Thermodynamics of analogue black holes in a non-Hermitian tight-binding model

Analogue Gravity in 2026, Benasque Science Center

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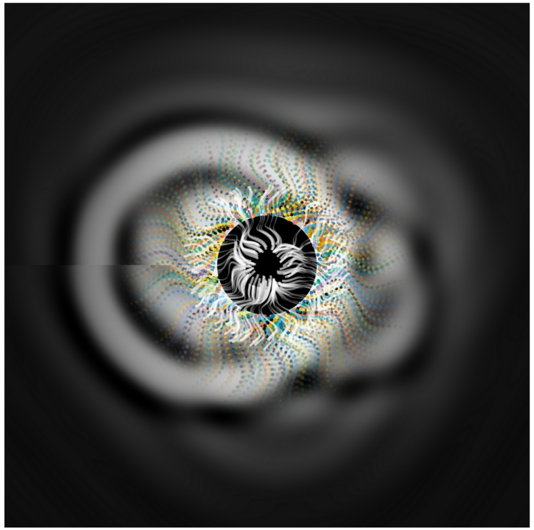
Motivation



Black hole (BH) evaporation process

Analogue gravity program

While the developments on analogue gravity have been grounded in Hermitian physics, their extension to non-Hermitian (nH) systems has opened new avenues for understanding spacetime analogues and causal structures in dissipative or driven settings

Many realistic experimental platforms, including those involving gain/loss and amplification, operate in nH regimes

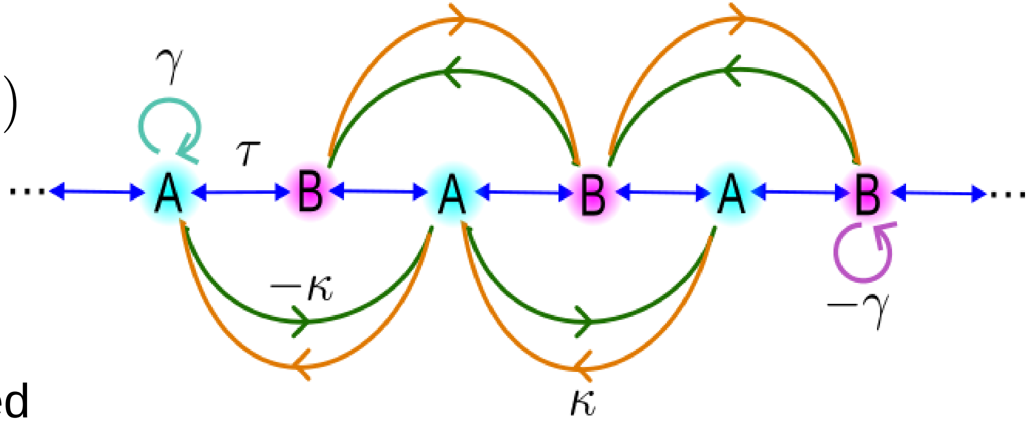
Recently, it was shown that nH Dirac-like systems with parity-time ($\mathcal{PT}$) symmetry can exhibit structures analogue to light cones in general relativity, with a well-defined effective line element that mimics causal flow near horizons

S. W. Hawking, Black hole explosions?, Nature 248, 30 (1974)
W. G. Unruh, Experimental Black-Hole Evaporation?, Phys. Rev. Lett. 46, 1351 (1981)
M. Stålhammar, J. Larana-Aragon, L. Rødland, and F. K. Kunst, $\mathcal{PT}$ symmetry-protected exceptional cones and analogue Hawking radiation New J. Phys. 25, 043012 (2023)

Extended non-Hermitian tight-binding model

The Hamiltonian of the model ($\mathcal{PT}$-symmetric)

$$H = \sum_{j=1}^{N} \Big[\tau\big(b_j^\dagger a_j + a_{j+1}^\dagger b_j + \mathrm{h.c.}\big) + i\gamma\big(a_j^\dagger a_j - b_j^\dagger b_j\big) - \kappa\big(a_{j+1}^\dagger a_j - a_j^\dagger a_{j+1} + b_j^\dagger b_{j+1} - b_{j+1}^\dagger b_j\big)\Big],$$



Bloch Hamiltonian with $\sin(k)$ and $\cos(k)$ expanded around π

$$h(\tilde{k}) = \begin{pmatrix} i(\gamma + \kappa\tilde{k}) & +i\tau\tilde{k} \\ -i\tau\tilde{k} & -i(\gamma + \kappa\tilde{k}) \end{pmatrix}$$

$$\tilde{k} = k - \pi,$$

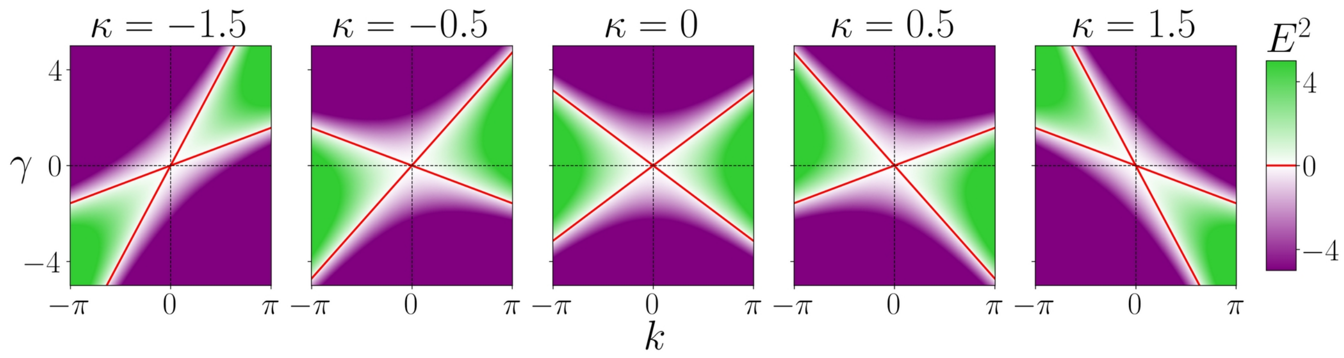
gain/loss potential (γ)

hopping parameter (τ)

NNN hopping (κ)

Eigenvalue equation

Eigenvalue equation $\epsilon_{\pm}(\tilde{k}) = \pm\sqrt{\tau^2\tilde{k}^2 - (\gamma + \kappa\tilde{k})^2}$



For κ=0 we see the usual Dirac-like cone.

For $|\kappa|>0$ cone is tilted

Exceptional points $\gamma = -\kappa\tilde{k} \pm |\tau\tilde{k}|$

Similar tilting profile can be found in the surroundings of a Schwarzschild BH event horizon in Painlevé-Gullstrand coordinates by tuning the free falling velocity!!

M. Stålhammar, J. Larana-Aragon, L. Rødland, and F. K. Kunst, New J. Phys. 25, 043012 (2023)
A. J. S. Hamilton and J. P. Lisle, Am. J. Phys. 76, 519 (2008).

Analogue spacetime

γ takes the form of the eigenvalues of a Dirac-like operator

$$\hat{\gamma} = -\kappa\tilde{k}\sigma^0 + \tilde{k}\sigma^x \rightarrow \hat{\gamma} = e^\mu_\alpha \tilde{k}_\mu \sigma^\alpha$$

e^μ_α are *vielbeins* or tetrad fields

Metric of the form

$$g^{\mu\nu} = e^\mu_\alpha e^\nu_\beta \eta^{\alpha\beta}$$

For the nH-TB model defined above, the *vielbeins* are

$$e^0_0 = 1,\ e^1_1 = \tau,\ e^0_1 = e^1_0 = -\kappa$$

Fixing $\tau{=}1$ the line element can be written as

$$ds^2 = -\left(1-\kappa^2\right)(dt)^2 + 2\kappa dt dx + (dx)^2$$

similar to the Schwarzschild metric in the PG coordinates

$$ds^2 = -\left(1 - \frac{2M_o}{r}\right)(dt_r)^2 + 2\sqrt{\frac{2M_o}{r}} dt_r dr$$

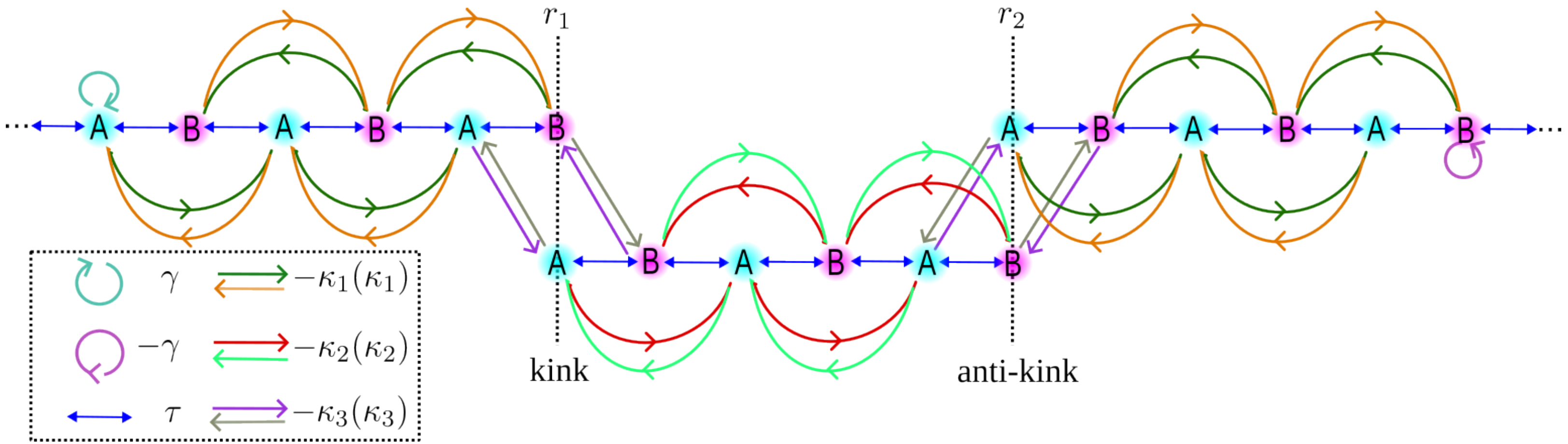
M_o mass at the event horizon for a freely falling observer

$\kappa{>}0$ analogue of a black hole

$\kappa{<}0$ analogue of a white hole

M. Stålhammar, J. Larana-Aragon, L. Rødland, and F. K. Kunst, New J. Phys. 25, 043012 (2023)
A. J. S. Hamilton and J. P. Lisle, Am. J. Phys. 76, 519 (2008).

Analogue event horizon setup



Bloch Hamiltonian of the coupled chains

$$h(k) = \begin{pmatrix} i\left[\gamma - f(r)\sin(k)\right] & 1 + e^{-ik} \\ 1 + e^{ik} & -i\left[\gamma - f(r)\sin(k)\right] \end{pmatrix}$$

Where $f(r)$ is

$$f(r) = \kappa_1 + \frac{\kappa_1 - \kappa_2}{2}\left[\mathrm{th}\frac{(r_1 - r)}{l} + \mathrm{th}\frac{(r - r_2)}{l}\right]$$

We find the condition

$$f(r_1) = f(r_2) = (\kappa_1 + \kappa_2)/2 = \kappa_3$$

Performing the same expansion

$$h(\tilde{k}) = -\tilde{k}\sigma^y + i\left[\gamma + f(r)\tilde{k}\right]\sigma^z$$

Exceptional points $\gamma = -f(r)\tilde{k} \pm |\tilde{k}|$

Constraints of the setup

The explicit spatial dependence of $f(r)$ makes the commutation relation non-vanishing

$$\left[\tilde{k}, h(\tilde{k})\right] = \tilde{k}\frac{\partial f(r)}{\partial r}$$

Neglecting these contributions when solving the eigenvalue equation

$$\left|\frac{\kappa_2 - \kappa_1}{2l}\tilde{k}\right| \ll |\gamma|, \quad \left|\frac{\kappa_2 - \kappa_1}{2l}\right| \ll |\tau|$$

Metric for this particular model

$$ds^2 = -\left[1 - f^2(r)\right] dt^2 + 2f(r)drdt + dr^2$$

Horizon in terms of the model parameters!!

$$\frac{\kappa_1 + \kappa_2}{2} - \frac{\kappa_1 - \kappa_2}{2l}(r - r_1) = 1 - \frac{1}{4M_o}(r - 2M_o)$$

$$\kappa_3 = \frac{\kappa_1 + \kappa_2}{2} = 1, \; r_1 = 2M_o, \text{ and } \frac{\kappa_1 - \kappa_2}{2l} = \frac{1}{2r_1}$$

This bounds the NNN hopping parameters as $\kappa_1 > 1$, $\kappa_2 < 1$, and $\kappa_3 = 1$

Semiclassical emision rate and Bekenstein-Hawking entropy

One contribution comes from a particle tunneling outwards the interior of the BH, and the other one comes from an antiparticle tunneling into the BH, which is quantified in terms of the semiclassical emission rate

$$\longrightarrow \quad \Gamma \propto e^{-8\pi M_o|\gamma|} = e^{\Delta S_{\text{B-H}}}$$

Leads to the change in the Bekenstein-Hawking entropy

$$\longrightarrow \quad \Delta S_{\text{B-H}} = -8\pi M_o|\gamma| = -4\pi|\gamma|\frac{l}{\kappa_1 - \kappa_2}$$

M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).

Thermodynamics of the analogue black hole

In general the thermal entropy is

$$dM = T_{\mathrm{H}} dS \quad \text{where} \quad T_{\mathrm{H}} \propto \frac{l}{\kappa_1 - \kappa_2}$$

$$\Delta S = \int_{M_o}^{M_o - \omega} 8\pi M dM = -\frac{3\pi}{2}\left(\frac{l}{\kappa_1 - \kappa_2}\omega + \omega^2\right)$$

Equating $\Delta S_{B-H} = \Delta S$,

we get $-8\pi M|\gamma| = -3\pi(2M\omega + \omega^2)/2$

In terms of the lattice parameters

$$\omega^{\pm} = -\frac{l}{2(\kappa_1 - \kappa_2)} \pm \sqrt{\frac{l^2}{4(\kappa_1 - \kappa_2)^2} + \frac{8l|\gamma|}{3(\kappa_1 - \kappa_2)}}$$

Taking into account the constraints

$$(\kappa_1 - \kappa_2)/2l \ll 1 \longrightarrow \omega^+ = 8|\gamma|/3$$

Conclusions

The nH-TB model with gain/loss and non-reciprocal NNN hopping gives rise to exceptional cones with tunable tilt

Dirac-like operators that mimic light cones in curved spacetime and correspondence with Schwarzschild black holes

Emulation of a black hole horizon with interior, exterior, and interface regions

Tunneling across the analogue horizon, applying a semiclassical approach inspired by the Parikh-Wilczek method

Derivation of an emission rate governed by the change in B-H entropy

Relation between the emission frequency and the on-site gain/loss parameter $\omega^{+}=8|\gamma|/3$

Outlook

Our results provide a concrete example of how nH topological models can simulate aspects of BH physics, including horizon dynamics and Hawking-like radiation.

The implementation of our model in an experimental setup is of great interest because it can yield insight into the nature of BH evaporation, the emission rate due to tunneling processes near the event horizon, and the thermodynamics of such systems.

By measuring the local density of states in the model described here, a relation between the probability of the emission rate of particles and antiparticles, the Hawking temperature, and the thermal entropy can be obtained.

CENTRO DE CIENCIAS
DE BENASQUE
PEDRO PASCUAL

Gracias
Thanks

Wave packet evolution

Initial Gaussian:

$$|\Psi(0)\rangle = e^{-\frac{(r-r_O)^2}{2\sigma^2}}$$

σ is the characteristic length scale

$$|\tilde{k}_r^{\pm}| = \frac{1}{\sigma} = \left\{ \frac{\gamma f(r)}{1-f^2(r)} \pm \sqrt{\frac{\gamma^2}{\left[1-f^2(r)\right]^2}} \right\}^{-1}$$

For r inside the black hole

$$\sigma = \frac{\kappa_1 + 1}{\gamma}$$

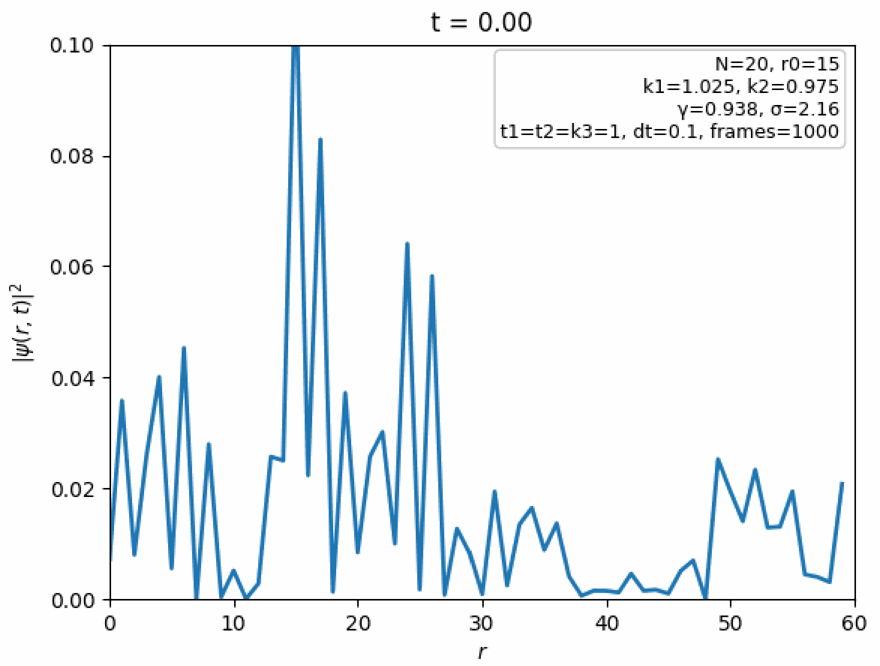
Time evolution:

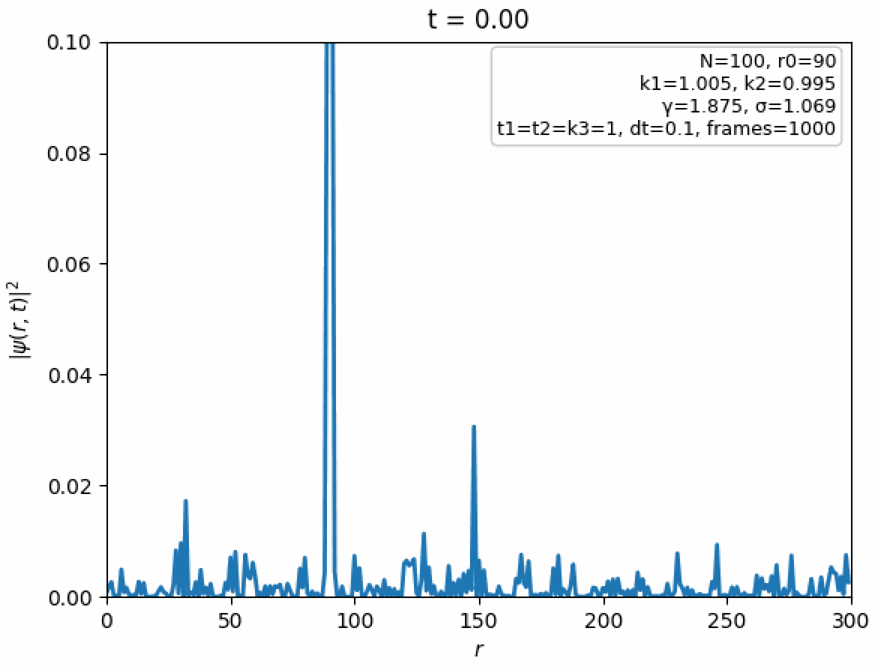
$$|\Psi(r,t)\rangle = \sum_{m=1}^{N} a_m \, |\phi_m^R\rangle \, e^{i\varepsilon_m t}$$

with $\quad a_m = \langle \phi_m^L | \Psi(0) \rangle$

Simulations

$$\kappa_1 = \frac{2r_h + 1}{2r_h} \qquad \kappa_2 = \frac{2r_h - 1}{2r_h} \qquad |\gamma| = \frac{3}{16}(r_h - r_o) \qquad \sigma = \frac{\kappa_1 + 1}{\gamma}$$





BH evaporation through Parikh-Wilczek method

particle tunneling outwards the interior of the BH, and the other one comes from an antiparticle tunneling into the BH, which is quantified in terms of the semiclassical emission rate

In the lattice model $\tilde{k}^{\pm}$ is the analogue of the conjugate momentum $p_r^{\pm}$

$$\tilde{k}_r^{\pm} = \frac{\gamma f(r)}{1 - f^2(r)} \pm \sqrt{\frac{\gamma^2}{[1 - f^2(r)]^2}}$$

In the analogue model the asymptotic infinity lies in $r_1 < r < r_2$

it is bound because of the PBC.

r_∞, such that $r_1 \ll r_\infty \ll r_2$, then $f(r_\infty) \simeq \kappa_2$

$$\tilde{k}_r^{\pm}(r_\infty) = \frac{\gamma \kappa_2}{1 - \kappa_2^2} \pm \left\{ \frac{\gamma^2}{[1 - \kappa_2^2]^2} \right\}^{1/2}$$

M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).

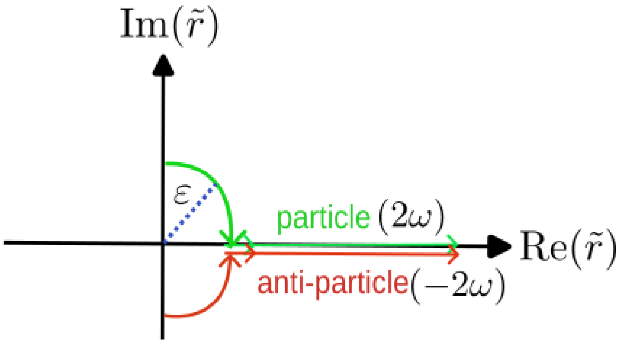
Semiclassical limit and emission rate

The classically forbidden process is obtained by computing the imaginary part of the action

$$\mathrm{Im}(S^{\pm}) = \mathrm{Im}\left(\int dr \tilde{k}_r^{\pm}\right) = I^{\pm}$$

$$I^{+} = \mathrm{Im}\left(\int_{r_1}^{r_1-2\omega} dr \frac{\gamma\left[1-\frac{\kappa_1-\kappa_2}{2l}(r-r_1)\right]}{\frac{1}{2r_1}(r-r_1)} + \left\{\frac{\gamma^2}{\left[\frac{1}{2r_1}(r-r_1)\right]^2}\right\}^{1/2}\right)$$

Contour deformation



$$I^{+} = \begin{cases} 2\pi r_1|\gamma| & \gamma < 0 \\ 0 & \gamma \geq 0. \end{cases}$$

$$I^{-} = \begin{cases} 2\pi r_1|\gamma| & \gamma > 0 \\ 0 & \gamma \leq 0. \end{cases}$$

Particles radiate out from the BH interior, causing the BH mass to decrease from M_o to $M_o - \omega$, which shrinks the horizon. The integration limits around the horizon are set as $r_{in} = r_1$ and $r_{out} = r_1 - 2\omega$, where $r_1 = 2M_o$

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The thermodynamics of the analogue black hole

The semiclassical emission rate

$$\Gamma \propto (\mathcal{A}^+ + \mathcal{A}^-)^2 = e^{-8\pi M_o|\gamma|} = e^{\Delta S_{\text{B-H}}}$$

where $\mathcal{A}^\pm = \exp(-I^\pm)$

Leads to the change in the Bekenstein-Hawking entropy

$$\Delta S_{\text{B-H}} = -8\pi M_o|\gamma| = -4\pi|\gamma|\frac{l}{\kappa_1 - \kappa_2}$$

In general the thermal entropy is

$dM = T_{\text{H}} dS$ where T_H is Hawking temperature

$$\Delta S = \int_{M_o}^{M_o-\omega} 8\pi M dM = -\frac{3\pi}{2}\left(\frac{l}{\kappa_1 - \kappa_2}\omega + \omega^2\right)$$

Equating $\Delta S_{B-H} = \Delta S$,
we get $-8\pi M|\gamma| = -3\pi(2M\omega + \omega^2)/2$

In terms of the lattice parameters

$$\omega^\pm = -\frac{l}{2(\kappa_1 - \kappa_2)} \pm \sqrt{\frac{l^2}{4(\kappa_1 - \kappa_2)^2} + \frac{8l|\gamma|}{3(\kappa_1 - \kappa_2)}}$$

Taking into account the constraints

$(\kappa_1 - \kappa_2)/2l \ll 1$ → $\omega^+ = 8|\gamma|/3$