Emergent Codes from Gauss Laws

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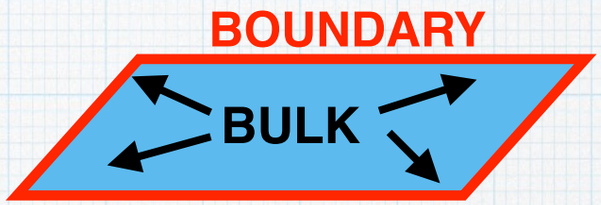
FWF Austrian Science Fund

ift Instituto de Física Teórica UAM-CSIC

The understanding and characterization of phases of matter is done through the boundary

- Topological insulators, fQHE,…



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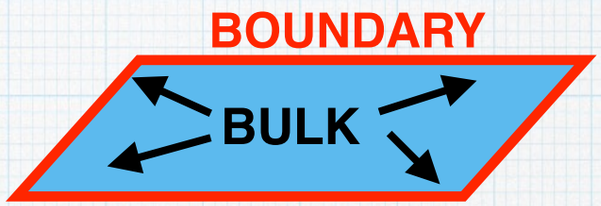

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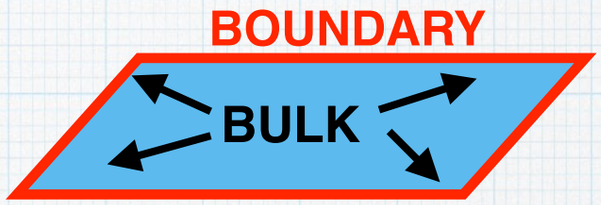

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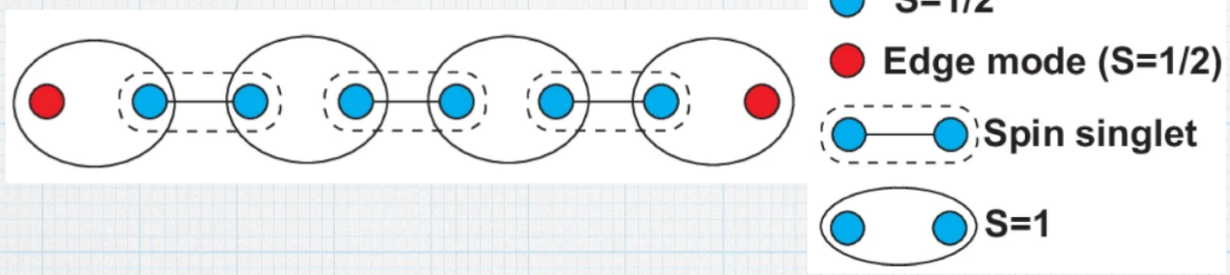

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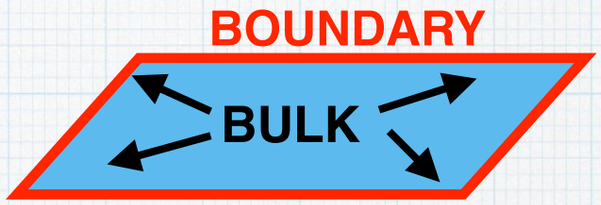

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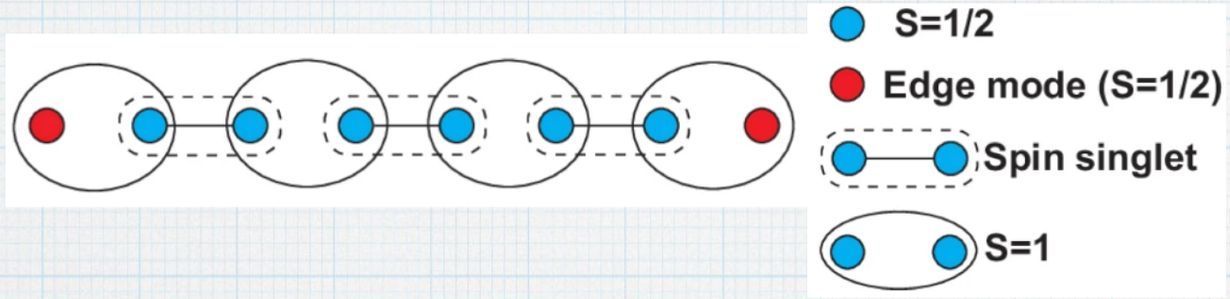

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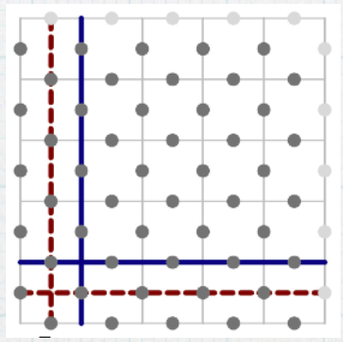

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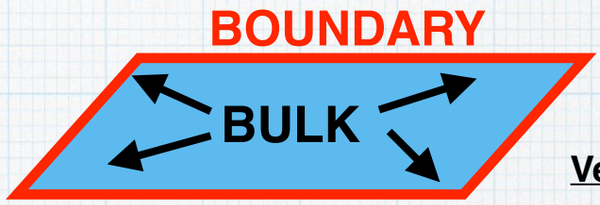

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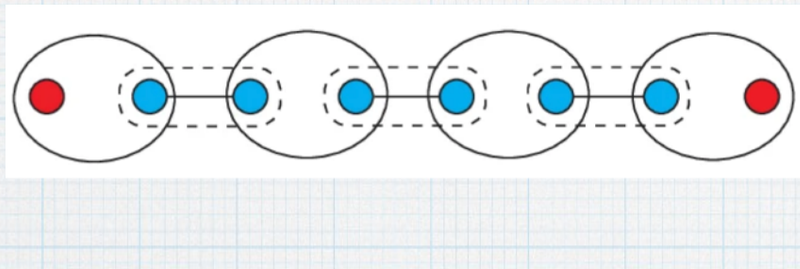

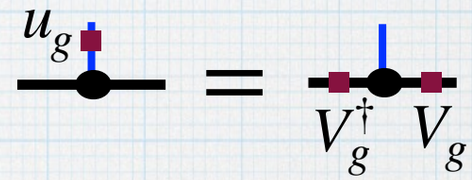

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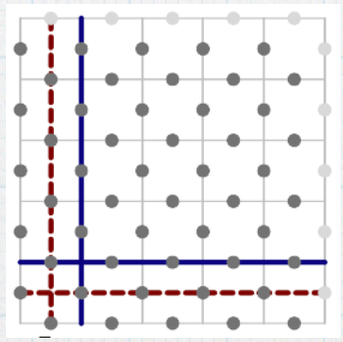

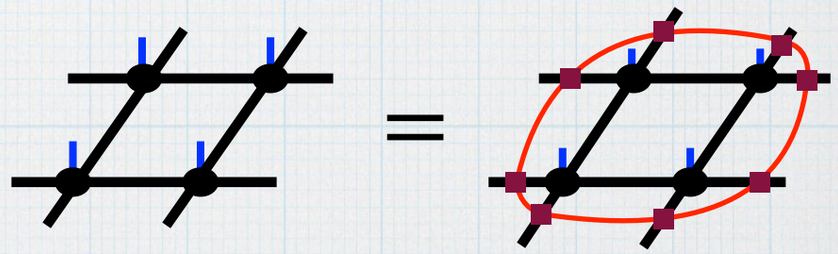

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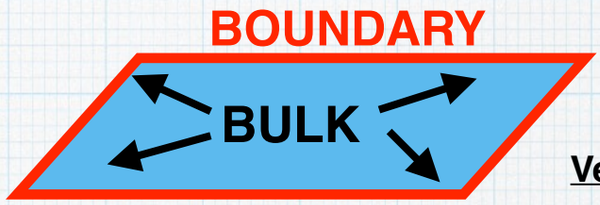

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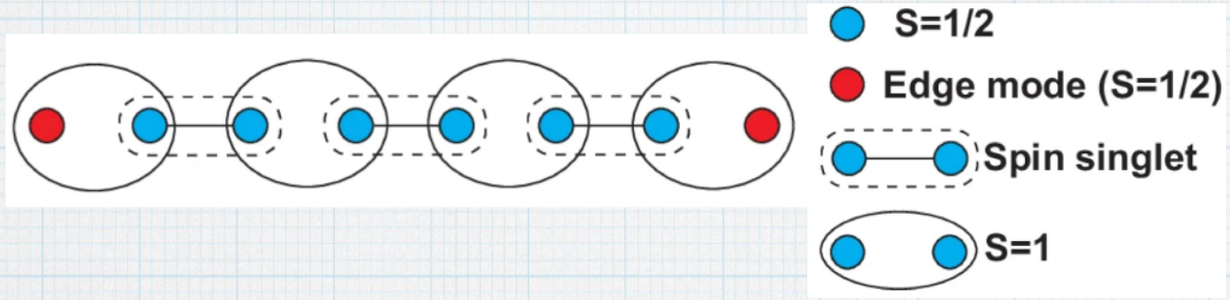

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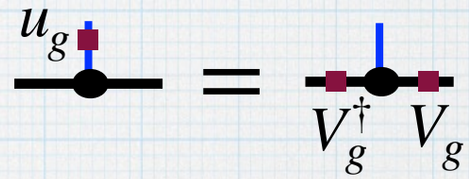

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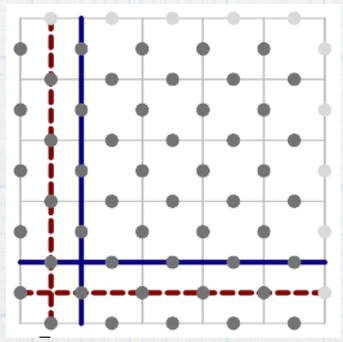

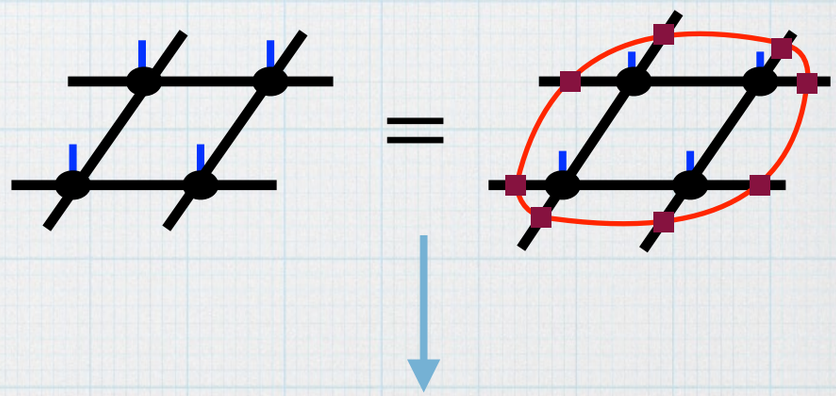

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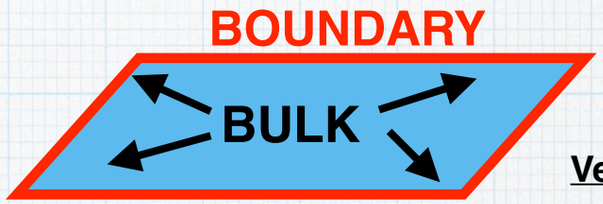

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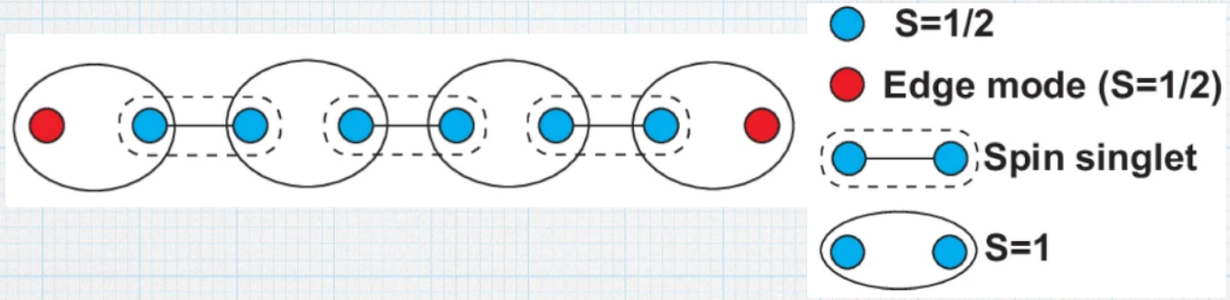

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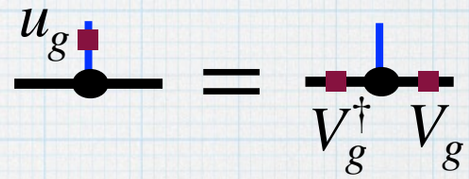

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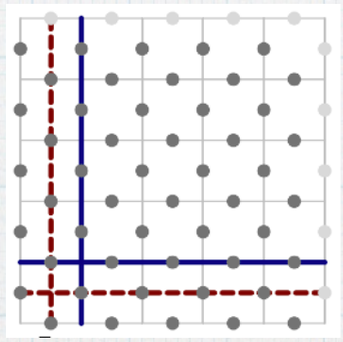

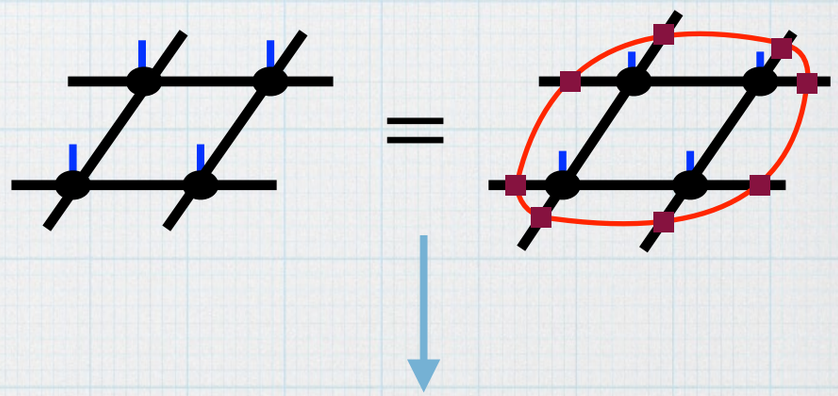

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THIS TALK: BOUNDARY —DETERMINES→ BULK

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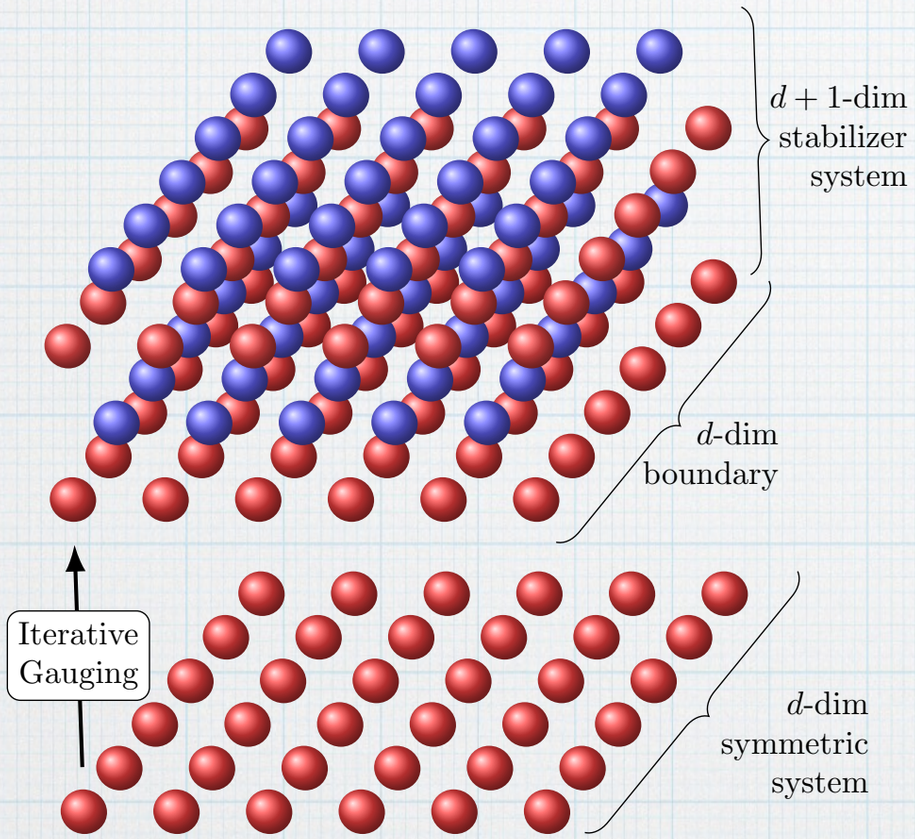

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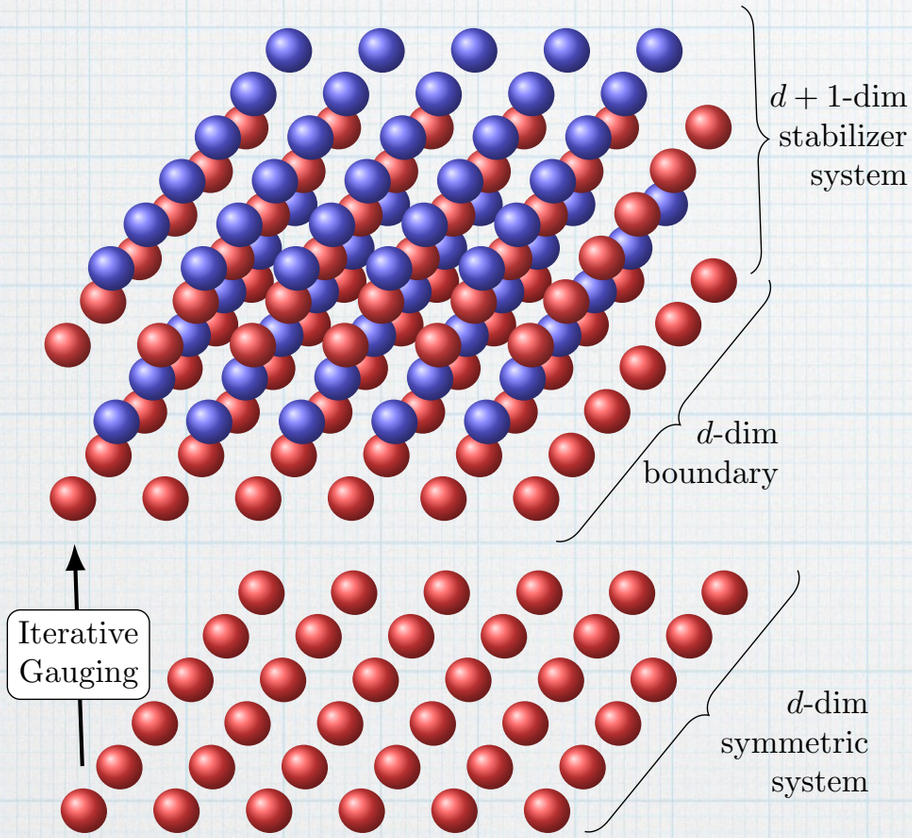

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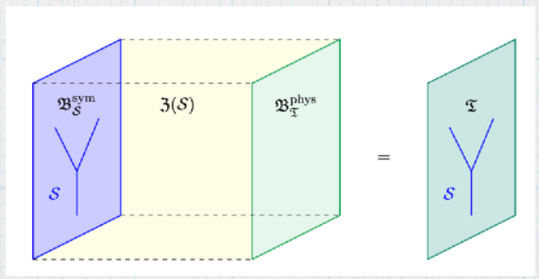
~ SymTFT in the lattice



Global symmetries and dualities/gauging

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[B. Vancraeynest-De Cuiper, **JGR**, F. Verstraete, K. Vervoort, D. Williamson, L. Lootens. *arXiv:2509.22051*]

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$\longrightarrow$ **What happens if we gauge again?**

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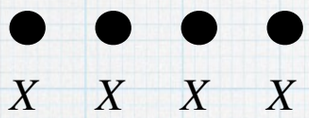

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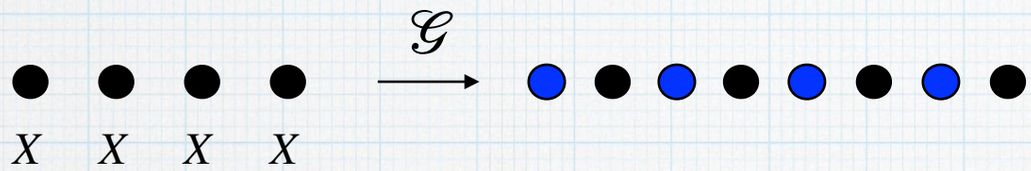

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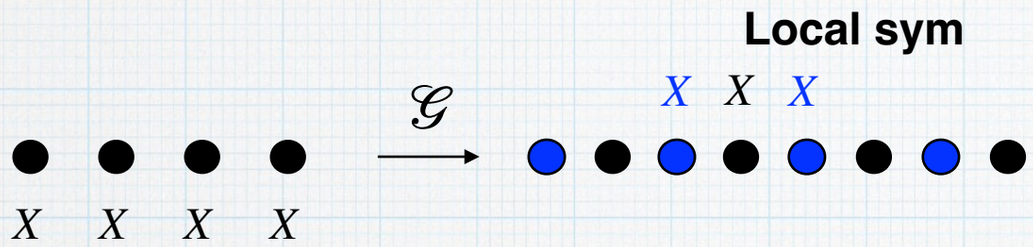

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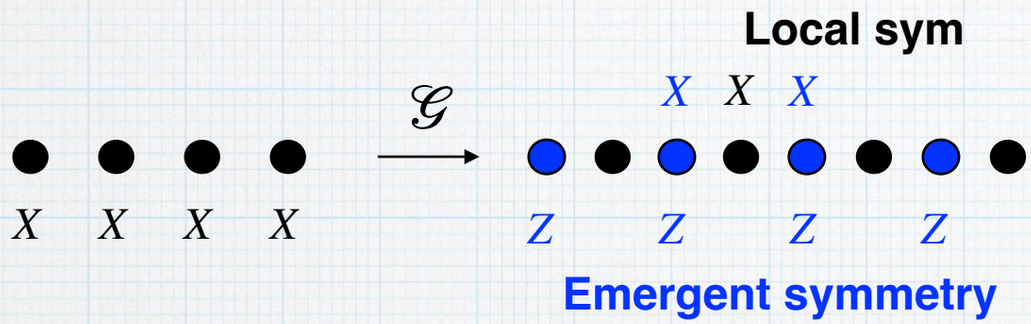

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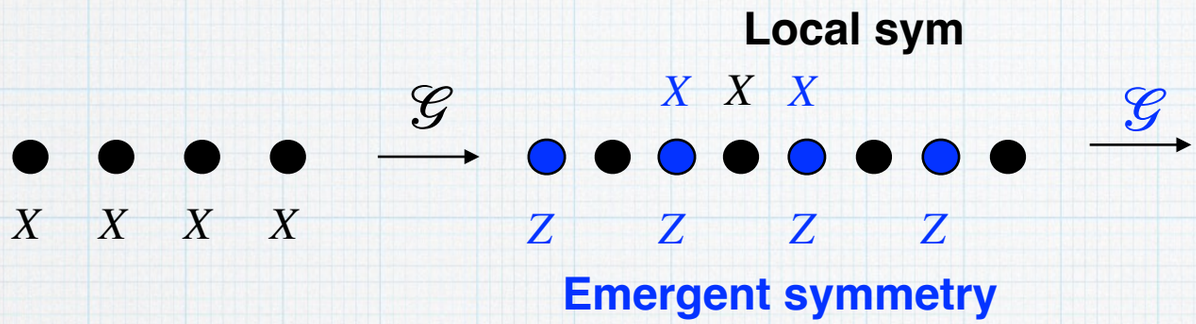

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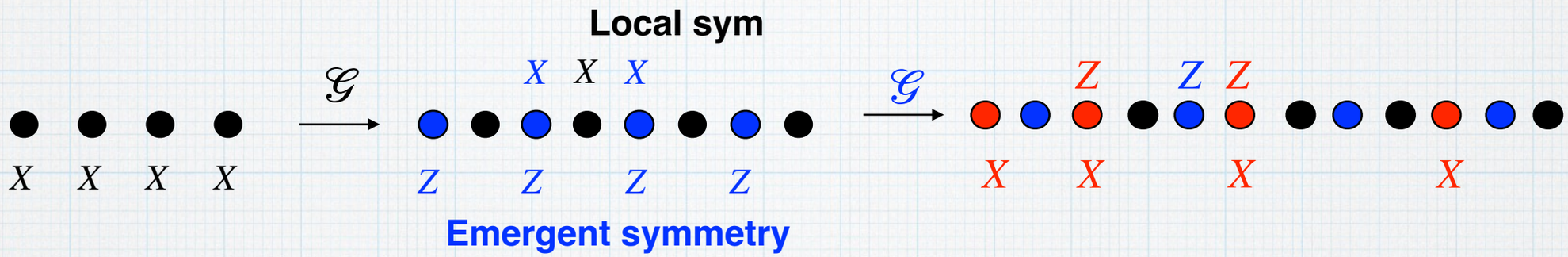

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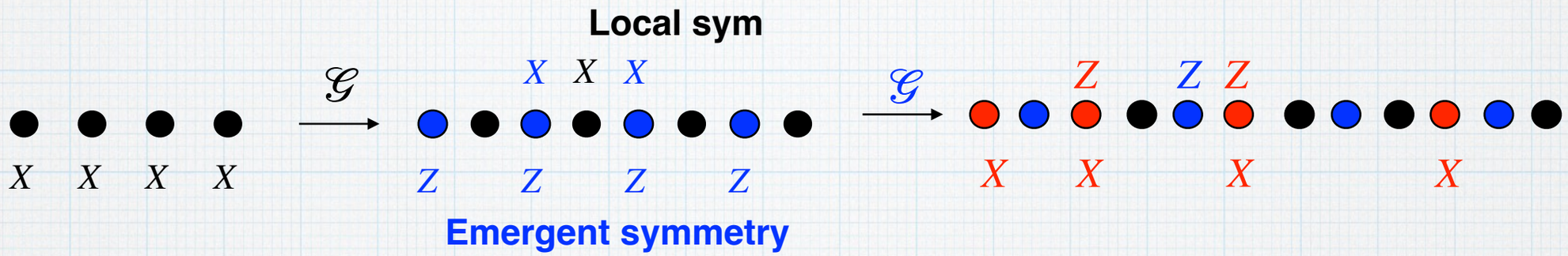

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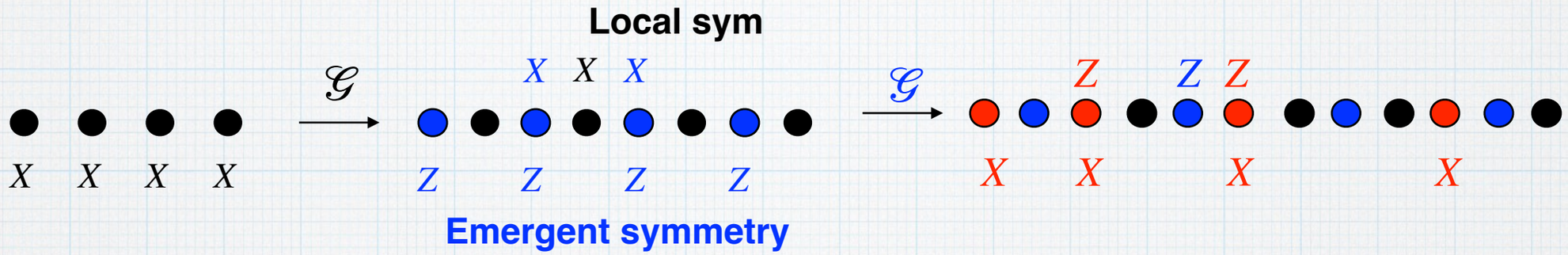

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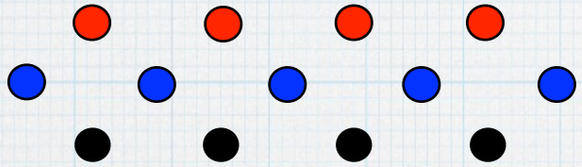

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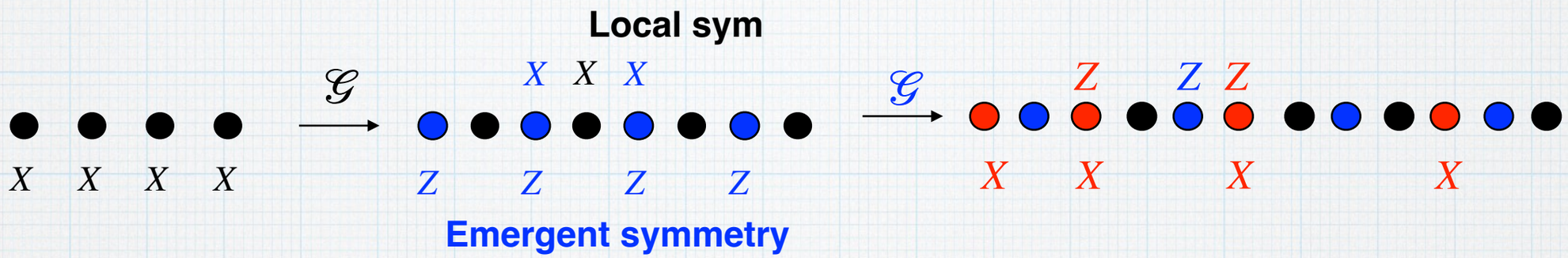

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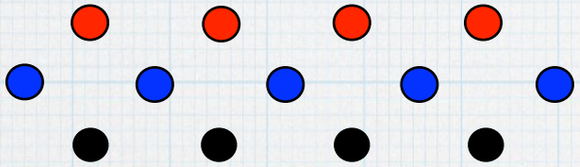

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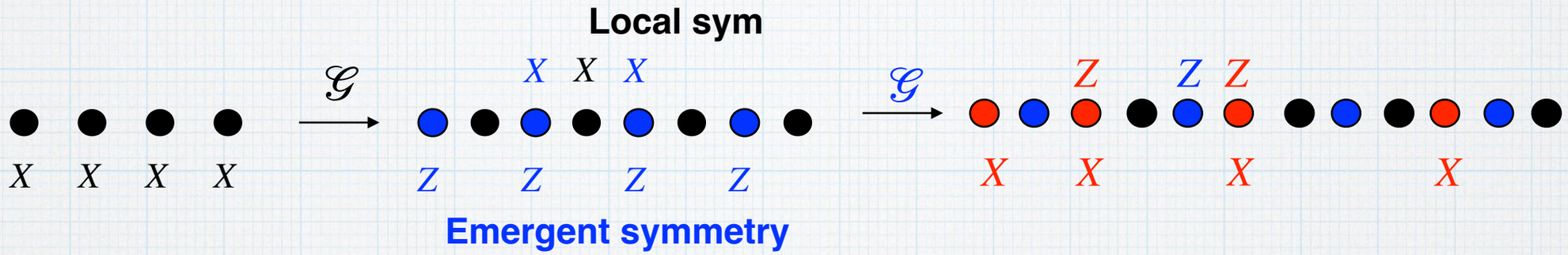

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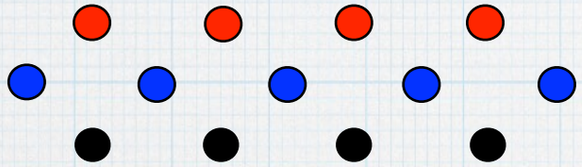

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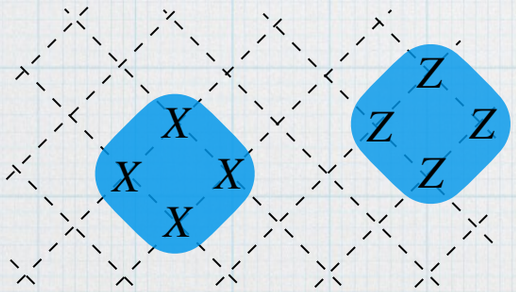

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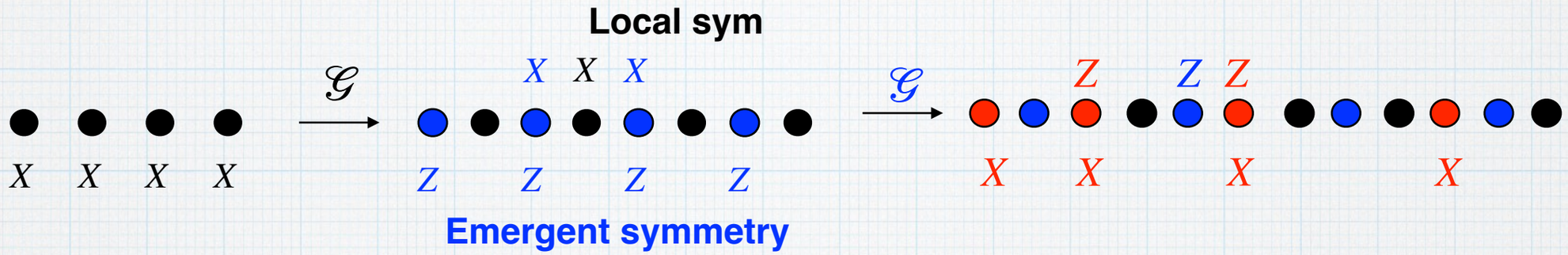

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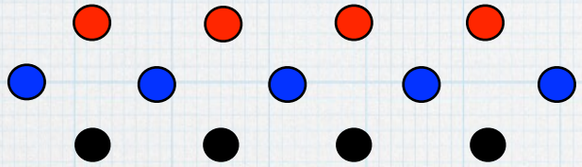

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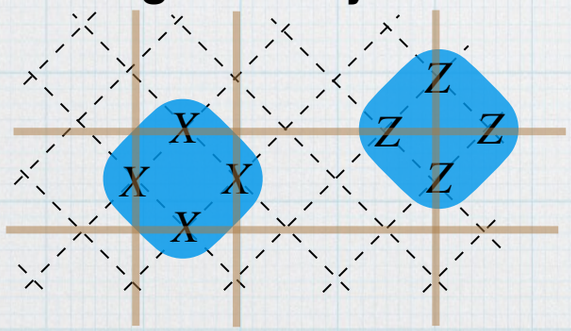

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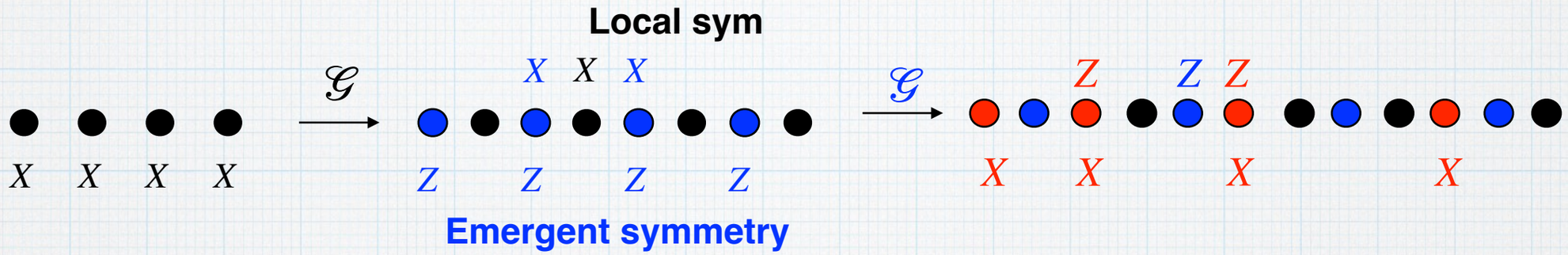
Toric Code

[A. Kitaev, *Annals Phys. 303* (2003)]

Gauging: New gauge fields + Gauss Law + Emergent global symmetry



1) Ordering?

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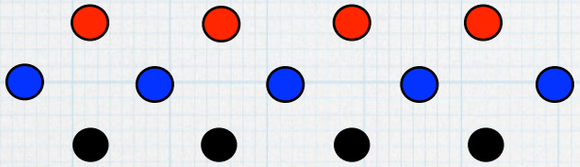

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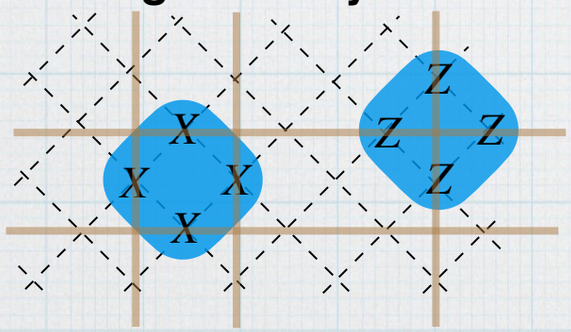

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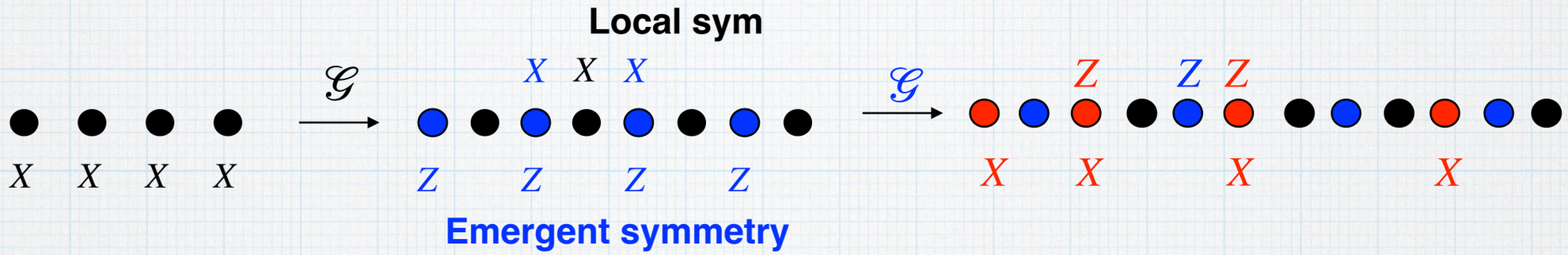

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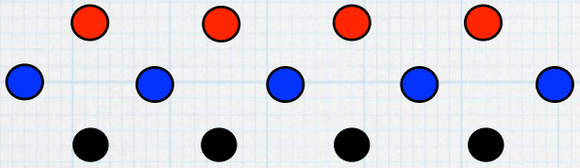

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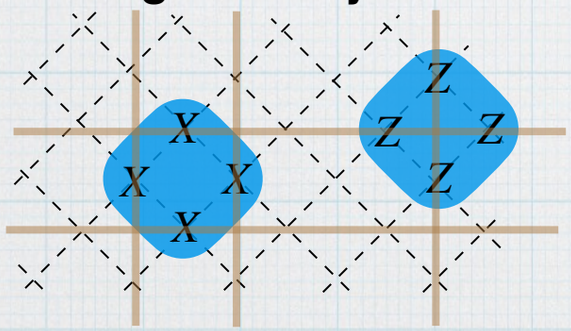

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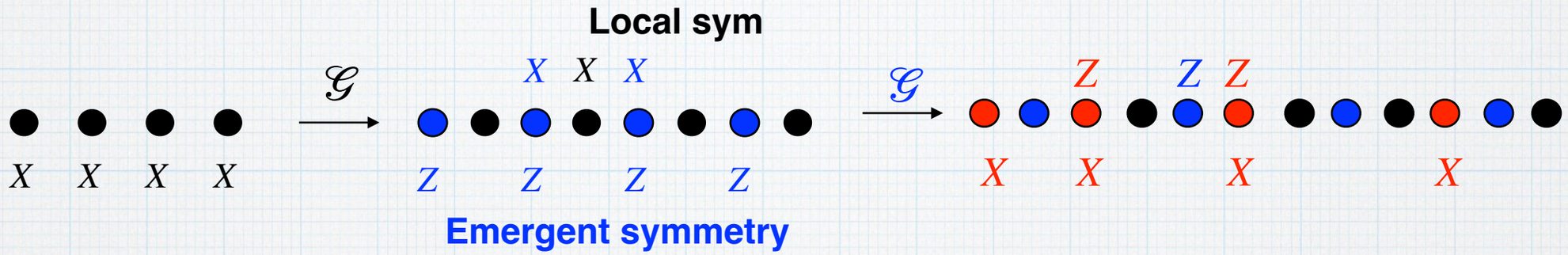

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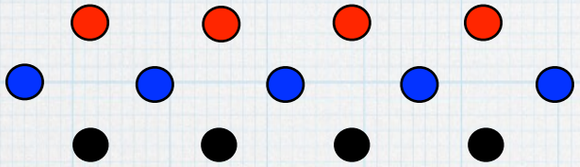

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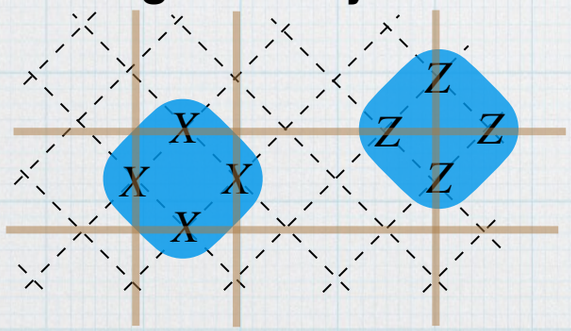

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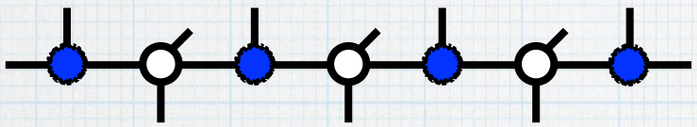
☑ **Non-abelian groups**

What happens at the boundary?

H_{2D}^{bulk} **only depends on the symmetry** $X^{\otimes n}$ **but the boundary terms depend on the phase of the initial Hamiltonian (the symmetries of the GS).** $H_{TFI} = -\sum_i (Z_i Z_{i+1} + \mu X_i)$

$\mathcal{G} =$ 

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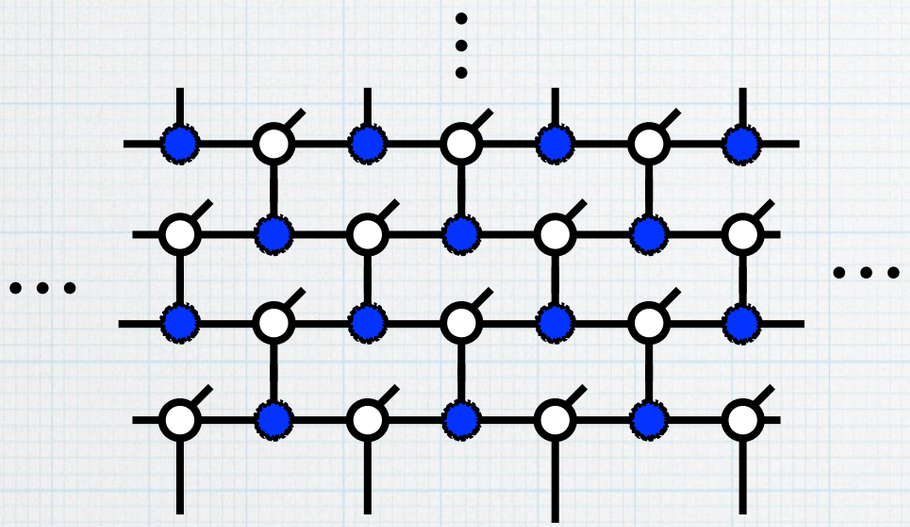

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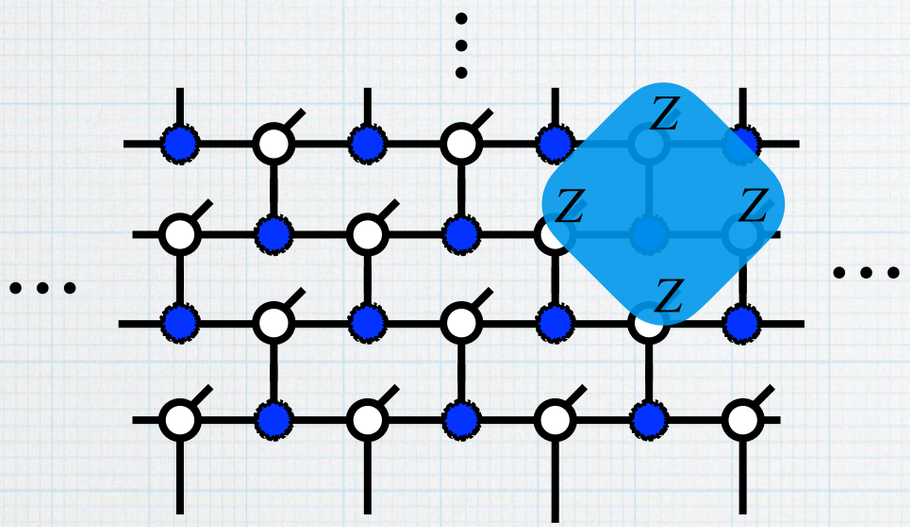

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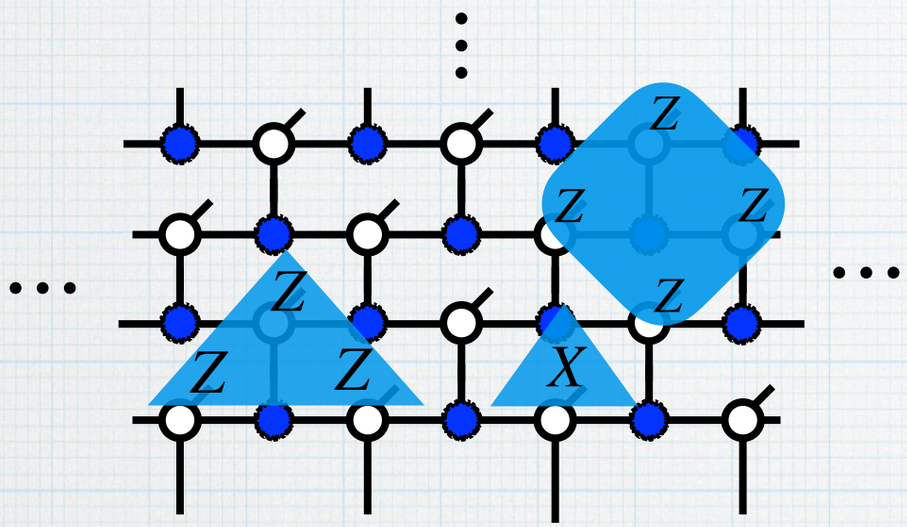

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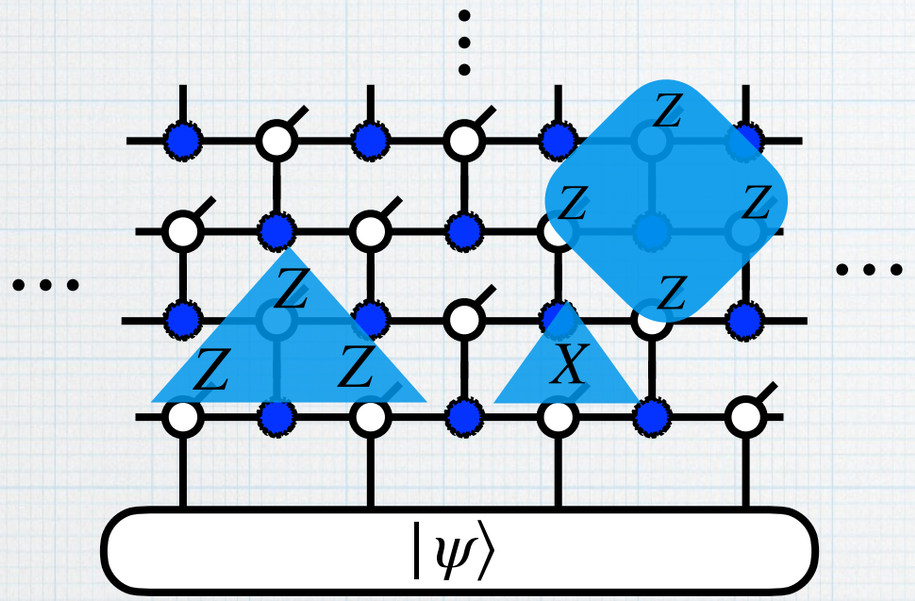

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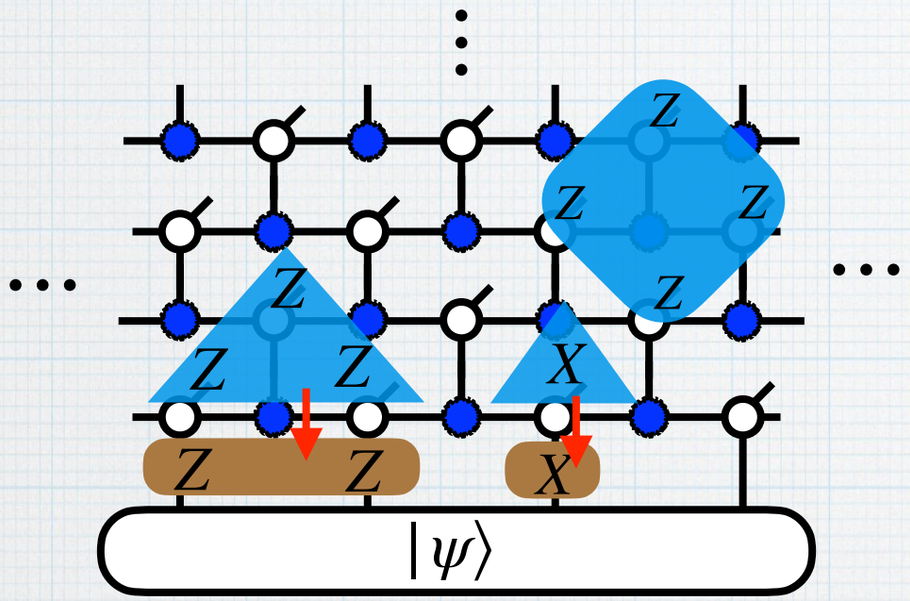

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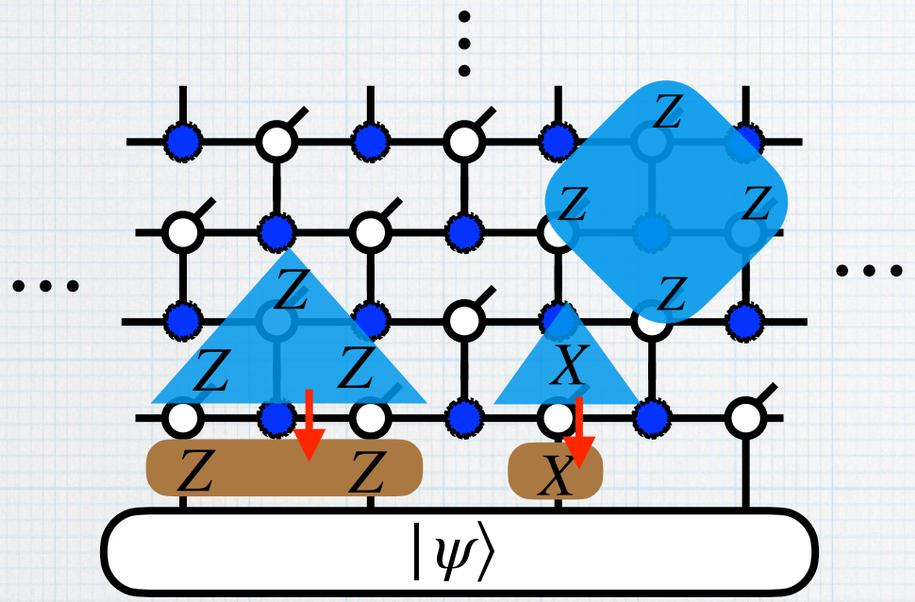

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Gapped boundaries of TC: condensing e or m

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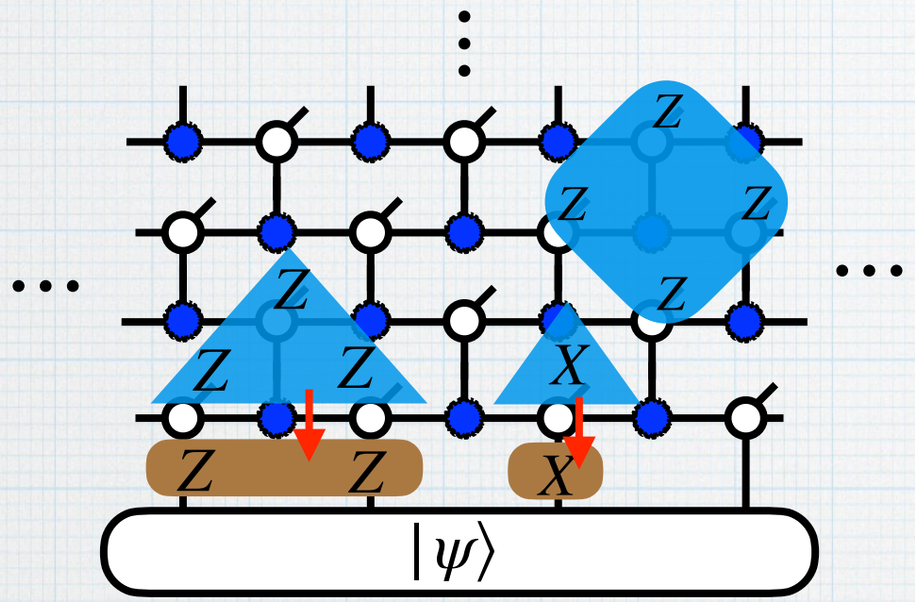

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☆ **Gapped boundaries of 2D** $\mathscr{D}(G)$ **= Quantum phases of global G symmetry =**
Module cat. over Vec_G

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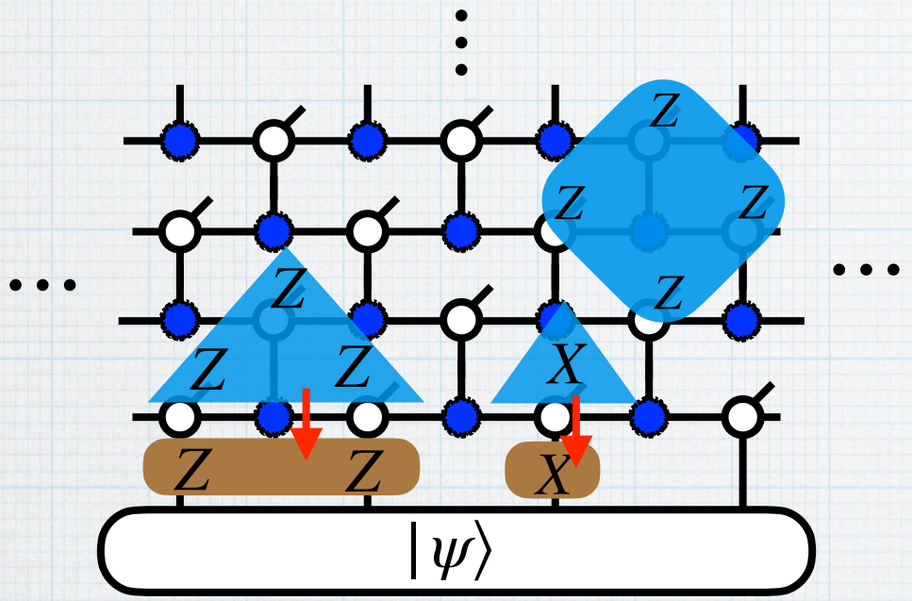

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$$H_{1D}^{sym} \longrightarrow \text{sym}\{\mathscr{G} \circ \cdots \mathscr{G} \circ \mathscr{G}|\psi\rangle\} \ = \ H_{\text{Emerg.}}^{2D} = H_{\text{bulk}} + H_{\text{bdry}}$$

Topological part: only G

Micro part: Q. Phase

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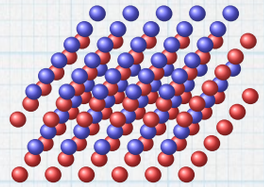
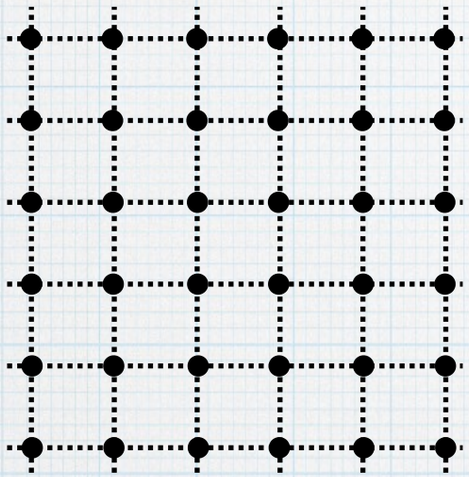
From 2D symmetries to 3D codes

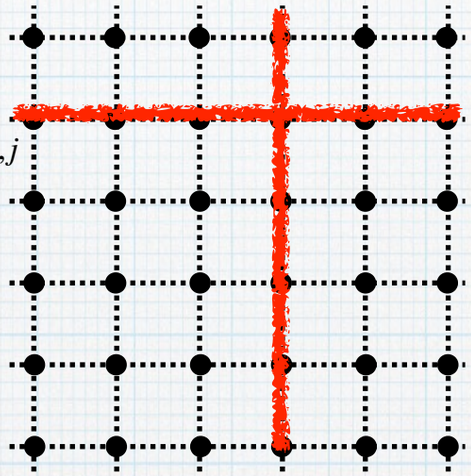


Global: $Z^{\otimes n}$



Linear Sub. Sym.

$$S^y_j = \prod_i Z_{i,j}$$



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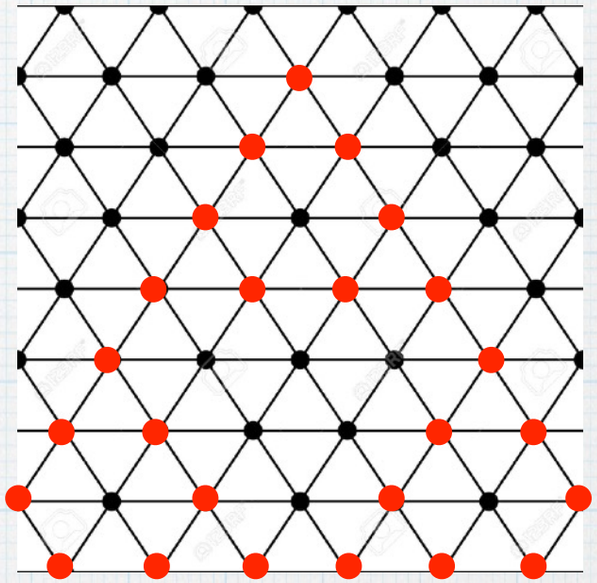

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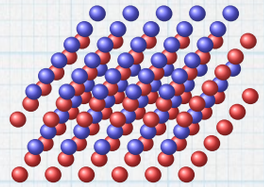

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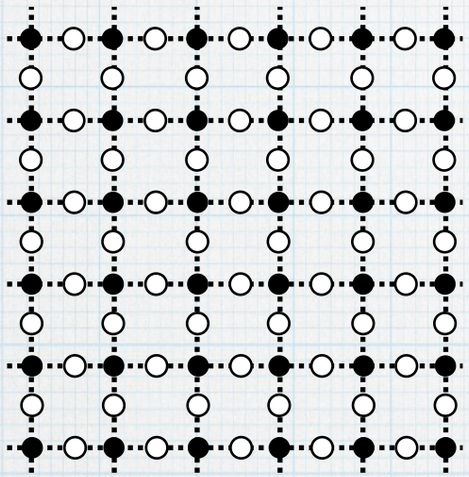

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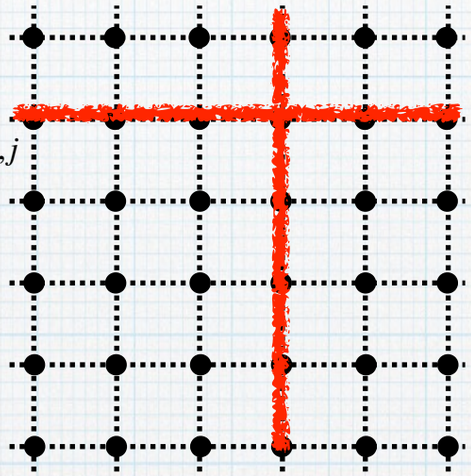

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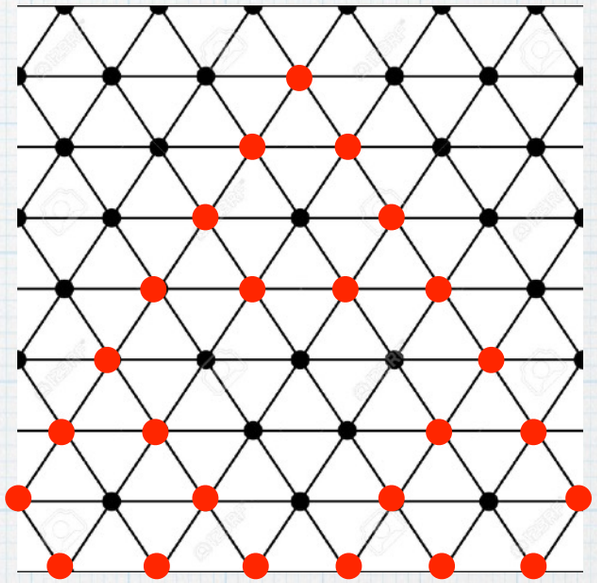

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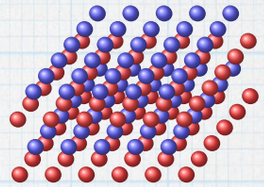

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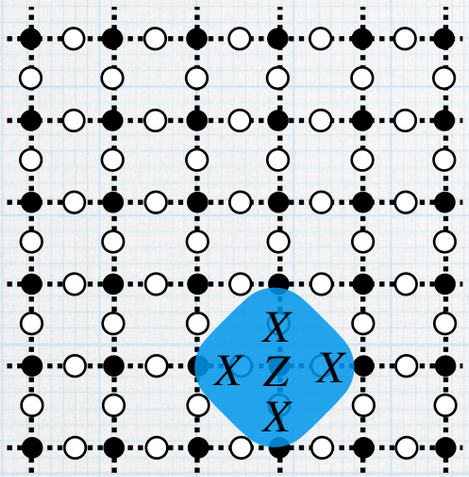

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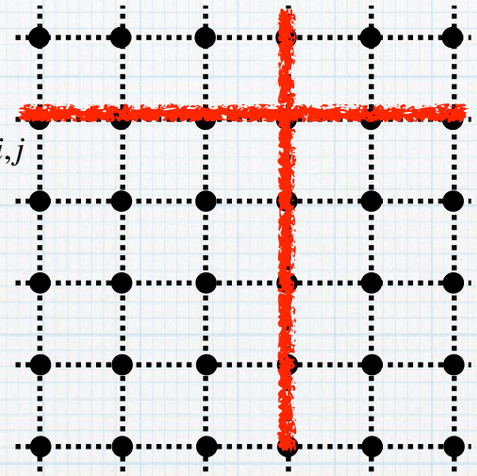

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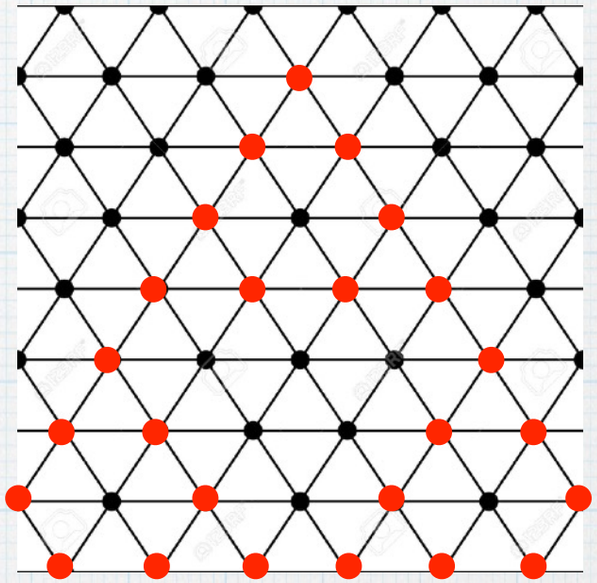

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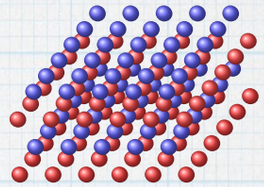

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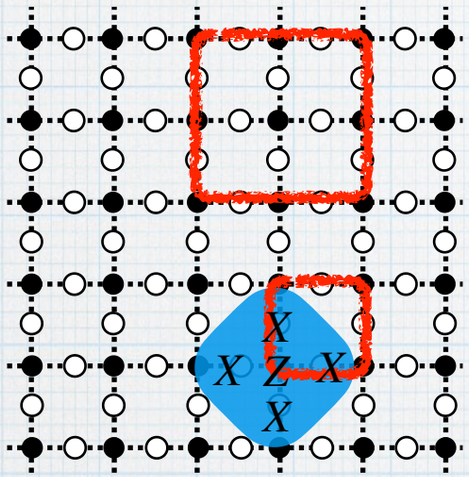

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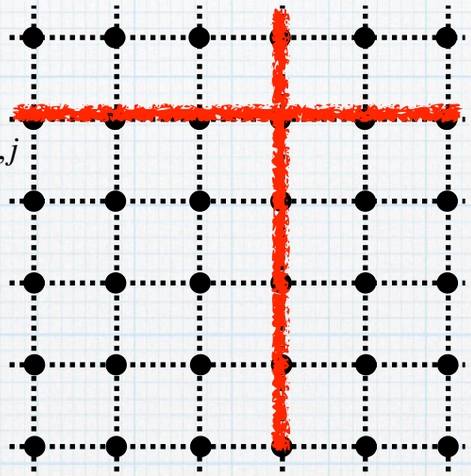

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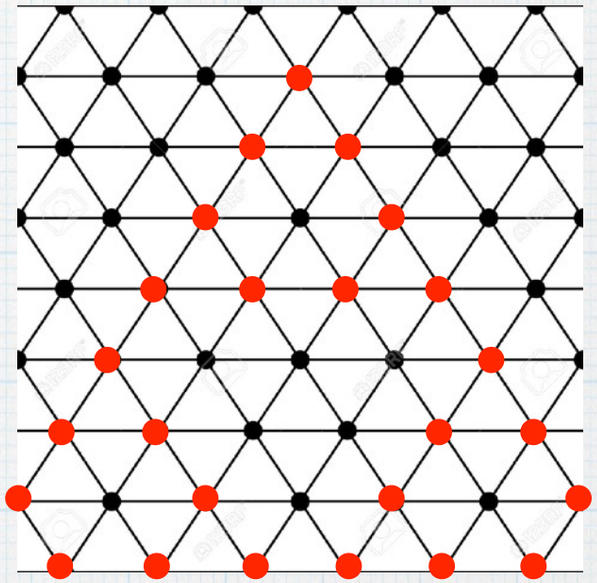

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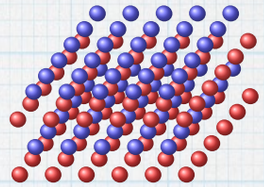

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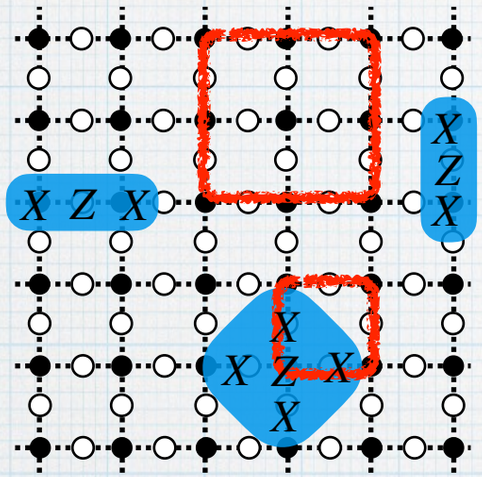

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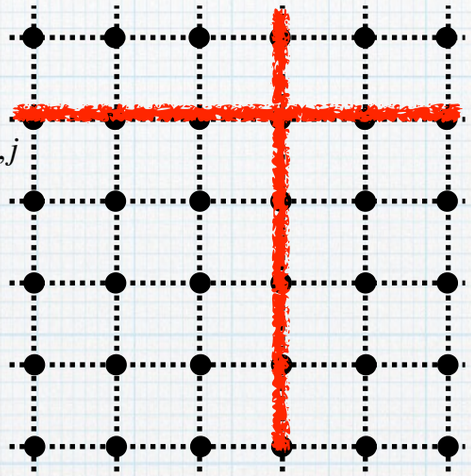

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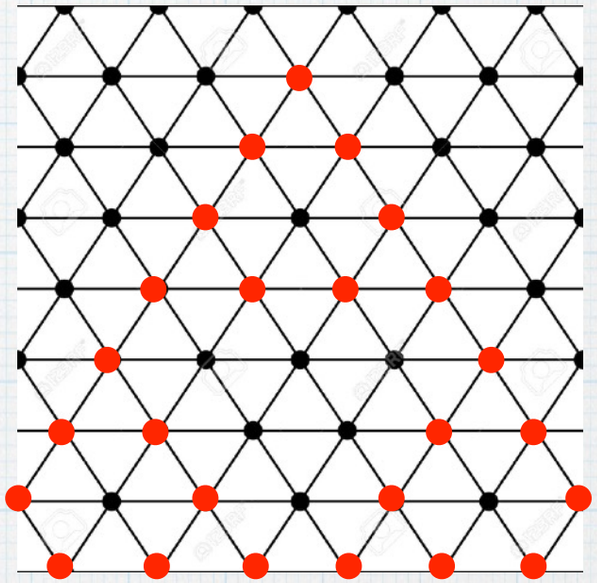

Ising, plaquette Ising & CS
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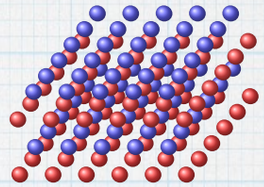

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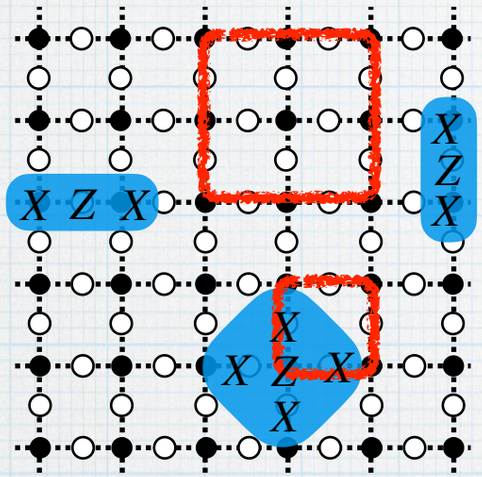

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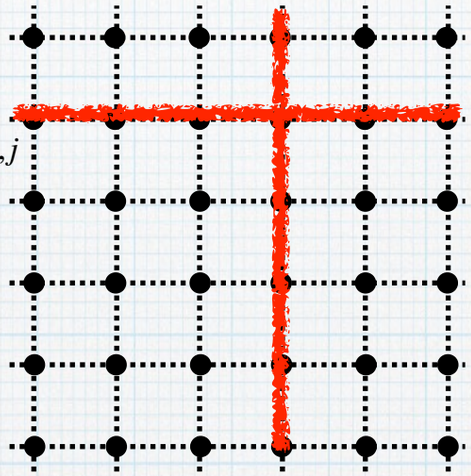

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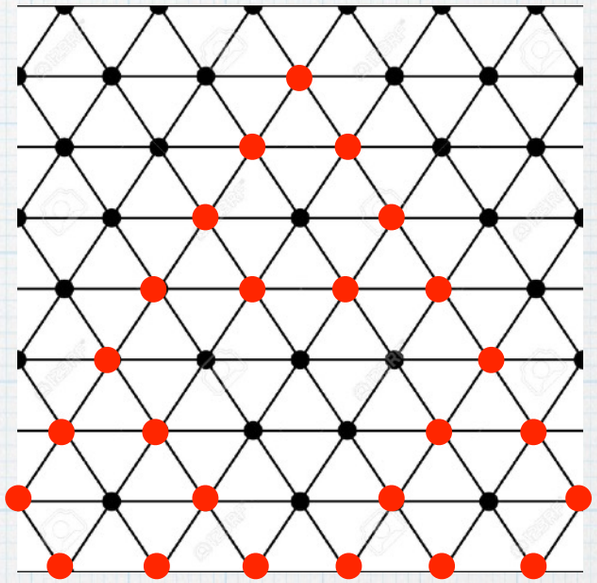

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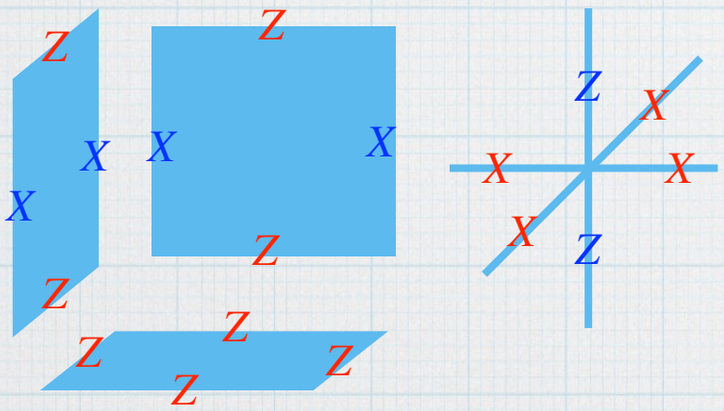

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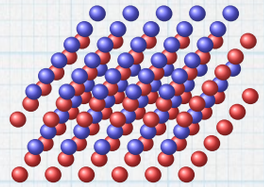

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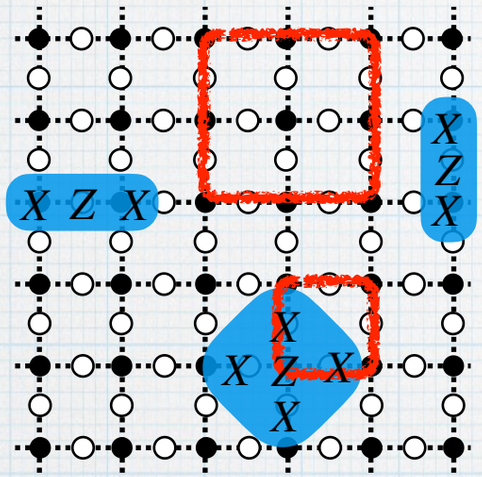

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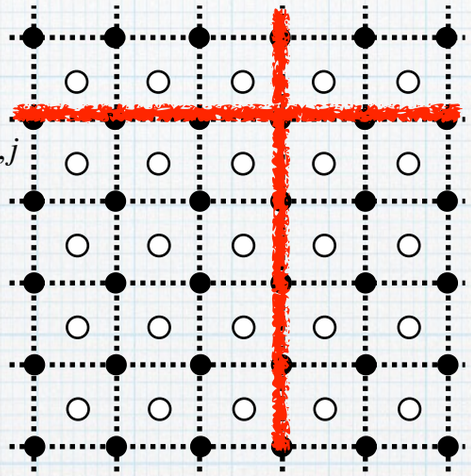

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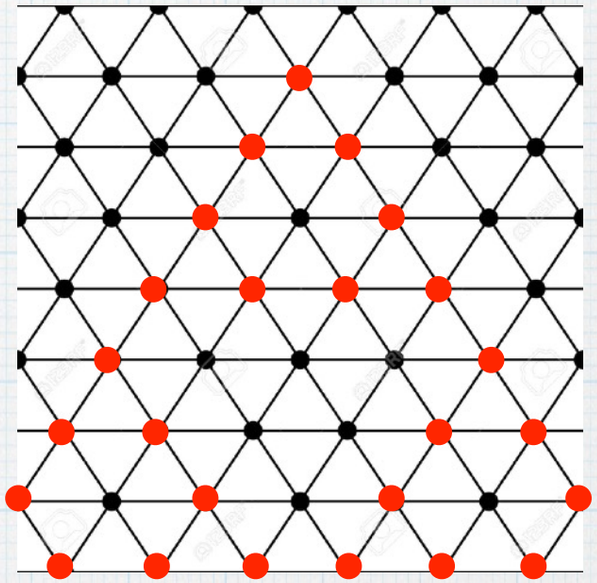

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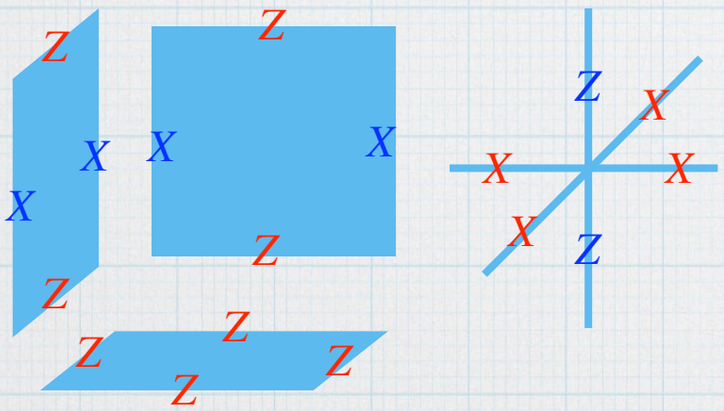

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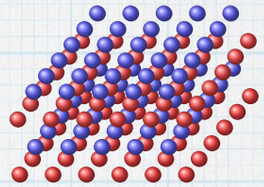

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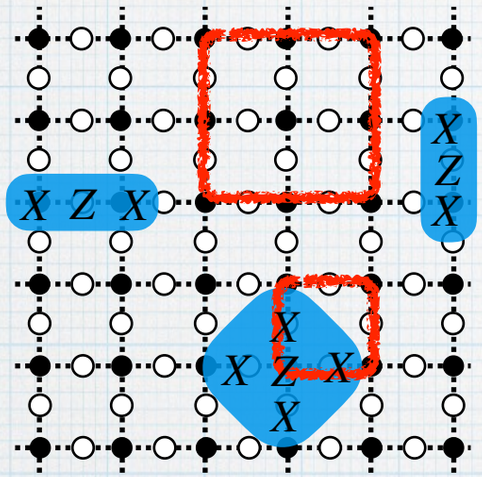

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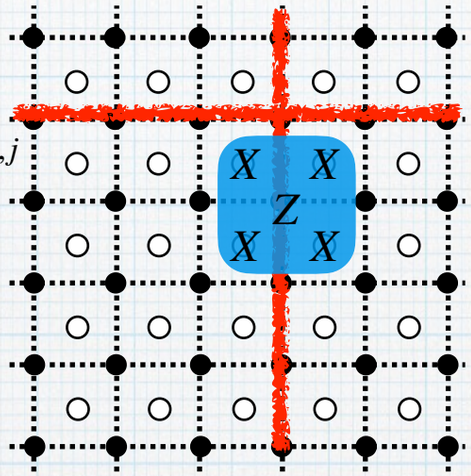

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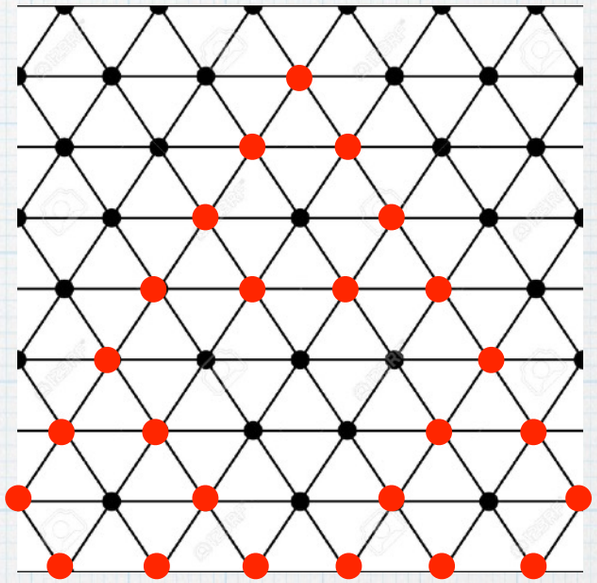

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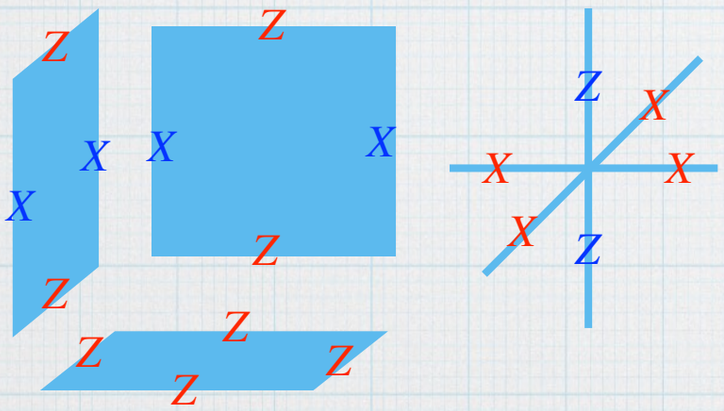

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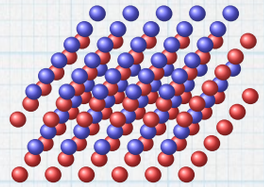

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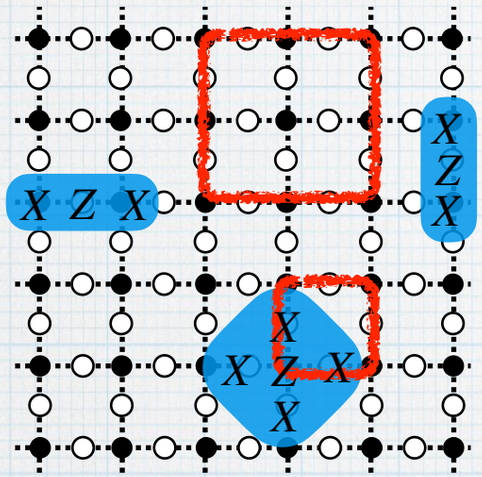

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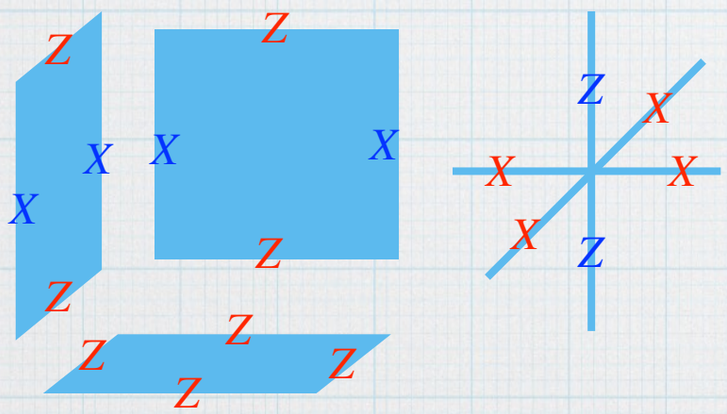

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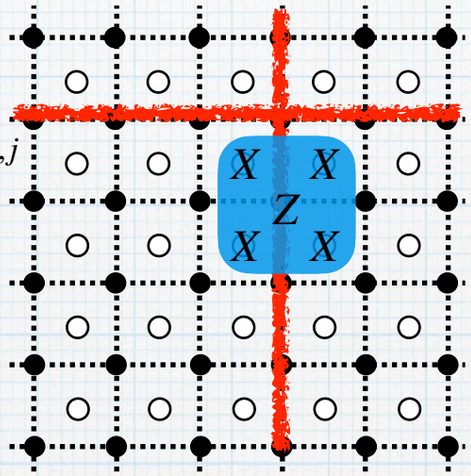

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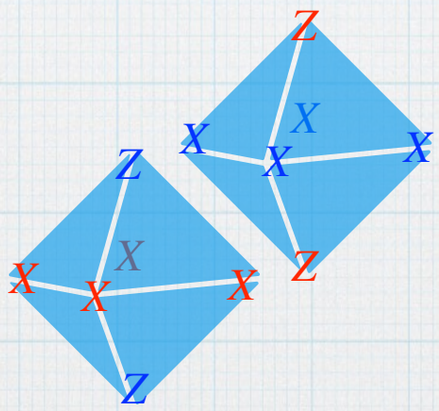

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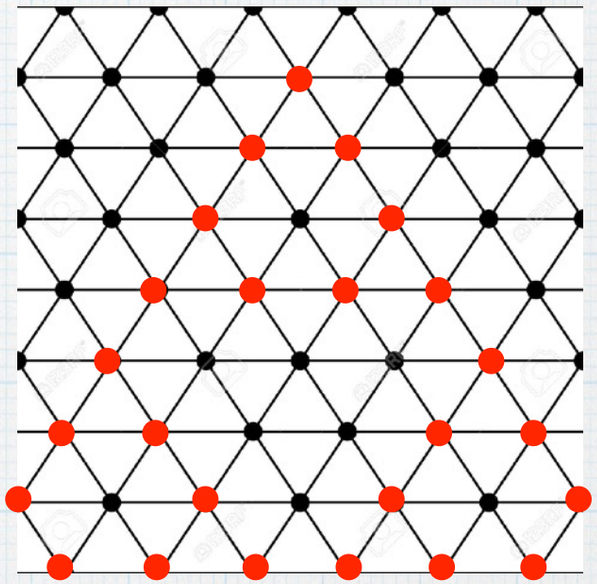

Ising, plaquette Ising & CS
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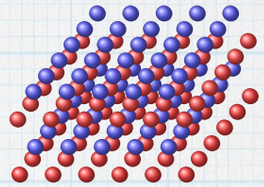

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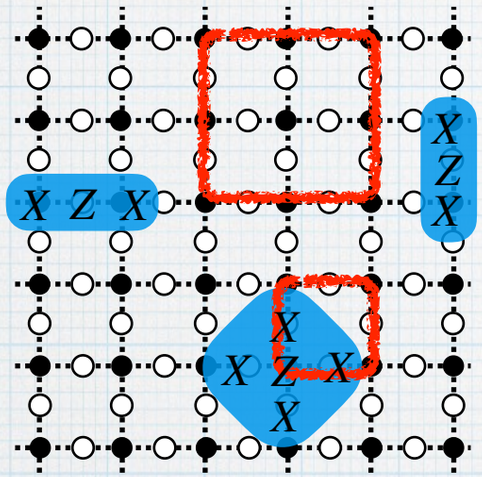

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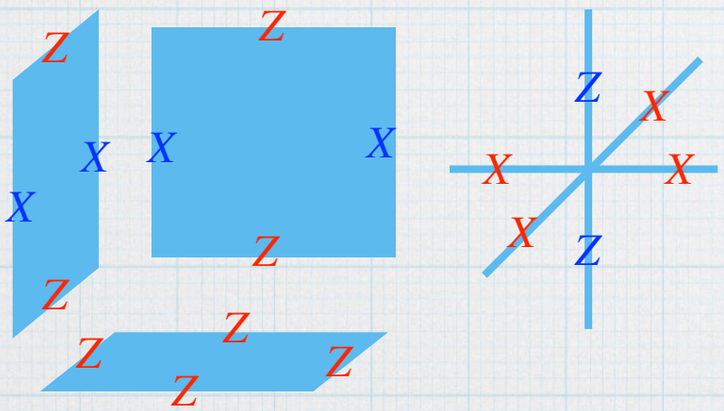

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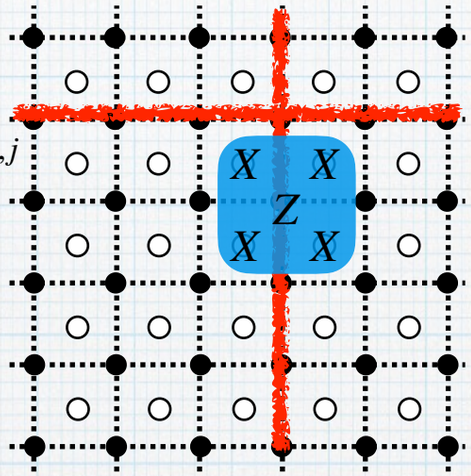

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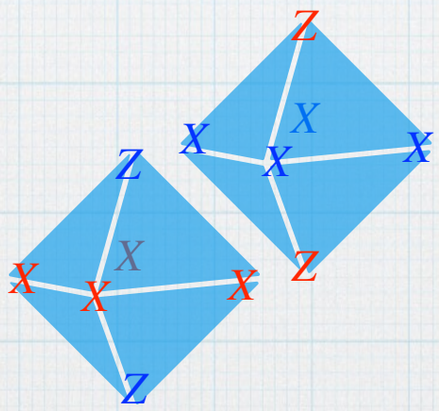

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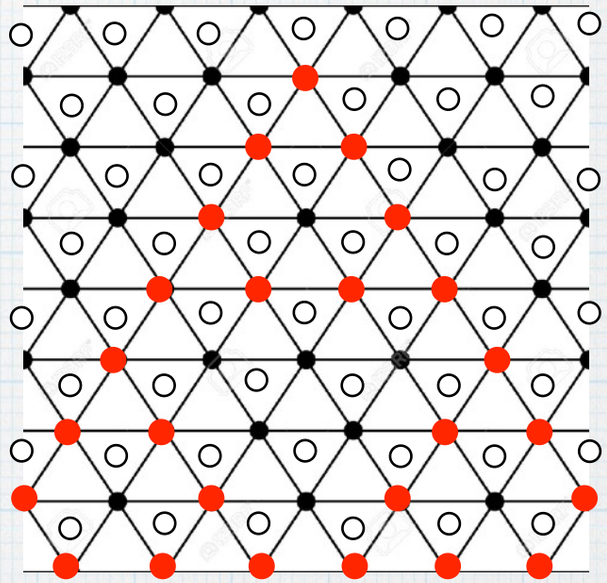

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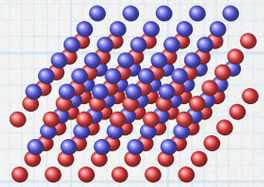

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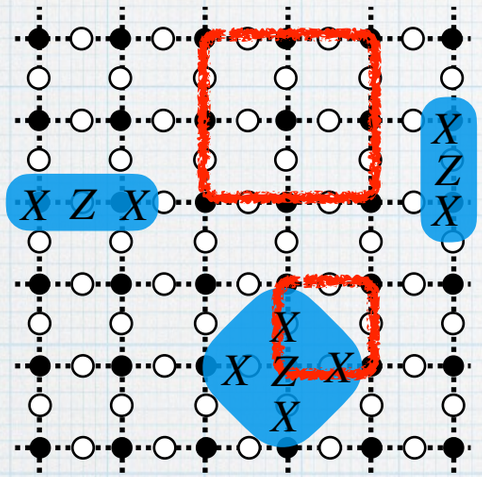

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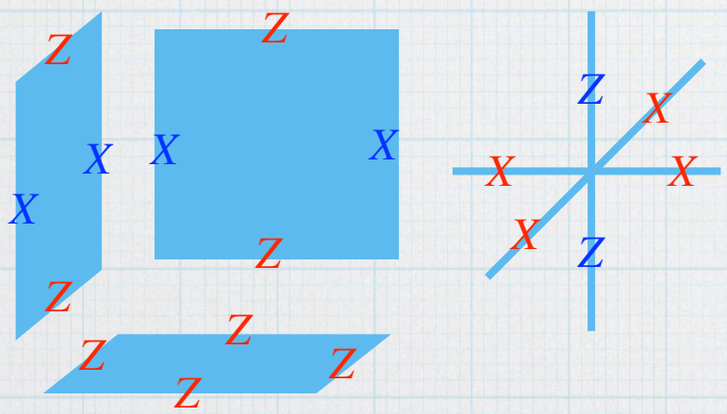

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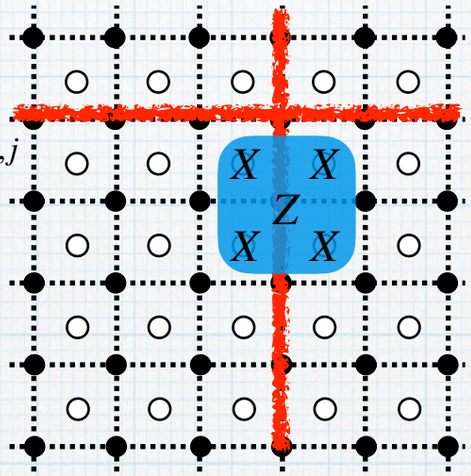

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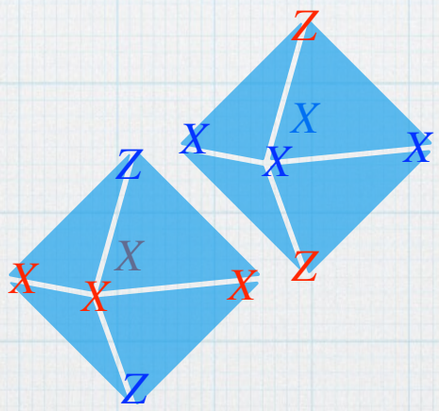

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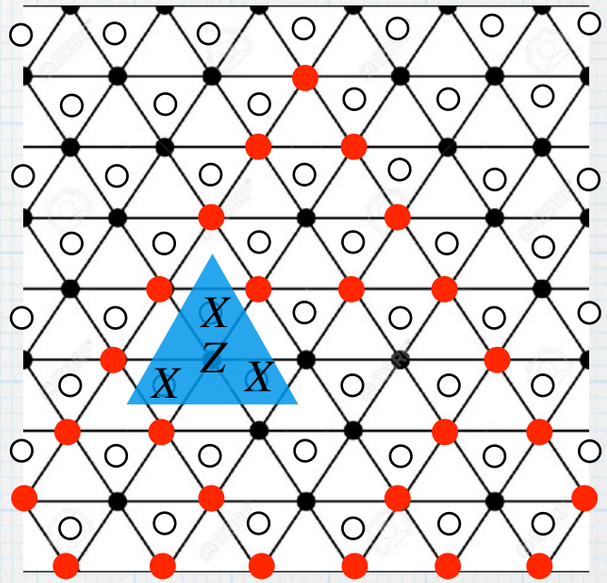

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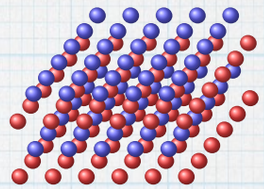

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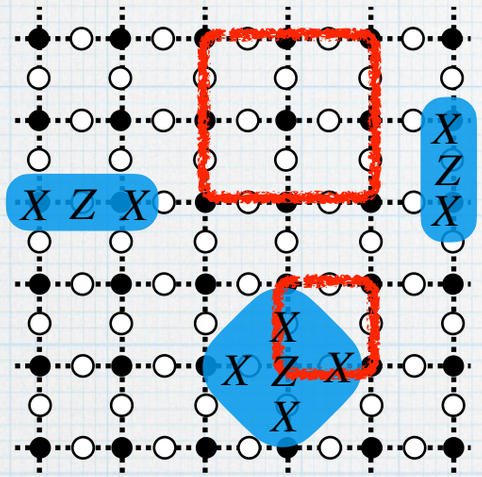

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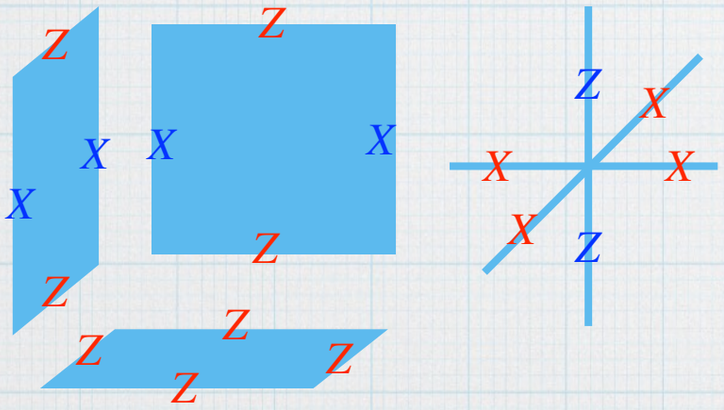

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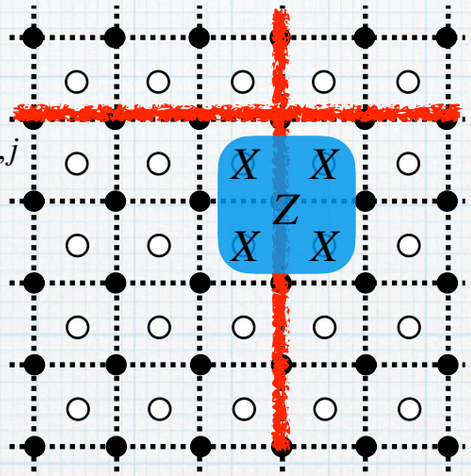

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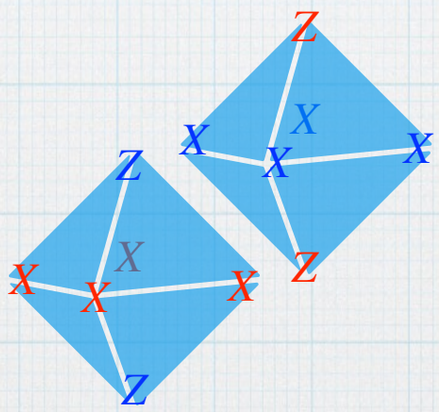

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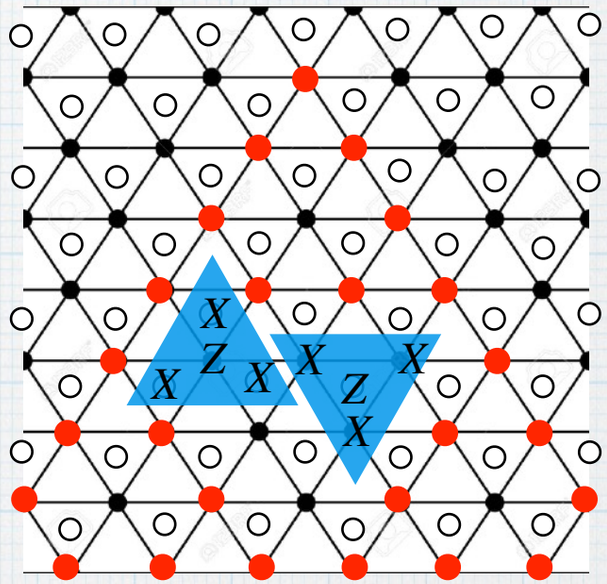

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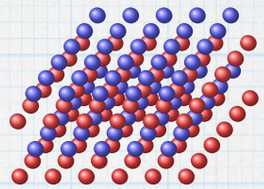

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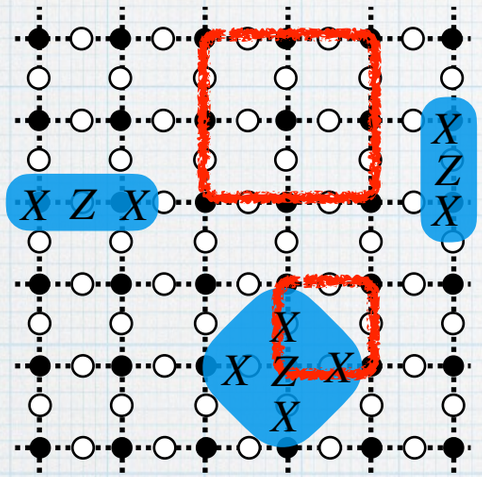

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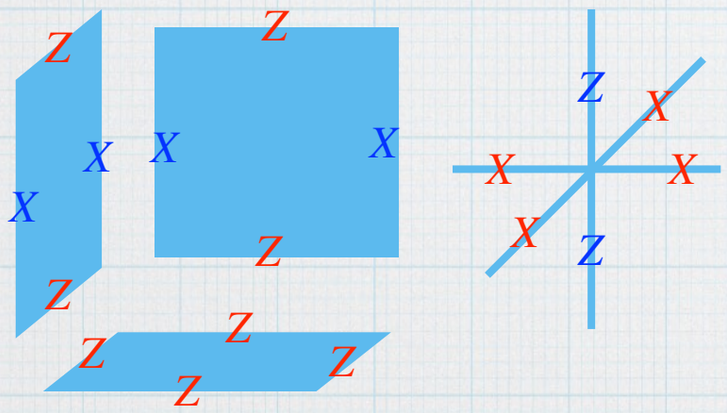

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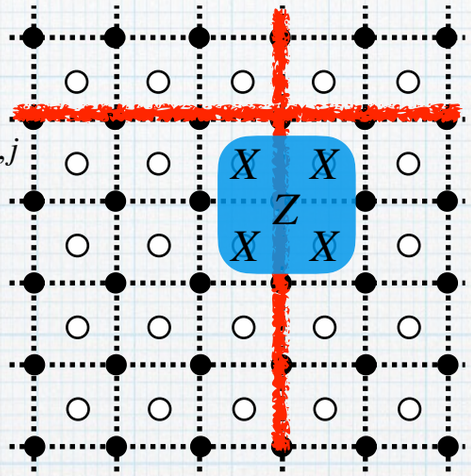

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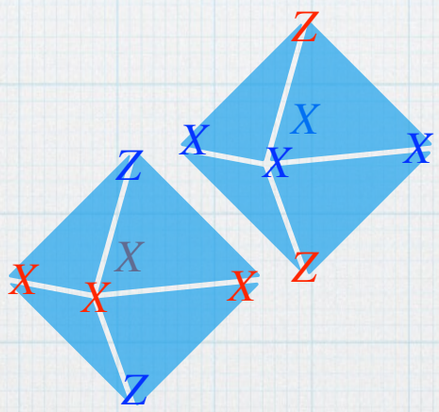

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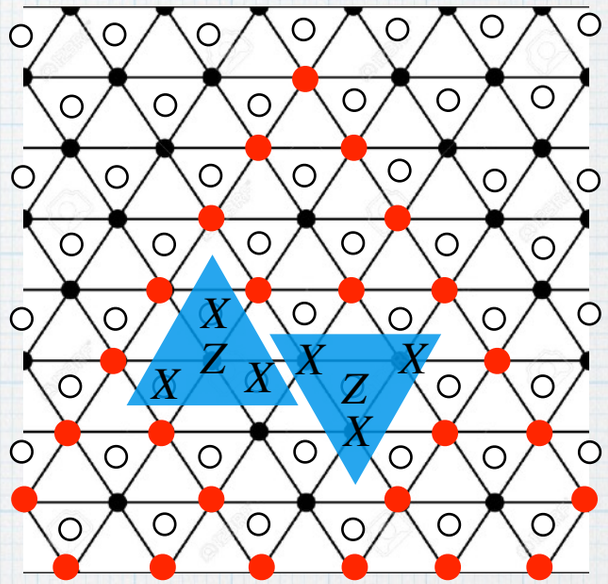

Sierpinski spin liquid
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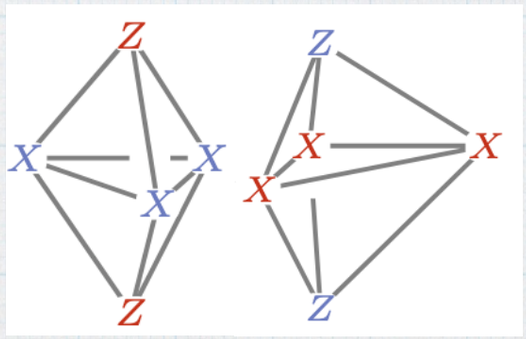

[Castelnovo & Chamon, Phil. Mag. 92, 304 (2012)]

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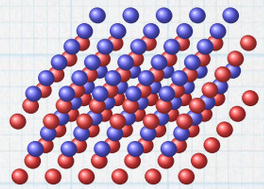

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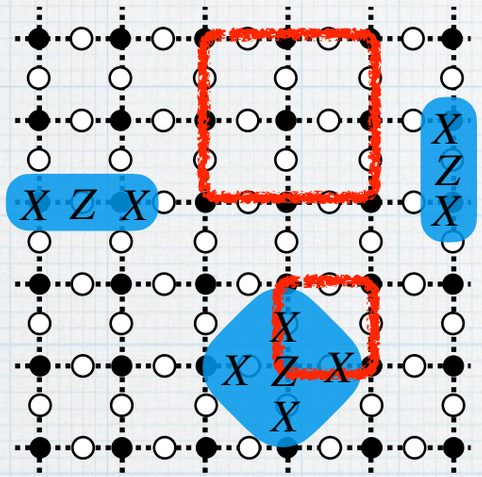

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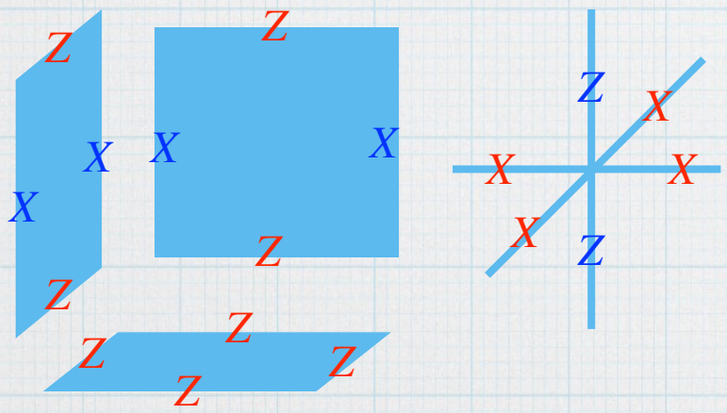
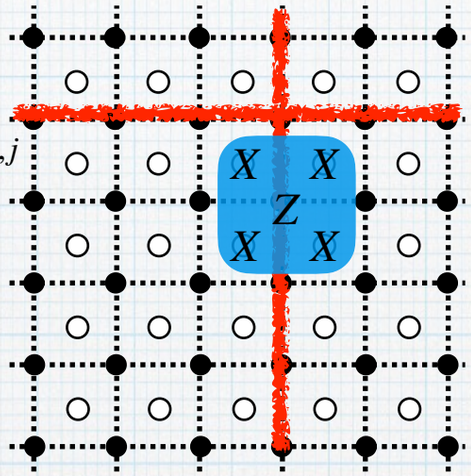


[Dennis et.al. JMP. **43**, 4452 (2002)]

Linear Sub. Sym.

$$S^y_j = \prod_i Z_{i,j}$$



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Anisotropic model:

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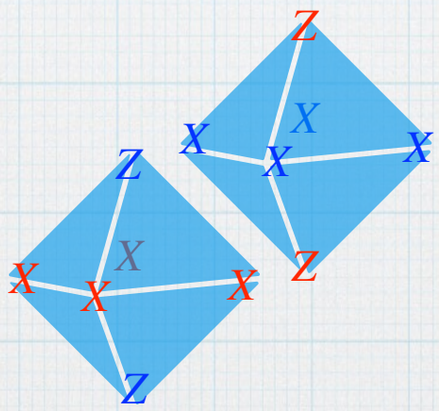
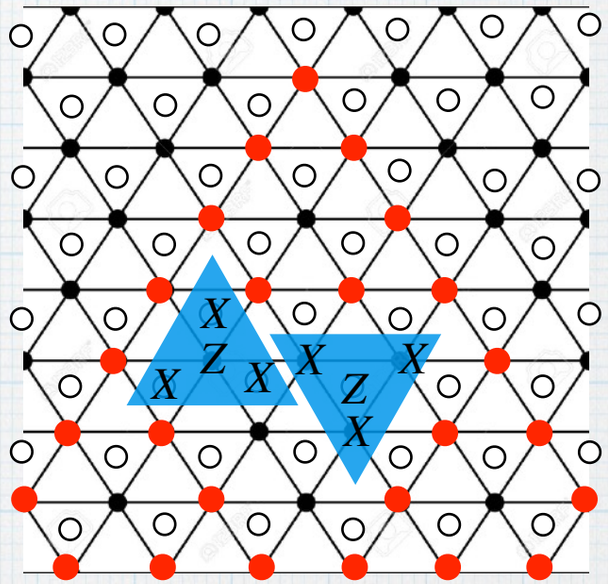


[C. Chamon, Phys. Rev. Lett. 94 (2005)]

Fractal Sub. Sym. $\prod_{i,j} \left(Z_{(i,j)}\right)^{f(i,j)}$

Sierpiński $f(i,j+1) = f(i,j) + f(i+1,j)$



Sierpinski spin liquid
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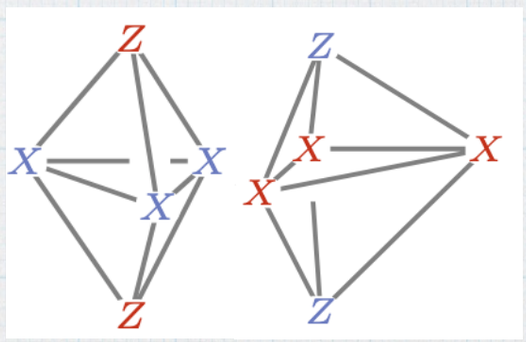

[Castelnovo & Chamon, Phil. Mag. 92, 304 (2012)]

☑ **Abelian groups, TN, symmetric phases & gapped boundaries…** *[B. Vancraeynest-De Cuiper & **JGR**, Quantum 9, 1852 (2025)]*

Twisted 1D gauging: confinement and dipoles

Kramers-Wannier $\longrightarrow$ Kennedy-Tasaki: Haldane phase to maximally broken phase of $\mathbb{Z}_2 \times \mathbb{Z}_2$

(Twisting the gauging map by a 2-cocycle α of G: $\mathcal{G} \to \mathcal{G}^{\alpha}$)

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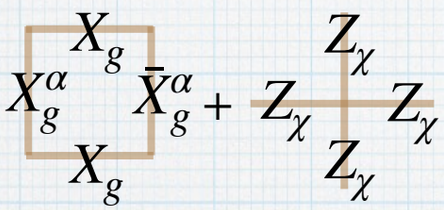

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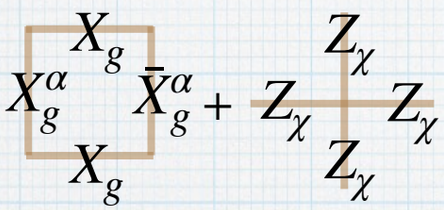

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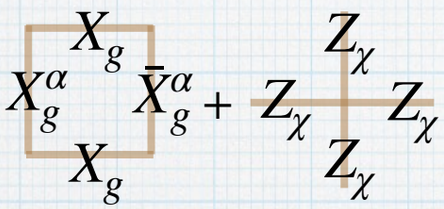

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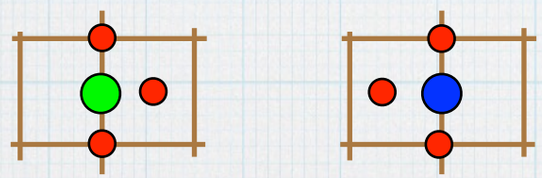

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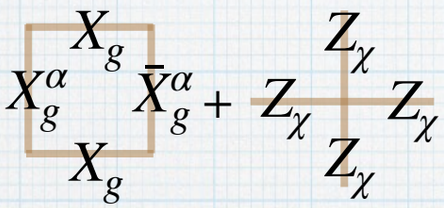

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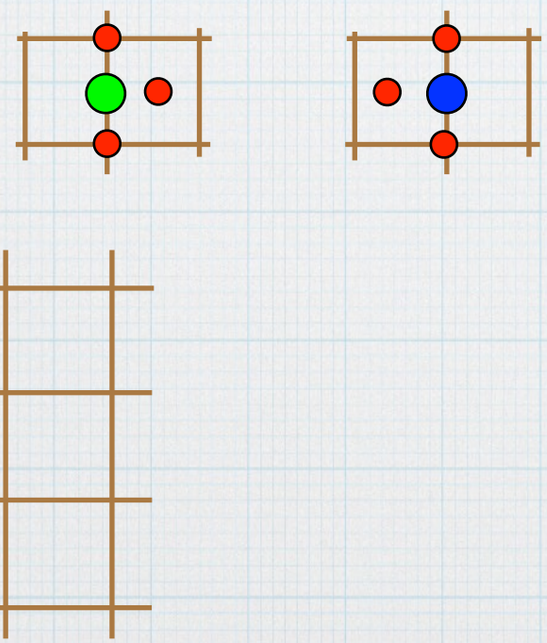

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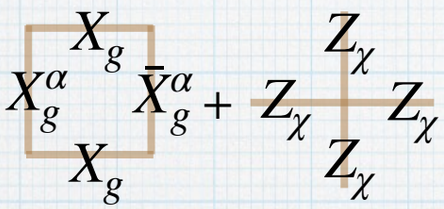

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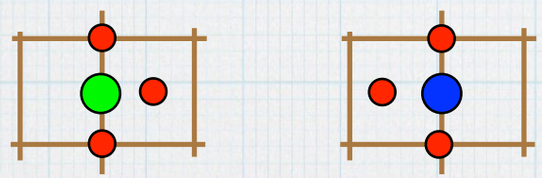

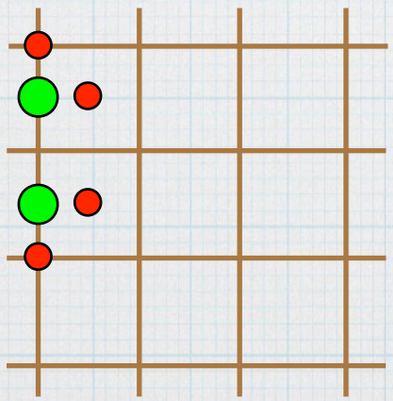

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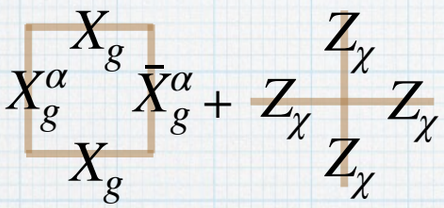

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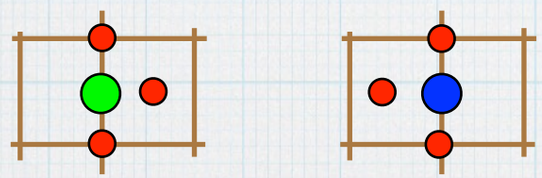

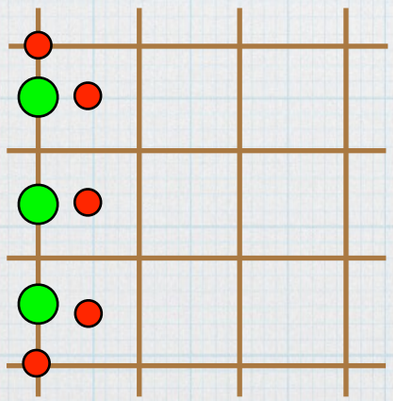

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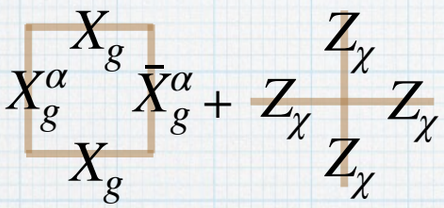

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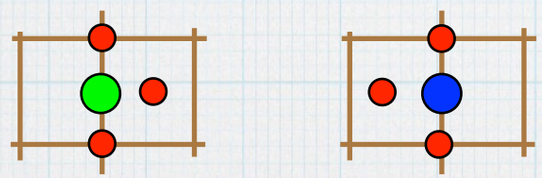

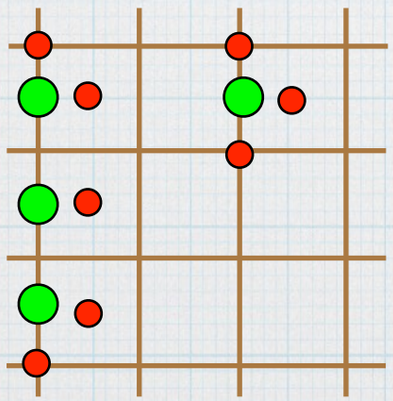

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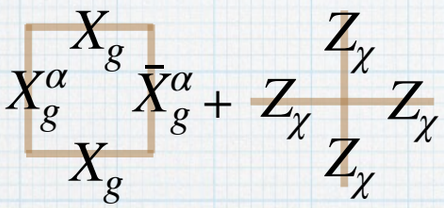
$$H_{2D}^{\mathrm{bulk}}(\alpha) = -\sum$$ 

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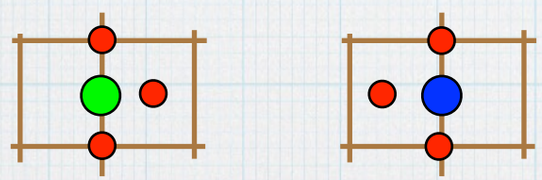

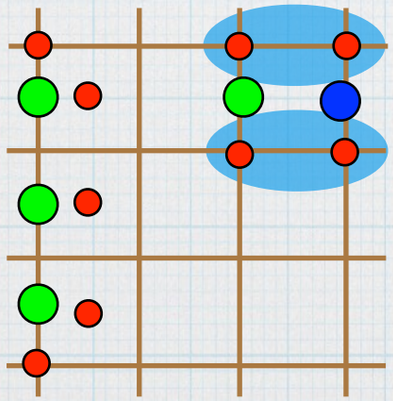

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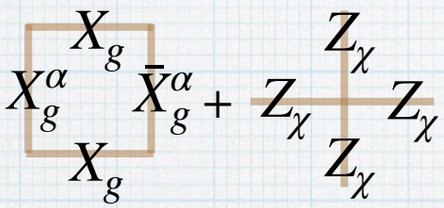

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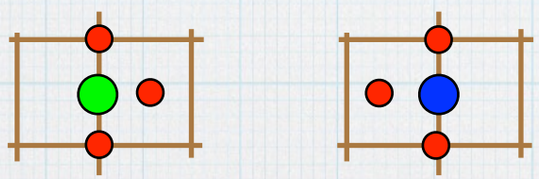

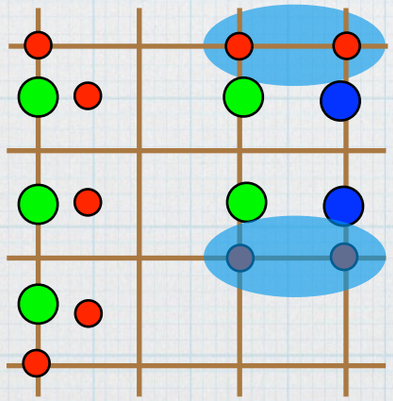

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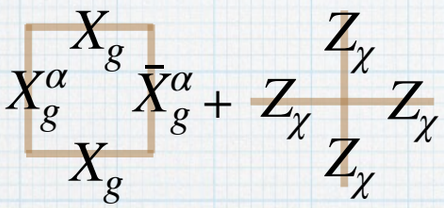
$$H_{2D}^{\mathrm{bulk}}(\alpha) = -\sum$$ 

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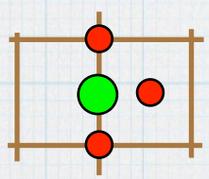

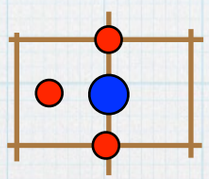

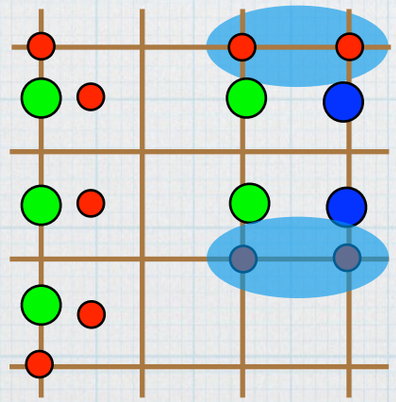

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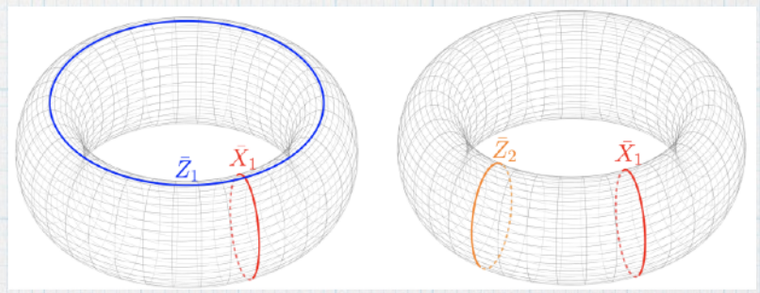
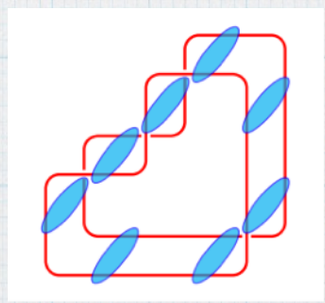
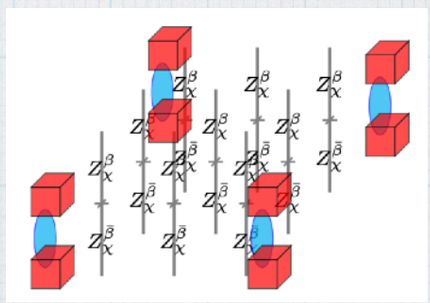
[**JGR,** *PRB 112,125134* (2025)]

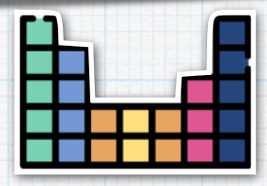






Outlook

* **Emergent principle to construct and order phases ~ Table of elements**



2D order:	Topological order
1D boundary symmetry	Global

3D order:	Topological (Surface codes)	Foliated type-I fractons	Fractal type-I fractons	Type-II fractons
2D boundary symmetry	Global & loops	Line Sub. Sym.	Fractal	??

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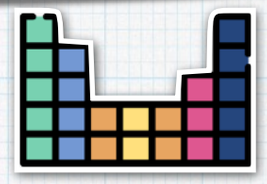

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4D order:	Topological	?	?	?	?	?
3D boundary symmetry	Global and loops	Closed membranes	Planar and loops in planes	Linear Subsystem	Fractal	?

[D. Aasen et. al., *Microsoft Quantum,* arXiv:2506.15130]

[D. Hayes et. al., Quantinuum, *PRA 110,062413 (2024)*]

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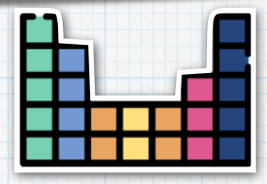

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* **These boundary symmetries correspond naturally with the virtual symmetries of the TN describing the bulk**

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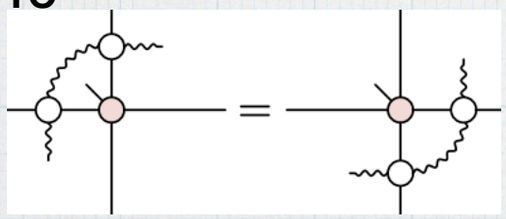
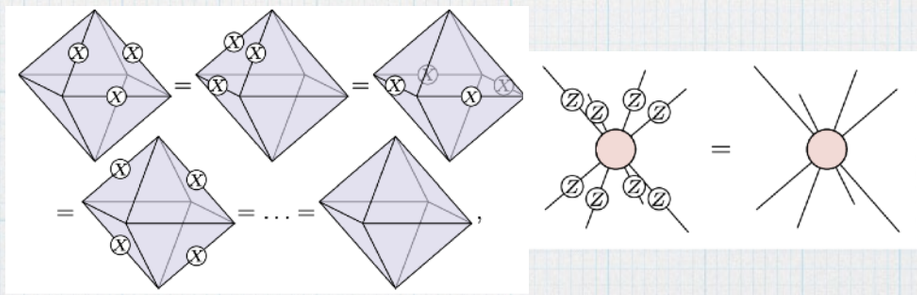


3D TO

[C. Delcamp & N. Schuch, *Quantum 5,604* (2021)]



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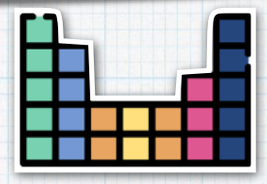

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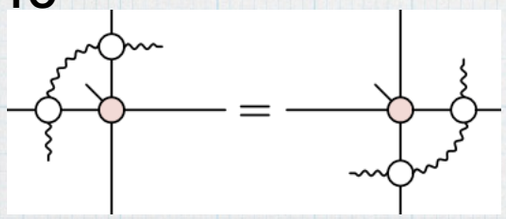

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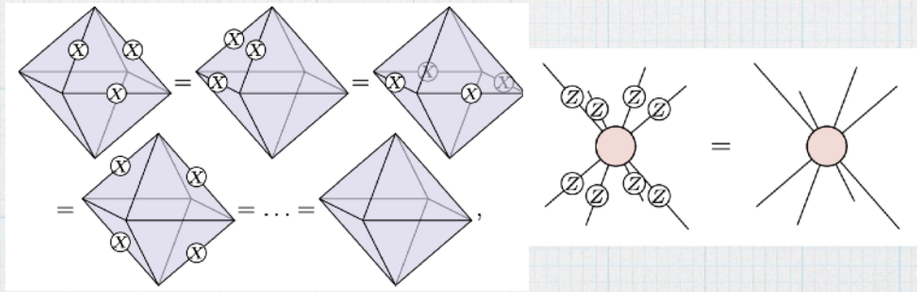

3D X-cube

[Poster of J. Wladika]

* **What about anomalous symmetries? 3D twisted codes, Gauging as a Quantum circuits …**