Quantum entanglement of Hawking-partner modes in expanding cavities

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Analogue Gravity in 2026

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Introduction

- **Hawking radiation:** one of the most important phenomena in QFT in Curved Spacetime related to unresolved issues such us the information loss problem.
- **Analogue gravity:** controlled framework to probe Hawking radiation and related phenomena in the laboratory.
- **Moving mirrors:** simple analogue model with enough complexity to study particle creation, entanglement generation, and purification.

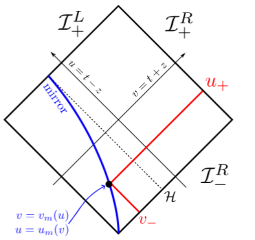


Figure: Evgenii Ievlev 2024 Class. Quantum Grav. 41 155009

Classical Scalar Field in an Expanding Cavity I

- Massless scalar field $\phi(t,x)$ in $1+1$ dimensions fulfilling the Klein–Gordon equation

$$\frac{1}{\sqrt{-\eta}}\partial_\mu\left(\sqrt{-\eta}\eta^{\mu\nu}\partial_\nu\phi\right)=0.$$

- Field confined between two moving boundaries satisfying Dirichlet conditions $\phi(t,x=f(t))=\phi(t,x=g(t))=0$. With $L(t)=g(t)-f(t)$
- Fourier mode decomposition

$$\phi(t,x)=\sum_{n=1}^{\infty}\phi_n(t)\sin\left[\frac{n\pi}{L(t)}(x-f(t))\right]$$

- Mode equations of motion

$$\ddot{\phi}_n+\sum_m R_{nm}\dot{\phi}_m+\sum_m S_{nm}\phi_m=0$$

- As long as $\dot{f}(t)\neq 0$ or $\dot{L}(t)\neq 0$, modes are coupled through R_{nm}, S_{nm}.

Classical Scalar Field in an Expanding Cavity II

- Solutions considered in the complexified phase space $\bar{\phi}_n(t) = \phi_n(t)$ are ordered as $\mathbf{U}(t) = \Big(\phi_1(t), \pi_1(t), \ldots, \phi_n(t), \pi_n(t)\Big)$.
- Canonical Poisson algebra:

$$\{\phi_n, \pi_m\} = 2\delta_{nm}.$$

- Klein–Gordon inner product:

$$\langle \mathbf{U}^{(1)}(t), \mathbf{U}^{(2)}(t)\rangle = \frac{i}{2}\sum_{n=1}^{\infty}\Big(\bar{\phi}_n^{(1)}(t)\pi_n^{(2)}(t) - \bar{\pi}_n^{(1)}(t)\phi_n^{(2)}(t)\Big).$$

- Choice of positive-definite basis $\{\mathbf{u}^{(I)} = (u_{1,\phi}, u_{1,\pi}, \ldots, u_{n,\phi}, u_{n,\pi})\}$:

$$\langle \mathbf{u}^{(I)}, \mathbf{u}^{(J)}\rangle = \delta^{IJ},\ \langle \mathbf{u}^{(I)}, \bar{\mathbf{u}}^{(J)}\rangle = 0,\ \langle \bar{\mathbf{u}}^{(I)}, \bar{\mathbf{u}}^{(J)}\rangle = -\delta^{IJ}.$$

- Solutions are expressed as

$$\mathbf{U}(t) = \sum_{I=1}^{\infty} a_I \mathbf{u}^{(I)}(t) + \bar{a}_I \bar{\mathbf{u}}^{(I)}(t). \Longrightarrow \phi_n(t) = \sum_{I=1}^{\infty} a_I u_{n,\phi}^{(I)}(t) + \bar{a}_I \bar{u}_{n,\phi}^{(I)}(t)$$

Bogoliubov Transformations

- Mode solutions at different times span different positive-frequency subspaces if there acceleration in the boundaries.
- Given another basis $\mathbf{w}^{(I)}(t)$, we have the Bogoliubov transformation

$$\mathbf{u}^{(I)}(t) = \sum_{J=1}^{\infty} \alpha_{IJ}\mathbf{w}^{(J)}(t) + \beta_{IJ}\bar{\mathbf{w}}^{(J)}(t).$$

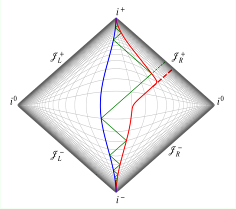
- Bogoliubov coefficients $\alpha_{IJ} = \langle \mathbf{w}^{(J)}(t), \mathbf{u}^{(I)}(t)\rangle, \beta_{IJ} = -\langle \bar{\mathbf{w}}^{(J)}(t), \mathbf{u}^{(I)}(t)\rangle$ encode particle creation and mode mixing.
- Define in-vacuum via canonical quantization as $\hat{a}_I|0\rangle_{\mathrm{in}} = 0$ for all I.
- Out-modes defined after boundary motion, with corresponding annihilation operators b_J. The corresponding out-vacuum defined by $\hat{b}_J|0\rangle_{\mathrm{out}} = 0$ for all J.
- Particle production encoded by non-zero β_{IJ} coefficients due to Bogoliubov transformation

$$\hat{a}_I = \sum_{J=1}^{\infty} \bar{\alpha}_{IJ}\hat{b}_J - \bar{\beta}_{IJ}\hat{b}_J^{\dagger} \Longrightarrow {}_{\mathrm{out}}\langle 0|\hat{a}_I^{\dagger}\hat{a}_I|0\rangle_{\mathrm{out}} = \sum_{J=1}^{\infty} |\beta_{IJ}|^2.$$

Specific Trajectory and Analogy

- Fixed left boundary and the right boundary accelerates and stops, symmetrically, at a final position (modified version of Fulling–Davies trajectory $g(t) = -t - 1/2\kappa e^{-2\kappa t} + 1/2\kappa$)

$$
\begin{aligned}
f(t) =& 0, \\
g(t) =& 1 + \frac{s}{2\kappa} + \frac{1}{2\kappa}\Big[\log\Big(\cosh\big(\kappa(t-t_0)\big)\Big) \\
& - \log\Big(\cosh\big(s - \kappa(t-t_0)\big)\Big)\Big].
\end{aligned}
$$



García Marín-Caro, A. et al. Phys. Rev. D 108, L061701 (2023)

- Nearly thermal spectrum of particles, with respect to the out vacuum, dictated by the (modified) Fulling–Davies spectrum is expected

$$
|\beta_{IJ}^{(\mathrm{f})}|^2 = \frac{2\Delta\omega_I \Delta\omega_J}{\pi\kappa\omega_I}\frac{\Gamma_\beta(\epsilon,\omega_J)}{e^{2\pi\omega_J/\kappa}-1},
$$

where $\Gamma_\beta(\epsilon,\omega_J)$ is a greybody factor depending on the final cavity length.

Gaussian States

- Define the phase space operators $\hat{q}_I = \frac{1}{\sqrt{2}}(\hat{a}_I + \hat{a}_I^\dagger)$, $\hat{p}_I = \frac{-i}{\sqrt{2}}(\hat{a}_I - \hat{a}_I^\dagger)$ and arrange them as $\hat{\mathbf{R}} = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^\mathsf{T}$.
- **Gaussian states:** Fully characterized by mean vector and covariance matrix

$$\mu^i \equiv \langle \hat{R}^i \rangle, \qquad \sigma^{ij} \equiv \frac{1}{2}\langle \{ \hat{R}^i - \mu^i,\, \hat{R}^j - \mu^j \} \rangle.$$

- Time evolution preserves Gaussianity under Bogoliubov transformations with $\boldsymbol{\sigma} \to \boldsymbol{S}\boldsymbol{\sigma}\boldsymbol{S}^\mathsf{T}$.
- Used here as initial states to probe entanglement generation. Particular states considered:
  - **Vacuum:** $\mu = 0, \quad \boldsymbol{\sigma} = \frac{1}{2}\mathbb{I}$.
  - **One-mode squeezed:** $\boldsymbol{\sigma} = S(r)\,\boldsymbol{\sigma}_0\,S^\mathsf{T}(r), \quad \langle \hat{a}\hat{a} \rangle \neq 0$.
  - **Two-mode squeezed:** $\langle \hat{a}_I \hat{a}_J \rangle \neq 0, \quad I \neq J$.
  - **Thermal (T):** $\langle \hat{a}_I^\dagger \hat{a}_I \rangle = n_T, \quad n_T = (e^{\omega_I/T} - 1)^{-1}$.

Entanglement in Gaussian States

- Covariance matrix of a Gaussian state encodes all entanglement properties. It must fulfill the condition $\boldsymbol{\sigma} + i\Omega \geq 0$ (Ω symplectic form) to represent a physical state.
- Partial transposition at covariance matrix level: $\tilde{\boldsymbol{\sigma}} = T\boldsymbol{\sigma}T$, where T changes sign of momentum operators of one subsystem.
- PPT criterion: if $\tilde{\boldsymbol{\sigma}} + i\Omega \not\geq 0$, the state is entangled.
- Entanglement in bipartitions of $1 \times (N-1)$ modes always detected by PPT criterion.
- Logarithmic negativity as an entanglement measure:

$$\mathrm{LogNeg}(\boldsymbol{\sigma}_{AB}) = \sum_{k=1}^{N} \max[0, -\log \tilde{v}_k],$$

  with $\tilde{v}_k$ the symplectic eigenvalues of $\tilde{\boldsymbol{\sigma}}_{AB}$.
- LogNeg is easily computable and is an entanglement monotone. It is also an upper bound to distillable entanglement.

Partner Particle: HSU Construction I

- Hawking radiation produces thermal spectrum which needs purification outside the Hilbert space of Hawking modes.
- This setup does not produce horizons, thus the purification must be encoded in the full multimode system.
- Given a single mode A with (quasi) thermal spectrum (Hawking mode), it can be seen as (fix J):

$$\hat{b}_H = \sum_{I=1}^{N} \alpha_I \hat{a}_I + \bar{\beta}_I \hat{a}_I^\dagger = \langle \bar{\boldsymbol{\alpha}} | \hat{\mathbf{a}} \rangle + \langle \hat{\mathbf{a}}^\dagger | \bar{\boldsymbol{\beta}} \rangle$$

- Define directions $\mathbf{n}_\parallel = \boldsymbol{\alpha}/|\boldsymbol{\alpha}|$ and $\mathbf{n}_\perp$ orthogonal to it. And new operators $\hat{b}_\parallel = \langle \mathbf{n}_\parallel | \hat{\mathbf{a}} \rangle$, $\hat{a}_\perp = \langle \mathbf{n}_\perp | \hat{\mathbf{a}} \rangle$ such that

$$\hat{b}_H = \alpha\, \hat{a}_\parallel + \bar{\beta}_\parallel \hat{a}_\parallel^\dagger + \bar{\beta}_\perp \hat{a}_\perp^\dagger$$

- Following Hotta–Schützhold–Unruh (2015), define the partner mode $\hat{b}_P$ such that the joint state of $(\hat{b}_H, \hat{b}_P)$ is pure.

Partner Particle: HSU Construction II

- Partner should be $\boxed{\hat{b}_P = \gamma_\parallel \hat{a}_\parallel + \gamma_\perp \hat{a}_\perp + \bar{\delta}_\parallel \hat{a}_\parallel^\dagger + \bar{\delta}_\perp \hat{a}_\perp^\dagger}$ with coefficients satisfying canonical commutation relations

$$|\alpha|^2 - |\beta_\parallel|^2 - |\beta_\perp|^2 = 1 \qquad |\gamma_\parallel|^2 + |\gamma_\perp|^2 - |\delta_\parallel|^2 - |\delta_\perp|^2 = 1$$
$$\bar{\gamma}_\parallel \alpha = \bar{\beta}_\parallel \delta_\parallel + \bar{\beta}_\perp \delta_\perp \qquad \bar{\alpha}\delta_\parallel = \bar{\gamma}_\parallel \beta_\parallel + \bar{\gamma}_\perp \beta_\perp$$

- Additional condition needed to fix the remaining freedom. Here we use **Criterion B1 (HSU):** $\hat{b}_P|0\rangle \propto \hat{b}_H^\dagger|0\rangle \Rightarrow \delta_\perp = 0$.

- If $\beta_\parallel = 0$, the effective transformation is a two-mode squeezing between Hawking and partner modes

$$\hat{b}_H \simeq \alpha\, \hat{a}_\parallel + \bar{\beta}\hat{a}_\perp^\dagger, \quad \hat{b}_P \simeq \alpha \hat{a}_\perp + \bar{\beta}\hat{a}_\parallel^\dagger.$$

- Advantage: No need to trace over all remaining modes explicitly, allowing high numerical efficiency in large multimode systems.

- Drawback: Construction only known for pure initial states and definition not strict enough to guarantee uniqueness (LogNeg not affected).

Hawking-partners Bogoliubov Coefficients

- For (relative) small accelerations of the boundaries, particle creation follows the Fulling-Davies law.
- Partner coefficients are also localized at low energies.

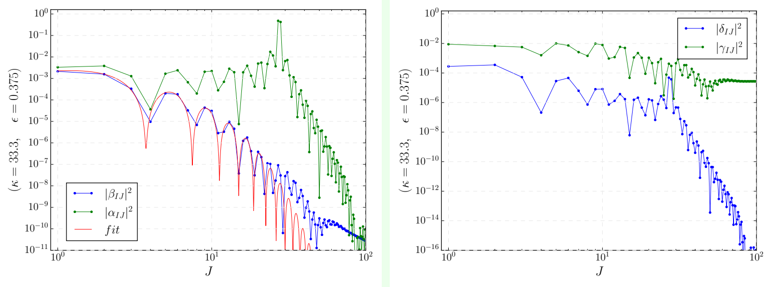


Figure: Hawking (left) and partner (right) Bogoliubov coefficients for small acceleration ($\kappa = 33$, $\epsilon = 0.375$) and *in* mode $I = 20$.

System as a Squeezing Device

- Hawking and partner modes are coupled via two-mode squeezing:

$$\hat{b}_H \simeq \alpha\, \hat{a}_{\parallel} + \bar{\beta}\hat{a}^{\dagger}_{\perp}, \quad \hat{b}_P \simeq \alpha\hat{a}_{\perp} + \bar{\beta}\hat{a}^{\dagger}_{\parallel}.$$

$$\Longrightarrow |\alpha_J|^2 - 1 \simeq |\beta_{\perp J}|^2, \quad |\gamma_{\perp J}|^2 - 1 \simeq |\delta_{\parallel J}|^2$$

- An expanding cavity acts as a squeezing device regarding a given Hawking mode and its partner

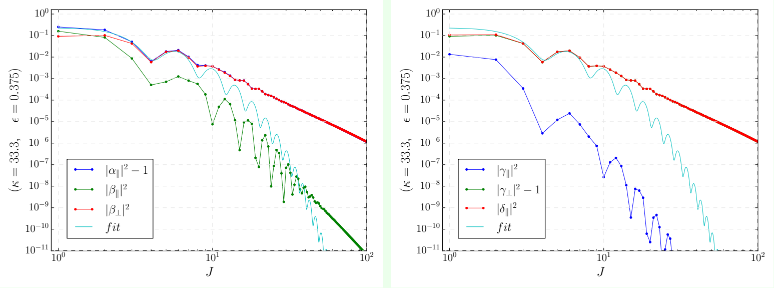


Figure: Bogoliubov coefficients for all out modes and partners.

- Logarithmic negativity computed between Hawking and partner coincides with the LogNeg computed as $1 \times (N-1)$ mode bipartition.
- Vacuum entanglement resembles the particle spectrum, decaying rapidly in the UV.
- Initial one-mode squeezing enhances entanglement and redistributes it towards higher frequencies.

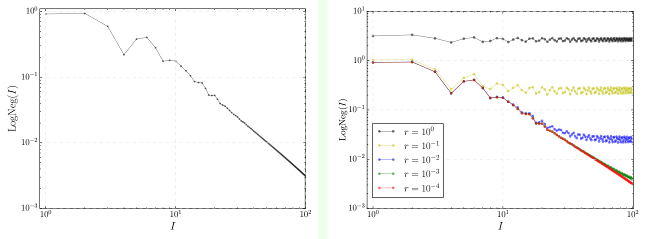


Figure: Logarithmic negativity between Hawking and partner modes (left) and $1 \times (N-1)$ Logarithmic negativity for one-mode squeezed states (right) for small acceleration ($\kappa = 120$, $\epsilon = 0.125$).

Entanglement vs Temperature

- Partner formula not known for mixed initial states, so we compute $1 \times (N-1)$ LogNeg only.
- For N fixed, there is a critical temperature, $\mathcal{T}_c(N)$, above which entanglement vanishes. For $N \to \infty$, $\mathcal{T}_c$ seems to converge to a finite value per mode $\mathcal{T}_c(N)/N \to 0.09145$.

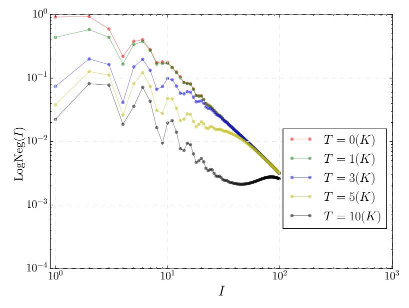


Figure: Logarithmic negativity between Hawking and partner modes vs initial temperature for small acceleration ($\kappa = 120$, $\epsilon = 0.125$).

Conclusions

- Fully non-perturbative numerical approach which allows to probe many different trajectories.
- The expanding cavity acts as a two-mode squeezing device for Hawking-partner pairs.
- Entanglement decays rapidly in the ultraviolet sector, consistent with the observed particle spectrum.
- Squeezing stimulates and redistributes entanglement across modes.
- Each mode exhibits a critical temperature above which entanglement vanishes, providing a possible upper bound.

Outlook

- We aim to further develop the partner formula so that it depends only on symplectic invariants and can be applied to non-pure initial states without ambiguities.
- We plan to explore alternative entanglement measures that can probe aspects such as multimode entanglement, which are not fully captured by logarithmic negativity.
- It is also necessary to extend the analysis to cover Robin boundary conditions instead of Dirichlet, as these are more suitable for experimental devices, as well as to consider other types of trajectories.
- Our setup could be implemented in a coplanar waveguide (CPW) connected to two superconducting quantum interference devices (SQUIDs).
- Although we have tentatively shown that the critical temperature per mode is higher than what is required in such systems, a more comprehensive study is needed to assess the feasibility of the implementation.

Thank you for your attention!

Questions?

Appendix: Classical Scalar Field in an Expanding Cavity: Equation of Motion Details

- The mode coupling matrices S_{mn} and R_{mn} are given by

$$S_{mn} = \delta_{mn}\Bigg[\left(\frac{n\pi}{L}\right)^2\left(1-\dot{f}^2-\dot{f}\dot{L}-\frac{\dot{L}^2}{3}\right) + \frac{\dot{L}^2}{2L^2}\left(1-\frac{2}{n\pi}\right)+\frac{\ddot{L}}{2n\pi L}\Bigg] + ((-1)^{m+n}-1) \times \left[\frac{2[\ddot{f}L+\ddot{L}L-2\dot{f}\dot{L}-2\dot{L}^2]m}{(m^2-n^2)\pi L^2} - \frac{8[\dot{f}\dot{L}+\dot{L}^2]mn^3}{(m^2-n^2)^2L^2}\right],$$

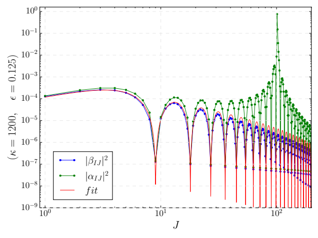
$$R_{mn} = -\delta_{mn}\frac{\dot{L}}{L} - (1-\delta_{mn})\frac{4[(-1)^{m+n}(\dot{f}+\dot{L})-\dot{f}]mn}{(m^2-n^2)L}.$$

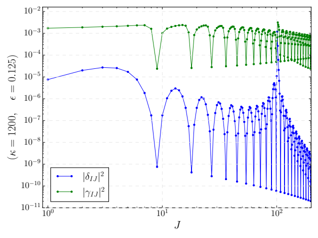
- The numerical method solves these coupled differential equations using plain waves as initial condition at t_0:

$$\mathbf{u}^{(l)}(t_0) = \left(0,0,\ldots,\tfrac{1}{\sqrt{\omega_l}},-i\sqrt{\omega_l},\ldots,0,0\right)$$

Appendix: Bogoliubov and Partner Coefficients (Large Acceleration)

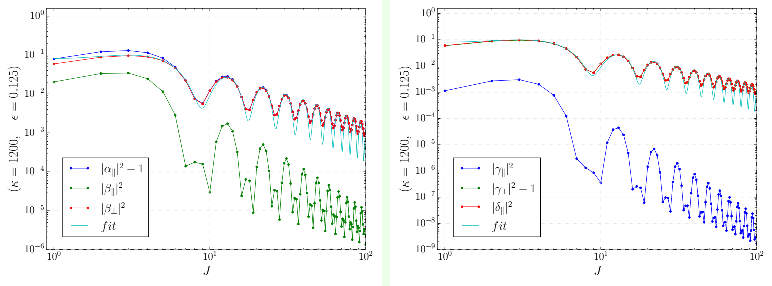
- For large acceleration ($\kappa = 1200$, $\epsilon = 0.125$), particle creation is strong and mode mixing is broadband.
- Hawking mode Bogoliubov coefficients β_{IJ} show a thermal spectrum at low frequencies, with a sharp decay in the UV.
- Partner coefficients δ_{IJ} are peaked near the Hawking mode and decay rapidly at high frequencies.
- Purification remains a low-energy process, even for sharp trajectories.





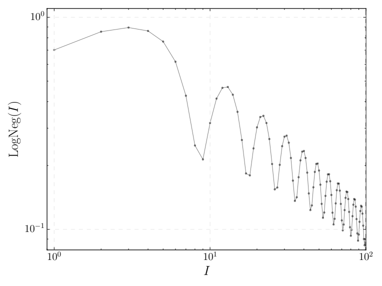
Appendix: System as a Squeezing Device (Large Acceleration)

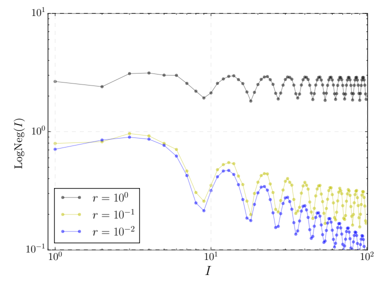
- Hawking and partner modes are coupled via two-mode squeezing, even for large acceleration.
- The system generates Hawking-partner pairs, with squeezing parameters determined by the Bogoliubov coefficients.
- The squeezing effect is more pronounced for sharp trajectories.



Appendix: Entanglement: Vacuum vs Squeezing (Large Acceleration)

- Logarithmic negativity decays in the UV for vacuum, but is enhanced and redistributed by initial squeezing.
- For large acceleration, squeezing leads to a broader and more robust entanglement distribution.
- Oscillatory structure persists, with higher entanglement at low frequencies.





Appendix: Entanglement vs Temperature (Large Acceleration)

- Entanglement decreases with increasing initial temperature, but is more robust for large acceleration.
- Each mode has a critical temperature above which entanglement vanishes; this threshold is higher for sharp trajectories.
- High-frequency modes retain entanglement up to higher temperatures.