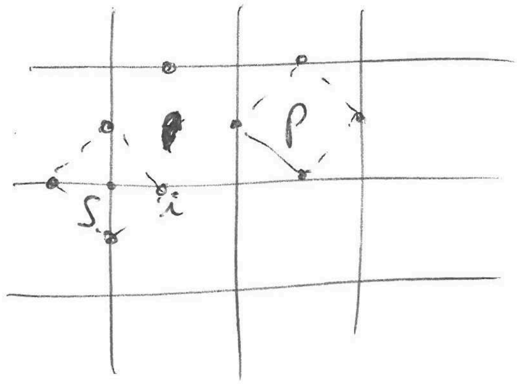
Toric Code : (Kitaev)

| fine tuned (matter less) $\mathbb{Z}_2$ lattice gauge theory



$\sigma_i^{x,y,z}$ Pauli matrices

$$H = -\sum_s A_s - \sum_p B_p$$

$[\sigma_i^\alpha, \sigma_j^\beta] = 0 \quad i \neq j \quad \forall \alpha, \beta$

$[\sigma_i^\alpha, \sigma_j^\beta] = 0 \ , \quad \{\sigma_i^\alpha, \sigma_i^\beta\} = 2\delta_{\alpha\beta} \quad \forall i$

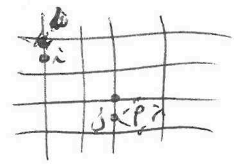
$$A_s = \prod_{i \in s} \sigma_i^z$$

$$B_p = \prod_{i \in p} \sigma_i^x$$

$[A_s, B_p] = 0$ | diagonalise simultaneously with Hamiltonian

ground state : $A_s |\psi_0\rangle = |\psi_0\rangle$

$B_p |\psi_0\rangle = |\psi_0\rangle$ | choose such that it includes j but not i



Correlations : $\langle \psi_0 | \sigma_i^z \sigma_j^z | \psi_0 \rangle$

$= \langle \psi_0 | \sigma_i^z \sigma_j^z B_p | \psi_0 \rangle$

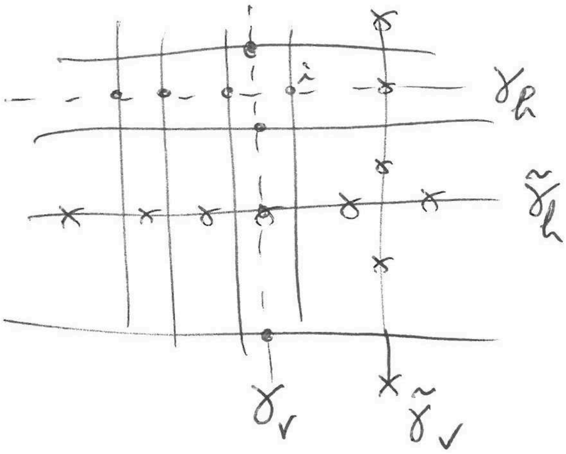
$\Downarrow$ $[\sigma_i^z, B_p] = 0$, $\{\sigma_j^z, B_p\} = 0$

$= -\langle \psi_0 | B_p \sigma_i^z \sigma_j^z | \psi_0 \rangle$ (Hermitian)

$= -\langle \psi_0 | \sigma_i^z \sigma_j^z | \psi_0 \rangle$

- all 2point correlations vanish $(i \neq j)$
- "all" multi-spin correlations vanish, unless they are products of A_s, B_p ops.

Trivial paramagnet? Nope: Topological correlations



$$\Gamma_h = \prod_{i \in \gamma_h} \sigma_i^z \qquad \Gamma_v = \dots$$

$$\tilde{\Gamma}_h = \prod_{i \in \tilde{\gamma}_h} \sigma_i^x \qquad \tilde{\Gamma}_v = \dots$$

$$[A_s, \Gamma] = [A_s, \tilde{\Gamma}] = [B_p, \Gamma] = [B_p, \tilde{\Gamma}] = 0 \quad \forall s, p$$

$\Longrightarrow$ commute with Hamiltonian (i.e., conserved charges)

Not all $\Gamma, \tilde{\Gamma}$ commute; max commuting = 2

eigenvalues : ± 1 $\longrightarrow$ $2^2 = 4$ eigenspaces

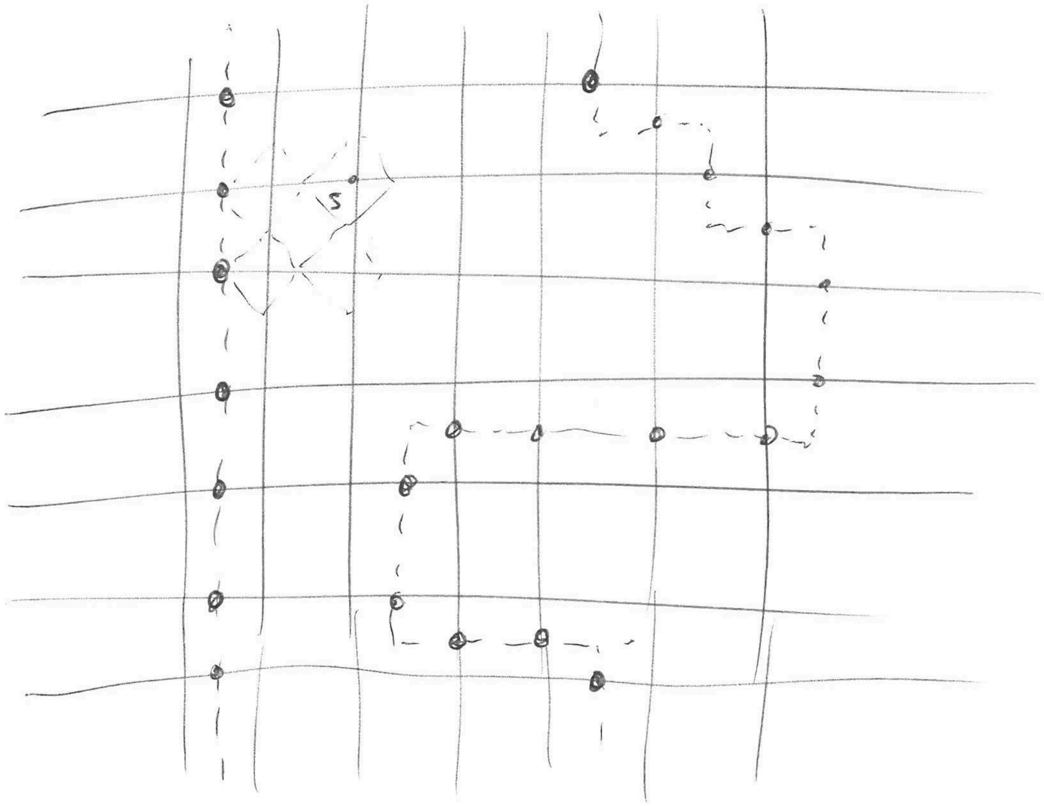
For example: $\langle \psi_0 | \Gamma_{h,v} | \psi_0 \rangle = \pm 1 \longrightarrow$ $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, $(-1, -1)$ — 4 distinct GSs

is this all? N sites, 2N bonds $\rightarrow |\mathcal{H}| = 2^{2N}$ ← Hilbert space

$\#_{A_s} = \#_{B_p} = N$
with eigenvalue ± 1
$\prod_s A_s = \prod_p B_p = \mathbb{1}$
$\} \rightarrow 2^{2(N-1)}$ eigenspaces of dim. 2^2 spanned by eigenstates of $\Gamma_{h,v}$

$\Longrightarrow$ all quantum numbers of $A_s, B_p, \Gamma_h, \Gamma_v$ uniquely identify 2^{2N} eigenstates : complete basis of $\mathcal{H}$

choice of $\gamma_{h,v}$?



$$\Gamma_h' = \prod_{i \in \gamma_h'} \sigma_i^z = \Gamma_h \cdot \prod_{\substack{s \\ \text{between} \\ \gamma_h \text{ and } \gamma_h'}} A_s$$

A_s → eigenvalues fixed and well defined in each eigenstate of H

Γ_h → $\prod_{i \in \gamma_h} \sigma_i^z$

$\Rightarrow$ { eigenvalue of Γ_h completely determines eigenvalue of Γ_h'

(recall $(\sigma_i^z)^2 = \mathbb{1}$)

(same for Γ_v ; $\tilde{\Gamma}_{h,v}$ using B_p instead)

left as exercise ...

four ground states $|\Psi_0(a,b)\rangle$, $a,b = \pm 1$

- trivial local correlations (vanishing)
- non-trivial topological correlations : $\Gamma_{h,v}$

> $\mathbb{Z}_2$ gauge theory is topologically ordered

{ parity of set of spins spanning the whole system; value known only if all are known (cf. magnetisation) largely known if you know most of the spins, statistical error reducing as $1/\sqrt{\#\text{spins known}}$

Example of GS: $|\Psi_0(+,+)\rangle \propto \prod_{p'} \left(\frac{1+B_{p'}}{2}\right) |\uparrow\uparrow\uparrow \ldots \uparrow\rangle$

$A_s |\Psi_0(+,+)\rangle = |\Psi_0(+,+)\rangle \qquad [A_s, B_p] = 0$

$\Gamma_{h,v} |\Psi_0(+,+)\rangle = |\Psi_0(+,+)\rangle \qquad [\Gamma_{h,v}, B_p] = 0$

$B_p |\Psi_0(+,+)\rangle = |\Psi_0(+,+)\rangle$ by projection

$\left\{ B_p \left(\frac{1+B_p}{2}\right) = \frac{B_p + B_p^2}{2} = \frac{B_p + 1}{2} \right.$

<u>the GS stores 2 qubits topologically</u>:

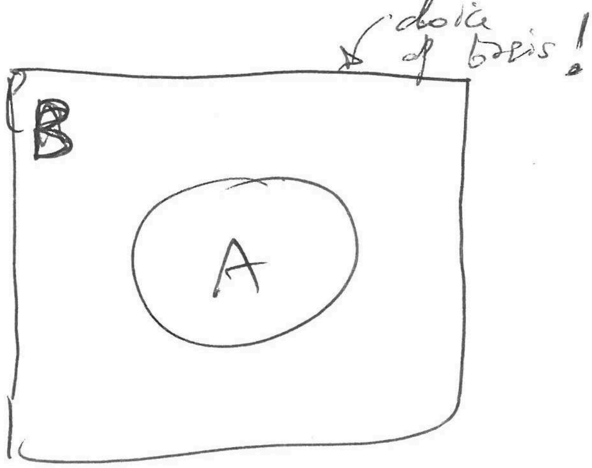
$[\Gamma_h, \hat{\Gamma}_v]$ and $[\Gamma_v, \hat{\Gamma}_h]$ are the same as for spin-1/2

→ topological quantum info and computing, protected against local perturbations!

Quantum vs classical correlations:

non-trivial (not order parameter) → use entanglement entropy (Von Neumann)
(agnostic measure of correlations)

density matrix $\rho = |\psi_0\rangle\langle\psi_0|$ or $e^{-\beta \hat{H}}$



$$\rho_A = \mathrm{Tr}_B\, \rho$$

$$S_{vN} = -\mathrm{Tr}\left[\rho_A \ln \rho_A\right]$$

$\left(= -\mathrm{Tr}\left[\rho_B \ln \rho_B\right]\right.$ only for pure state $\left.\right)$

- classically: entropy of A given B is in any of its possible states
- "measure" of how much A knows ~~about~~ B (only for pure states)

Always trivial "surface" component due to H matrix elements ~~terms~~ (interactions) across A, — unless diagonal, e.g., eigenbasis bipartition

Toric code: additional non-trivial contribution due to topological order

expected to be of order 1 → need for careful subtraction of trivial surface contribution

→ Levin + Wen
→ Kitaev + Preskill
'06 (S_{topo})

Toric code $|\Psi_0\rangle$ "simple" $\Rightarrow$ $S_{topo} = \ln 2$
can be computed exactly
(exercise ?)

Quantum or classical ?

How can one tell ? Naively, compare with correlation of "equivalent" classical system (not easy and not necessarily well-defined; basis dependent)

diagonal density matrix

Consider $H = -\sum_s A_s$ classical (in $|\sigma^z_i\rangle$ basis)

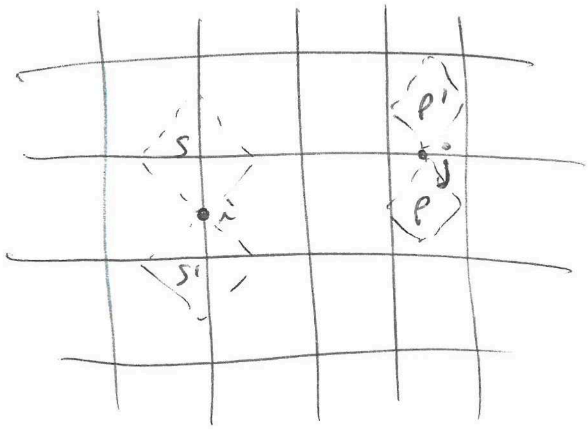
(8 vertex model)

$S_{topo}(8\text{-vertex}) = \frac{1}{2} S_{topo}(\text{Toric code})$

$\Rightarrow$ not all topo entropy has quantum origin!

(for toric code one can convince oneself that the rest is Q.M. contribution, but very difficult in general systems)

Toric code excitations : DW → point-like quasiparticles



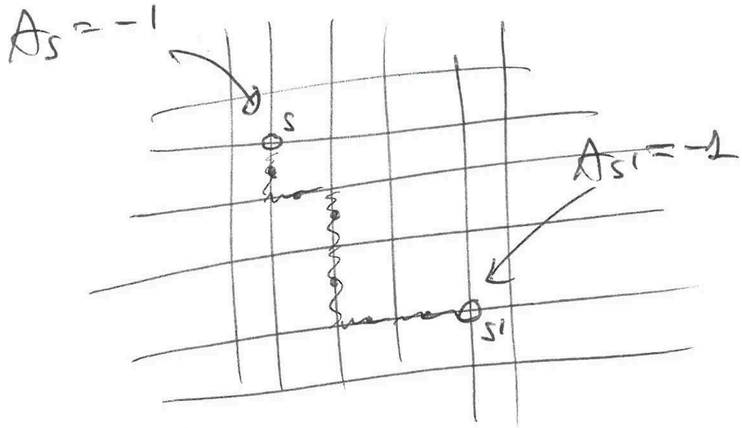
$\sigma_i^z \to -\sigma_i^z$

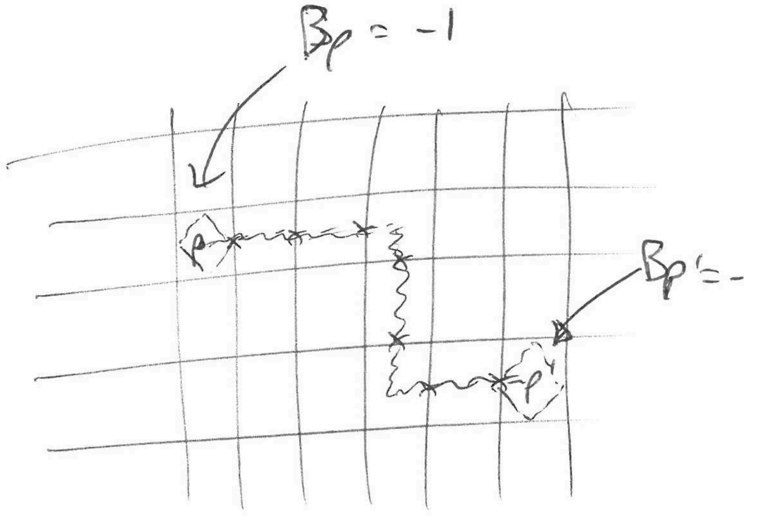
$\Rightarrow A_s, A_{s'} \to -A_s, -A_{s'}$

$\sigma_j^x \to -\sigma_j^x$

$\Rightarrow B_p, B_{p'} \to -B_p, -B_{p'}$

domain nucleation ?



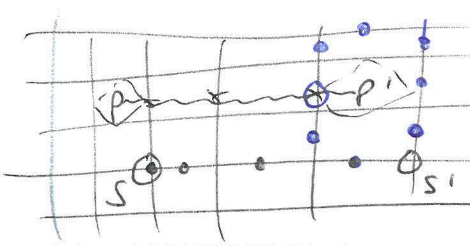


pair-defect creation and propagation : point-like quasiparticle excitations; spin-flip fractionalises into two q.p. (Oshikawa)

(dual aspect of topological order)

(Note: creation, winding, annihilation of defect pair ⟺ winding loop operators)

Trivially bosonic but non-trivial mutual quantum statistics :



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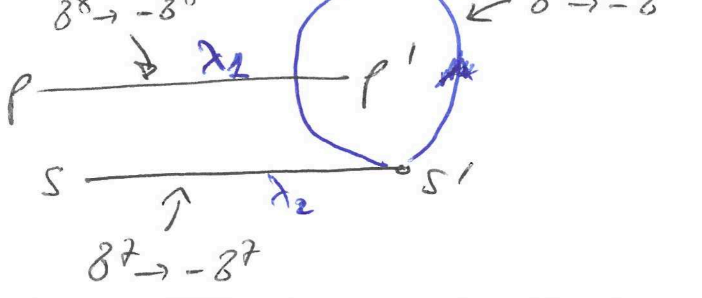


flips $\sigma_i^x \to -\sigma_i^x$; flips $\sigma_i^z \to -\sigma_i^z$

$$\left(\prod_{i\in\lambda_1} \sigma_i^z\right)\left(\prod_{i\in\lambda_2} \sigma_i^x\right)|\Psi_0\rangle$$

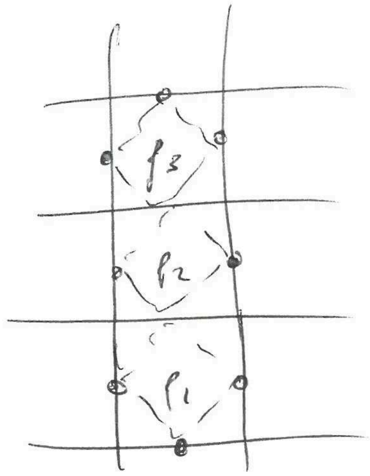
$$\left(\prod_{i\in\lambda_3} \sigma_i^x\right)\left(\prod_{i\in\lambda_2} \sigma_i^z\right)\left(\prod_{i\in\lambda_2} \sigma_i^x\right)|\Psi_0\rangle =$$

anticommute (λ_3 and λ_2 must intersect an odd number of times)

$$= -\left(\prod_{i\in\lambda_1} \sigma_i^z\right)\left(\prod_{i\in\lambda_2} \sigma_i^x\right)\left(\prod_{i\in\lambda_3} \sigma_i^x\right)|\Psi_0\rangle$$

$\left(\prod_{i\in\lambda_3} \sigma_i^x\right)|\Psi_0\rangle = |\Psi_0\rangle$

closed loop $= \prod_{p \text{ inside loop}} B_p$



$$\prod_{i\in\lambda_3} \sigma_i^x = B_{p_1} \cdot B_{p_2} \cdot B_{p_3}$$

(recall $(\sigma_i^x)^2 = \mathbb{1}$)

$\Longrightarrow$ mutual fermionic statistics

(same as e charge moving about π-flux: Aharonov-Bohm phase $= \pi$)

anyonic quasiparticles $\longleftrightarrow$ Topological order

$S_{topo} \sim \ln \sum_a d_a^2$ → Hilbert space dim of q.p. labelled by a

statistical angle is genuinely Q.M.

classical toric code (8-vertex), $H = -\sum_s A_s$:
point-like excitations but no statistical angle (only "bosonic" behaviour)

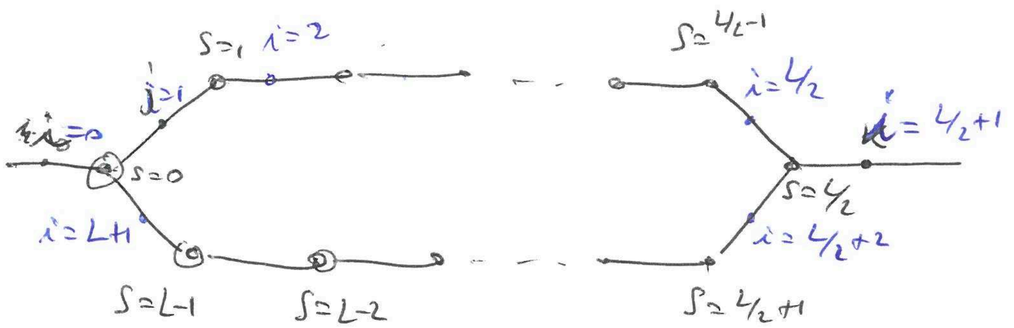
{ each q.p. can be understood classically but both present requires superposition; and stat. angle requires coherence.

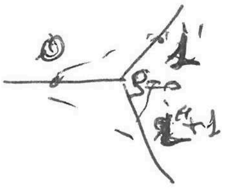
given a system supposed to be toric code how can one check quantum vs. classical behaviour (e.g., in presence of coupling to bath)?

Non-trivial task (see negativity ... only bounds and hard to compute) → exercise

→ anyon interferometry: not easy to implement but if visible then crisp!

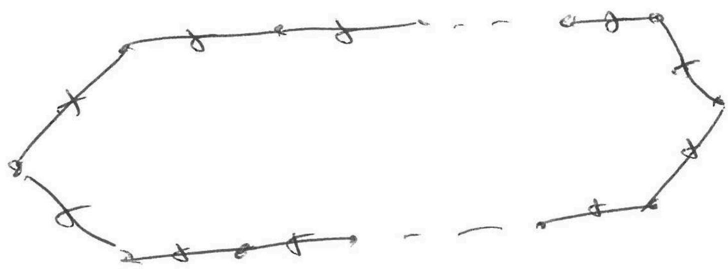
Simple example: single plaquette → proposal to use to assess quantum coherence in programmable quantum hardware



$A_{s=0} =$  $= \sigma_0^z \sigma_1^z \sigma_{L+1}^z$

$A_{s=1} =$  $= \sigma_1^z \sigma_2^z$

etc...

$B_p = \prod_{\substack{i \neq 0 \\ i \neq L/2+1}} \sigma_i^x =$ 

boundary spins fixed ($i = 0, L/2+1$)

$[A_s, B_p] = 0$ $H = -\sum A_s - B_p$

$\Updownarrow$

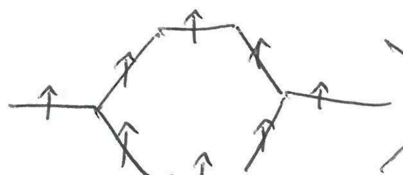
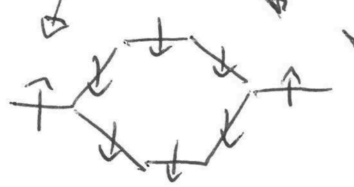
Toric code

plan: 1) suggest a protocol
2) discuss how it can be used
3) Lindblad testing (if time allows)

$L=6$

ground state:

fixed boundary spins

$\alpha \Big|$  $\Big\rangle + \Big|$  $\Big\rangle$

$\langle A_s \rangle = +1$

$\langle B_p \rangle = +1$

excited state of B_p:

$\langle A_s \rangle = +1$

$\langle B_p \rangle = -1$

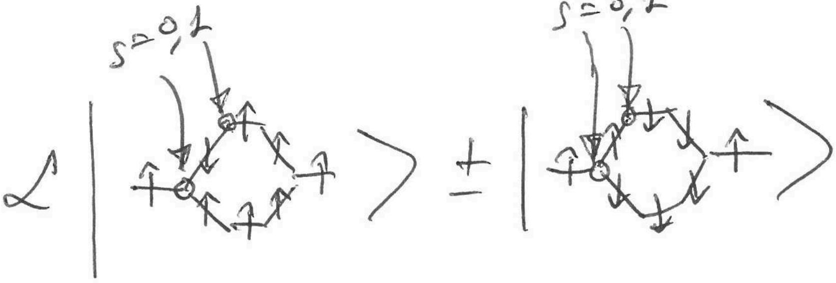
$\propto \; | \ldots \rangle - | \ldots \rangle$

excited state of A_s: (created in pairs)

$\langle A_0 \rangle = \langle A_1 \rangle = -1$

$\langle A_{s \neq 0,1} \rangle = 1$

$\langle B_p \rangle = \pm 1$

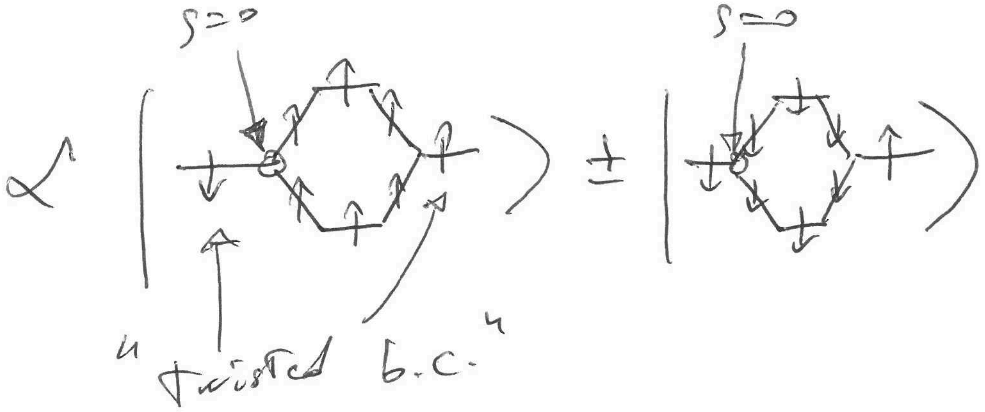


single A_s excited state by <u>twisting b.c.</u>

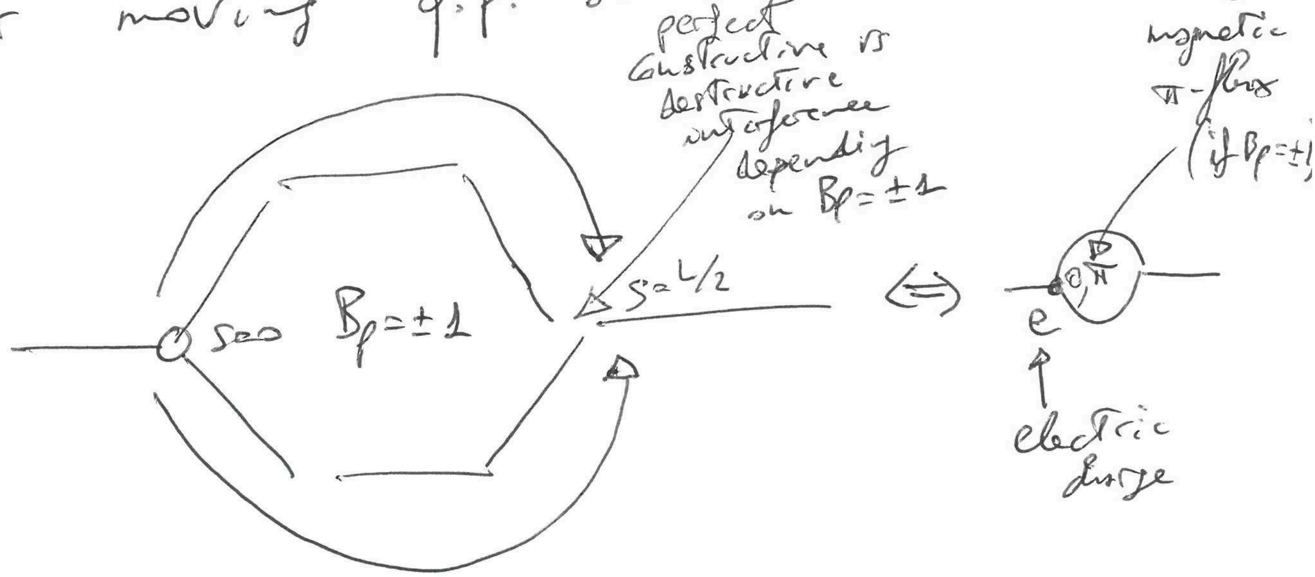
$\langle A_0 \rangle = -1$

$\langle A_{s \neq 0} \rangle = +1$

$\langle B_p \rangle = \pm 1$



Consider moving q.p. at $s=0$ around the ring:



Recall: - simultaneous presence of A_s and B_p excitations required quantum superposition

- statistical angle and interference strictly require phase coherence being preserved during the motion

$\Rightarrow$ Test of many-body quantum coherence

(Note: single particle picture is emergent, but truly supported by many body state of all qubits/spins interacting system)

... but in Toric code q.p. are static?

Motion: add small transverse field

Commute: B_p excitation preserved

$$H = -\sum A_s - B_p - \Gamma \sum_{i \neq 0, \frac{L+1}{2}} \sigma_i^x$$

(not on boundary spins)

→ creation and annihilation of q.p. pairs (prevented by gap)

⟷ (justified as long as q.p. gap >> bandwidth)

→ tight binding motion of q.p. (A_s excitations only)

initial condition:

$\langle A_0 \rangle_0 = -1$

$\langle A_{s \neq 0} \rangle_0 = +1$

$\langle B_p \rangle_0 = \pm 1$

time evolve ⟶ measure $\langle A_{s=L/2} \rangle$

↗ 0 $\langle B_p \rangle_0 = -1$

↘ $\neq 0$ $\langle B_p \rangle_0 = +1$

loss of coherence brings the values of $\langle A_{s=L/2} \rangle$ for the two cases ($\langle B_p \rangle_0 = +1$ and $\langle B_p \rangle_0 = -1$) towards one another

(no coherence ⇒ same value)

see arXiv: 2503.12573

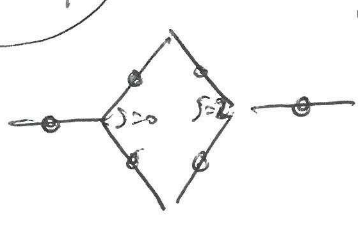
- numerical demonstration using Lindblad coupling to bath
- "experimental" demonstration on circuit-based quantum hardware

<u>Example</u>: isotropic, infinite-T bath

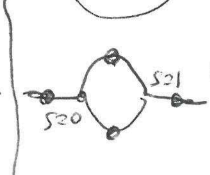
strength of coupling to bath

$$\dot{\rho} = -i[H,\rho] + \gamma \sum_{\substack{i \neq 0 \\ i \neq \frac{L}{2}+1}} \left[\left(\sigma_i^x \rho \sigma_i^x + \dots \right)^{y,z} - 3\rho \right]$$

L = 4



L = 2



analytically

larger systems:

$\dot{\rho} = \mathcal{L}\rho$

- $\mathcal{L}$: Lindbladian superoperator (linear in ρ)
- ρ: represent as "vector"

+ diagonalisation or Trotter evolution