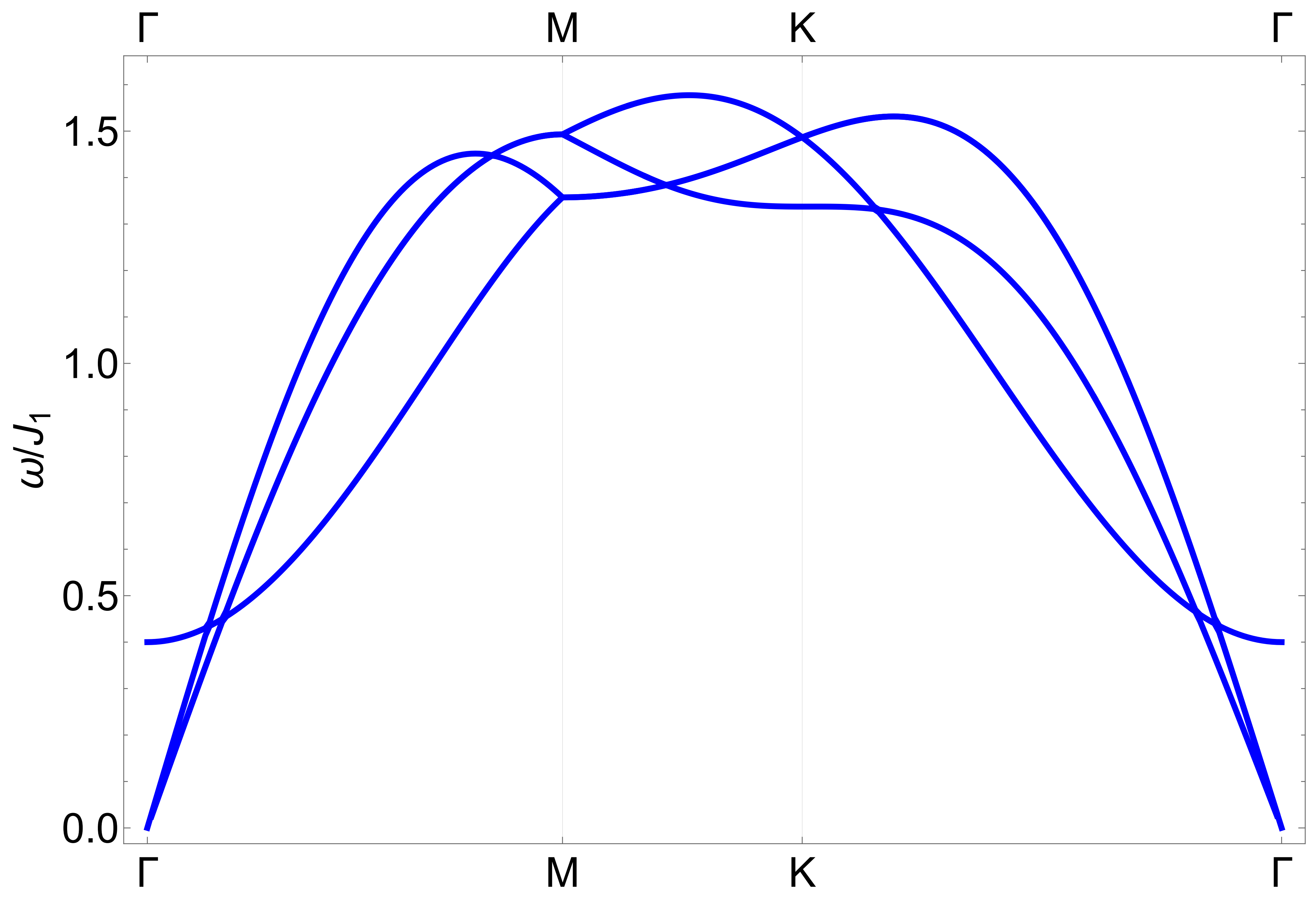
(ii) $\Delta(\mathbf{q}=0)=4K$.

(iii) Finite Berry curvature

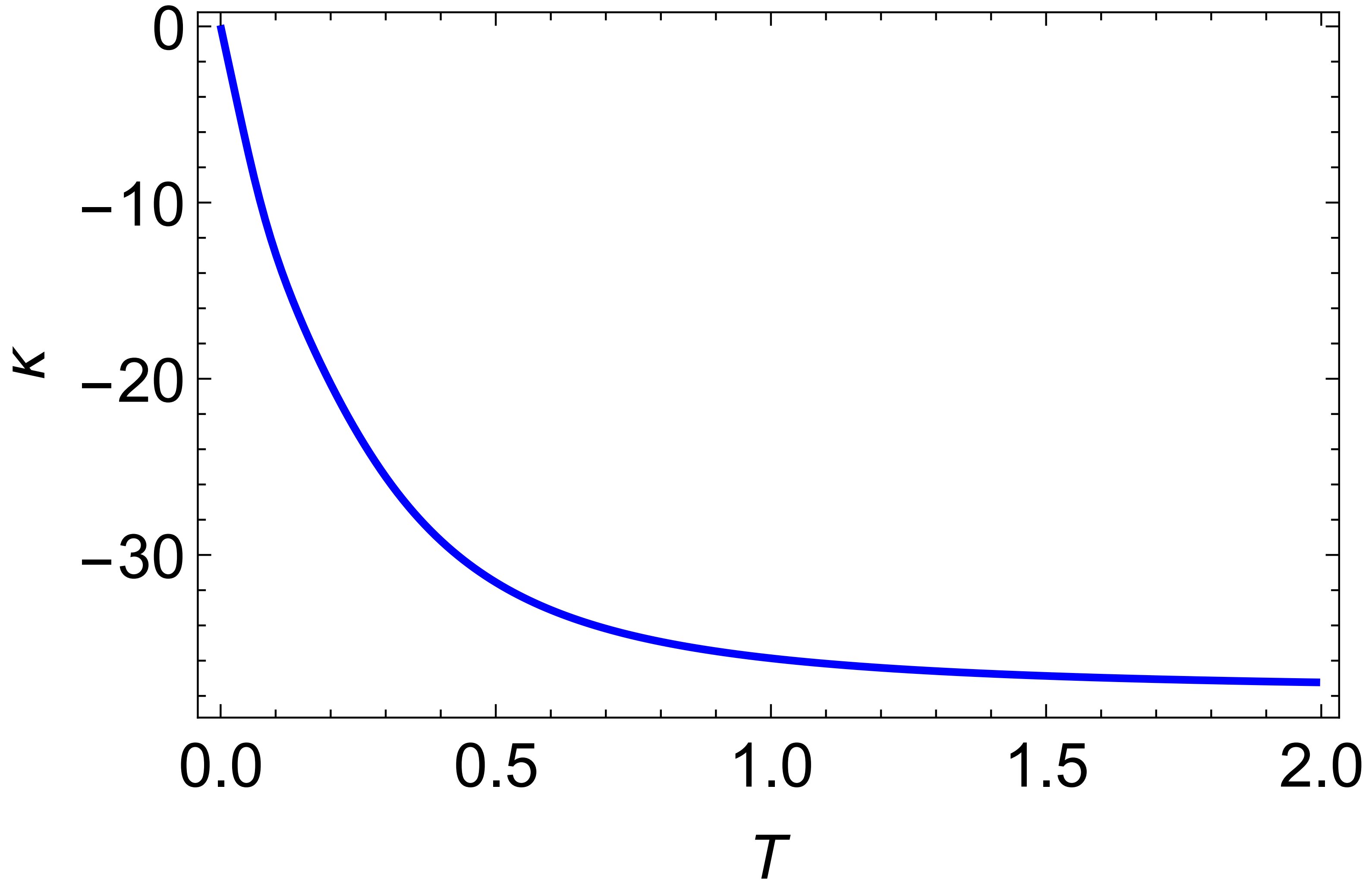
$J_1 = 1, \ J_2 = 0.5,$ and $K = 0.1$

Finite Berry curvature





Thermal Hall effect:



Quantum effects:

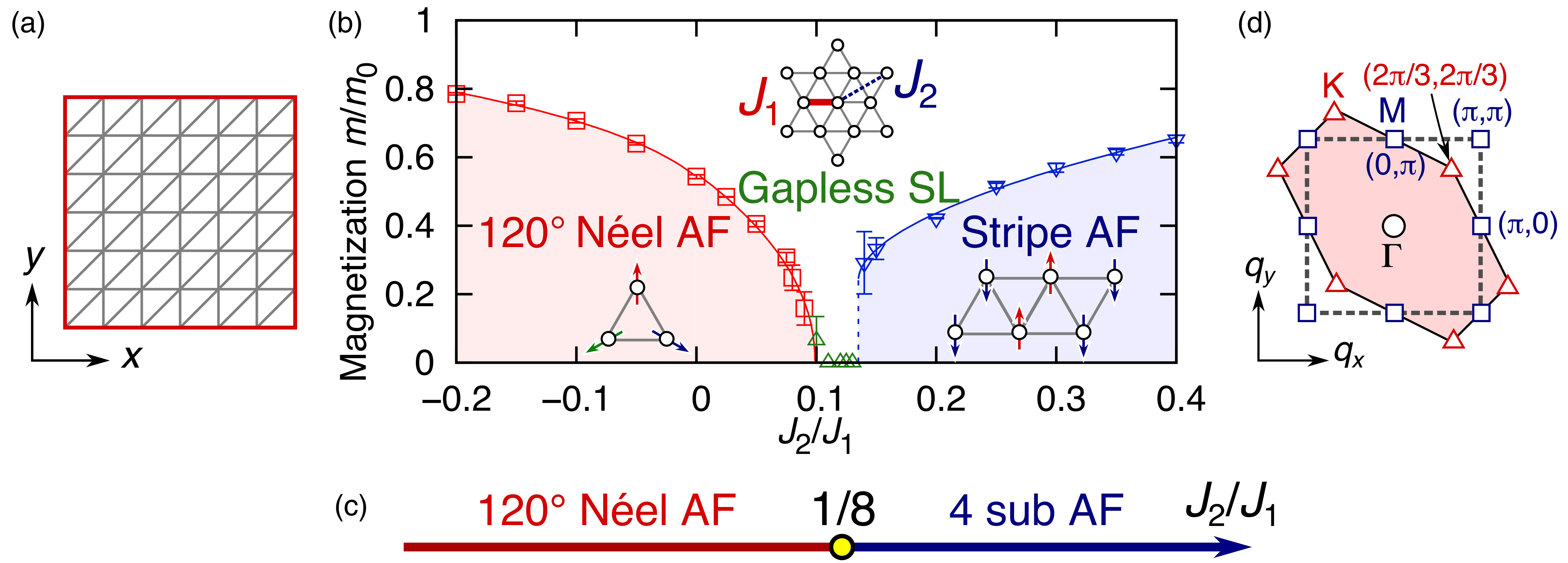
S=1/2 triangular lattice

Journal of the Physical Society of Japan **83**, 093707 (2014) **Letters**
http://dx.doi.org/10.7566/JPSJ.83.093707

Gapless Spin-Liquid Phase in an Extended Spin 1/2 Triangular Heisenberg Model

Ryui Kaneko[1*], Satoshi Morita[2], and Masatoshi Imada[3]

ground states and low-energy excitations up to 18×18 sites
by a many-variable variational Monte Carlo (mVMC) method



Journal of the Physical Society of Japan **83**, 093707 (2014) **Letters**
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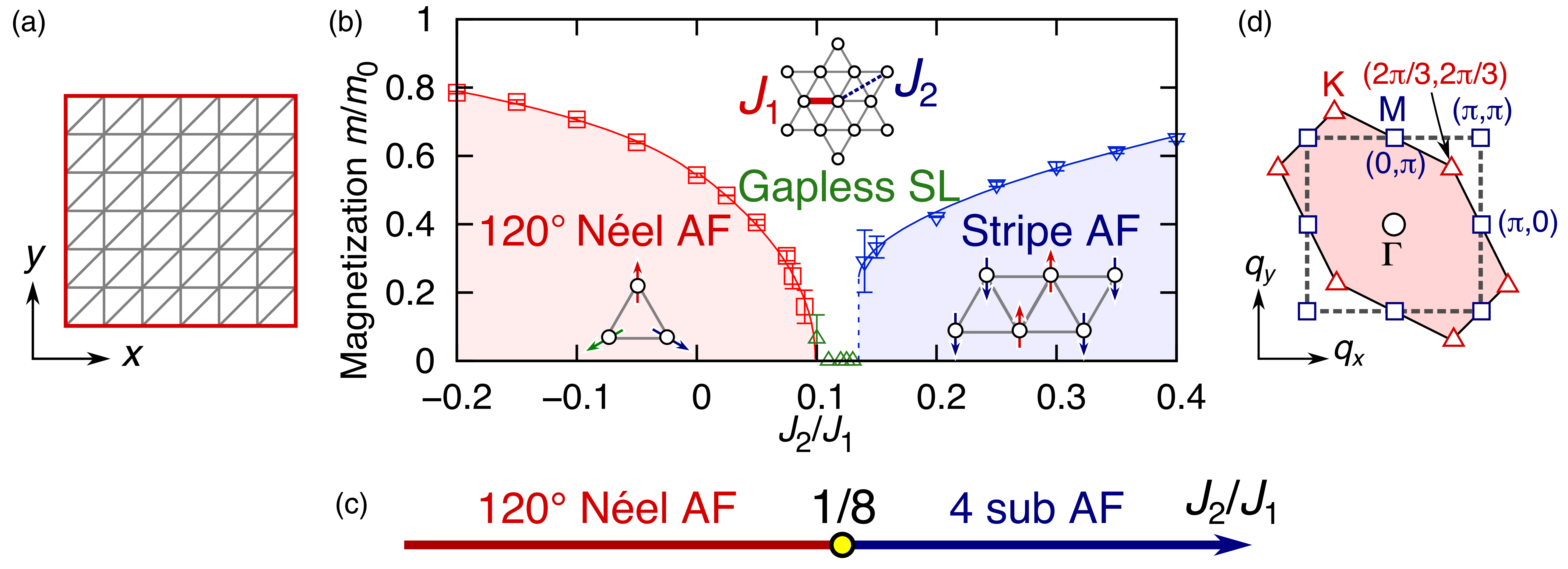
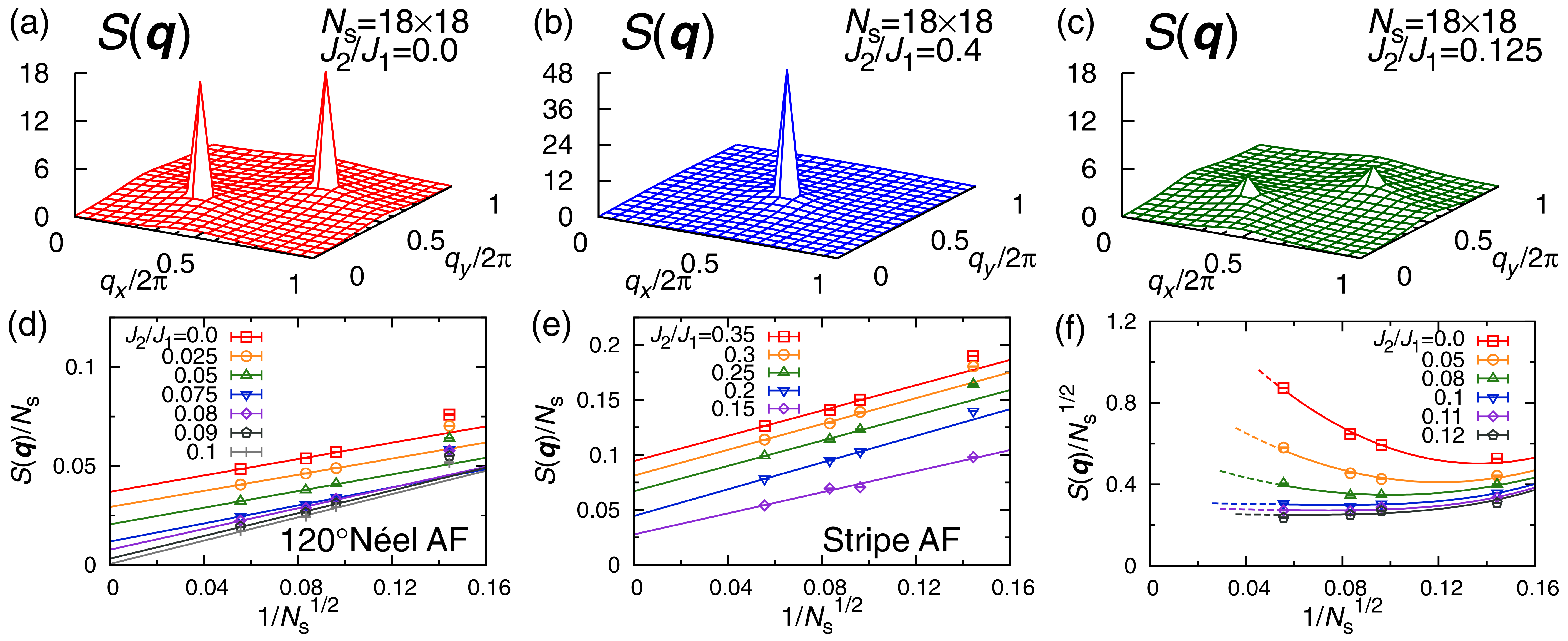
Ryui Kaneko[1*], Satoshi Morita[2], and Masatoshi Imada[3]





PHYSICAL REVIEW B **91**, 014426 (2015)

Quasiclassical magnetic order and its loss in a spin-$\frac{1}{2}$ Heisenberg antiferromagnet on a triangular lattice with competing bonds

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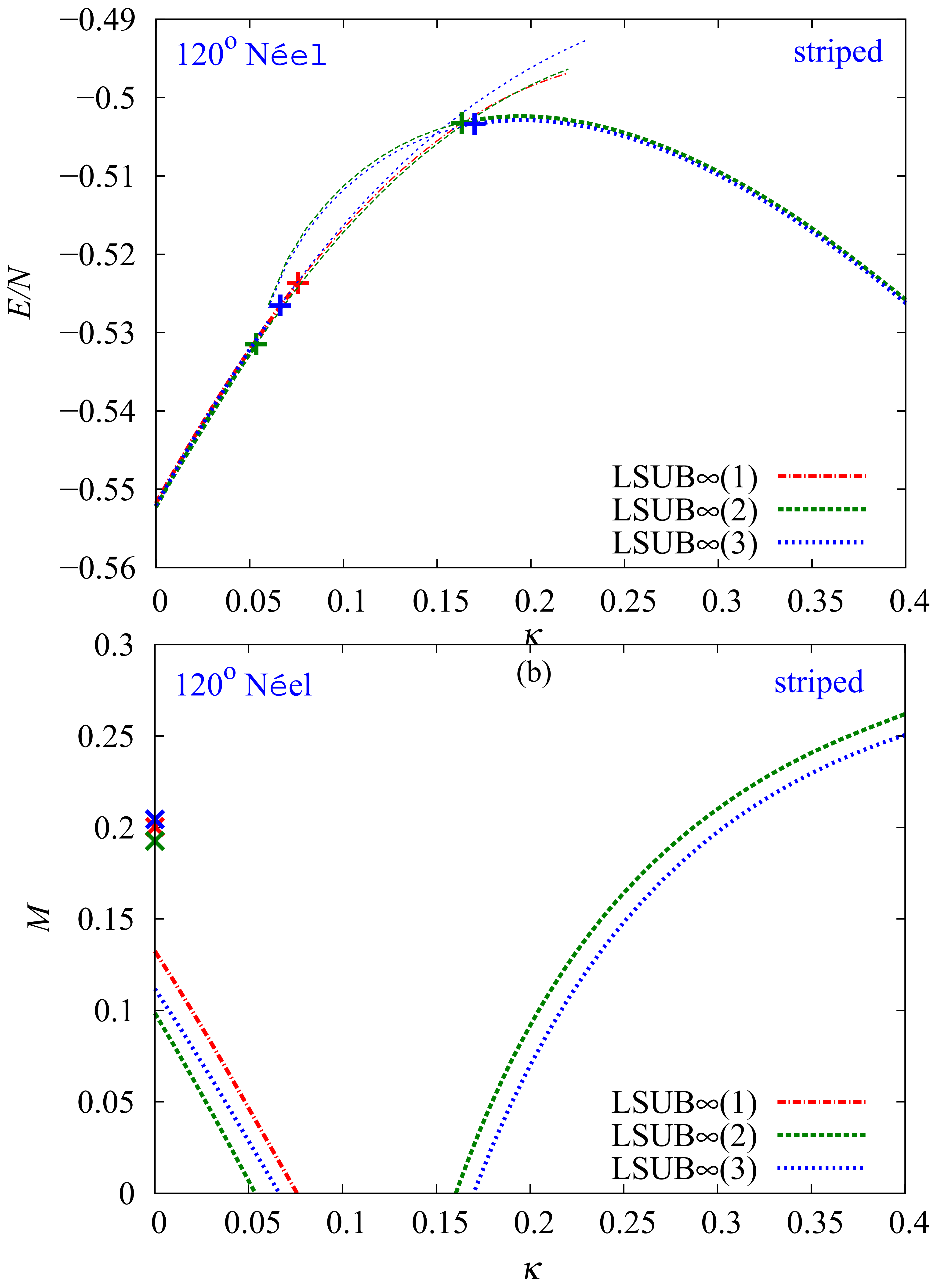
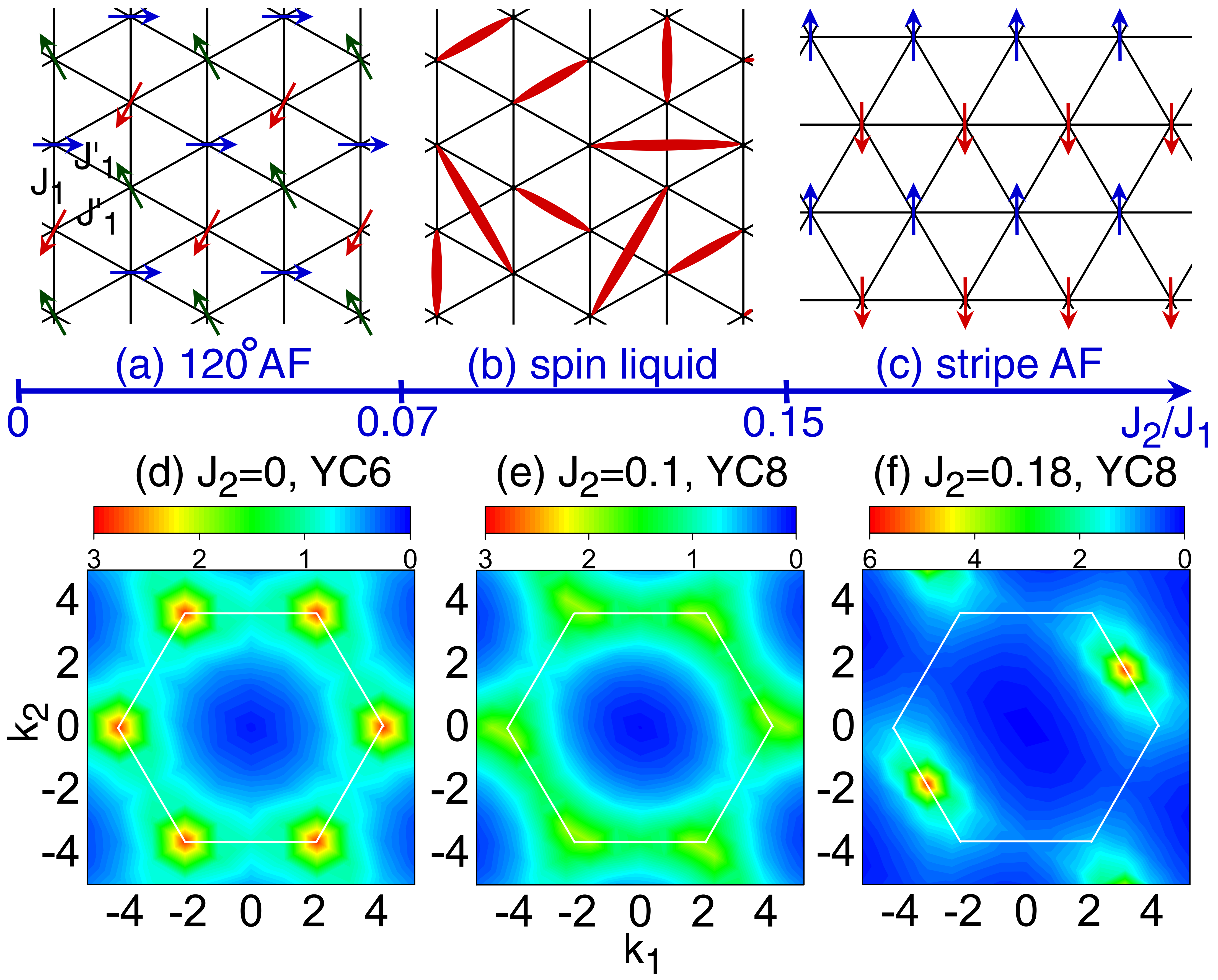


Competing spin-liquid states in the spin-$\frac{1}{2}$ Heisenberg model on the triangular lattice

Wen-Jun Hu, Shou-Shu Gong,* Wei Zhu, and D. N. Sheng



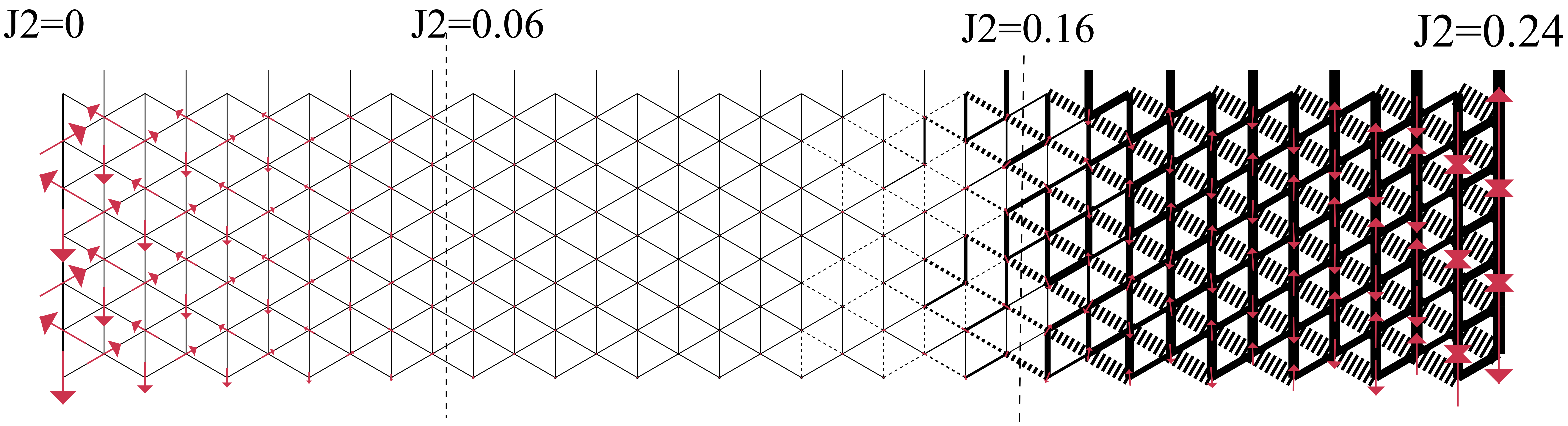
Summary and discussions. By means of DMRG calculations on wide cylinder systems of the spin-$\frac{1}{2}$ J_1-J_2 Heisenberg model on a triangular lattice, we find a SL region bordered by a 120° Néel AF phase for $J_2 \lesssim 0.07$ and a stripe AF phase for $J_2 \gtrsim 0.15$. The spin and dimer correlations all decay fast for wider systems with small correlation lengths comparable to lattice constants with large spin and singlet excitation gaps. The ES in the odd sector could be consistent with the fermionic spinon in Z_2 SL. However, the long-range chiral correlation is observed in the even sector for a space isotropic model. By tuning the anisotropic bond coupling, we find that the possible gapped Z_2 SL is stabilized by some weak anisotropy ($J_1' \sim 1.02$). The chiral correlations are enhanced in the even sector by opposite anisotropy ($J_1' \sim 0.98$), which may be stabilized in both sectors by TRS breaking terms, and we leave this open issue for future studies.

PHYSICAL REVIEW B **92**, 041105(R) (2015)

Spin liquid phase of the $S = \frac{1}{2}$ $J_1 - J_2$ Heisenberg model on the triangular lattice

Zhenyue Zhu and Steven R. White

We study the $S = 1/2$ Heisenberg model on the triangular lattice with nearest- and next-nearest-neighbor interactions J_1 and J_2 with the density matrix renormalization group, on long open cylinders with widths up to nine lattice spacings. In an intermediate J_2 region $0.06 \lesssim J_2/J_1 \lesssim 0.17$, we find evidence for a spin liquid (SL) state with short range spin-spin, bond-bond, and chiral correlation lengths, bordered by a classical 120° Néel ordered state at small J_2 and by a two sublattice collinear magnetically ordered state at larger J_2. Focusing on $J_2/J_1 = 0.1$, we find a number of signatures of a gapped SL phase.

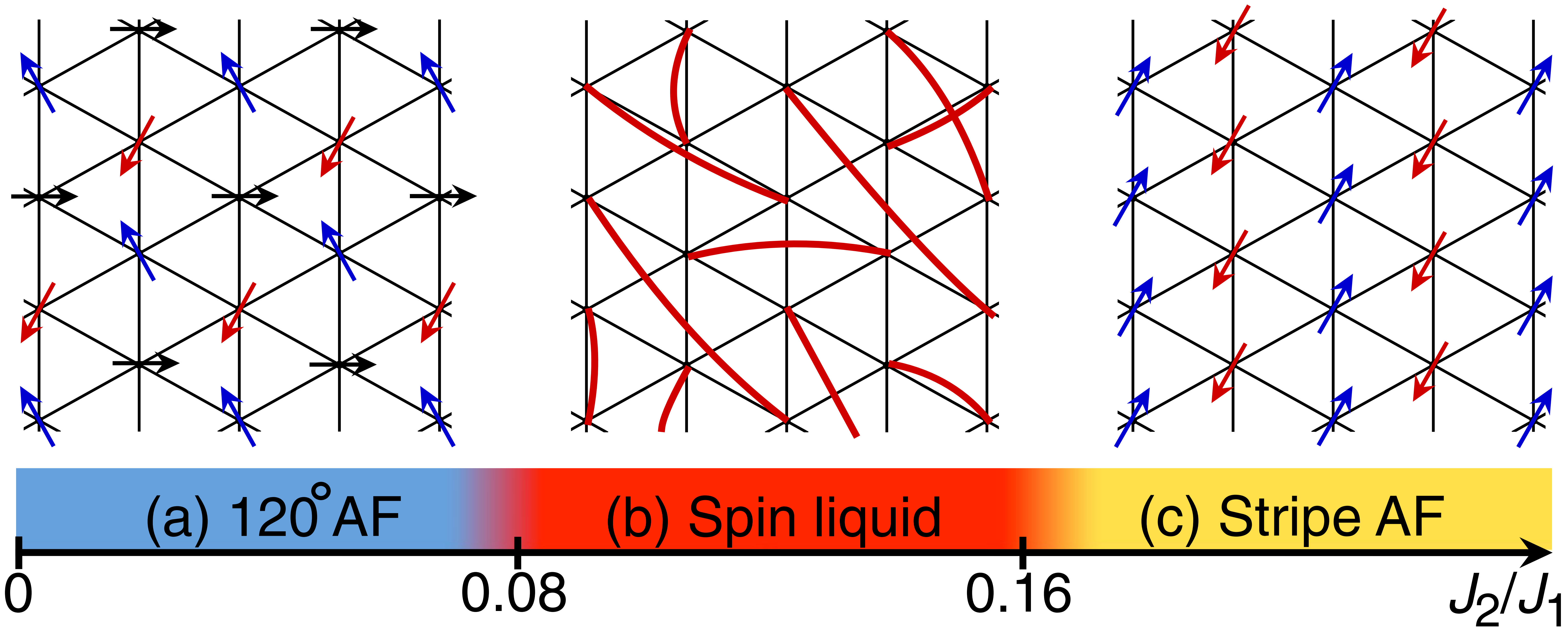


PHYSICAL REVIEW B **93**, 144411 (2016)

Spin liquid nature in the Heisenberg J_1-J_2 triangular antiferromagnet

Yasir Iqbal,[1,*] Wen-Jun Hu,[2,†] Ronny Thomale,[1,‡] Didier Poilblanc,[3,§] and Federico Becca[4,‖]

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$



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$$\mathbf{S}_i = \tfrac{1}{2} c^{\dagger}_{i,\alpha} \boldsymbol{\sigma}_{\alpha\beta} c_{i,\beta}$$

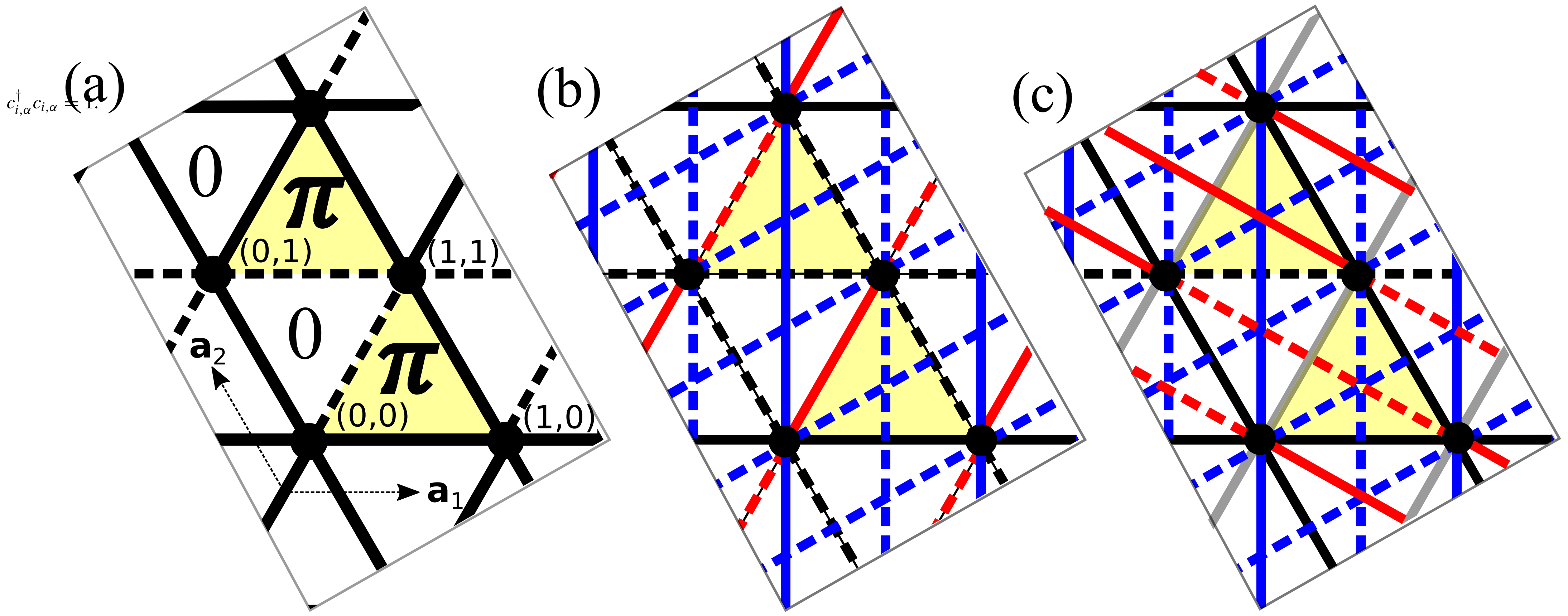
$$\mathcal{H} = \frac{J_1}{2} \sum_{\langle i,j \rangle} \left(c^{\dagger}_{i,\alpha} c_{i,\beta} c^{\dagger}_{j,\beta} c_{j,\alpha} - \tfrac{1}{2} c^{\dagger}_{i,\alpha} c_{i,\alpha} c^{\dagger}_{j,\beta} c_{j,\beta} \right) + \frac{J_2}{2} \sum_{\langle\langle i,j \rangle\rangle} \left(c^{\dagger}_{i,\alpha} c_{i,\beta} c^{\dagger}_{j,\beta} c_{j,\alpha} - \tfrac{1}{2} c^{\dagger}_{i,\alpha} c_{i,\alpha} c^{\dagger}_{j,\beta} c_{j,\beta} \right).$$

$$\mathcal{H}_{\rm MF} = \sum_{(i,j),\alpha} \chi_{ij} c^{\dagger}_{i,\alpha} c_{j,\alpha} + \sum_{(i,j)} \Delta_{ij} (c^{\dagger}_{i,\uparrow} c^{\dagger}_{j,\downarrow} + \text{H.c.}) + \sum_i \left[\mu \sum_{\alpha} c^{\dagger}_{i,\alpha} c_{i,\alpha} + \zeta (c^{\dagger}_{i,\uparrow} c^{\dagger}_{i,\downarrow} + \text{H.c.}) \right],$$

$$|\Psi_{\rm var}\rangle = \mathcal{J}_z \mathcal{P}_G |\Phi_0\rangle.$$

TABLE II. A list of states whose wave functions are studied.

State	Gapped	Unit cell	Parent state	No. in Ref. [53]
uRVB	No	1×1		
DSL	No	1×2		
$Z_2\{0\}\mathcal{A}$	Yes	1×1	uRVB	1
$Z_2\{\pi\}\mathcal{A}$	Yes	1×2	DSL	20
$Z_2\ \mathcal{C}$	Yes	1×2	None	6
$Z_2\{\pi\}\mathcal{B}$	No	1×2	DSL	18
VBC_2	Yes	1×2	DSL	
VBC_4	Yes	2×2	DSL	



PHYSICAL REVIEW B **93**, 144411 (2016)

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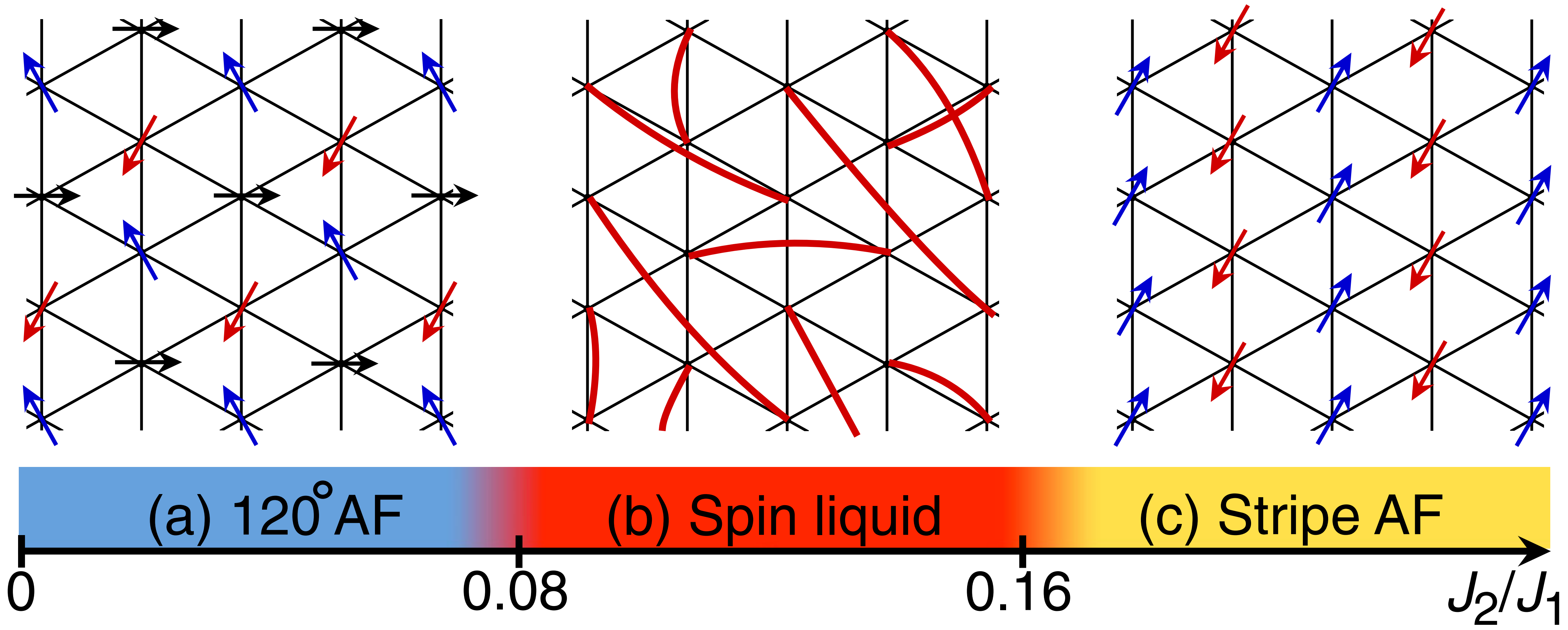
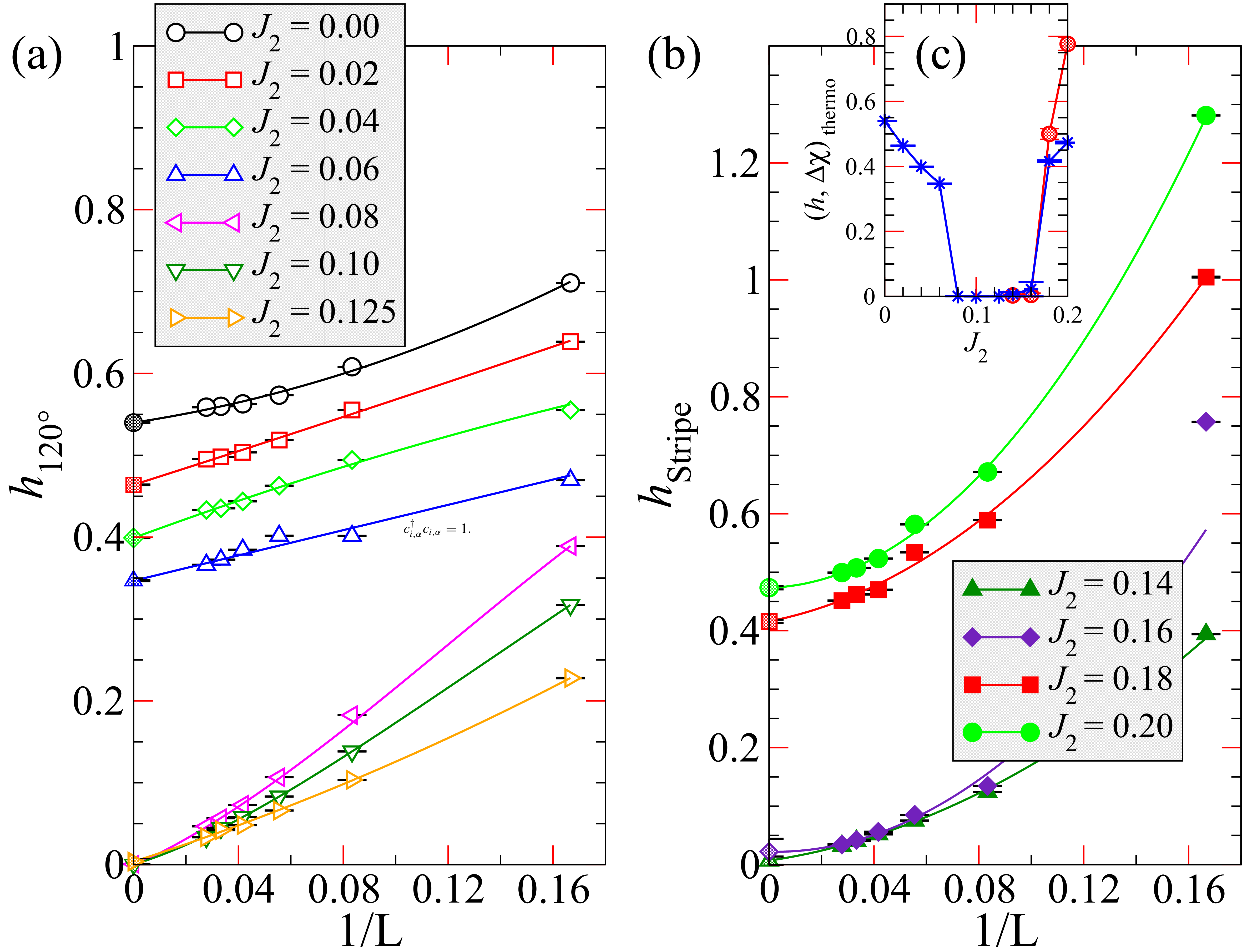




Chiral spin states and superconductivity

X. G. Wen, Frank Wilczek,* and A. Zee

$$E_{123} \equiv \langle \boldsymbol{\sigma}_1 \cdot (\boldsymbol{\sigma}_2 \times \boldsymbol{\sigma}_3) \rangle$$

scalar spin chirality

$$\mathcal{P}l_{123} = \langle \chi_{12}\chi_{23}\chi_{31} \rangle \ ,$$

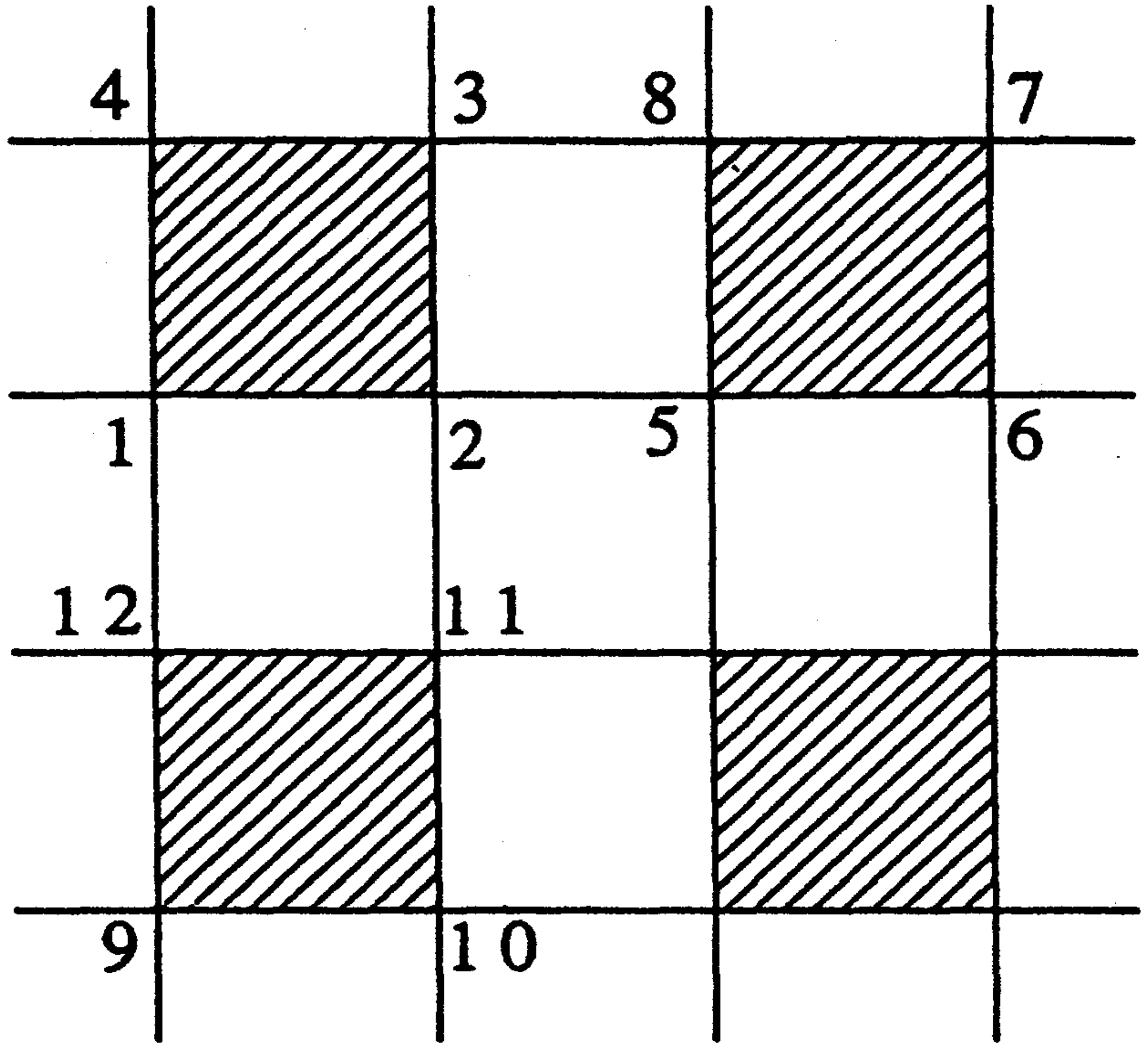
$$\chi_{ij} \equiv c^{\dagger}_{i\sigma} c_{j\sigma}$$

or

$$\mathcal{P}l_{1234} = \langle \chi_{12}\chi_{23}\chi_{34}\chi_{41} \rangle$$

$$\mathcal{P}l_{123} - \mathcal{P}l_{132} = -\frac{i}{2} E_{123}$$

$$\mathcal{P}l_{1234} - \mathcal{P}l_{1432} = \frac{i}{4}(-E_{123} - E_{134} - E_{124} + E_{234})$$



Novel Local Symmetries and Chiral-Symmetry-Broken Phases in $S=\frac{1}{2}$ Triangular-Lattice Heisenberg Model

G. Baskaran[a]

$$H=J\sum_{\text{nn}}\mathbf{S}_i\cdot\mathbf{S}_j+\alpha J\sum_{\text{nnn}}\mathbf{S}_i\cdot\mathbf{S}_j\ ,$$

$$\hat{\chi}_{ijk}^2=-\tfrac{1}{16}(\mathbf{S}_i+\mathbf{S}_j+\mathbf{S}_k)^2+\tfrac{15}{64}$$

$$H=-8J\sum_{\langle ijk\rangle}\hat{\chi}_{ijk}^2+\alpha J\sum_{\text{nnn}}\mathbf{S}_i\cdot\mathbf{S}_j+\text{const}\,.$$

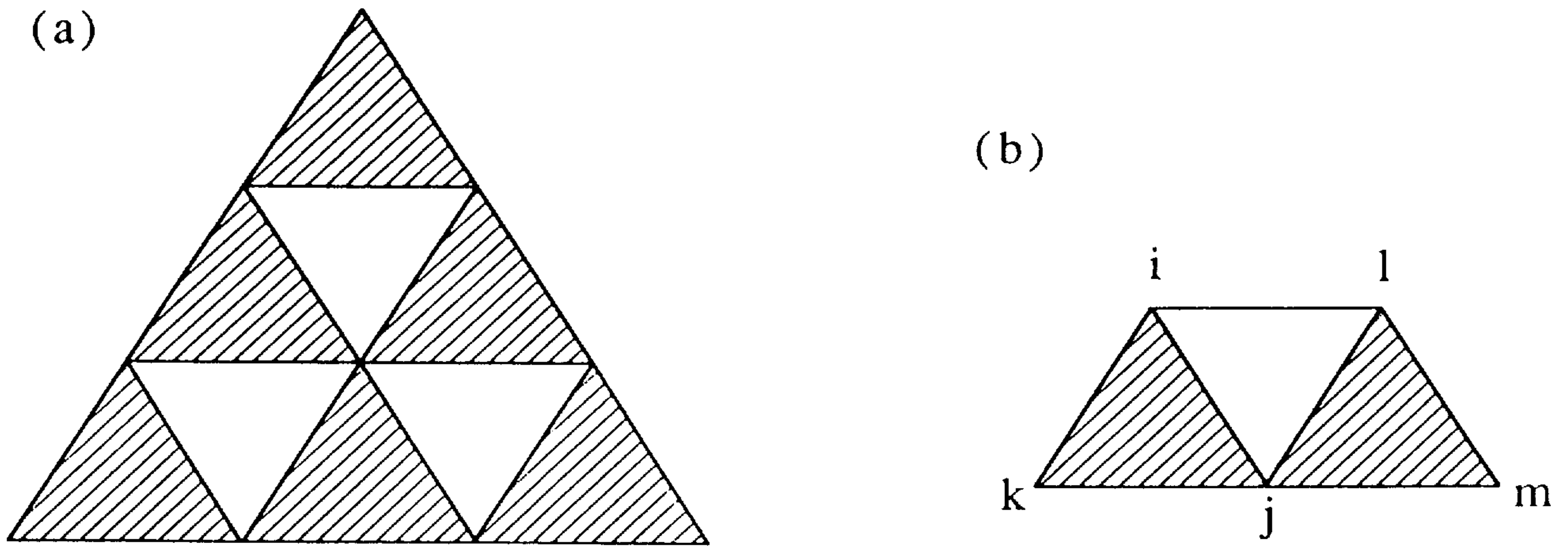


FIG. 1. (a) The shaded triangles are the ones on which $\hat{\chi}_{ijk}$ and m_{ijk} are defined. (b) Two triangles sharing a corner. The first nontrivial Ising-like coupling of chiral variables occurs between these two triangles.

Four-Spin Terms and the Origin of the Chiral Spin Liquid in Mott Insulators on the Triangular Lattice

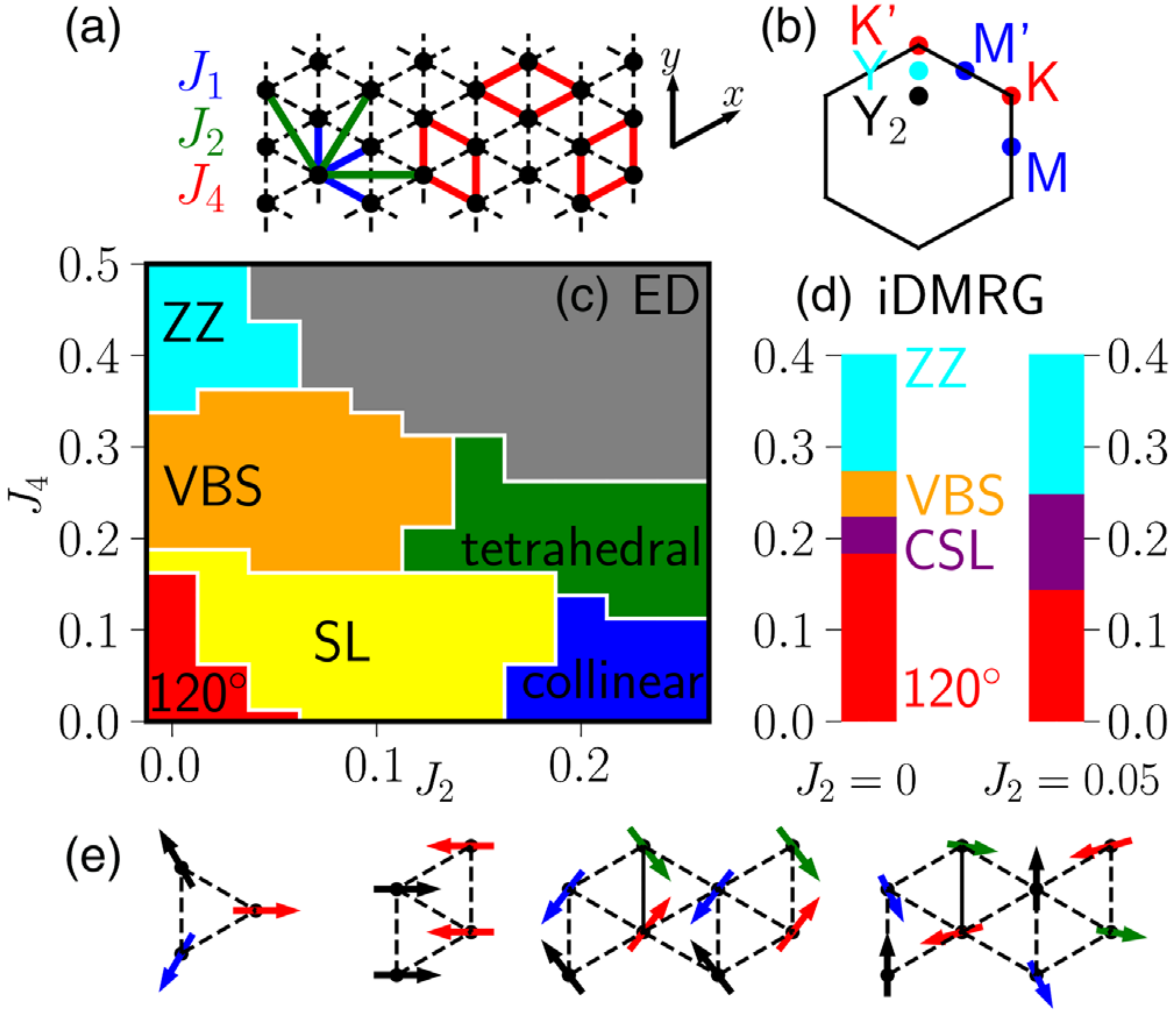
Tessa Cookmeyer,[1,2*] Johannes Motruk,[1,2,3] and Joel E. Moore[1,2]

$$H = J_1 \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + H_4,$$

$$H_4 = J_4 \sum_{\langle i,j,k,l \rangle} [(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)(\boldsymbol{S}_k \cdot \boldsymbol{S}_l) + (\boldsymbol{S}_i \cdot \boldsymbol{S}_l)(\boldsymbol{S}_j \cdot \boldsymbol{S}_k) - (\boldsymbol{S}_i \cdot \boldsymbol{S}_k)(\boldsymbol{S}_j \cdot \boldsymbol{S}_l)],$$

$J_1 = 4(1 - 7t^2/U^2)t^2/U$, $J_2 = 4t^4/U^3$, $J_3 = 4t^4/U^3$, and $J_4 = 80t^4/U^3$, where J_3 is a next-next-nearest-neighbor

DMRG study

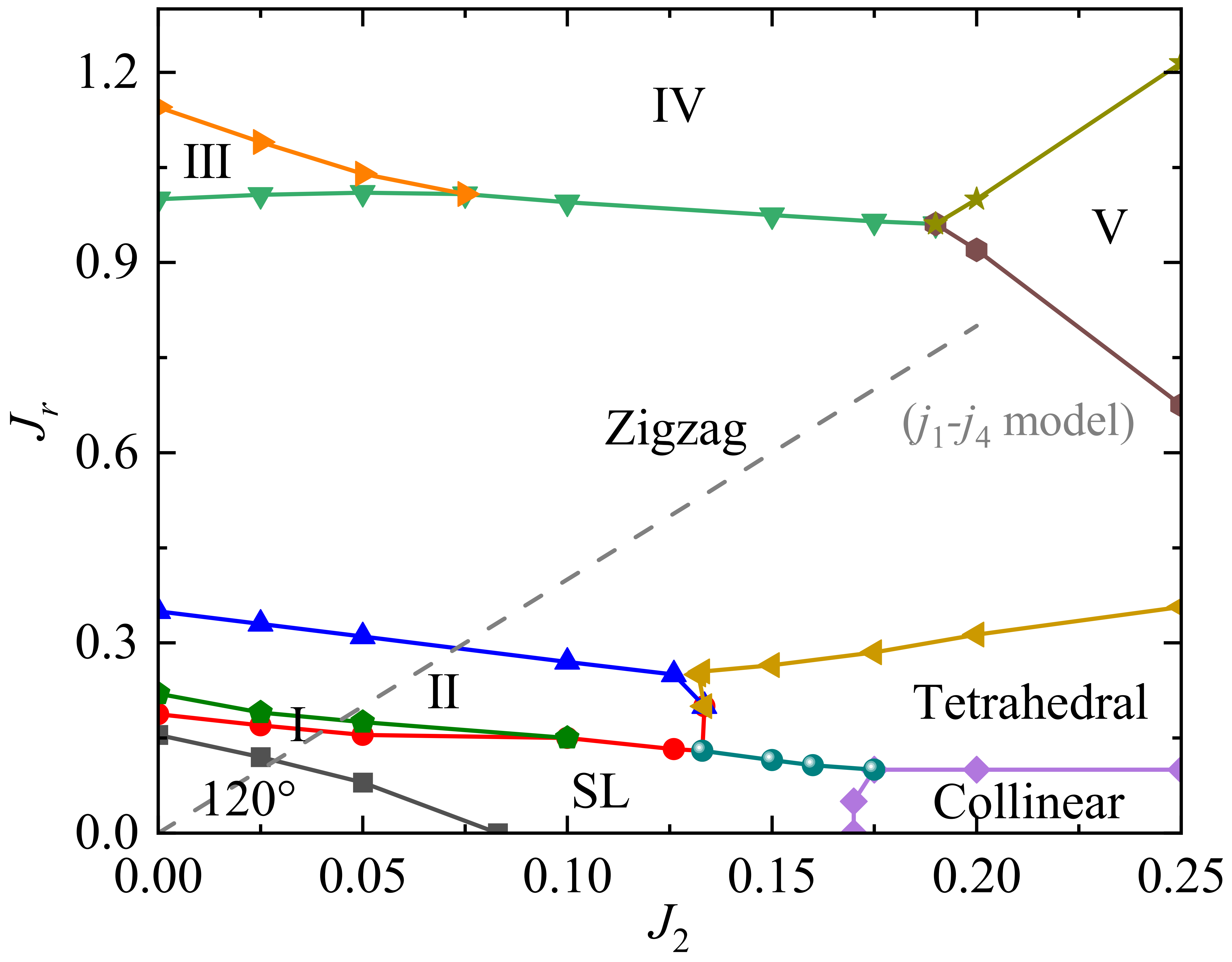


Exact diagonalization study of triangular Heisenberg model with four-spin ring-exchange interaction

Yuchao Zheng, Muwei Wu, Dao-Xin Yao,* and Han-Qing Wu†

$$H = J_1 \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_r \sum_{\langle i,j,k,l \rangle} [(\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l) + (\mathbf{S}_i \cdot \mathbf{S}_l)(\mathbf{S}_j \cdot \mathbf{S}_k) - (\mathbf{S}_i \cdot \mathbf{S}_k)(\mathbf{S}_j \cdot \mathbf{S}_l)], \quad ($$

Exact diagonalization of a 36-site cluster



Projective symmetry group classification of chiral spin liquids

Samuel Bieri,[1,2,*] Claire Lhuillier,[1] and Laura Messio[1]

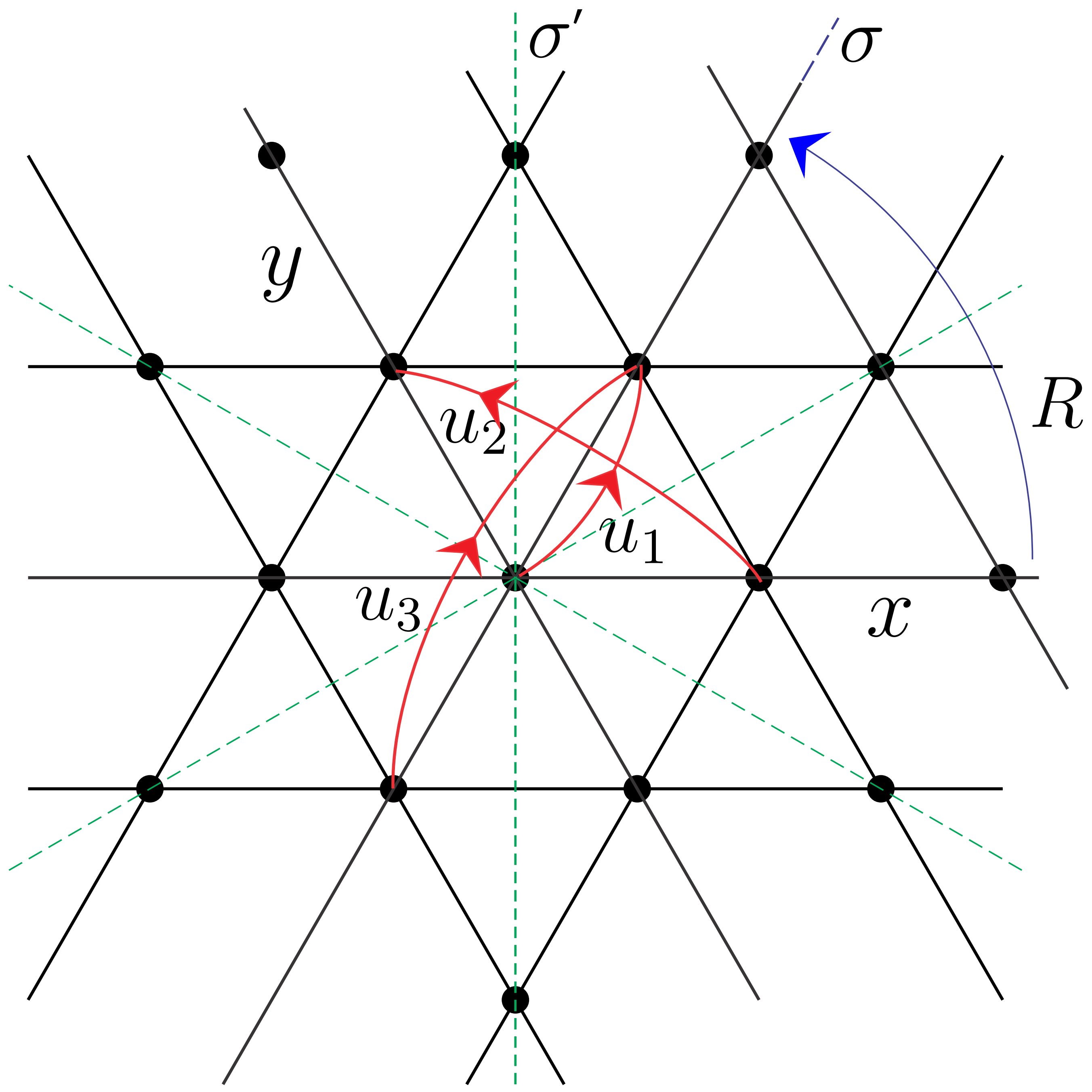


FIG. 1. Symmetry generators and *ansatz* parameters u for the triangular lattice.

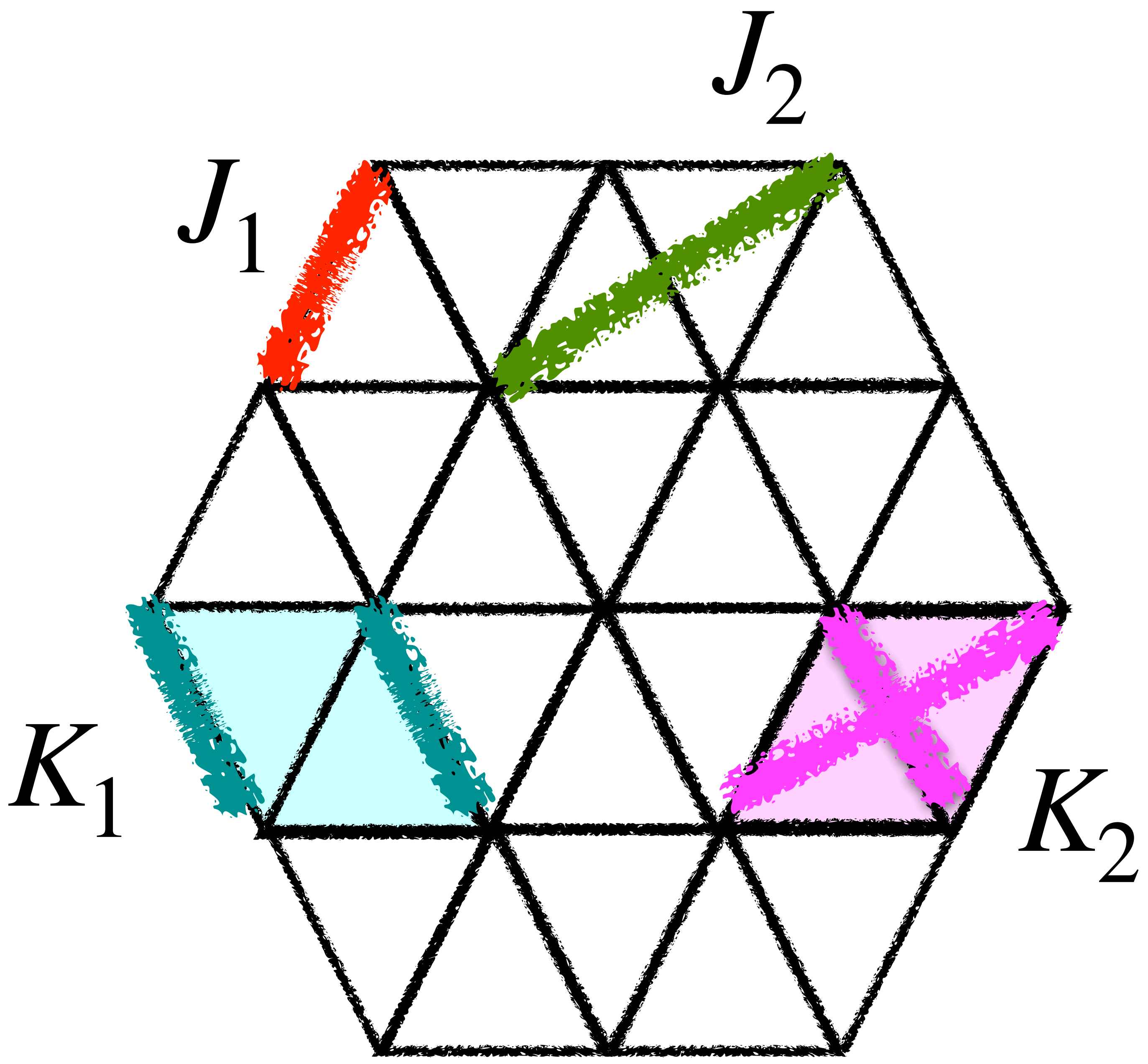
$$\hat{u}_{ij} = \frac{1}{2}\begin{pmatrix} \boldsymbol{f}_i^\dagger \boldsymbol{f}_j & \boldsymbol{f}_i^T \varepsilon \boldsymbol{f}_j \\ \boldsymbol{f}_j^\dagger \varepsilon \boldsymbol{f}_i^* & \boldsymbol{f}_j^\dagger \boldsymbol{f}_i \end{pmatrix}.$$

TABLE III. Quantum spin liquids on the triangular lattice respecting rotation symmetry ($\tau_R = 0$). All lattice symmetries are respected for $\tau_\sigma = 0$; states Nos. 1a to 6 also respect time reversal, so they are *symmetric* QSLs [70]. The reflection symmetries are broken in the Kalmeyer-Laughlin CSLs Nos. 7 to 10d ($\tau_\sigma = 1$). Column "PSG" refers to the point group representations in Table II. λ is the on-site field, and u_a is the *ansatz* on links shown in Fig. 1, in the notation of allowed real components $(\tau_\mu) = (i\mathbb{1}_2, \sigma_a)$; "x" means that the field must vanish by symmetry. $a = \exp(i\pi\sigma_3/3)$ and $b = \exp(i\pi\sigma_3/6)$.

No.	τ_σ	τ_R	ϵ_2	PSG	g_σ	g_R	$\lambda[\sigma_a]$	$u_1[\tau_\mu]$	$u_2[\tau_\mu]$	$u_3[\tau_\mu]$
1	0	0	+	1	$\mathbb{1}_2$	$\mathbb{1}_2$	1,2,3	3	1,3	1,2,3
1a	0	0	+	2	$i\sigma_3$	$\mathbb{1}_2$	3	3	3	3
2	0	0	+	6	$i\sigma_1$	a	x	1	1	1
3	0	0	−	4	$i\sigma_3$	$i\sigma_3$	3	1	x	3
4	0	0	−	3	$\mathbb{1}_2$	$i\sigma_3$	3	x	1	3
5	0	0	−	5	$i\sigma_2$	$i\sigma_3$	x	1	2	x
6	0	0	−	7	$i\sigma_2$	b	x	1	2	x
7	1	0	+	6	$i\sigma_2$	a	3	1,3	1,3	1,3
8	1	0	−	7	$i\sigma_1$	b	3	0,1	0,2	3
9	1	0	−	6	$i\sigma_2$	a	3	0	0	1,3
10	1	0	−	5	$i\sigma_1$	$i\sigma_3$	3	0,1	0,2	3
10a	1	0	−	2	$i\sigma_2$	$\mathbb{1}_2$	3	0	0	3
10b	1	0	−	3	$\mathbb{1}_2$	$i\sigma_2$	x	0,3	0	x
10c	1	0	−	4	$i\sigma_2$	$i\sigma_2$	x	0	0,3	x
10d	1	0	−	1	$\mathbb{1}_2$	$\mathbb{1}_2$	x	0	0	x

How do we stabilise a non-coplanar (chiral) state?

The 4-site Hamiltonian we consider:

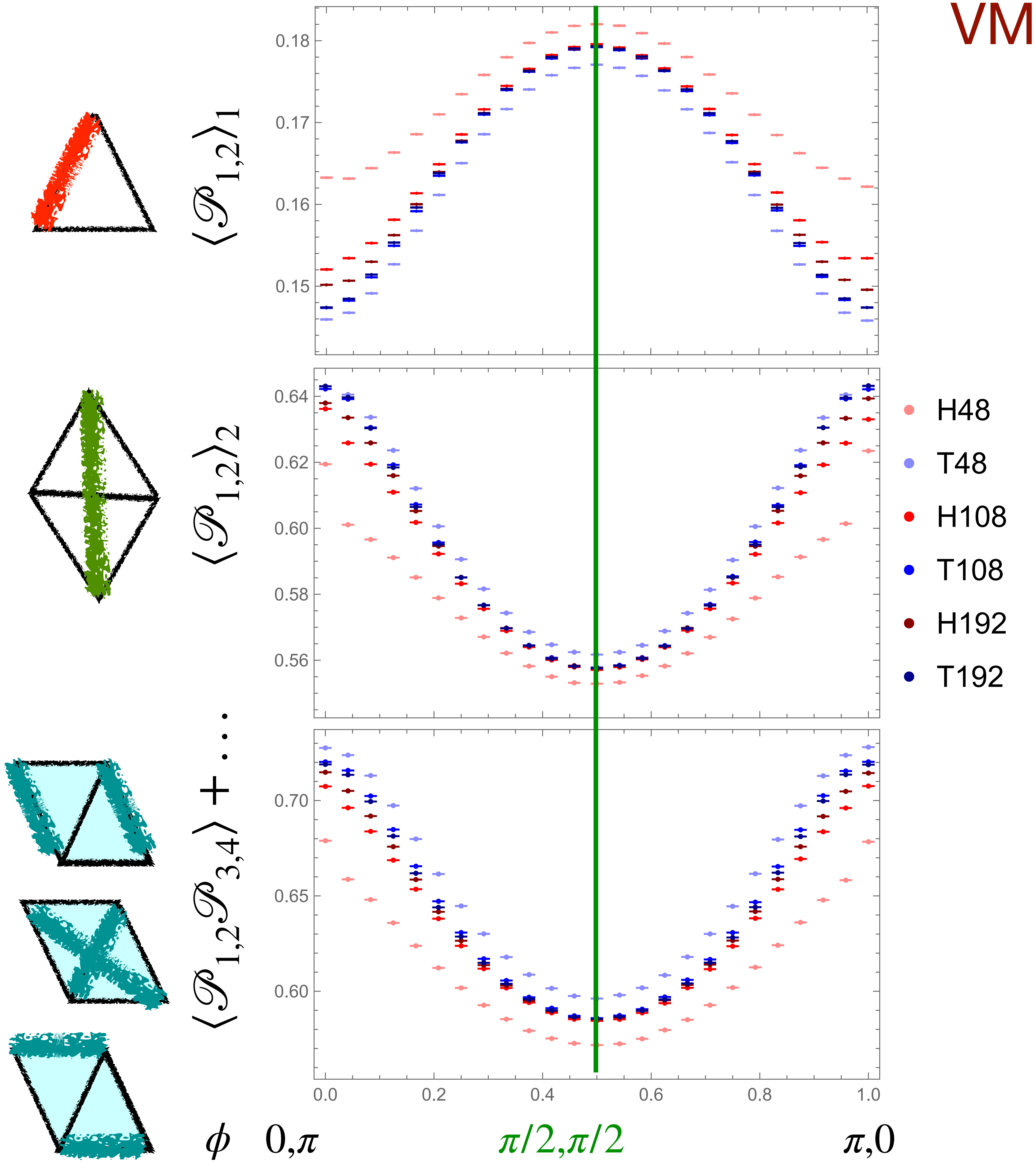


$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle_{\rm 1st}} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle i,j \rangle_{\rm 2nd}} \mathbf{S}_i \cdot \mathbf{S}_j$$
$$+ K_1 \sum_{\langle i,j,k,l \rangle_{\rm tet}} \left[(\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l) + (\mathbf{S}_i \cdot \mathbf{S}_k)(\mathbf{S}_j \cdot \mathbf{S}_l) \right]$$
$$+ K_2 \sum_{\langle i,j,k,l \rangle_{\rm tet}} (\mathbf{S}_i \cdot \mathbf{S}_l)(\mathbf{S}_j \cdot \mathbf{S}_k)$$

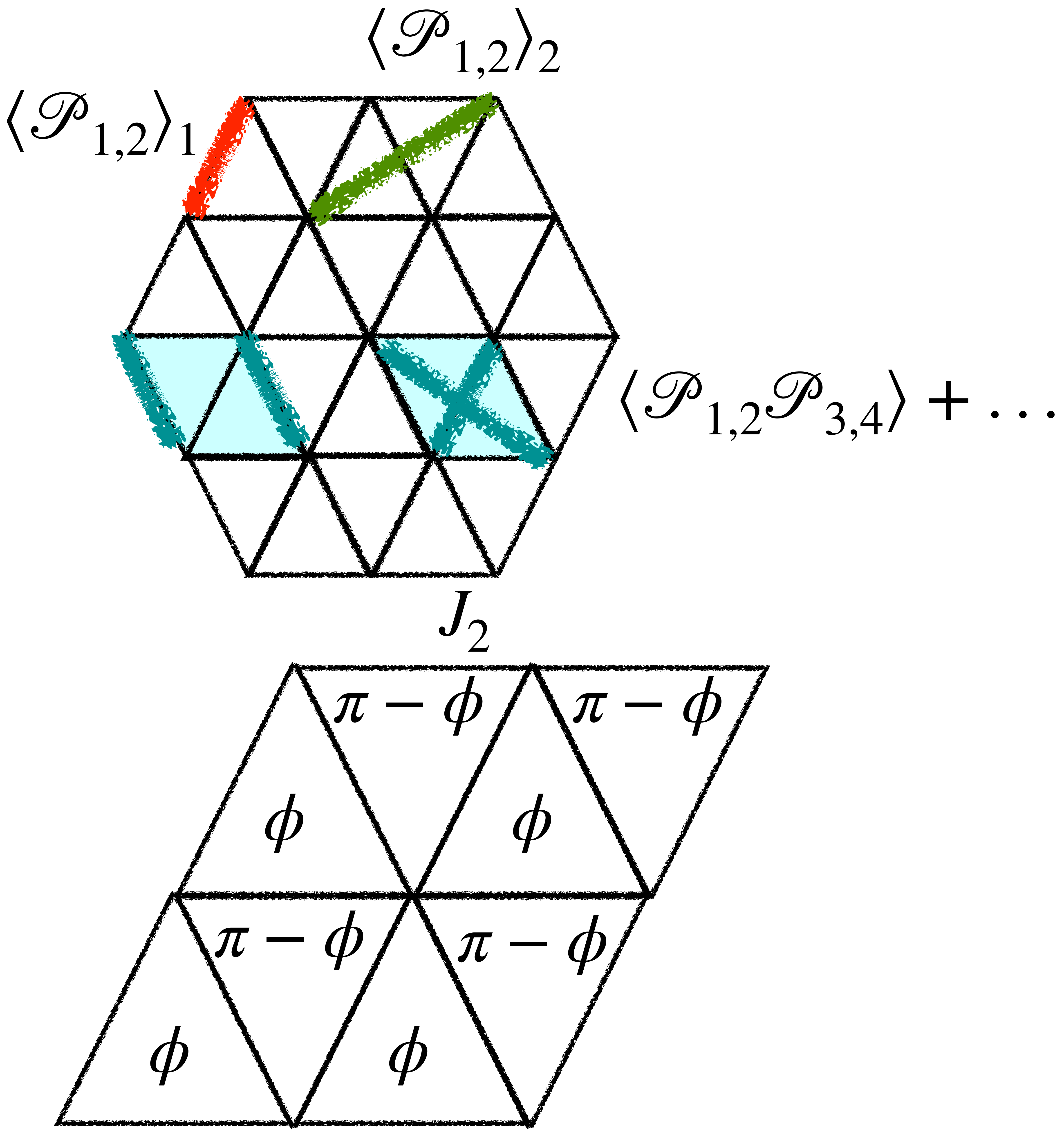
Remember:

$$\mathcal{P}_{1,2,3,4} + \mathcal{P}_{1,4,3,2} = \frac{1}{4} + \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$
$$+ 4 \sum_{\langle i,j,k,l \rangle_{\rm tet}} \left[(\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{S}_3 \cdot \mathbf{S}_4) + (\mathbf{S}_1 \cdot \mathbf{S}_4)(\mathbf{S}_2 \cdot \mathbf{S}_3) - (\mathbf{S}_1 \cdot \mathbf{S}_3)(\mathbf{S}_2 \cdot \mathbf{S}_4) \right]$$

VMC results

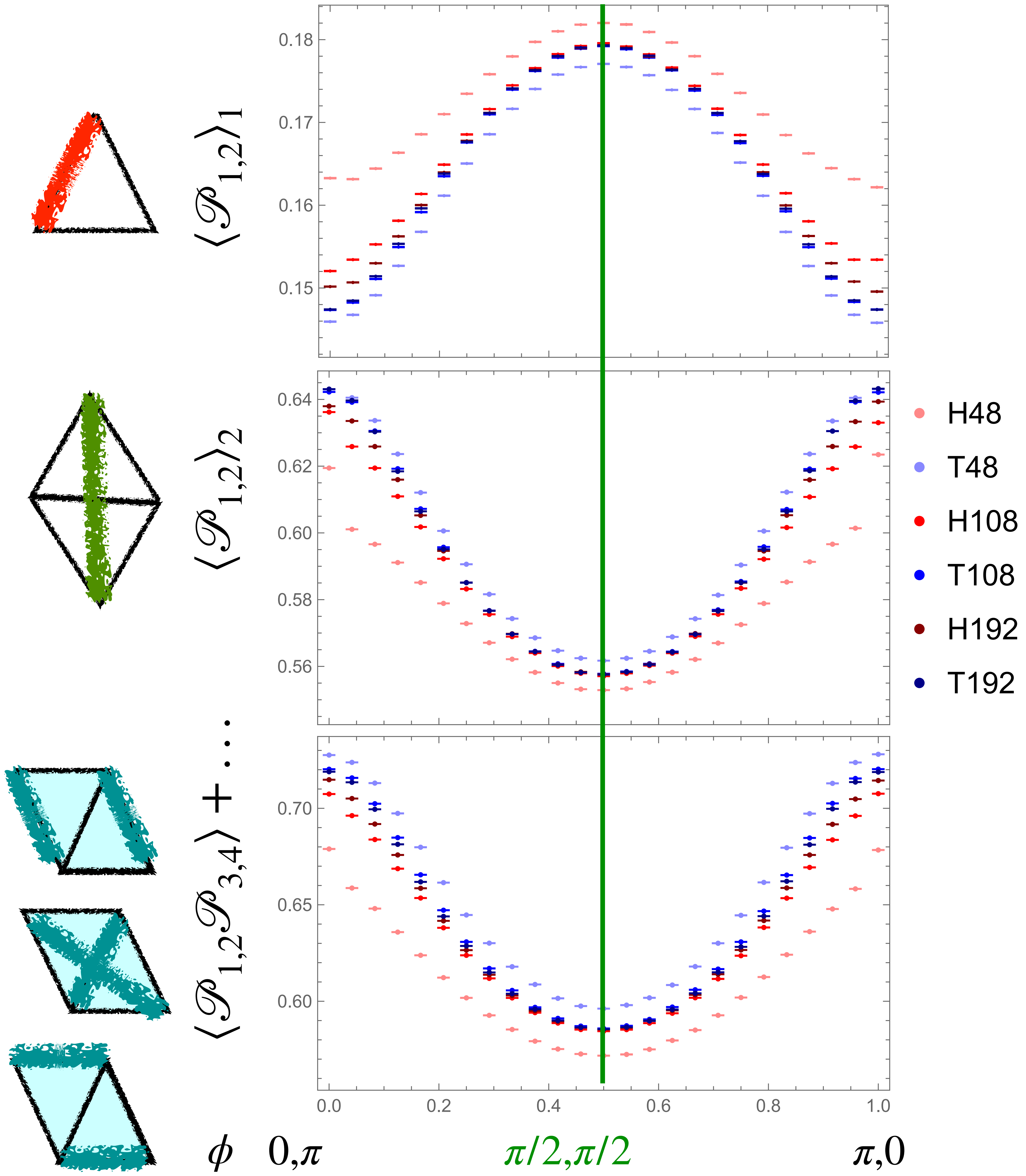


$\langle \mathscr{P}_{1,2} \rangle_1$
$\langle \mathscr{P}_{1,2} \rangle_2$
$\langle \mathscr{P}_{1,2}\mathscr{P}_{3,4} \rangle + \ldots$
H48
T48
H108
T108
H192
T192
ϕ 0,π $\pi/2,\pi/2$ $\pi,0$

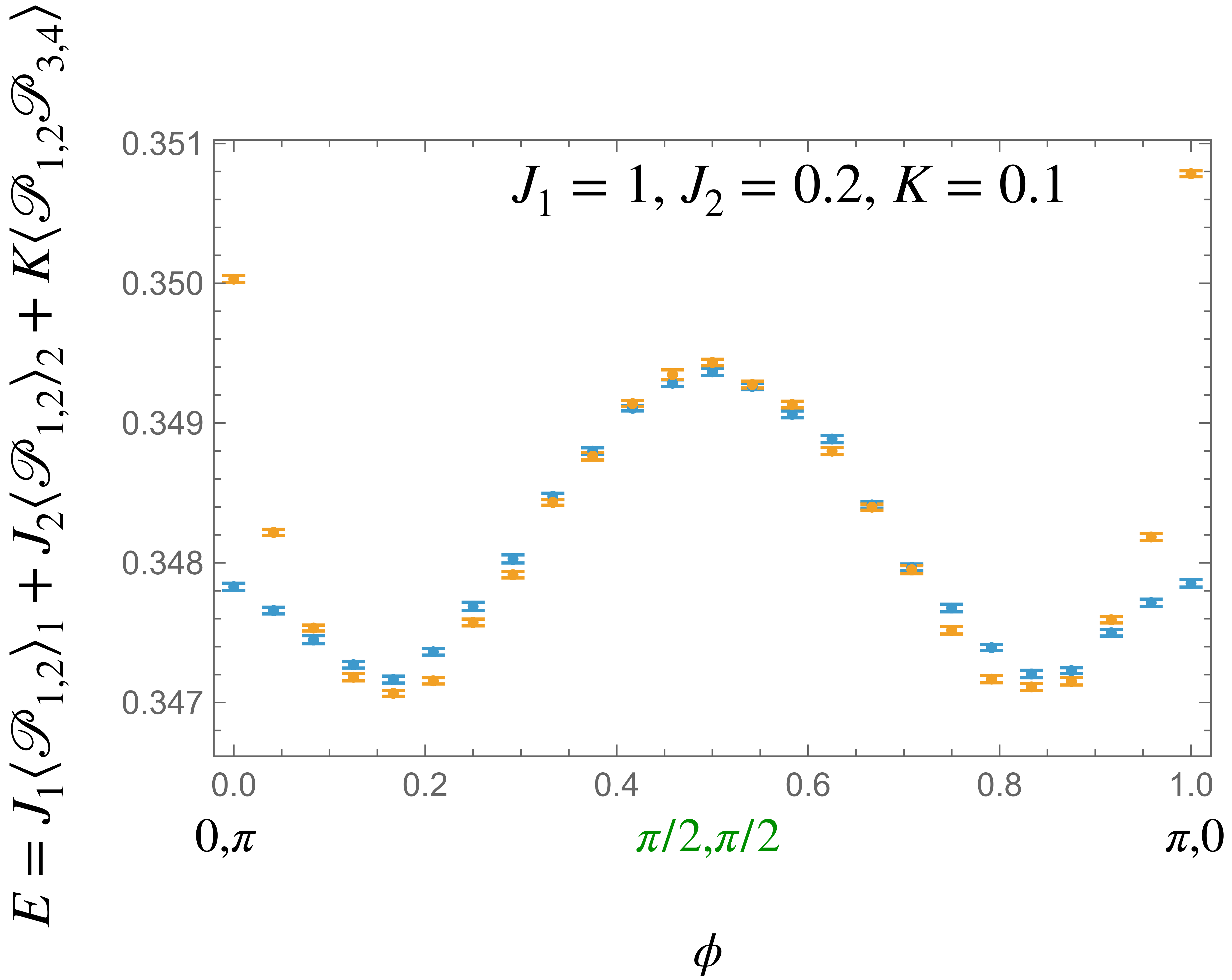


$\langle \mathscr{P}_{1,2} \rangle_2$
$\langle \mathscr{P}_{1,2} \rangle_1$
$\langle \mathscr{P}_{1,2}\mathscr{P}_{3,4} \rangle + \ldots$
J_2
$\pi - \phi$
ϕ

VMC results



$\langle \mathcal{P}_{1,2} \rangle_1$
$\langle \mathcal{P}_{1,2} \rangle_2$
$\langle \mathcal{P}_{1,2}\mathcal{P}_{3,4} \rangle + \ldots$
0.18
0.17
0.16
0.15
0.64
0.62
0.60
0.58
0.56
0.70
0.65
0.60
0.0
0.2
0.4
0.6
0.8
1.0
ϕ
$0,\pi$
$\pi/2,\pi/2$
$\pi,0$
H48
T48
H108
T108
H192
T192



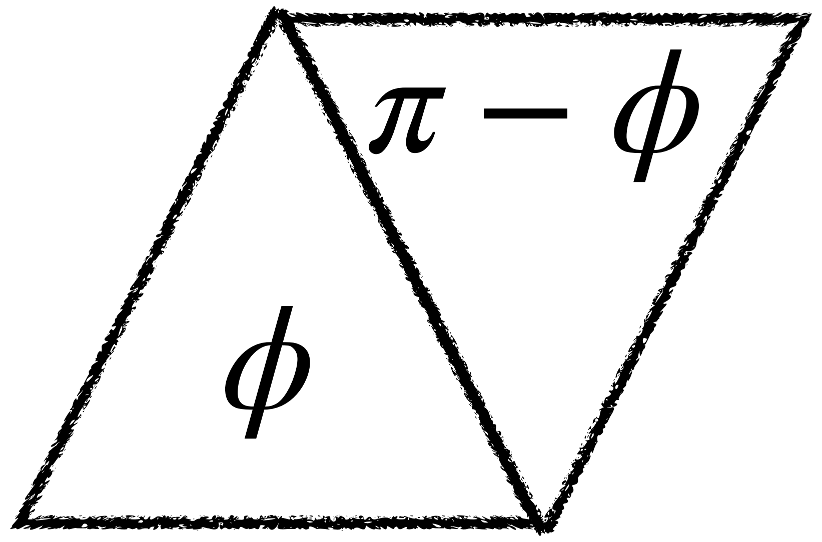
$E = J_1\langle \mathcal{P}_{1,2} \rangle_1 + J_2\langle \mathcal{P}_{1,2} \rangle_2 + K\langle \mathcal{P}_{1,2}\mathcal{P}_{3,4} \rangle$
$J_1 = 1, J_2 = 0.2, K = 0.1$
0.351
0.350
0.349
0.348
0.347
0.0
0.2
0.4
0.6
0.8
1.0
$0,\pi$
$\pi/2,\pi/2$
$\pi,0$
ϕ

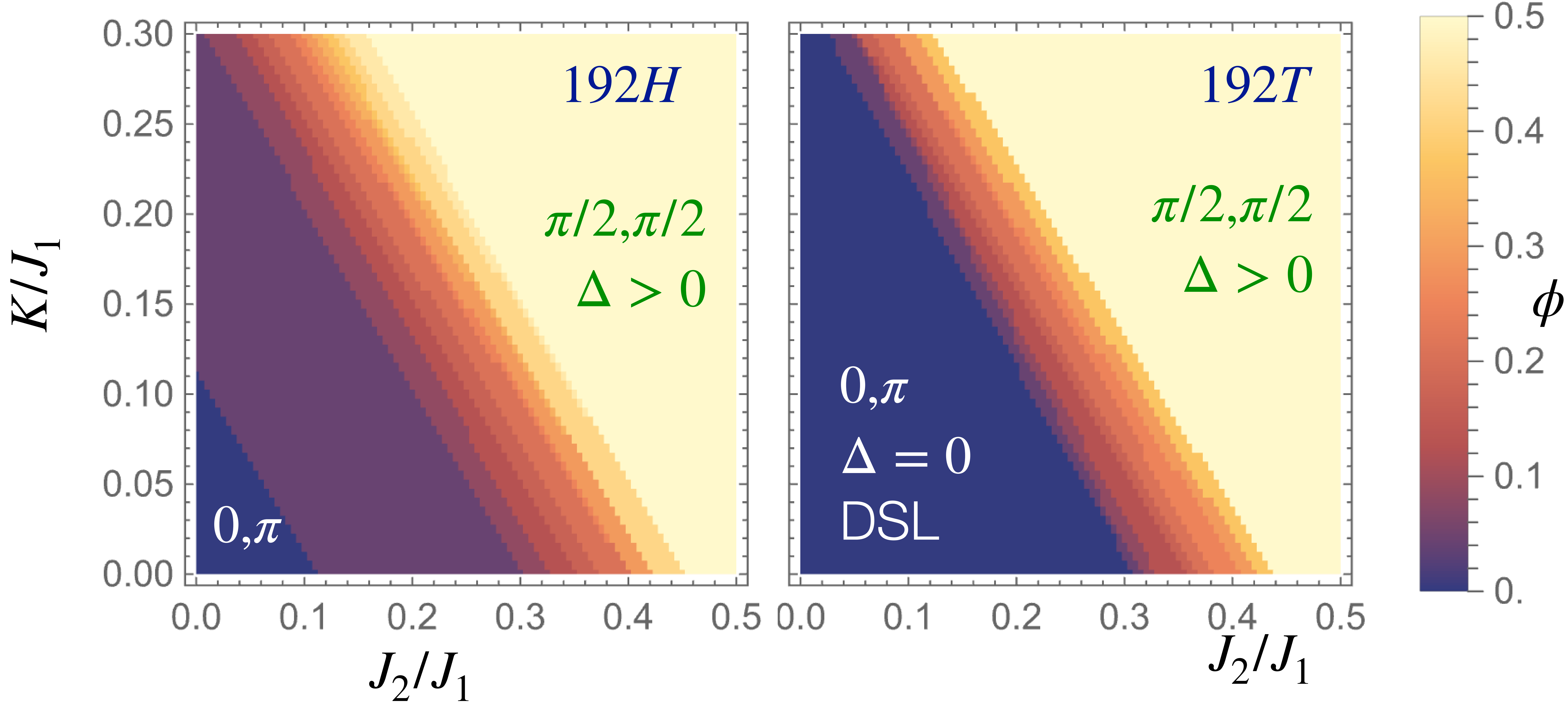
VMC results - spontaneous breaking of time reversal symmetry

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle_{1st}} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{\langle i,j \rangle_{2nd}} \mathbf{S}_i \cdot \mathbf{S}_j$$

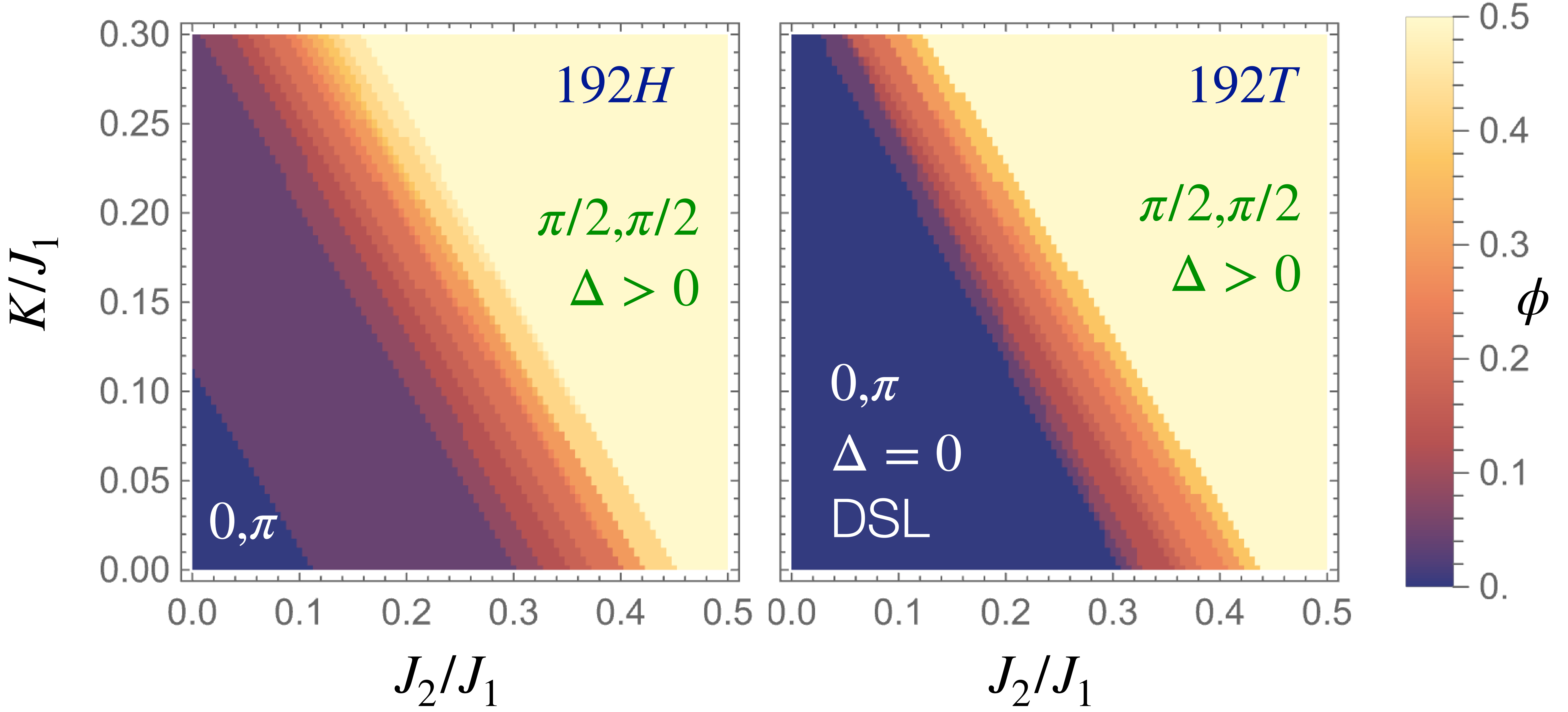
$$+ K \sum_{\langle i,j,k,l \rangle_{tet}} \left[(\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l) + (\mathbf{S}_i \cdot \mathbf{S}_k)(\mathbf{S}_j \cdot \mathbf{S}_l) + (\mathbf{S}_i \cdot \mathbf{S}_l)(\mathbf{S}_j \cdot \mathbf{S}_k) \right]$$

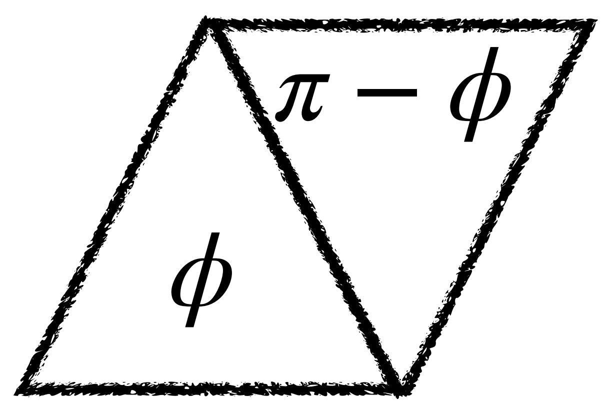
c.f. A. Wietek, J. Knolle



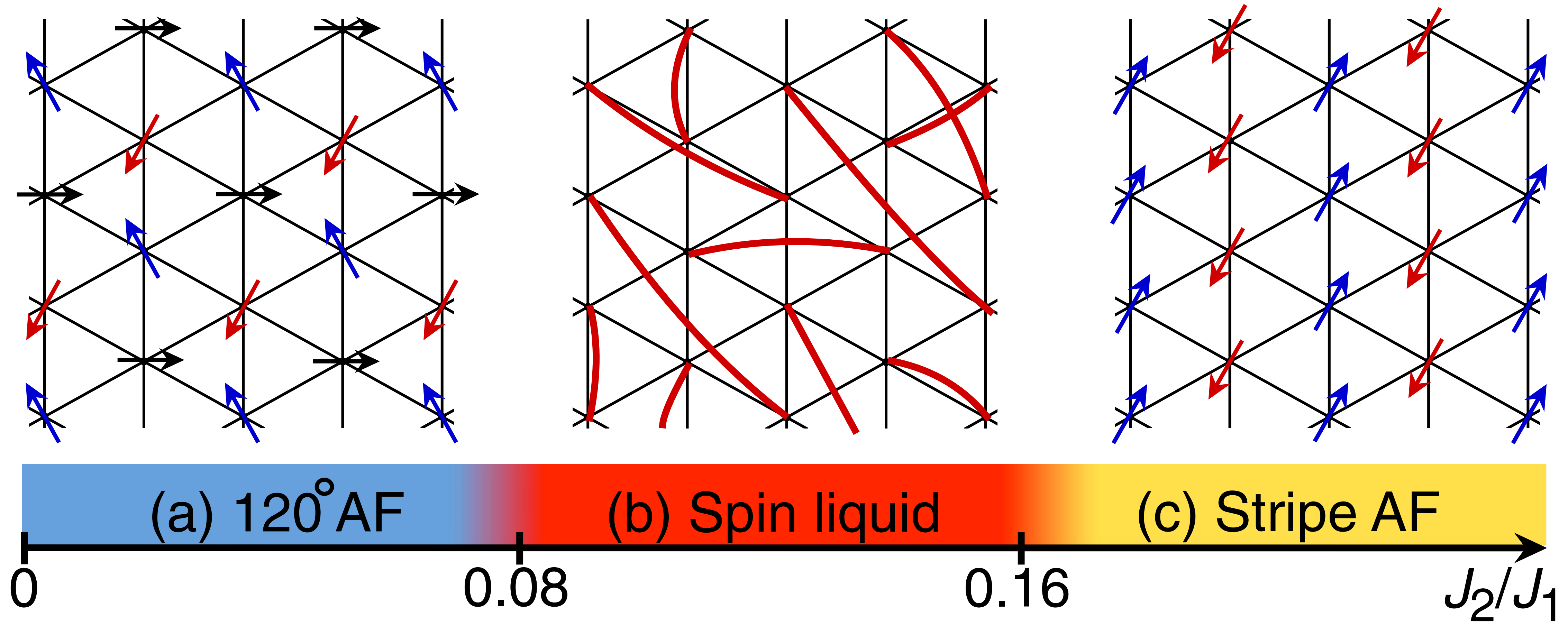


VMC results - spontaneous breaking of time reversal symmetry

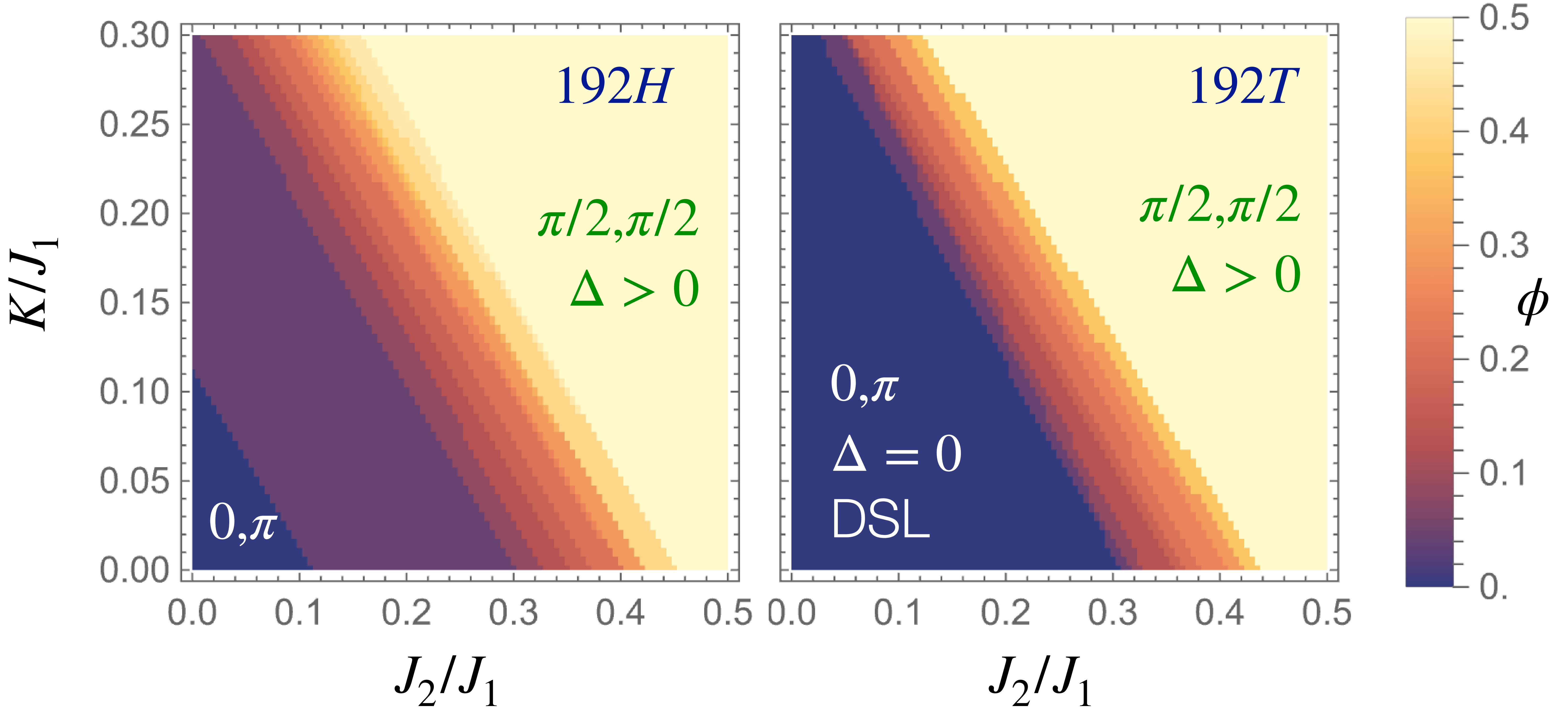


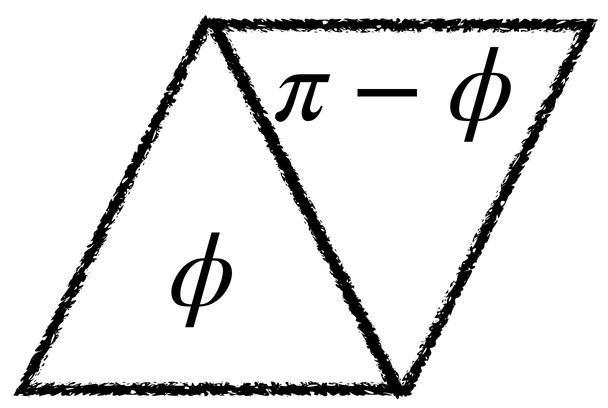


$$\mathbf{S}_i = \frac{1}{2} c^\dagger_{i,\alpha} \boldsymbol{\sigma}_{\alpha\beta} c_{i,\beta}$$



VMC results - spontaneous breaking of time reversal symmetry



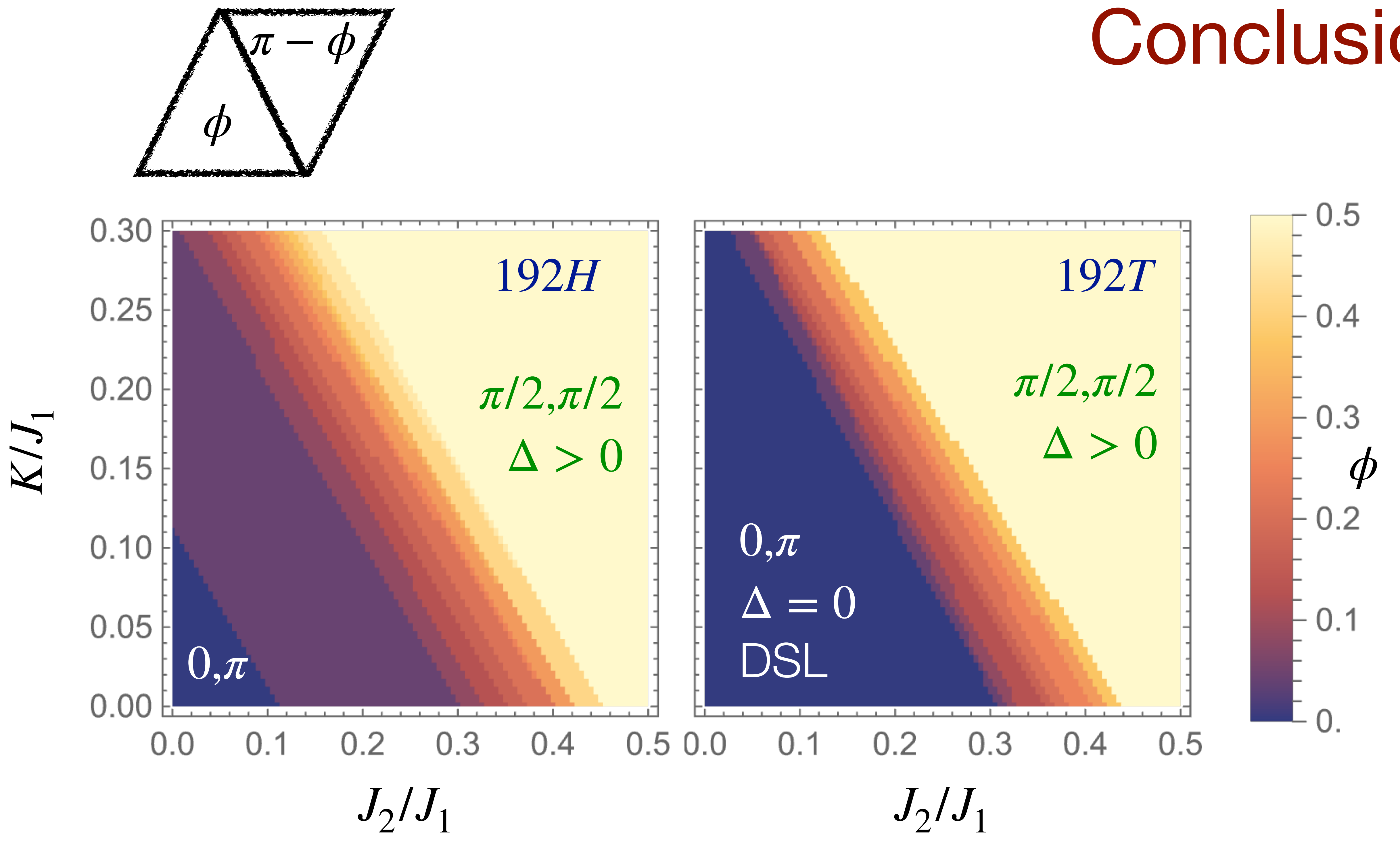


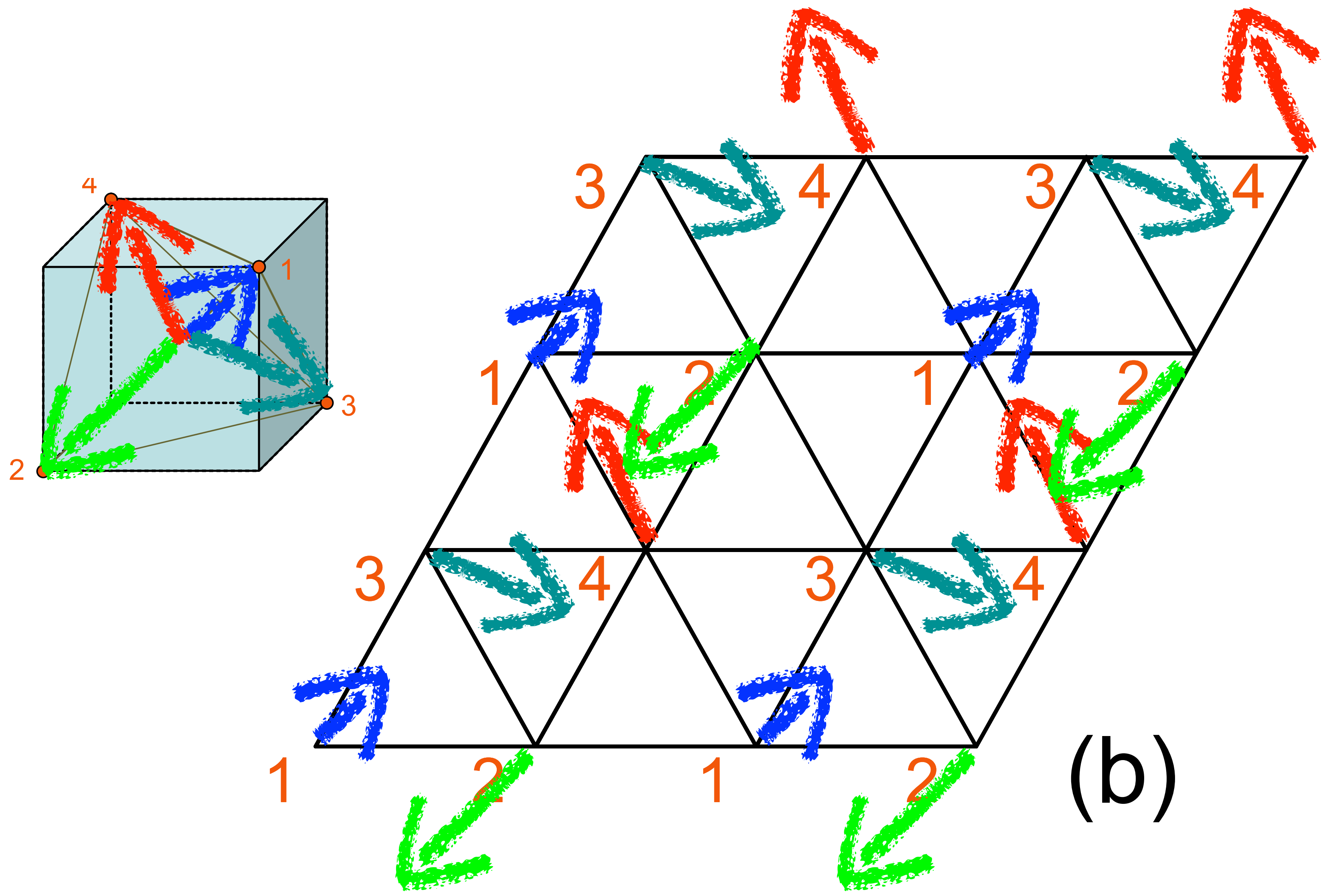
$$\mathbf{S}_i = \frac{1}{2} c^\dagger_{i,\alpha} \boldsymbol{\sigma}_{\alpha\beta} c_{i,\beta}$$

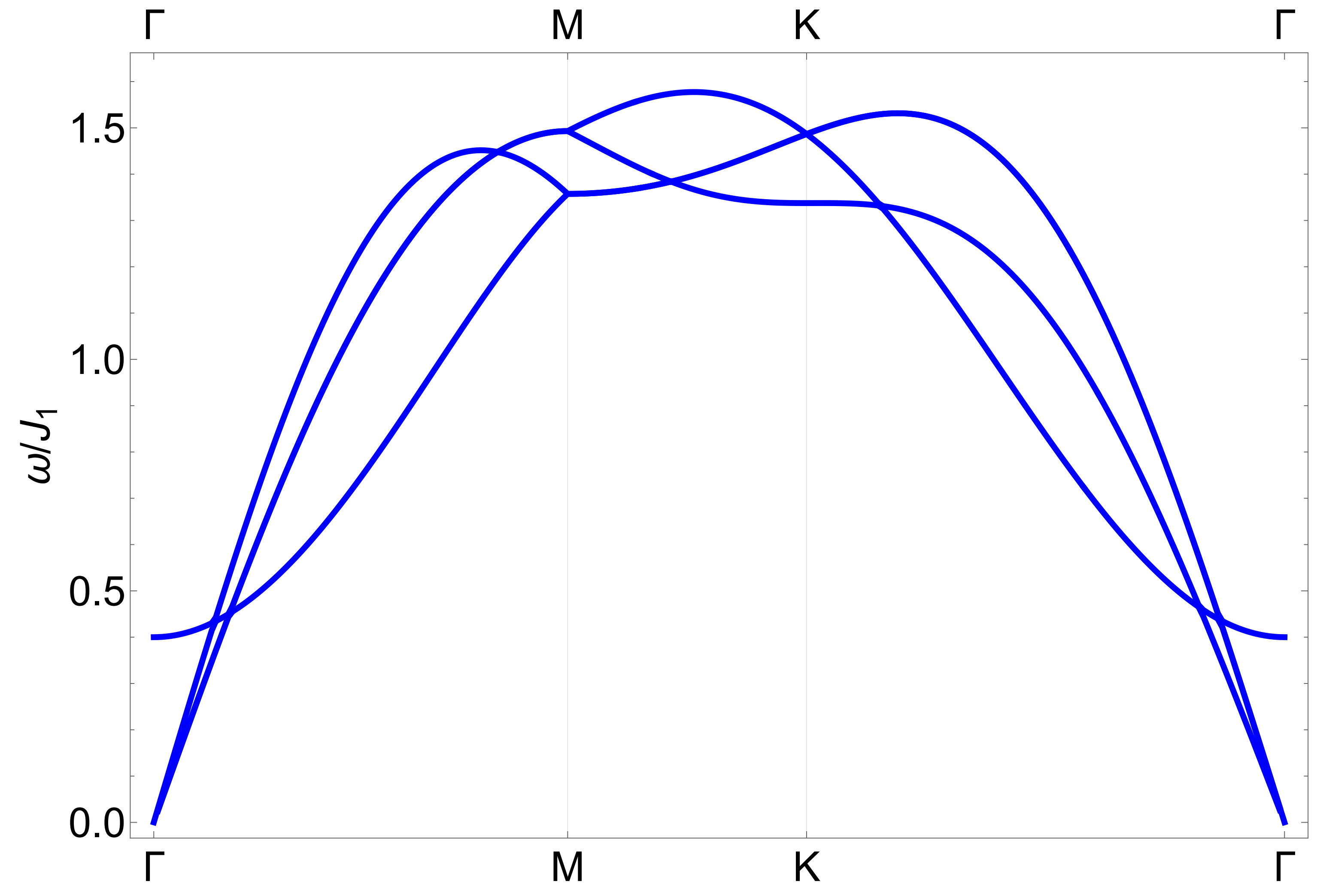
(i) Parton bands separated by a gap. Lower bands are entirely filled.

(ii) Excitations are parton-hole pairs (spinons) -> Thermal Hall response (to be calculated)

Conclusions







$$\mathbf{S}_i = \frac{1}{2} c_{i,\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{i,\beta}$$

(i) Thermal Hall in a spin liquid and in a tetratic state?

(ii) What is nature of the chiral spin liquid? How is it connected to the tetratic state?

(iii) How to avoid the collinearity ? Triple-**q** states…

nature communications

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A classical chiral spin liquid from chiral interactions on the pyrochlore lattice

Received: 14 May 2024

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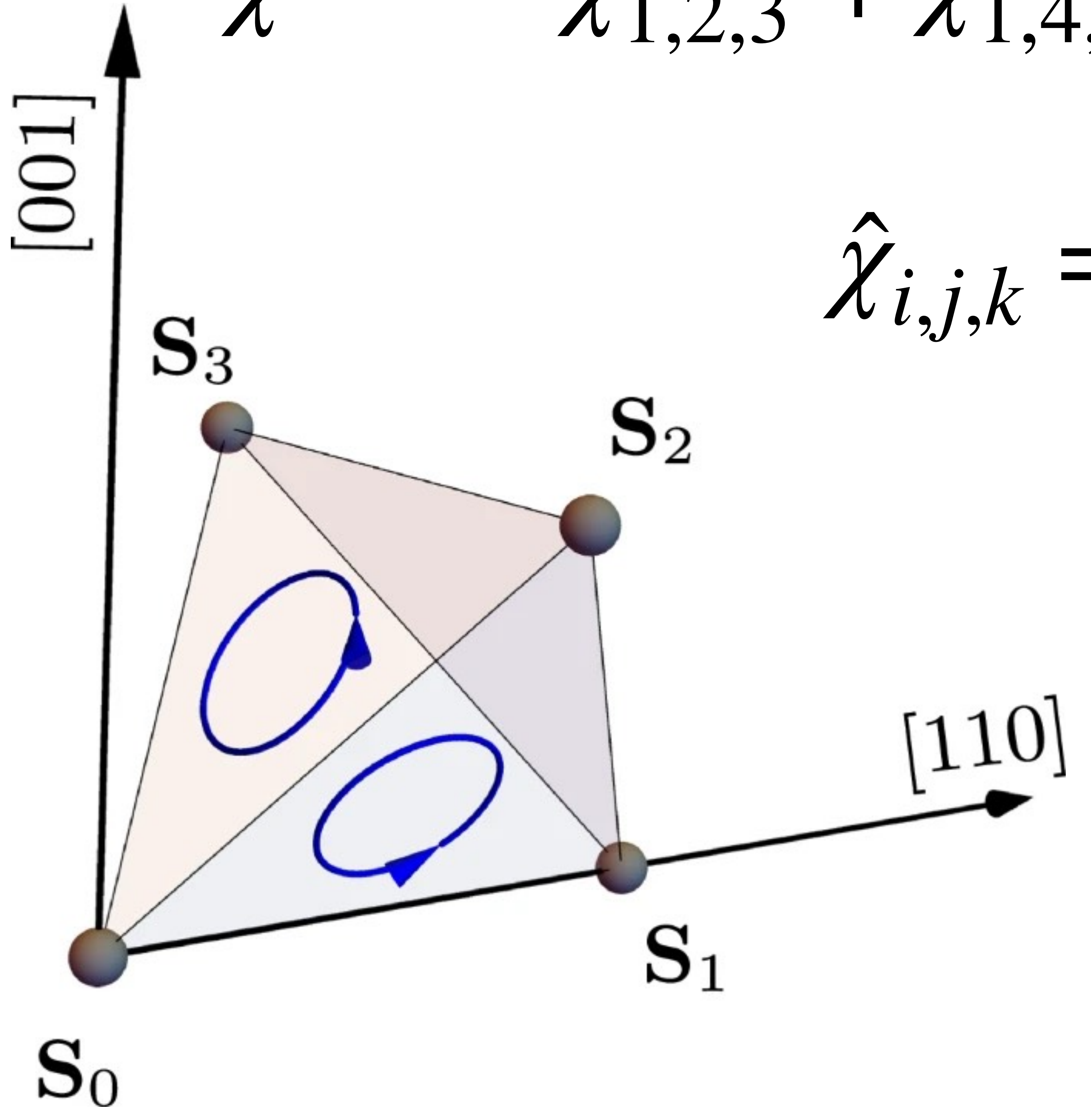
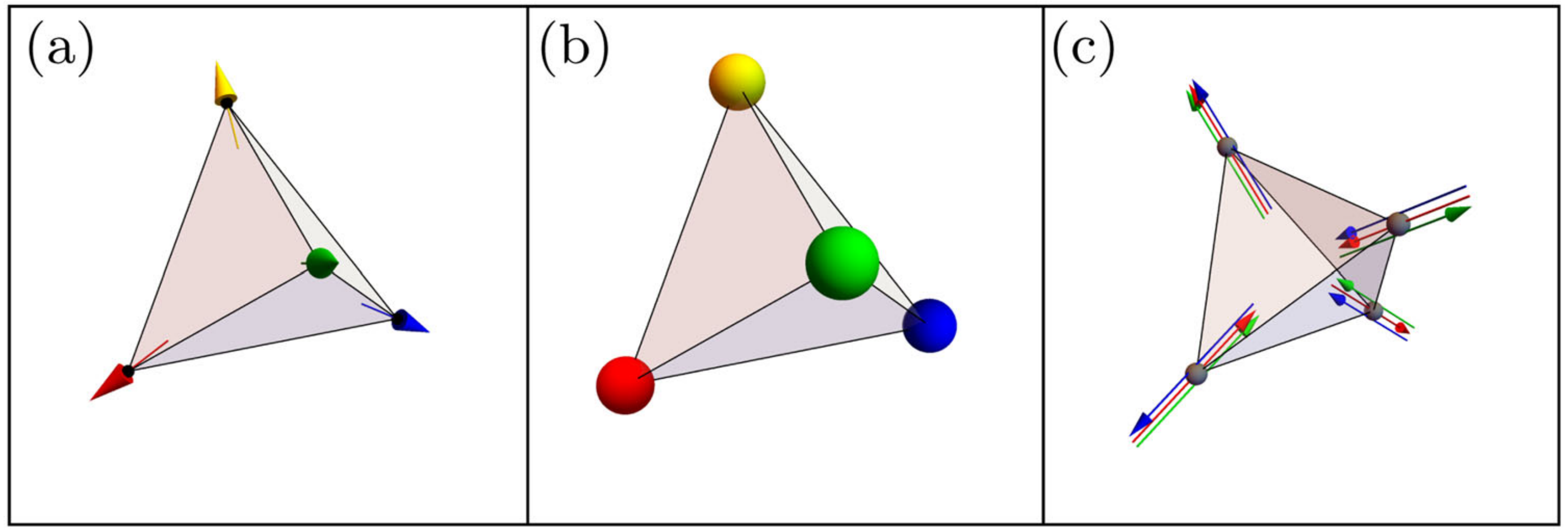
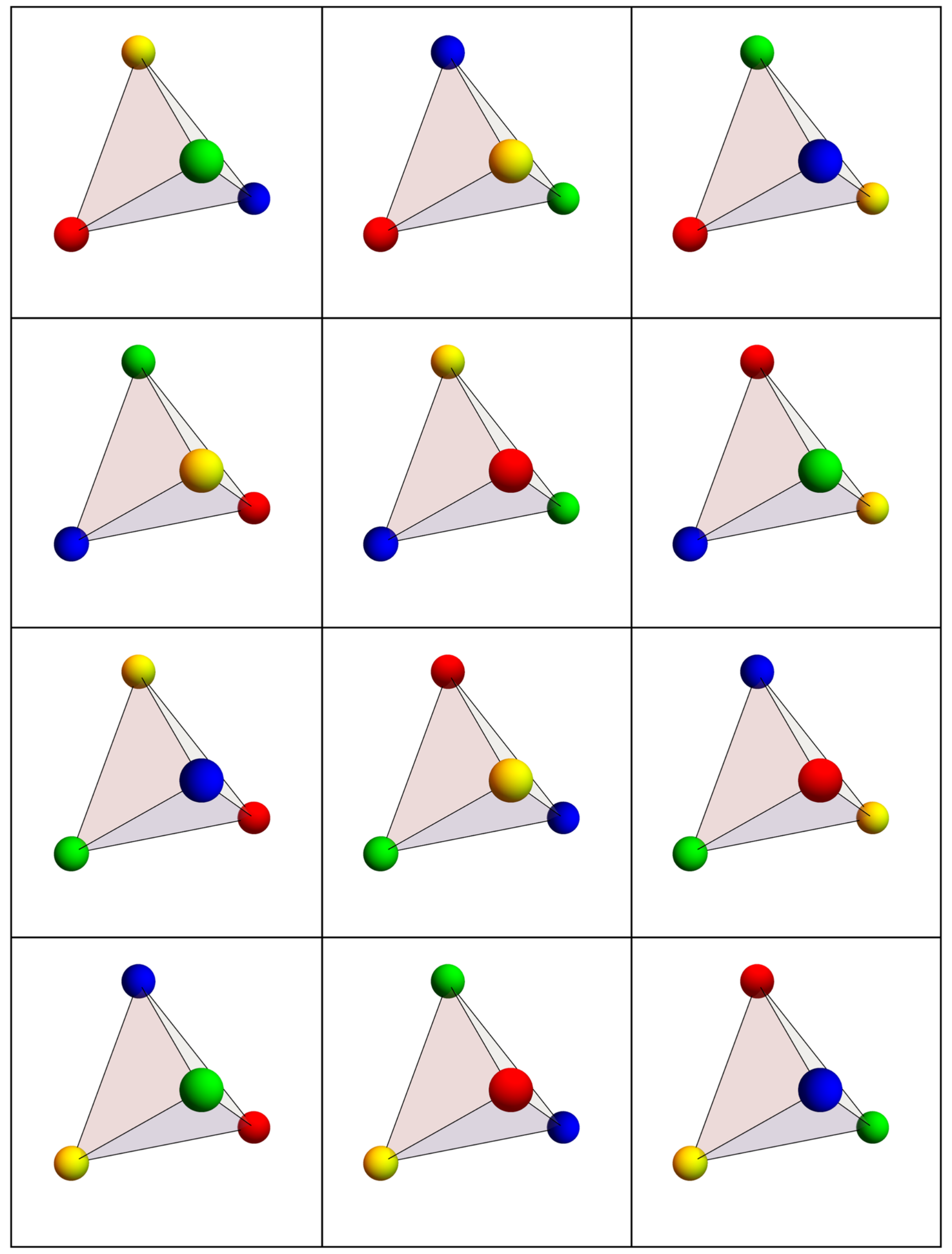
Daniel Lozano-Gómez [1], Yasir Iqbal [2] & Matthias Vojta [1]

$$\chi^{\text{tet}} = \chi_{1,2,3} + \chi_{1,4,2} + \chi_{1,3,4} + \chi_{2,4,3}$$

$$\hat{\chi}_{i,j,k} = \hat{\mathbf{S}}_i \cdot \left(\hat{\mathbf{S}}_j \times \hat{\mathbf{S}}_k \right)$$

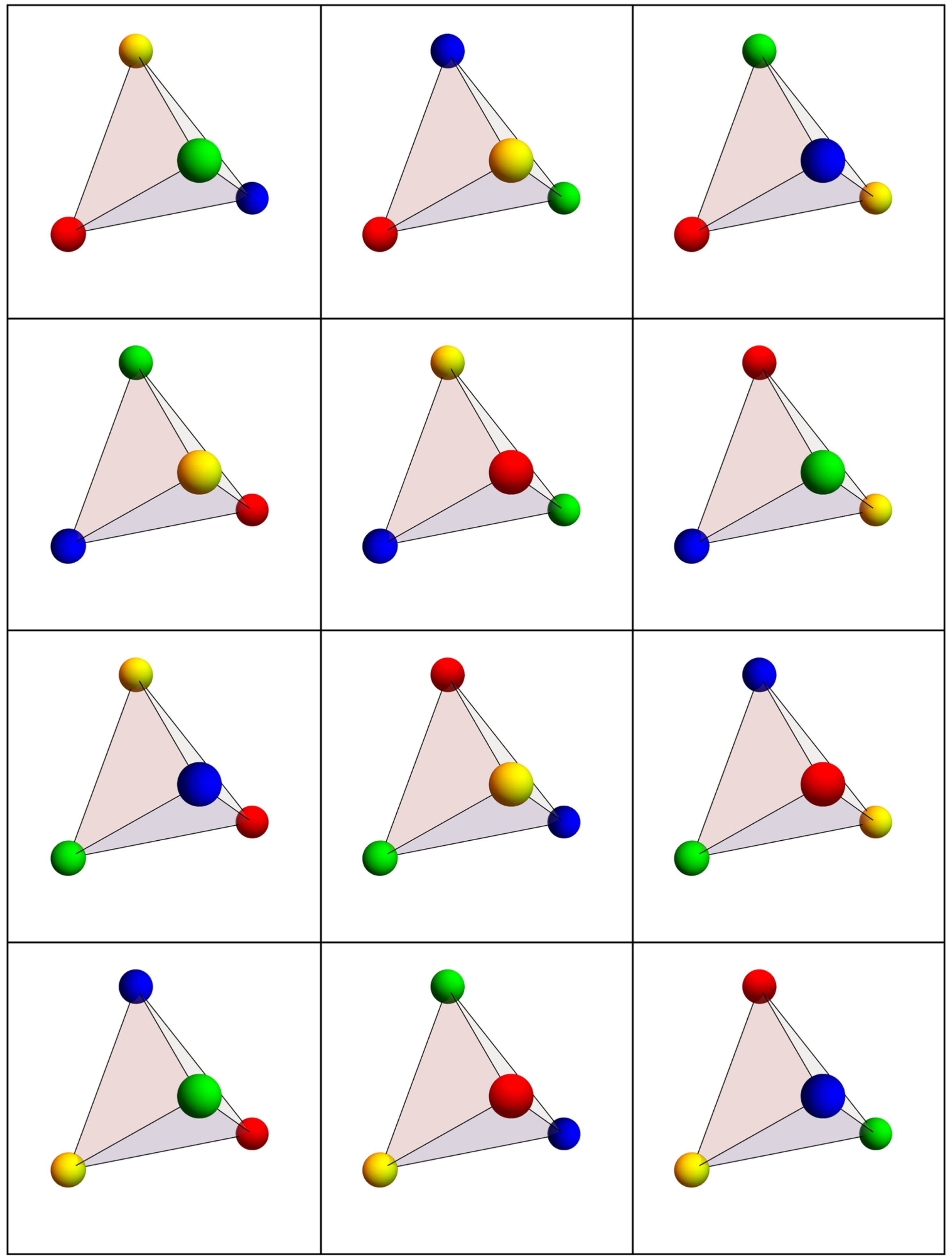


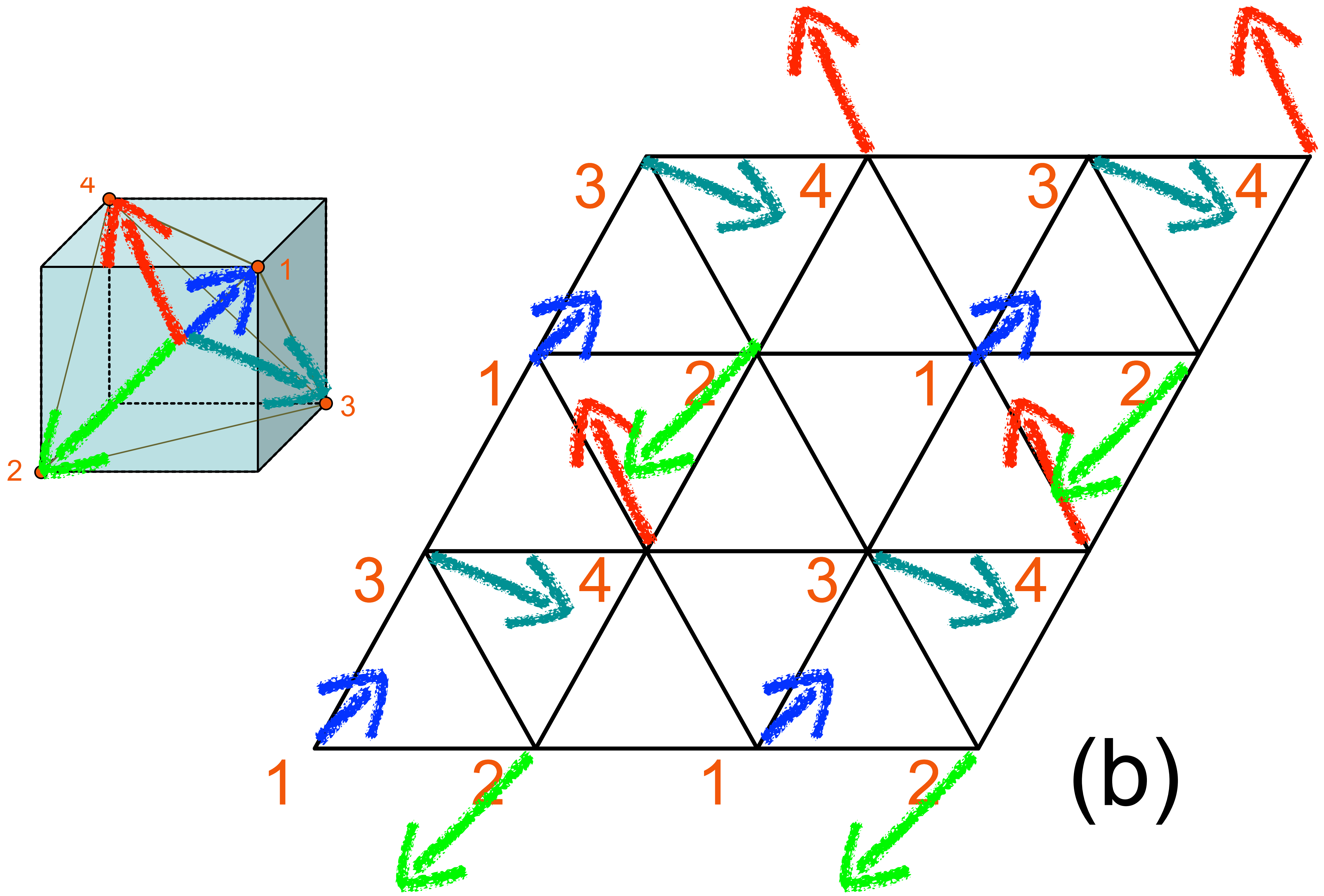




Color-ice states -> pinch points, gauge theory

Is there a parent quantum-spin Hamiltonian for the 4-coloring states?





The parent Hamiltonian for the 3-coloring states (coplanar 120-configurations):

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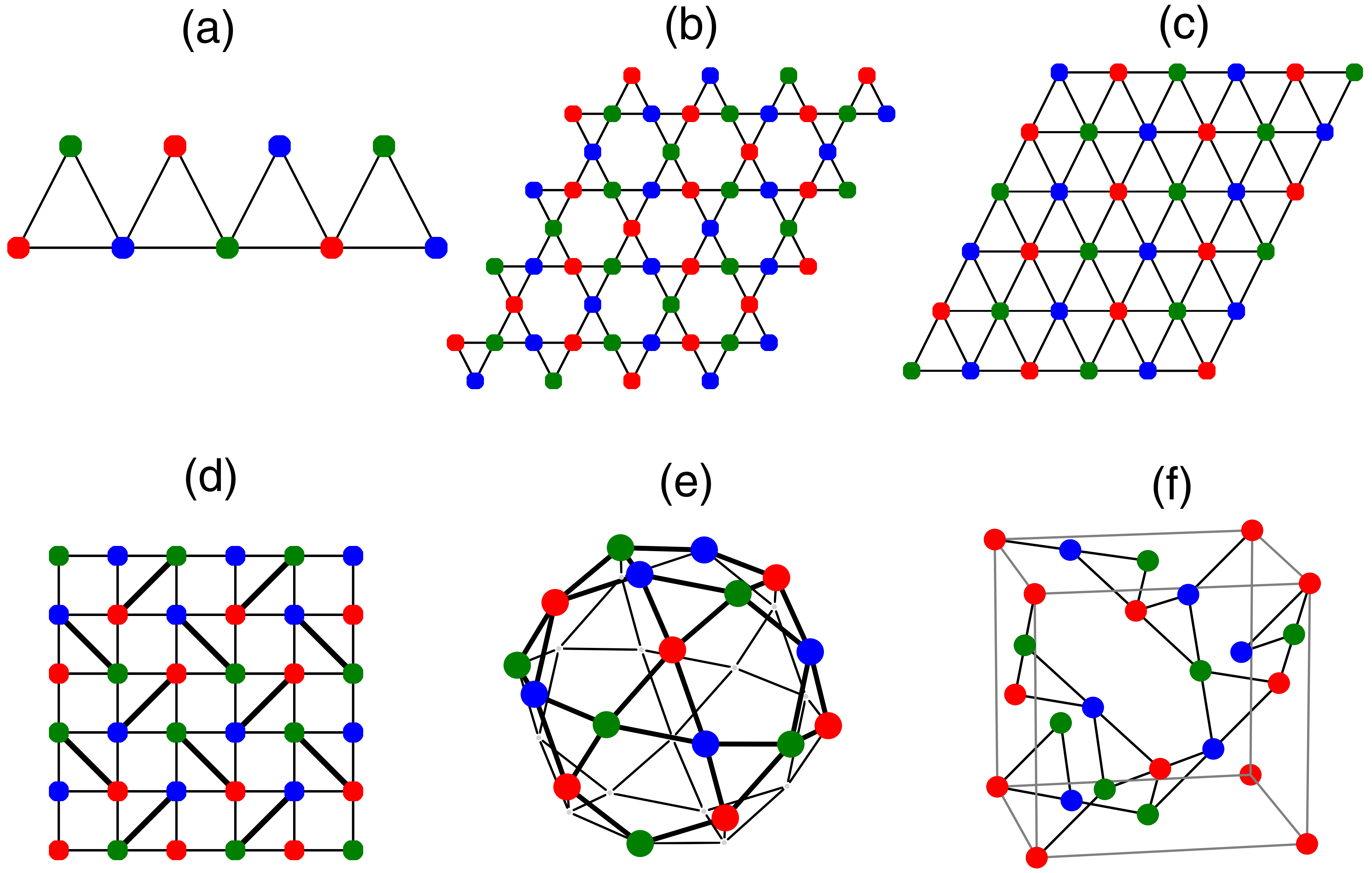
Editors' Suggestion

Macroscopically Degenerate Exactly Solvable Point in the Spin-1/2 Kagome Quantum Antiferromagnet

Hitesh J. Changlani,[1,2,3] Dmitrii Kochkov,[3] Krishna Kumar,[3] Bryan K. Clark,[3] and Eduardo Fradkin[3]



$$\mathcal{H}_{XXZ} = \sum_{\rangle i,j \langle} \left(S_i^x S_j^x + S_i^y S_j^y - \frac{1}{2} S_i^z S_j^z \right)$$

Exact product wave function:

$$|C\rangle \equiv P_{S_z} \left(\prod_{\text{valid}} \otimes |\gamma_s\rangle \right),$$

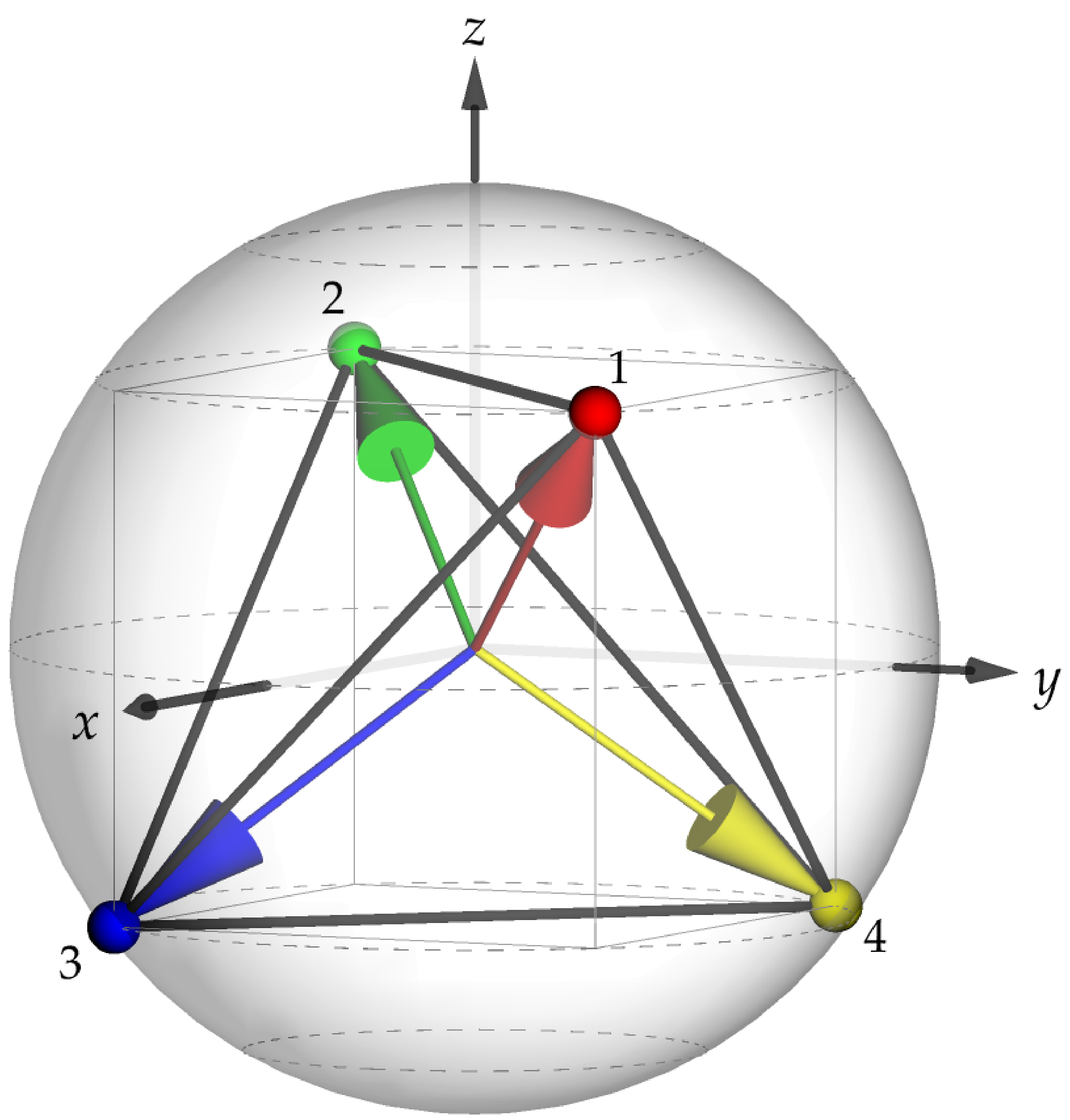
$$|a\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + |\downarrow\rangle \right)$$

$$|b\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + e^{-2\pi i/3} |\downarrow\rangle \right)$$

$$|c\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + e^{2\pi i/3} |\downarrow\rangle \right)$$

Parent Hamiltonian on a tetrehedron for the chiral tetrahedral states

$$|\Psi_{4c}\rangle = |\mathbf{n}_1\rangle \otimes |\mathbf{n}_2\rangle \otimes |\mathbf{n}_3\rangle \otimes |\mathbf{n}_4\rangle$$



and maximize the sum of the four chirality operators

$$\chi^{\text{tet}} = \chi_{1,2,3} + \chi_{1,4,2} + \chi_{1,3,4} + \chi_{2,4,3}$$

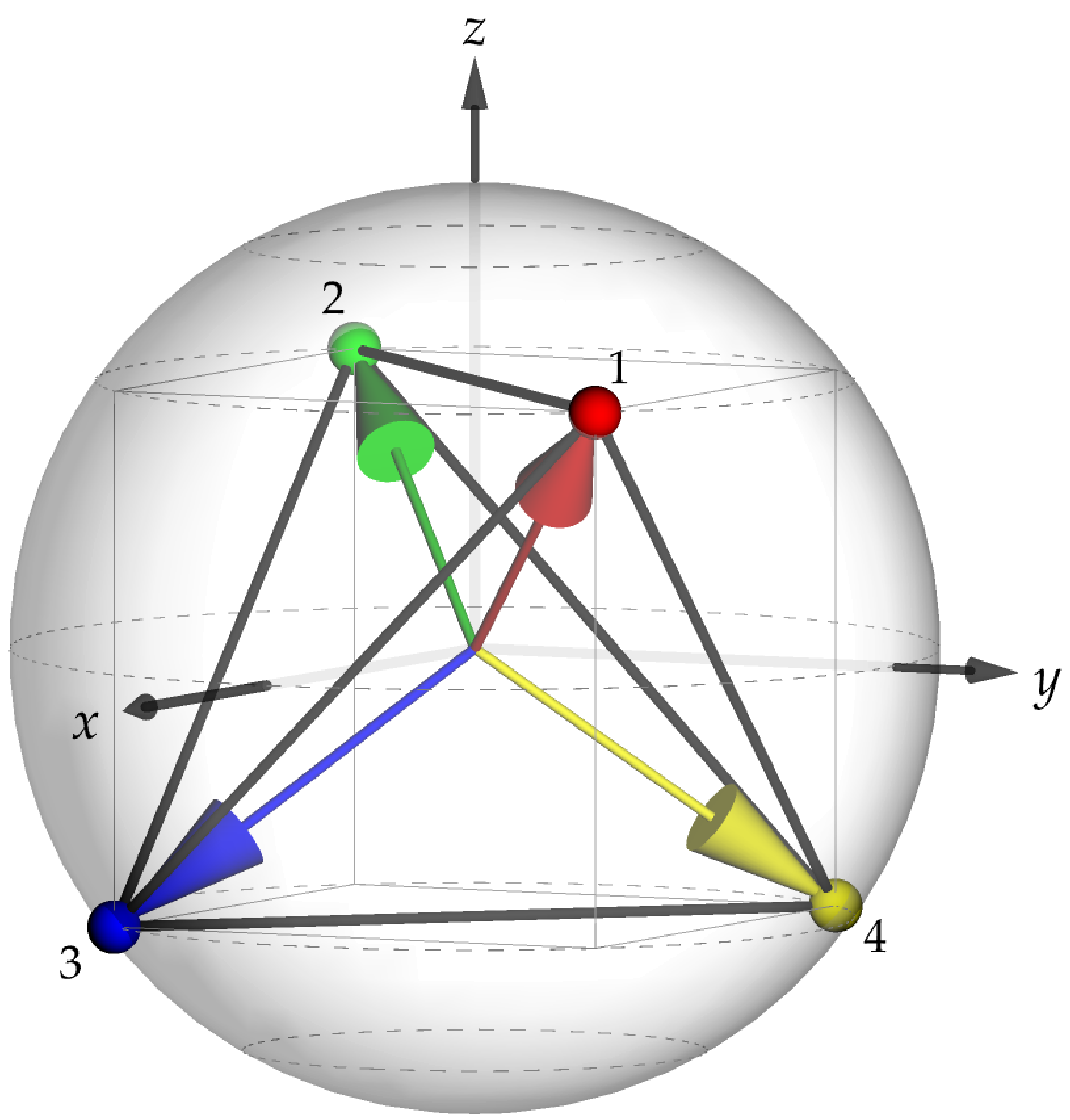
$$\hat{\chi}_{i,j,k} = \hat{\mathbf{S}}_i \cdot \left(\hat{\mathbf{S}}_j \times \hat{\mathbf{S}}_k\right)$$

We search for a positive semidefinite $\mathscr{H}_{4c}$ such that

$$\mathscr{H}_{4c}|\Psi_{4c}\rangle = 0$$

The recipe:

$$|\Psi_{4c}\rangle = |\mathbf{n}_1\rangle \otimes |\mathbf{n}_2\rangle \otimes |\mathbf{n}_3\rangle \otimes |\mathbf{n}_4\rangle$$



1. Sample chiral tetrahedral configurations and construct the associated product states numerically for a given S=1/2, 1, 3/2, etc.
2. Form the Hilbert subspace defined by the linear span of the sampled states.
3. Determine the complete set of operators that annihilate this subspace, i.e., whose nullspace is exactly the spanned Hilbert space.

Happy birthday, Sriram!

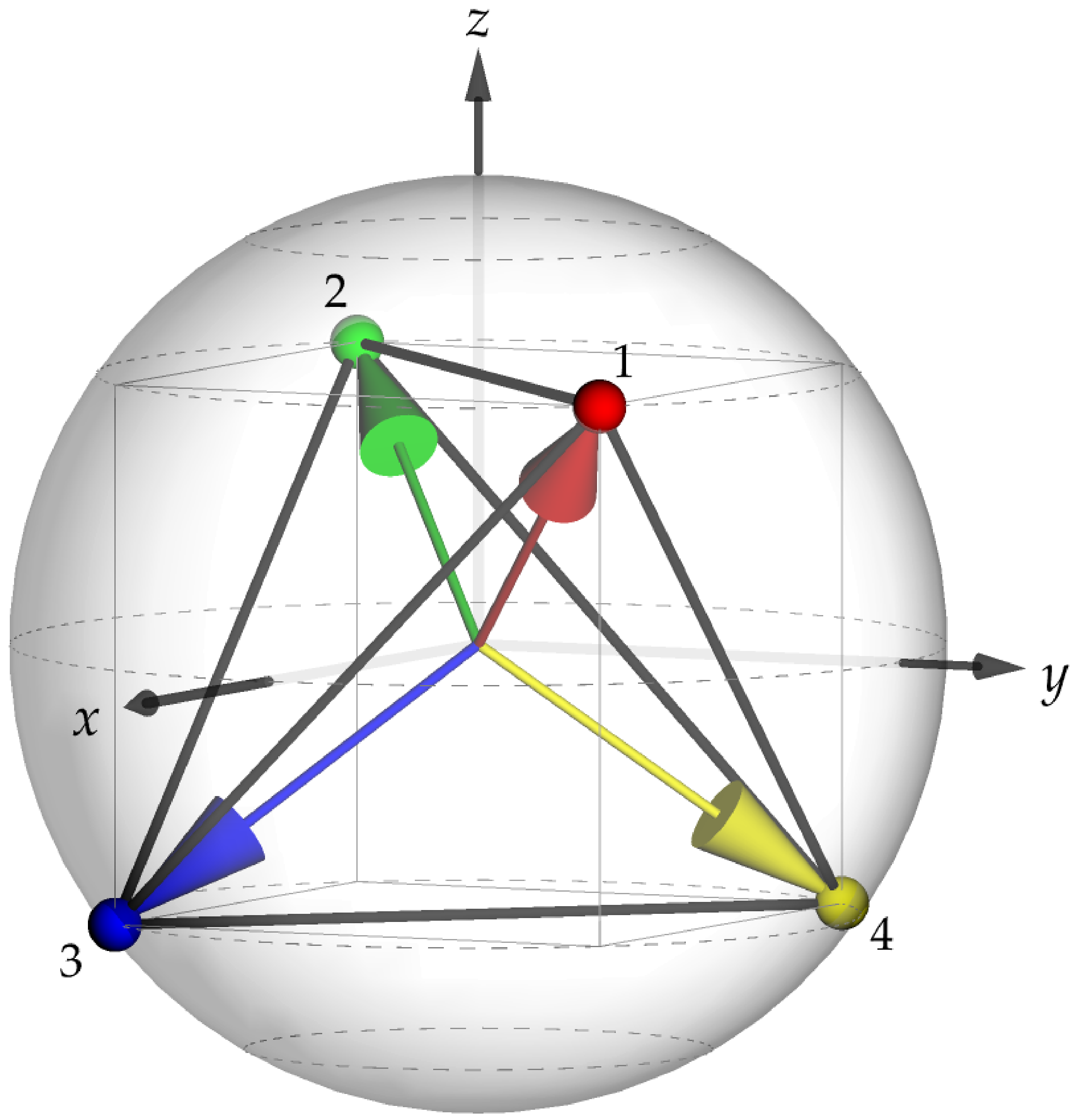
$$\mathscr{H}_{4c} = \frac{\sqrt{3}}{2} J \sum_{\langle ijk \rangle} \mathbf{S}_i \left(\mathbf{S}_j \times \mathbf{S}_k \right) - \frac{1}{4} J \sum_{\langle ij \rangle} \mathbf{S}_i \mathbf{S}_j + J \sum_{\langle ijkl \rangle} (\mathbf{S}_i \mathbf{S}_j)(\mathbf{S}_k \mathbf{S}_l),$$

A gift from Péter Kránitz, Yasir Iqbal, and me.

Parent Hamiltonian for S=1/2

$|\Psi_{4c}\rangle = |\mathbf{n}_1\rangle \otimes |\mathbf{n}_2\rangle \otimes |\mathbf{n}_3\rangle \otimes |\mathbf{n}_4\rangle$

$\mathscr{H}_{4c}|\Psi_{4c}\rangle = 0$



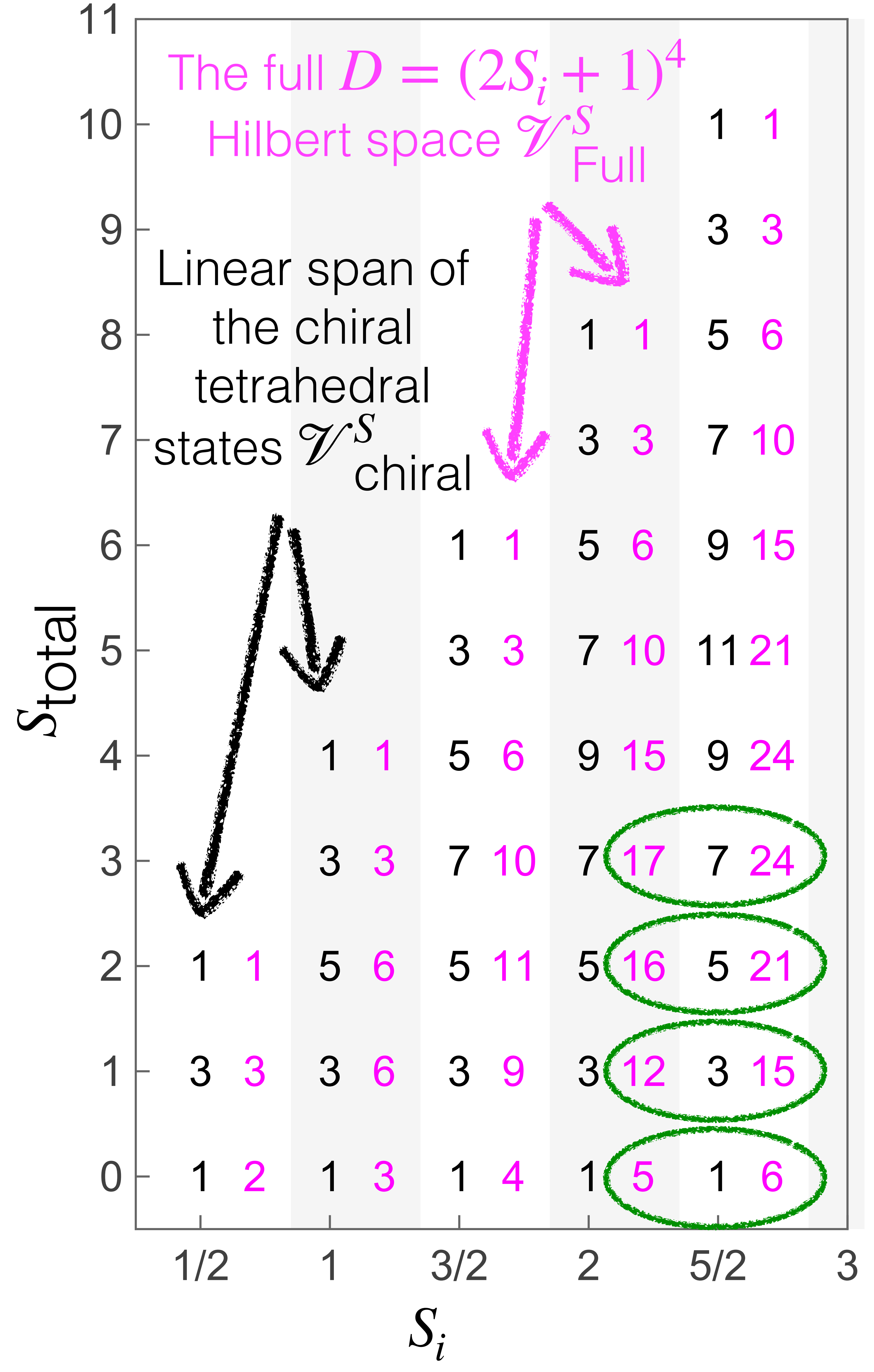
Chiral, $\hat{\chi}^{\text{tet}}|C^{\pm}_{0,0}\rangle \propto \pm|C^{\pm}_{0,0}\rangle$

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = 0 \oplus \underbrace{0 \oplus 1 \oplus 1 \oplus 1 \oplus 2}_{\text{finite overlap with}\,\Psi_{4C}}$$

$$\mathcal{H}_{4c}(\lambda) = 3|C^{+}_{0,0}\rangle\langle C^{+}_{0,0}|$$

$$|C^{\pm}_{0,0}\rangle = |\uparrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\uparrow\rangle + e^{\pm 2\pi i/3}(|\downarrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle) + e^{\mp 2\pi i/3}(|\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle)$$

$$\mathscr{H}_{4c} = -\frac{1}{4}\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j + \frac{\sqrt{3}}{2}\sum_{\langle ijk\rangle}\mathbf{S}_i\cdot\left(\mathbf{S}_j\times\mathbf{S}_k\right) + \sum_{\langle ijkl\rangle}(\mathbf{S}_i\cdot\mathbf{S}_j)(\mathbf{S}_k\cdot\mathbf{S}_l)$$



Parent Hamiltonian for S>1/2 on a tetrahedron

$$\mathscr{H}_{4c} = \sqrt{3}S\sum_{\langle ijk\rangle}\hat{\mathbf{S}}_i\cdot\left(\hat{\mathbf{S}}_j\times\hat{\mathbf{S}}_k\right) + \sum_{\langle ijkl\rangle}(S^2-\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j)(S^2-\hat{\mathbf{S}}_k\cdot\hat{\mathbf{S}}_l),$$

$$\mathcal{H}_{4c} = \mathcal{B}^\dagger\mathcal{B}$$

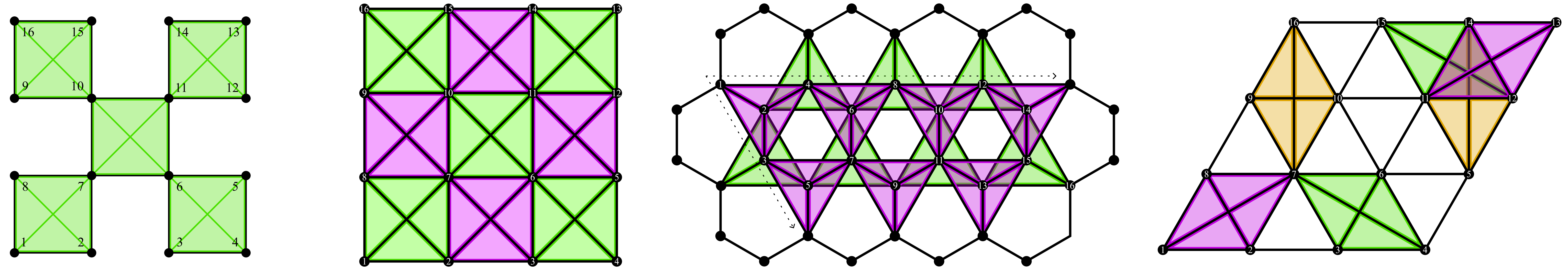
$$\mathcal{B} = \frac{1}{\sqrt{2}}\left[a_1a_2b_3b_4 + b_1b_2a_3a_4 + e^{2\pi i/3}\left(a_1b_2a_3b_4 + b_1a_2b_3a_4\right) + e^{-2\pi i/3}\left(a_1b_2b_3a_4 + b_1a_2a_3b_4\right)\right]$$

$|C^{+}_{0,0}\rangle = \mathcal{B}^\dagger|0\rangle$ The “wrong” chiral state that is not member of the tetrahedral states

$$\mathscr{V}^{S+1/2}_{\text{Full}} = \mathscr{V}^{S+1/2}_{\text{Chiral}} \oplus \left(\mathscr{V}^{S}_{\text{Full}} \otimes \{\,|C^{+}_{0,0}\rangle\}\right)$$

Degeneracies for the S=1/2 model on different lattices

$$\mathscr{H}_{4c} = -\frac{1}{4}\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{\sqrt{3}}{2}\sum_{\langle ijk\rangle} \mathbf{S}_i \cdot \left(\mathbf{S}_j \times \mathbf{S}_k\right) + \sum_{\langle ijkl\rangle} (\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l)$$

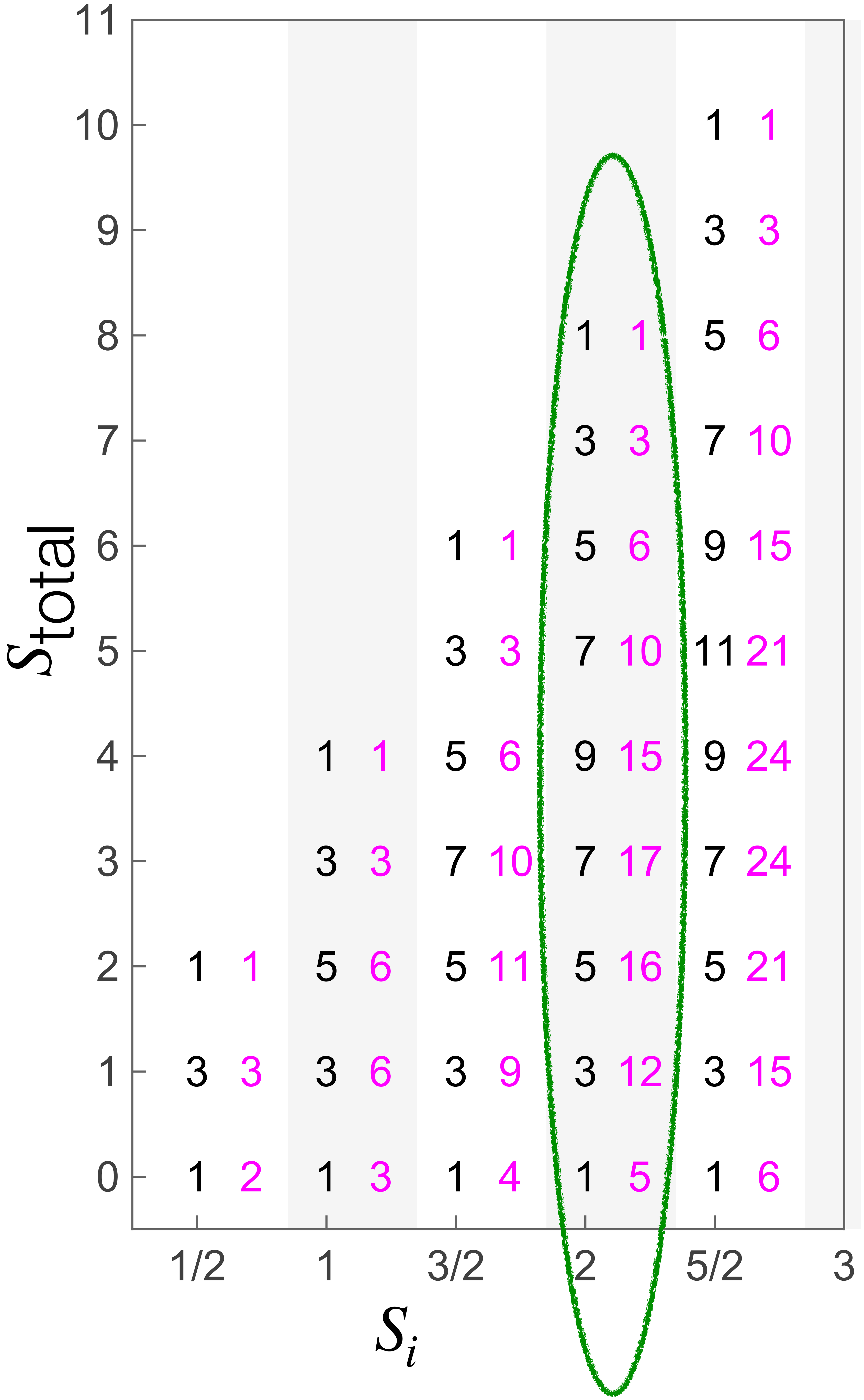


(a) (b) (c)

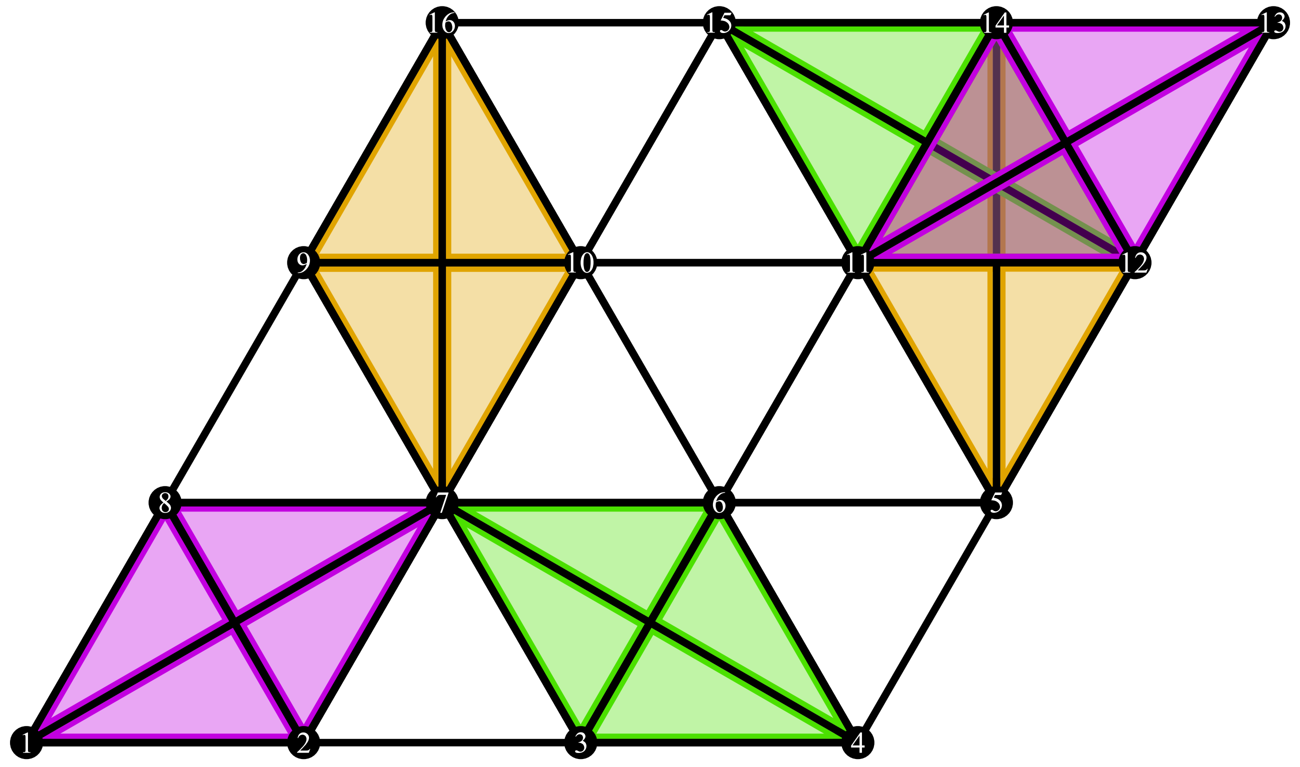
Frustration-free Hamiltonian:
Highly degenerate ground state manifold on pyrochlore, checkerboard, honeycomb, square, triangular lattice, etc. …

$L = 16$		Checkerboard	Square		Honeycomb	Triangular
Chirality:		uniform	uniform	alternating	uniform	uniform
S_{tot}	Dim.					
0	1430	528	101	56	114	1
1	3432	1368	244	118	263	3
2	3640	1672	382	312	364	5
3	2548	1400	486	476	478	7
4	1260	840	453	452	452	21
5	440	352	264	264	264	45
6	104	96	88	88	88	56
7	15	15	15	15	15	15
8	1	1	1	1	1	1

Degeneracies for the S=1/2 model on triangular lattices



4×4 = 16 sites
Four-sublattice states

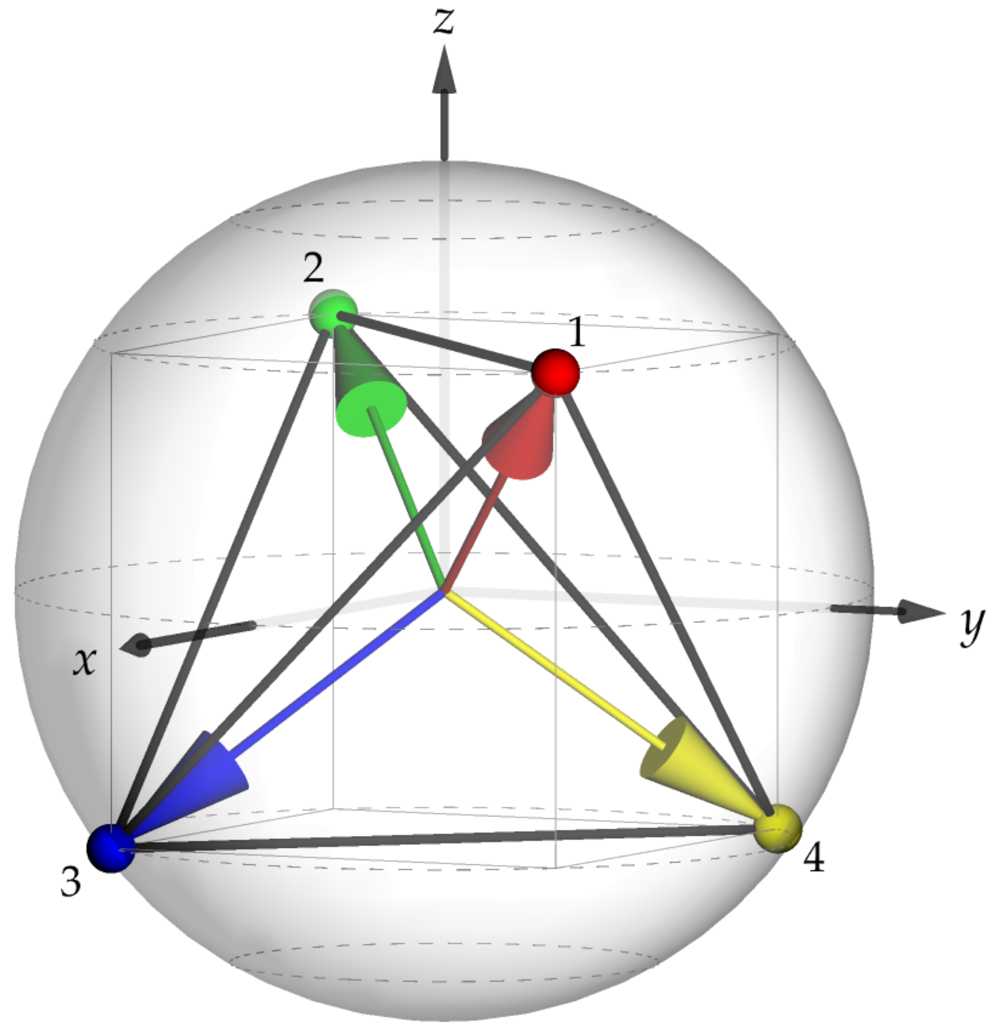


(c)

$L = 16$		Triangular
Chirality:		uniform
S_{tot}	Dim.	
0	1430	1
1	3432	3
2	3640	5
3	2548	7
4	1260	21
5	440	45
6	104	56
7	15	15
8	1	1

4×4 = 16 sites
Additional states appear for $S \geq 4$

Classical limit



$$|\{\mathbf{n}_i\}\rangle = |\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3\mathbf{n}_4\rangle$$

$$\langle\{\mathbf{n}_i\}|\mathscr{H}_{4c}|\{\mathbf{n}_i\}\rangle = JS^4 \underbrace{\left(\sqrt{3}\sum_{\langle ijk\rangle}\mathbf{n}_i\cdot\left(\mathbf{n}_j\times\mathbf{n}_k\right) + \sum_{\langle ijkl\rangle}(1-\mathbf{n}_i\cdot\mathbf{n}_j)(1-\mathbf{n}_k\cdot\mathbf{n}_l)\right)}_{\mathscr{H}_{4c}^{\mathrm{cl}}}$$

It is linear in $\mathbf{n}_i$'s,
the effect of $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ is to
impose an effective field for $\mathbf{n}_4$

$$\mathscr{H}_{4c}^{\mathrm{cl}} = b_{123} - \mathbf{B}_{123}\cdot\mathbf{n}_4 = b_{123}\left(1-\hat{\mathbf{B}}_{123}\cdot\mathbf{n}_4\right)$$

$$\mathbf{n}_4 = \hat{\mathbf{B}}_{123}$$

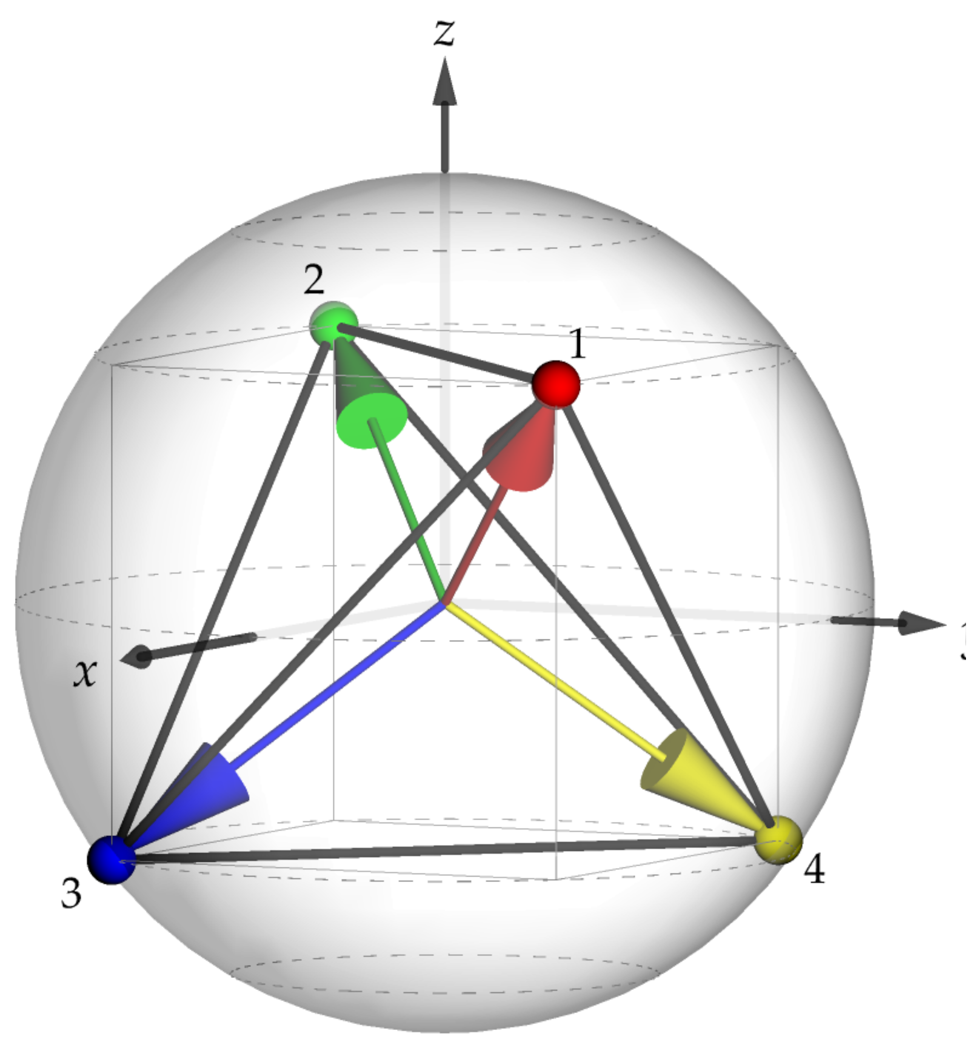
$$b_{123} = (1-\mathbf{n}_1\mathbf{n}_2) + (1-\mathbf{n}_1\mathbf{n}_3) + (1-\mathbf{n}_2\mathbf{n}_3) + \sqrt{3}\mathbf{n}_1(\mathbf{n}_2\times\mathbf{n}_3)$$

$$\mathbf{B}_{123} = (1-\mathbf{n}_2\mathbf{n}_3)\mathbf{n}_1 + (1-\mathbf{n}_1\mathbf{n}_3)\mathbf{n}_2 + (1-\mathbf{n}_1\mathbf{n}_2)\mathbf{n}_3 + \sqrt{3}\left(\mathbf{n}_1\times\mathbf{n}_2 + \mathbf{n}_2\times\mathbf{n}_3 + \mathbf{n}_3\times\mathbf{n}_1\right)$$

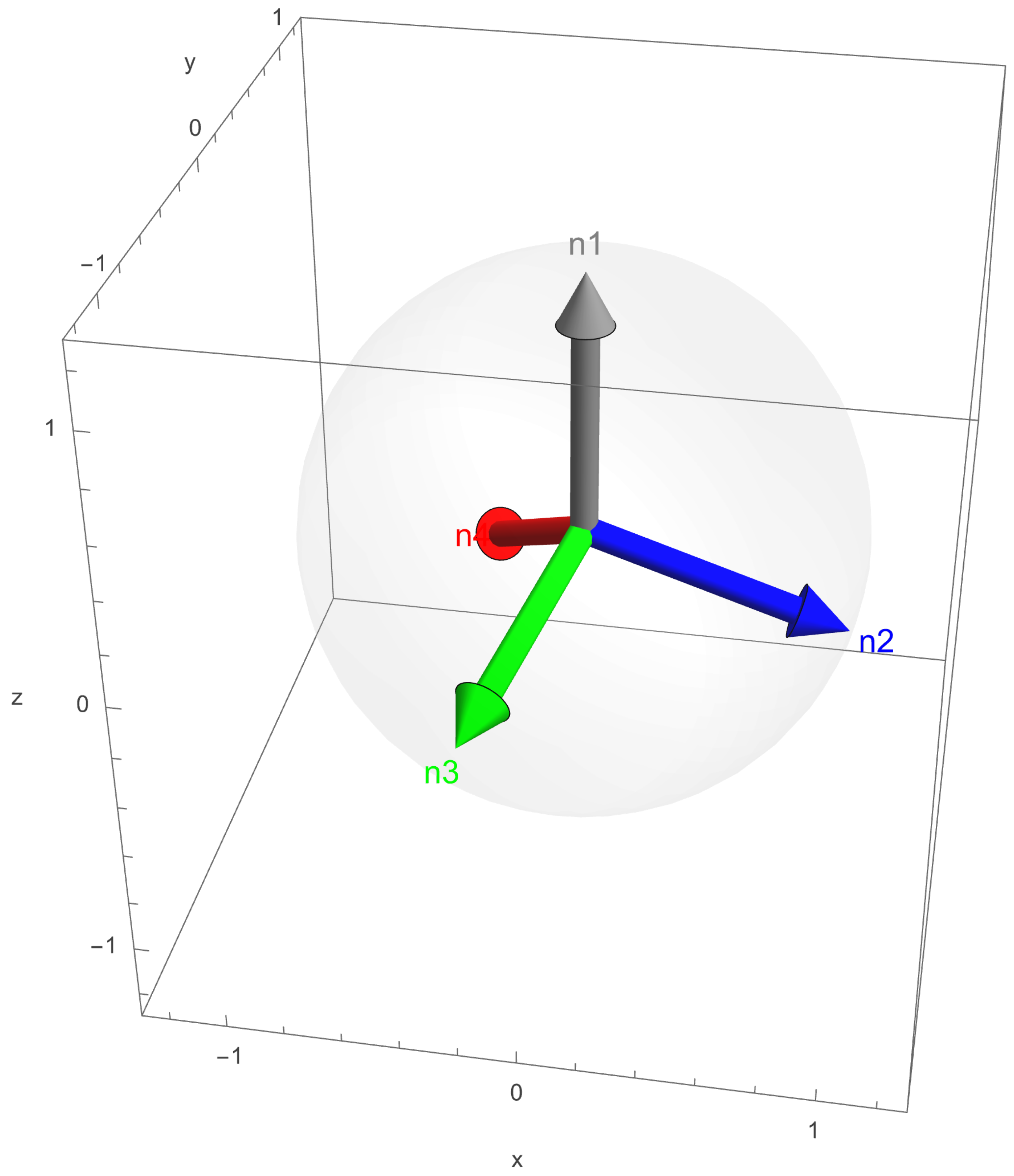
$$b_{123}^2 = \|\mathbf{B}_{123}\|^2 \qquad \frac{16}{3} \geq b_{123} \geq 0$$

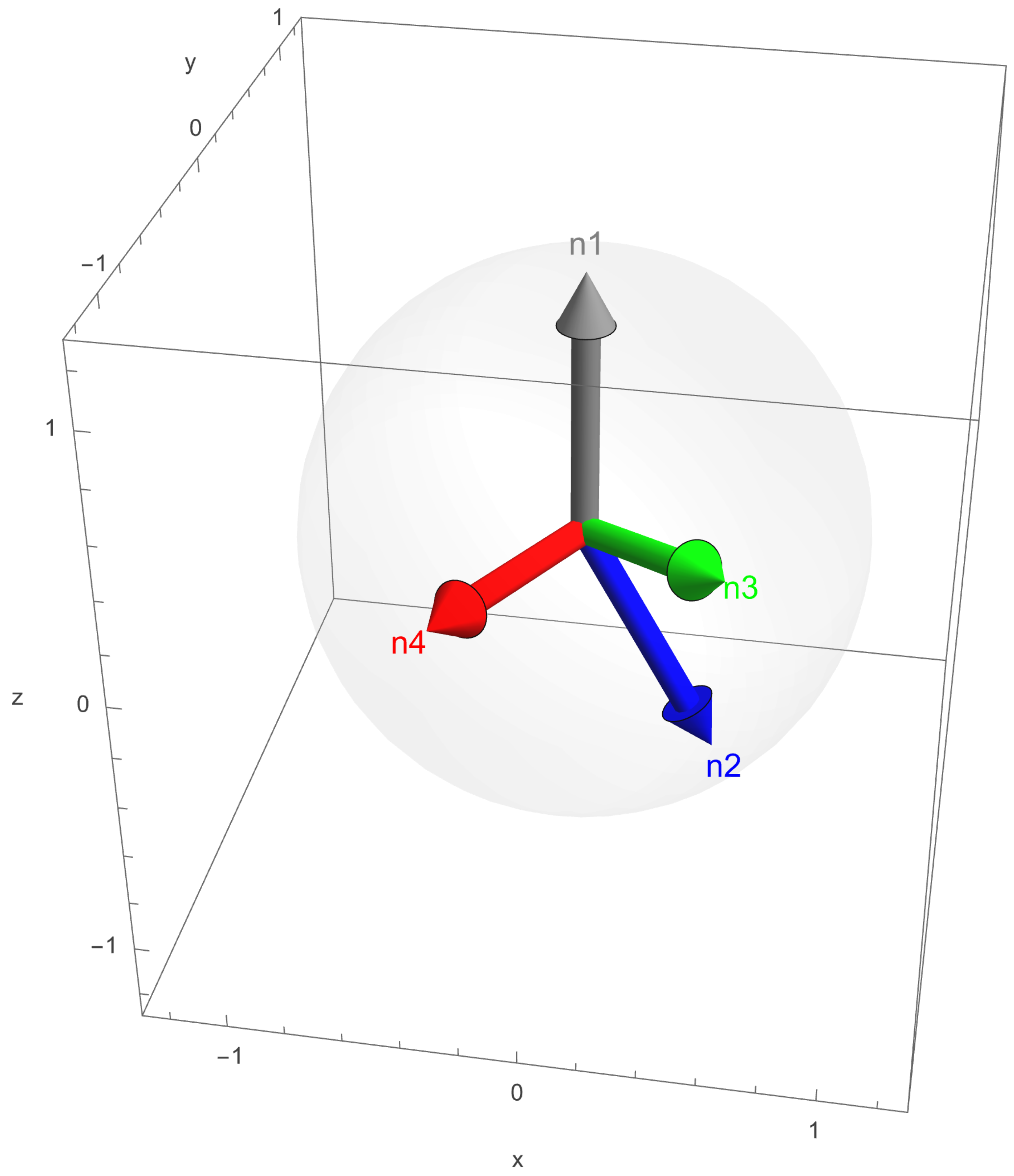
Classical limit

$$\mathbf{n}_4 \propto (1-\mathbf{n}_2\mathbf{n}_3)\mathbf{n}_1 + (1-\mathbf{n}_1\mathbf{n}_3)\mathbf{n}_2 + (1-\mathbf{n}_1\mathbf{n}_2)\mathbf{n}_3 + \sqrt{3}\left(\mathbf{n}_1\times\mathbf{n}_2 + \mathbf{n}_2\times\mathbf{n}_3 + \mathbf{n}_3\times\mathbf{n}_1\right)$$

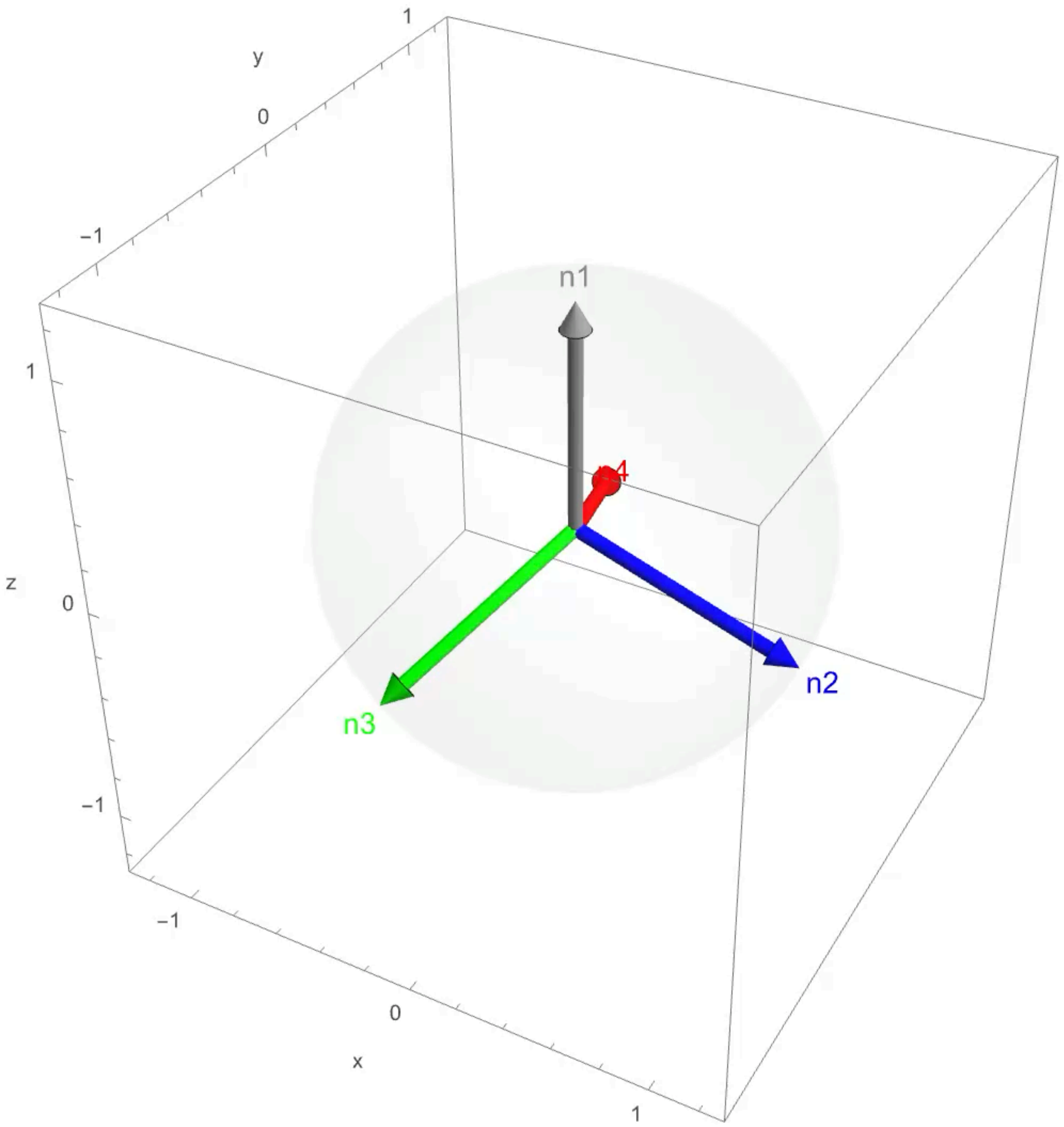


There are many classical states beyond the tetrahedral states, even the ferro-aligned one is a good ground state





Classical limit



Degenerate classical manifold on the tetrahedron,

c.f. degenerate manifold of the Heisenberg model on a tetrahedron:

$$\mathscr{H} = \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{2}\left(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4\right)^2 - \text{const.}$$

, where the ground state manifold satisfies

$$\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4 = 0$$

Parent Hamiltonian on a tetrehedron for the chiral tetrahedral states

Works for S=1/2

$$\mathscr{H}_{4c} = -\frac{1}{4}\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j + \frac{\sqrt{3}}{2}\sum_{\langle ijk\rangle}\mathbf{S}_i\cdot\left(\mathbf{S}_j\times\mathbf{S}_k\right) + \sum_{\langle ijkl\rangle}(\mathbf{S}_i\cdot\mathbf{S}_j)(\mathbf{S}_k\cdot\mathbf{S}_l)$$

Works for S=1, 3/2, etc…

$$\mathcal{H}_{4c} = J\sqrt{3}S\sum_{\langle ijk\rangle}\hat{\mathbf{S}}_i\left(\hat{\mathbf{S}}_j\times\hat{\mathbf{S}}_k\right) + J\sum_{\langle ijkl\rangle}(S^2-\hat{\mathbf{S}}_i\cdot\hat{\mathbf{S}}_j)(S^2-\hat{\mathbf{S}}_k\cdot\hat{\mathbf{S}}_l)$$

The positive semidefinite form: $\mathscr{H}_{4c} = \mathscr{B}^\dagger\mathscr{B}$, with

$$\mathscr{B} = \frac{1}{\sqrt{2}}\left[a_4a_3b_2b_1 + b_4b_3a_2a_1 + \omega\left(a_4b_3a_2b_1 + b_4a_3b_2a_1\right) + \omega^2\left(a_4b_3b_2a_1 + b_4a_3a_2b_1\right)\right].$$

Works for classical spins:

$$\mathcal{H}^{\mathrm{cl}}_{4c} = \sqrt{3}\sum_{\langle ijk\rangle}\mathbf{n}_i\left(\mathbf{n}_j\times\mathbf{n}_k\right) + \sum_{\langle ijkl\rangle}(1-\mathbf{n}_i\mathbf{n}_j)(1-\mathbf{n}_k\mathbf{n}_l),$$

Many open questions, more to come…