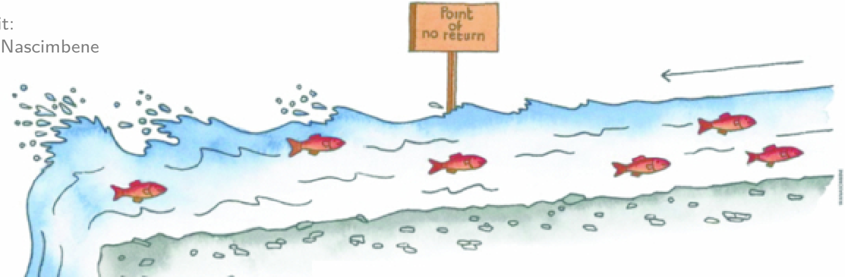
credit:
Yan Nascimbene



Quantum features of a BEC analog black hole

Benasque, january 2026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J. C. Maxwell (1870)

The recognition of the formal analogy between two systems leads to a knowledge of both, more profound than could be obtained by studying each system separately.

J. Bouveresse (1999)

Prodiges et vertiges de l'analogie

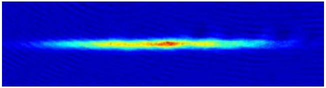
(Wonders and miracles of analogy)

Chez les déconstructionnistes, on fait de la théorie à peu près comme on fait de la poésie ou de la musique.

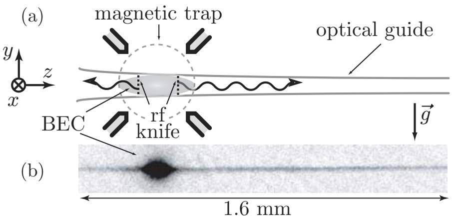
Among deconstruction philosophers theory is performed in much the same way as poetry or music.

State of play

	general relativity	**BEC analogues**
Hawking radiation ?	YES $\neq \|0\rangle$ for incoming and outgoing modes	YES for the same reason relies on kinematic features
Thermality ?	YES black hole thermodynamics	roughly YES (but disputed) if $\lambda \to \infty$ is sound-like
Entanglement	YES $\|BH\rangle = 2$ modes squeezed state for Schwarzchild observer	YES rich structure because 3 modes
quantum backreaction	YES conceptual and technical difficulties	YES technical difficulties



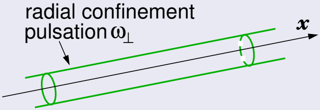
quasi-1D condensate
longitudinal size $\sim 10^2 \mu$m
transverse size $\sim 1 \mu$m



Guerin *et al.*, Phys. Rev. Lett. (2006)

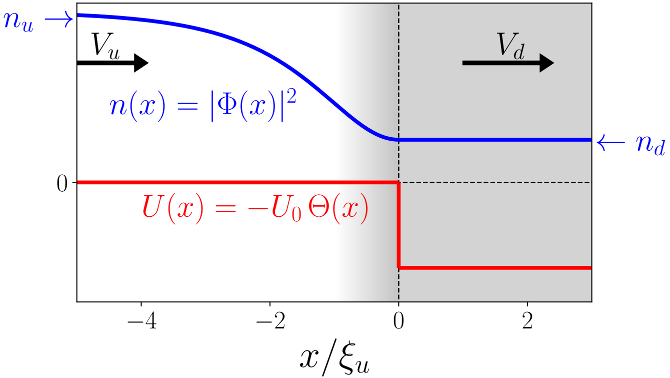
low T, quantum,
low c, 1D.

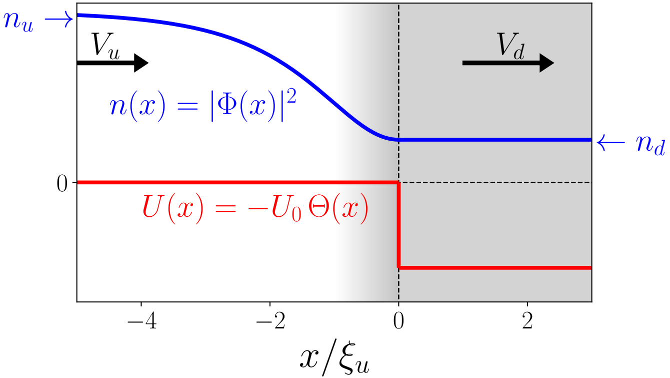
tight harmonic radial confinement :



$$V_\perp(\vec{r}_\perp) = \frac{1}{2} m \omega_\perp^2 r_\perp^2 \; .$$

$\rightarrow$ **1D model** : $\hat{\Psi}(x, t)$

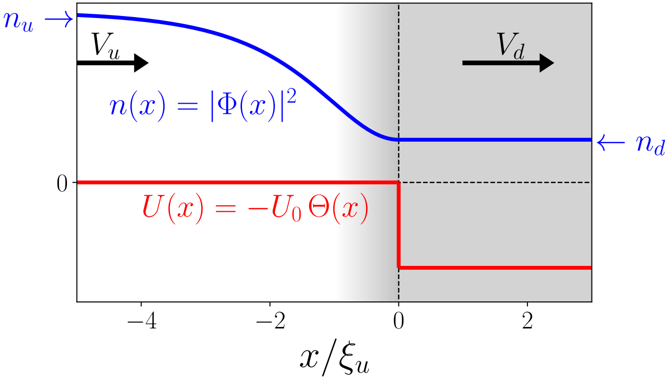

$n_u \rightarrow$
V_u
V_d
$n(x) = |\Phi(x)|^2$
$\leftarrow n_d$
0
$U(x) = -U_0\,\Theta(x)$
−4
−2
0
2
x/ξ_u



$$-\frac{\hbar^2}{2m}\Phi_{xx} + \left(U(x) + g|\Phi|^2\right)\Phi = \mu\Phi$$

$$\frac{V_d}{V_u} = \frac{n_u}{n_d} = \left(\frac{c_u}{V_u}\right)^2 = \frac{V_d}{c_d} = \left(\frac{c_u}{c_d}\right)^2$$

$$M_{u/d} = \frac{V_{u/d}}{c_{u/d}} : \qquad M_d = M_u^{-2}$$



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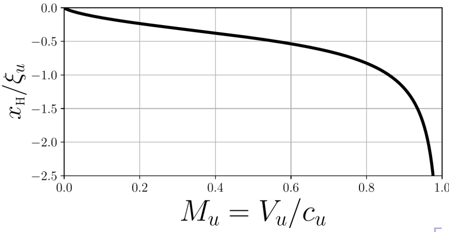
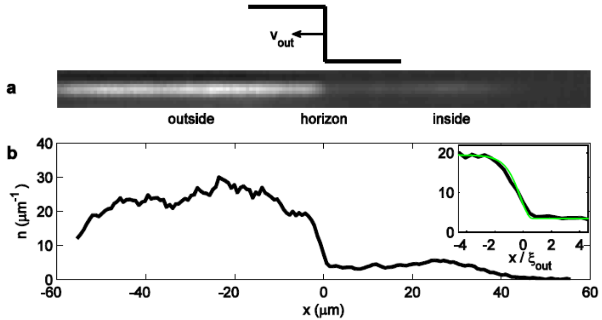
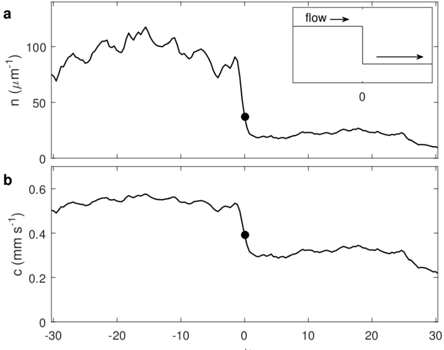
$$M_{u/d} = \frac{V_{u/d}}{c_{u/d}}: \qquad M_d = M_u^{-2}$$





2016

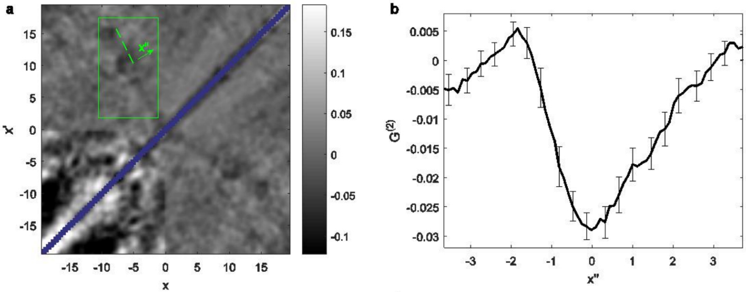


2019

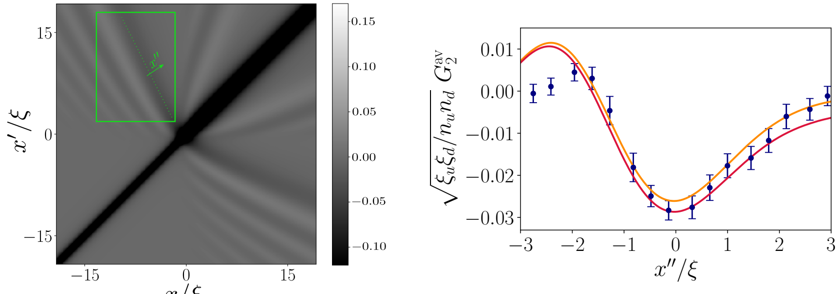
$$M_d = V_d / c_d$$

$$M_d|_{exp} = 2.9 = M_d|_{theo}$$

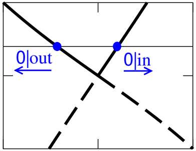
$$T_H/(gn_u)|_{exp} = 0.124 \quad (0.35\ \text{nK})$$
$$T_H/(gn_u)|_{theo} = 0.106$$

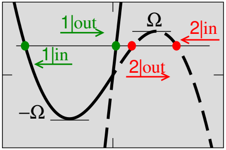


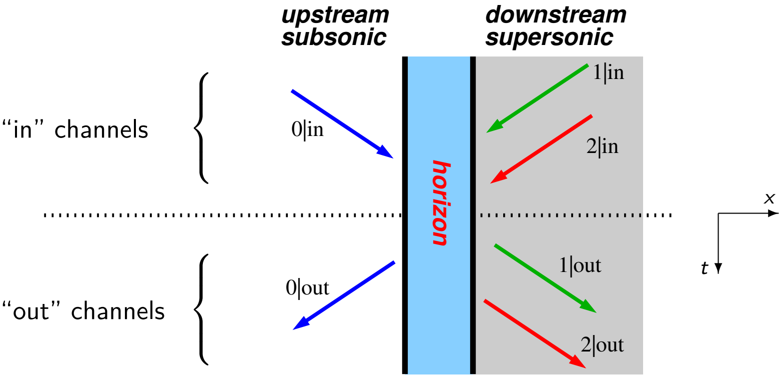
$T_H|_{exp} = 0.35$ nK $\quad T_H/(gn_u)|_{exp} = 0.124 \quad T_H/(gn_u)|_{theo} = 0.106 \quad \left(\frac{V_d}{c_d} = 2.9\right)$



Propagation channels

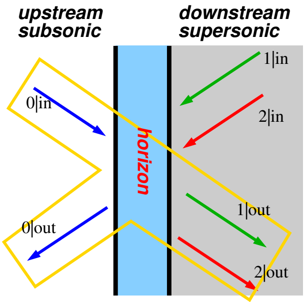




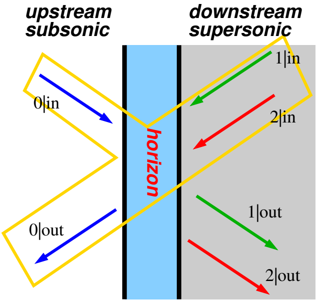


quantization modes: either 3 ingoing modes or 3 outgoing ones

example of ingoing mode: $\hat{b}_0$

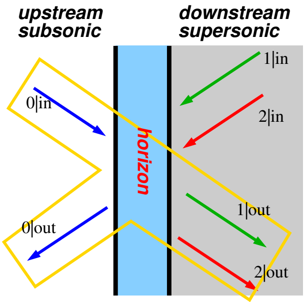


example of outgoing mode: $\hat{c}_0$

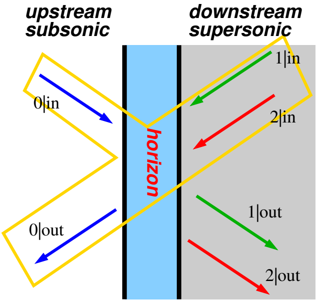


quantization modes: either 3 ingoing modes or 3 outgoing ones

example of ingoing mode: $\hat{b}_0$



example of outgoing mode: $\hat{c}_0$



$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix}_{out} = \begin{pmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix}_{in}$$

$$\begin{cases} \hat{c}_0 = \text{Hawking mode} \\ \hat{c}_1 = \text{Comoving=Spectator=Witness=Companion} \\ \hat{c}_2 = \text{Partner} \end{cases}$$

Bogoliubov modes are related through:

$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2{}^{\dagger} \end{pmatrix}_{out} = \begin{pmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2{}^{\dagger} \end{pmatrix}_{in}$$

Theoretical tool: 6×6 covariance matrices $\sigma^{(b)}$ and $\sigma^{(c)}$:

$$\boldsymbol{\sigma_{ij} = \tfrac{1}{2}\langle \hat{\xi}_i \hat{\xi}_j + \hat{\xi}_j \hat{\xi}_i \rangle - \langle \hat{\xi}_i \rangle \langle \hat{\xi}_j \rangle} \quad \text{with} \quad \xi = \sqrt{2}(\hat{q}_1, \hat{p}_1, \cdots, \hat{q}_n, \hat{p}_n)$$

$$\hat{q}_i = \frac{1}{\sqrt{2}}(\hat{b}_i + \hat{b}_i^{\dagger}) \ \text{ for } \sigma^{(b)} \quad \text{and} \quad \hat{q}_i = \frac{1}{\sqrt{2}}(\hat{c}_i + \hat{c}_i^{\dagger}) \ \text{ for } \sigma^{(c)}$$

at $T = 0$, $\boldsymbol{\rho = |0\rangle_{b\,b}\langle 0|}$, $\sigma^{(b)} = \mathbb{1}_6$ and $\sigma^{(c)}$ is 6×6: $\sigma^{(c)} = \mathscr{S}\sigma^{(b)}\mathscr{S}^T$
where $\mathscr{S}$ is a simplectic matrix: $\mathscr{S}^T \Omega \mathscr{S} = \Omega$ with $\Omega = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}$.

By another (local unitary) simplectic transform one can write

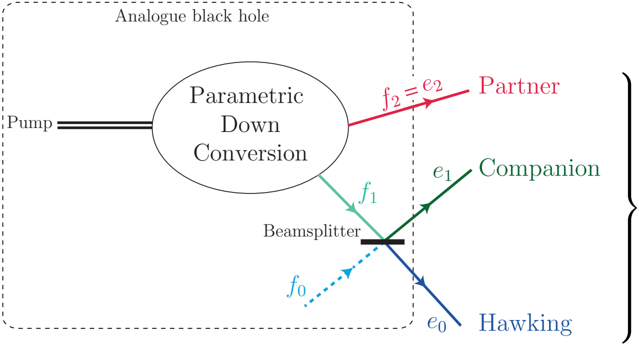
$$\mathcal{S}\,\sigma^{(c)}\,\mathcal{S}^{\mathrm{T}} = \mathbb{1}_2 \oplus \sigma_{\mathrm{sq}}, \quad \text{where} \quad \mathcal{S} = \mathcal{S}_{01} \oplus \mathcal{S}_2$$

where σ_{sq} is the 4×4 covariance matrix of a two mode squeezed state

Alternative approach to that of Busch and Parentani PRD 2014, Finazzi and Carusotto PRA 2014 and de Nova, Zapata and Sols NJP 2015.

$$|0\rangle_b = \exp\left\{ r_2 \left(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_1 \hat{f}_2 \right) \right\} |0\rangle_c \; , \text{ where } \quad \sinh^2 r_2 = |S_{22}|^2 - 1 = |S_{02}|^2 + |S_{12}|^2$$

$$\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_0 \\ \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} , \quad \cos\theta = \frac{|S_{02}|}{\sqrt{|S_{22}|^2 - 1}} \quad \text{and} \quad \begin{aligned} \hat{e}_0 &= (|S_{02}|/S_{02})\, \hat{c}_0 \\ \hat{e}_1 &= (|S_{12}|/S_{12})\, \hat{c}_1 \\ \hat{e}_2 &= (S_{22}/|S_{22}|)\, \hat{c}_2 \end{aligned}$$



3-mode Gaussian state

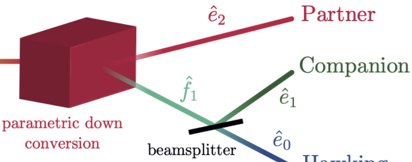
see also Agullo, Brady and Kranas PRL 2022

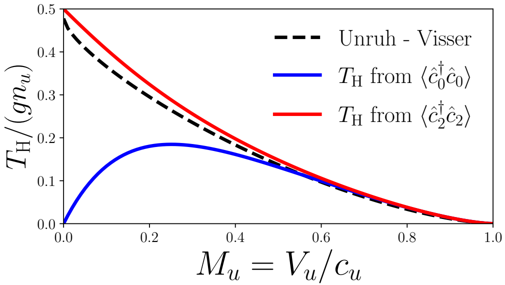
$$|0\rangle_b = \exp\left\{ r_2 \left(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_1 \hat{f}_2 \right) \right\} |0\rangle_c$$

Hawking temperature

$$\langle \hat{f}_2^\dagger \hat{f}_2 \rangle = \langle \hat{f}_1^\dagger \hat{f}_1 \rangle = \frac{1}{\exp(\hbar\omega / T_{eff}(\omega)) - 1}$$

$$T_H = \lim_{\omega \to 0} T_{eff}(\omega) = \lim_{\omega \to 0} \hbar\omega |S_{22}(\omega)|^2$$





Gray body factor

$$\Gamma_0 = \lim_{\omega \to 0} \underbrace{\cos^2\theta}_{\text{transmission}} = \frac{4M_u}{(1+M_u)^2}$$

← identical to Anderson, Fabbri, Balbinot PRD 2015 and Linder, Schützhold, Unruh PRD 2016

non monotonous (qualitative: Yes/No)

– Cauchy Schwarz criterion
– PPT criterion (Generalized Peres-Horodecki parameter)

de Nova, Sols, Zapata PRA 2014
Busch, Carusotto, Parentani PRA 2014
Busch Parentani, PRD 2014
Finazzi, Carusotto, PRA 2014
Steinhauer PRD 2015
Boiron, Fabbri, Larré, Pavloff, Westbrook, PRL 2015
de Nova, Sols, Zapata, NJP 2015
Coutant, Weinfurtner, PRD 2018

entanglement monotones (quantitative)

–entanglement entropy (pure state) generalizes to entanglement of formation
–(squared) logarithmic negativity generalizes to **Gaussian contangle**[a] which verifies the monogamy inequalities

$$G^{(i|jk)} \geq G^{(i|j)} + G^{(i|k)}$$

[a] Adesso, Illuminati, NJP 2006

Giovanazzi, PRL 2011
Horstmann, Schützhold, Reznik, Fagnocchi, Cirac, NJP 2011
Bruschi, Friis, Fuentes, Weinfurtner, NJP 2013
Jacquet, König, SciPost 2020
Nambu, Osawa, PRD 2021

$\rho_{(i)}$: single mode reduced density matrix.
Purity= $tr[\rho_{(i)}^2]$
local mixedness $a_i = 2\langle \hat{c}_i^\dagger \hat{c}_i \rangle + 1 = (\text{purity})^{-1}$

bipartite entanglement (say, between 0 and 2)

–trace out mode 1
–partial transpose $\sigma^{\rm PT} = \Lambda\,\sigma\,\Lambda\,,\quad$ with $\quad \Lambda = \sigma_z \oplus \mathbb{1}_2$
– compute the simplectic eigenvalues $\nu^{\rm PT}_{(\pm)}$
– separability: $\rho^{\rm PT} \geq 0 \iff$ all $\nu^{\rm PT} \geq 1$
– $\nu^{\rm PT}_+ > 1$, hence criterion for non-separability: $\nu^{\rm PT}_- < 1$

$$\nu^{\rm PT}_- = \frac{a_0 + a_2}{2} - \sqrt{\left(\frac{a_0 - a_2}{2}\right)^2 + 4\,|\langle \hat{c}_0\,\hat{c}_2\rangle|^2}$$

The corresponding value of the "**PPT measure**" is $\Lambda^{(0|2)} = 1 - \nu^{\rm PT}_-$

recipe for continuous variables — Martin & Vennin, PRA 2016

(1) define a pseudo-spin operator $\hat{\boldsymbol{S}}^{(i)}(\omega)$ for each mode ($i = 0, 1, 2,\ \omega \geq 0$)

(2) use this pseudo-spin to write a two-mode CHSH operator $\hat{\mathcal{B}}^{(i|j)}$

(3) show that the expectation value $\langle \hat{\mathcal{B}}^{(i|j)} \rangle$ may violate a Bell inequality.

(1) Gour-Khanna-Mann-Revzen pseudo-spins — Phys. Lett. A 2004

$|n\rangle$: eigenvalue of $\hat{c}_i^\dagger(\omega)\hat{c}_i(\omega)$ associated with eigenvalue n.
$|q\rangle$: eigenvector of $\hat{q}_i(\omega) = \frac{1}{\sqrt{2}}\left(\hat{c}_i(\omega) + \hat{c}_i^\dagger(\omega)\right)$ associated with eigenvalue q.

$\hat{S}_z^{(i)} = \sum_n \left(|2n\rangle\langle 2n| - |2n+1\rangle\langle 2n+1|\right) = \int dq\, |q\rangle \langle -q|$
$\hat{S}_y^{(i)} = \cdots \qquad \hat{S}_x^{(i)} = \cdots$

$$\left[\hat{S}_x^{(i)}, \hat{S}_y^{(i)}\right] = 2i\hat{S}_z^{(i)}$$

$\hat{\boldsymbol{S}} \cdot \boldsymbol{n}$ squares to unity for any normalized vector $\boldsymbol{n} \rightsquigarrow$ eigenvalues ± 1.

Beware: infinitely degeneracy! all $|2n\rangle$ ($|2n+1\rangle$) are eigenvectors of $\hat{S}_z^{(i)}$ with eigenvalue $+1$ (-1)

Proof of nonlocality

(2) CHSH operator

choose 4 measurement directions (normalized vectors $\boldsymbol{a}$, $\boldsymbol{a}'$, $\boldsymbol{b}$, $\boldsymbol{b}'$) and define

$$\hat{\mathscr{B}}^{(i|j)}(\omega) = (\boldsymbol{a} + \boldsymbol{a}') \cdot \hat{\boldsymbol{S}}^{(i)} \otimes \boldsymbol{b} \cdot \hat{\boldsymbol{S}}^{(j)} + (\boldsymbol{a} - \boldsymbol{a}') \cdot \hat{\boldsymbol{S}}^{(i)} \otimes \boldsymbol{b}' \cdot \hat{\boldsymbol{S}}^{(j)} \tag{1}$$

Local realism (λ : hidden variable)

the outcome of the measurement of $\boldsymbol{a} \cdot \hat{\boldsymbol{S}}^{(i)} \otimes \boldsymbol{b} \cdot \hat{\boldsymbol{S}}^{(j)}$ is the product

- of the measurement $M^{(i)}(\lambda, \boldsymbol{a}) = \pm 1$ of $\boldsymbol{a} \cdot \hat{\boldsymbol{S}}^{(i)}$
- with the measurement $M^{(j)}(\lambda, \boldsymbol{b}) = \pm 1$ of $\boldsymbol{b} \cdot \hat{\boldsymbol{S}}^{(j)}$

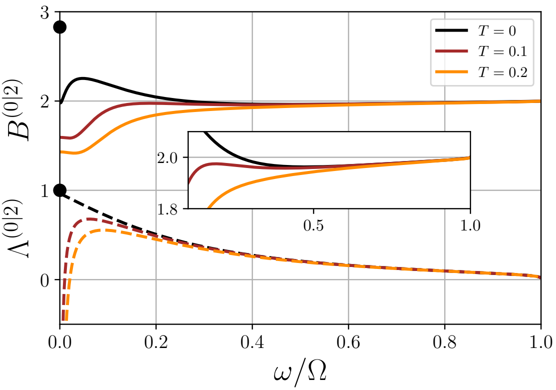
There are only two possible cases: either $M^{(i)}(\lambda, \boldsymbol{a})$ and $M^{(j)}(\lambda, \boldsymbol{a}')$ are equal (their difference cancels and they sum to ± 2) either they are opposite (their sum cancels and their difference is ± 2).

It follows from (1) that such a measurement of $\hat{\mathscr{B}}^{(i|j)}$ yields a result ± 2.

$$\text{local realism} \rightsquigarrow \text{Bell inequality} : \quad -2 \leq \langle \hat{\mathscr{B}}^{(i|j)}(\omega) \rangle \leq 2$$

(3) Violation of Bell inequality

$$B^{(i|2)}(\omega) \equiv \max_{a,a',b,b'} \left\langle \hat{\mathscr{B}}^{(i|2)}(\omega) \right\rangle \qquad (i = 0, 1)$$

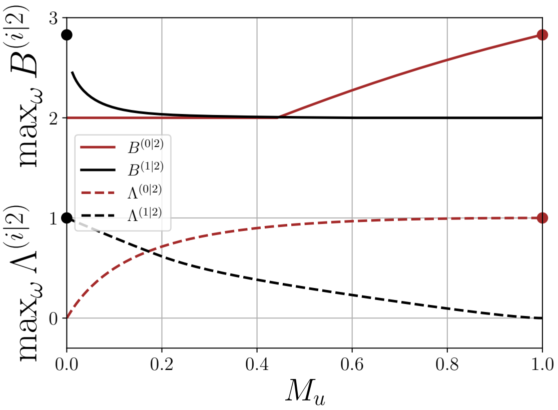


0 = Hawking
1 = Companion
2 = Partner

$M_u = 0.59$

(3) Violation of Bell inequality

$$B^{(i|2)}(\omega) \equiv \max_{\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}'} \left\langle \hat{\mathscr{B}}^{(i|2)}(\omega) \right\rangle \qquad (i = 0, 1)$$



0 = Hawking
1 = Companion
2 = Partner

$T = 0$

Experimental input

assumption: Gaussian state

$$\Lambda^{(0|2)} = 1 - \frac{a_0 + a_2}{2} + \sqrt{\left(\frac{a_0 - a_2}{2}\right)^2 + 4|\langle \hat{c}_0\, \hat{c}_2 \rangle|^2}$$

$$B^{(0|2)} = 2\sqrt{\frac{4}{\pi^2}\arctan^2\left(\frac{2|\langle \hat{c}_0 \hat{c}_2\rangle|}{\sqrt{A_{02}}}\right) + \frac{1}{A_{02}^2}}\,.$$

where

$$a_i = 2\langle \hat{c}_i^\dagger\, \hat{c}_i \rangle + 1 \qquad A_{02} = a_0 a_2 - 4\left|\langle \hat{c}_0 \hat{c}_2 \rangle\right|^2,$$

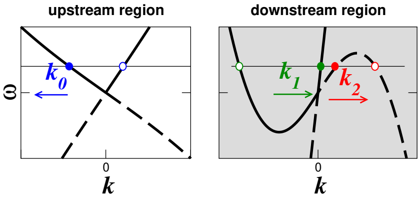
$$\begin{pmatrix} \hat{c}_0 \\ \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix}_{out} = \begin{pmatrix} S_{0,0} & S_{0,1} & S_{0,2} \\ S_{1,0} & S_{1,1} & S_{1,2} \\ S_{2,0} & S_{2,1} & S_{2,2} \end{pmatrix} \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2^\dagger \end{pmatrix}_{in}$$

Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega)\hat{c}_2(\omega)\rangle = S_{0,2}(\omega)S^*_{2,2}(\omega)$$

$$\propto \int_{-L_u}^{0} dx \int_{0}^{L_d} dx'\; e^{-i(k_0(\omega)x + k_2(\omega)x')}\, G_2(x, x')$$

G_2: density-density correlation function



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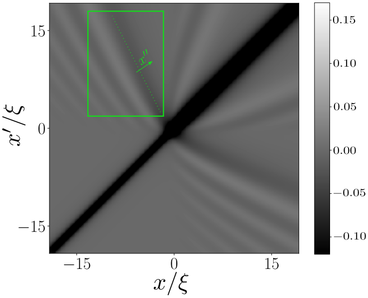

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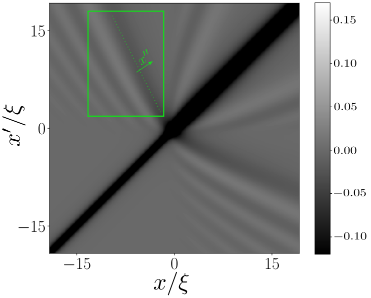
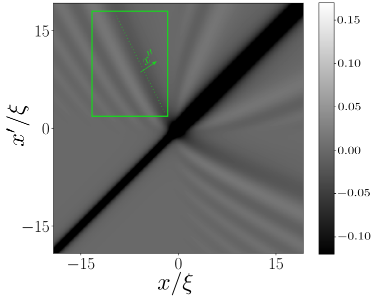


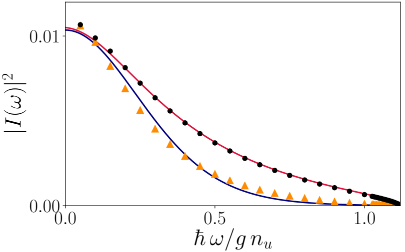
de Nova, Sols, Zapata, NJP 2015:

$$\frac{L_u}{|\partial\omega/\partial q(k_0)|} = \frac{L_d}{\partial\omega/\partial q(k_2)}$$

Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega)\hat{c}_2(\omega)\rangle = S_{0,2}(\omega)S^*_{2,2}(\omega)$$
$$\propto \int_{-L_u}^{0} dx \int_0^{L_d} dx'\, e^{-i(k_0(\omega)x + k_2(\omega)x')} G_2(x,x')$$





blue curve: fully thermal radiation

- $|S_{0,2}(\omega)|^2 = n_{th}$ at $T_H|_{exp}$
- discard mode 1
- $|\langle \hat{c}_0\hat{c}_2\rangle|^2 \simeq n_{th}(1+n_{th})$

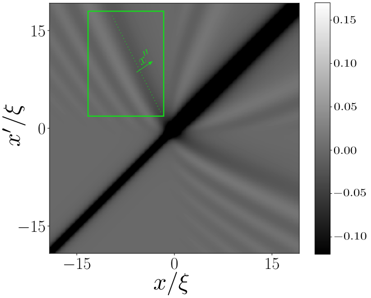
de Nova, Sols, Zapata, NJP 2015:

$$\frac{L_u}{|\partial\omega/\partial q(k_0)|} = \frac{L_d}{\partial\omega/\partial q(k_2)}$$

Steinhauer, PRD 2015:

$$\langle \hat{c}_0(\omega)\hat{c}_2(\omega)\rangle = S_{0,2}(\omega)S^*_{2,2}(\omega)$$

$$\propto \int_{-L_u}^{0} dx \int_{0}^{L_d} dx'\, e^{-i(k_0(\omega)x+k_2(\omega)x')}\, G_2(x,x')$$



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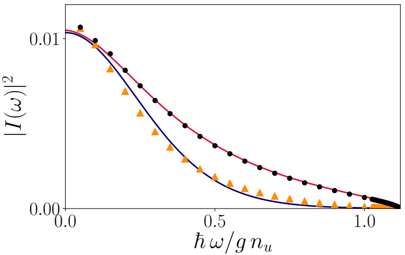

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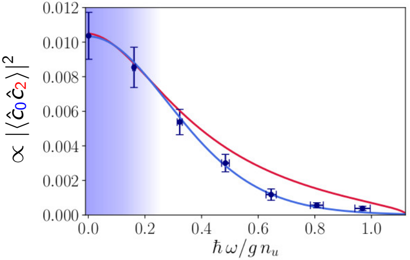


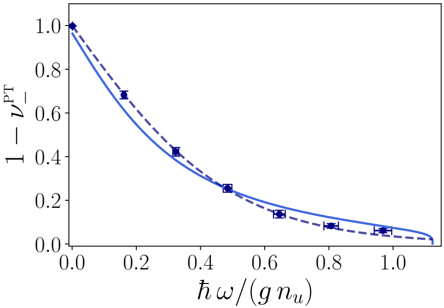
Assume:

– that the state is Gaussian and that $T = 0$

–that the occupation of the companion is negligeable (then $\langle \hat{c}_0^\dagger \hat{c}_0 \rangle = \langle \hat{c}_2^\dagger \hat{c}_2 \rangle$)

–that $\langle \hat{c}_0 \hat{c}_2 \rangle$ is accurately extracted from the density-density correlation signal

$$\nu_-^{\mathrm{PT}} \simeq \sqrt{1 + 4|\langle \hat{c}_0 \hat{c}_2 \rangle|^2} - 2\,|\langle \hat{c}_0 \hat{c}_2 \rangle|$$



points from de Nova *et al.* Nature 2019

solid line: waterfall model

dashed line:
$|\langle \hat{c}_0 \hat{c}_2 \rangle|^2 \simeq n_{th}(1 + n_{th})$ at $T_H|_{exp}$

$$\hat{S}_y^0 \otimes \hat{S}_y^1 \otimes \hat{S}_z^2 |0_{\omega=0}\rangle^{\rm in} = +|0_{\omega=0}\rangle^{\rm in}$$
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$|x_\pm\rangle$ eigenvector of $\hat{S}_x$ with eigenvalue ± 1

GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|x_+ x_- x_-\rangle + |x_- x_+ x_+\rangle\right)$$

$$\hat{S}_y^0 \otimes \hat{S}_y^1 \otimes \hat{S}_z^2 |0_{\omega=0}\rangle^{\text{in}} = +|0_{\omega=0}\rangle^{\text{in}}$$
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$|x_\pm^{\boldsymbol{n}}\rangle$ eigenvector of $\hat{S}_x$ with eigenvalue ± 1 $\quad \boldsymbol{n} \in \mathbb{N}$

degenerate GHZ state

$$|0_{\omega=0}\rangle^{\text{in}} = \sum_{n,l,m} C_{n,l,m} \left(|x_+^n x_-^l x_-^m\rangle + |x_-^n x_+^l x_+^m\rangle \right)$$

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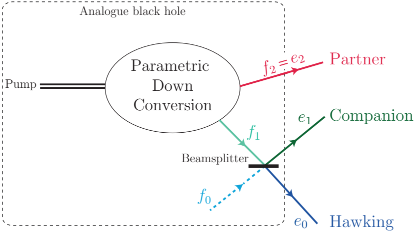
effect of degeneracy

GHZ state whose entanglement resists partial tracing !

Braunstein, van Hoock 2003

Adesso, Illuminati 2006

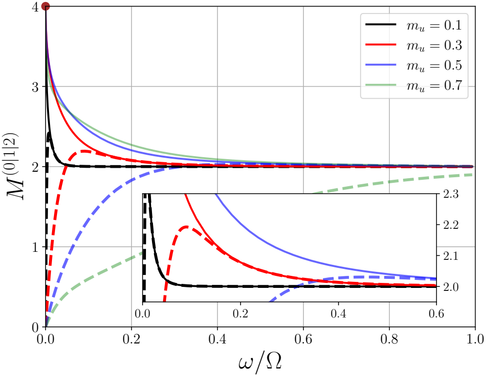


Mermin parameter

$$M^{(0|1|2)}(\omega) \equiv \max_{\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}',\boldsymbol{c},\boldsymbol{c}'} \left|\left\langle \hat{\mathscr{M}}^{(0|1|2)}(\omega)\right\rangle\right|$$

where

$$\hat{\mathscr{M}}^{(0|1|2)}(\omega) = -\boldsymbol{a}\cdot\hat{\boldsymbol{S}}^0 \otimes \boldsymbol{b}\cdot\hat{\boldsymbol{S}}^1 \otimes \boldsymbol{c}\cdot\hat{\boldsymbol{S}}^2 + \boldsymbol{a}\cdot\hat{\boldsymbol{S}}^0 \otimes \boldsymbol{b}'\cdot\hat{\boldsymbol{S}}^1 \otimes \boldsymbol{c}'\cdot\hat{\boldsymbol{S}}^2 + \boldsymbol{a}'\cdot\hat{\boldsymbol{S}}^0 \otimes \boldsymbol{b}\cdot\hat{\boldsymbol{S}}^1 \otimes \boldsymbol{c}'\cdot\hat{\boldsymbol{S}}^2 + \boldsymbol{a}'\cdot\hat{\boldsymbol{S}}^0 \otimes \boldsymbol{b}'\cdot\hat{\boldsymbol{S}}^1 \otimes \boldsymbol{c}\cdot\hat{\boldsymbol{S}}^2$$



solid: $T = 0$

dashed: $T = 0.1\, g n_u$

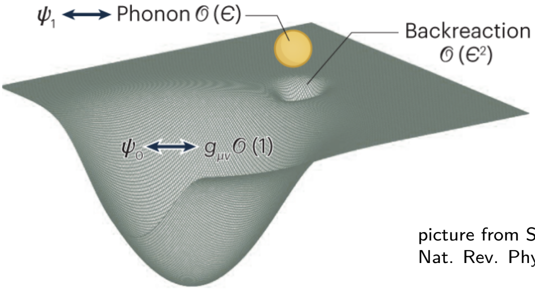
Quantum backreaction

Schützhold, Uhlmann, Xu, Fischer, PRD 2005
Liberati, Tricella, Trombettoni, Appl. Sci. 2020
Baak, Ribeiro, Fischer, PRA 2022
Pal, Fischer, PRD 2024

- leading order: Gross-Pitaevskii mean-field (classical background).
- next order: Bogoliubov (quantum fluctuations).

☞ backreaction: modification of the background flow caused by quantum fluctuations (beyond mean-field corrections + impact of Hawking radiation).

- Next order: modification of Hawking radiation.



picture from S.L. Braunstein *et al.* Nat. Rev. Phys. **5**, 612 (2023)

Perturbative backreaction

$$i\partial_t \hat{\Psi}(\boldsymbol{r}, t) = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\hat{\Psi} + \left(U(\boldsymbol{r}) + g\hat{\Psi}^\dagger\hat{\Psi} - \mu\right)\hat{\Psi}, \qquad (1)$$

broken symmetry $\quad \hat{\Psi} = \underbrace{\langle\hat{\Psi}\rangle}_{\Phi(\boldsymbol{r},t)} + \hat{\psi}(\boldsymbol{r}, t) \quad$ with $\quad \langle\hat{\psi}\rangle = 0. \qquad (2)$

solve perturbatively $\langle(1)\rangle$ and $(1) - \langle(1)\rangle$ for the fields Φ and $\hat{\psi}$.

- lowest order: neglect quantum contributions in $\langle(1)\rangle \rightsquigarrow \Phi_{GP}$
- next order: first order in $\hat{\psi}$ in $(1) - \langle(1)\rangle \rightsquigarrow$ Bogoliubov

☞ next order: include averages of Bogoliubov fields at second order such as $\langle\hat{\psi}^\dagger\hat{\psi}\rangle$ and $\langle\hat{\psi}\hat{\psi}\rangle$ in $\langle(1)\rangle \rightsquigarrow$ modification of $\Phi = \Phi_{GP} + \delta\Phi$.

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This fails in 1D (Hohenberg-Mermin-Wagner infrared divergences).
Popov 1972 approach: replace (2) by

$$\hat{\Psi} = \exp(i(\Theta + \hat{\theta}))\sqrt{\rho + \delta\hat{\rho}} \quad \text{with} \quad \langle\hat{\theta}\rangle = 0 = \langle\delta\hat{\rho}\rangle. \tag{3}$$

the classical field Φ is not identical to $\varphi = \sqrt{\rho}\exp(i\Theta)$:

$$\Phi = \varphi\left(1 + \tfrac{i}{2}\langle\hat{\theta}\hat{\eta}\rangle - \tfrac{1}{2}\langle\hat{\theta}\hat{\theta}\rangle - \tfrac{1}{8}\langle\hat{\eta}\hat{\eta}\rangle + \cdots\right) \quad \text{where} \quad \hat{\eta} = \delta\hat{\rho}/\rho.$$

$$\Theta = \Theta_{GP} + \delta\Theta,\ \rho = \rho_{GP} + \delta\rho,\ \boldsymbol{V}_{GP} = \frac{\hbar}{m}\boldsymbol{\nabla}\Theta_{GP}$$

$$\hbar(\partial_t + \boldsymbol{V}_{GP}\cdot\boldsymbol{\nabla})\left(\delta\Theta - \tfrac{1}{2}\Re\langle\hat{\eta}\hat{\theta}\rangle\right) + g(\delta\rho + \tfrac{1}{2}\rho_{GP}g^{(2)}) = \frac{-\hbar^2}{4m\rho_{GP}}\boldsymbol{\nabla}\cdot\left[\rho_{GP}\boldsymbol{\nabla}\left(\frac{\delta\rho}{\rho_{GP}} - \langle\hat{\theta}^2\rangle + \frac{\delta(\mathbf{0})}{4\rho_{GP}} - \frac{g^{(2)}}{4}\right)\right]. \quad (1)$$

$g^{(2)} = \langle\hat{\eta}^2\rangle - \delta(\mathbf{0})/\rho_{GP}$ where $\hat{\eta} = \delta\hat{\rho}/\rho_{GP}$.

$$\partial_t\delta\rho + \boldsymbol{\nabla}\cdot\delta\boldsymbol{J} = 0, \quad \text{where} \quad \delta\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{V}_{GP}\delta\rho + \rho_{GP}\delta\boldsymbol{V} + \langle\hat{\boldsymbol{j}}\rangle \quad (2)$$

$$\delta\boldsymbol{V} = \frac{\hbar}{m}\boldsymbol{\nabla}\delta\Theta, \qquad \hat{\boldsymbol{v}} = \frac{\hbar}{m}\boldsymbol{\nabla}\hat{\theta}, \qquad \hat{\boldsymbol{j}} = \tfrac{1}{2}(\hat{\rho}\hat{\boldsymbol{v}} + \hat{\boldsymbol{v}}\hat{\rho}), \qquad \langle\hat{\boldsymbol{j}}\rangle \overset{here}{=} \Re\langle\rho_{GP}\hat{\eta}\hat{\boldsymbol{v}}\rangle$$

$\langle\hat{\theta}^2\rangle - \frac{1}{4}\delta(\mathbf{0})/\rho_{GP} + \frac{1}{4}g^{(2)} = \langle\hat{\psi}^\dagger\hat{\psi}\rangle/\rho_{GP}$ is infrared divergent but harmless

Anderson, Fabbri, Balbinot, Phys. Rev. D 2015

- Uniform and stationary configuration. Density $\rho = \rho_{GP} + \delta\rho$, $\boldsymbol{V}_{GP} = 0$

$$-\hbar\partial_t\Theta = \mu = g\rho + \tfrac{1}{2} g\rho_{GP} g^{(2)}, \qquad \text{Hartree-Fock-Bogoliubov result}$$

$\rightsquigarrow$ Lee-Huang-Yang expression: $\mu \overset{3D}{=} g\rho + \dfrac{4g}{3\pi^2\xi^3}$ (Phys. Rev. 1957)

where $\xi = \hbar/\sqrt{mg\rho}$

- 1D speed of sound

$$\mu \overset{1D}{=} g\rho - \frac{g}{\pi\xi}, \quad mc^2 = \rho\frac{\partial\mu}{\partial\rho} = g\rho - \frac{g}{2\pi\xi}.$$

cf. Mora & Castin, Phys. Rev. A 2003, Lieb, Phys. Rev. 1963.

corrective term, **1D small parameter :** $(\rho\xi)^{-1} \approx 0.02$

Consider a stationary blackhole configuration (say, "waterfall").
Asymptotic regions $x \to \mp\infty$, $\alpha = u/d$

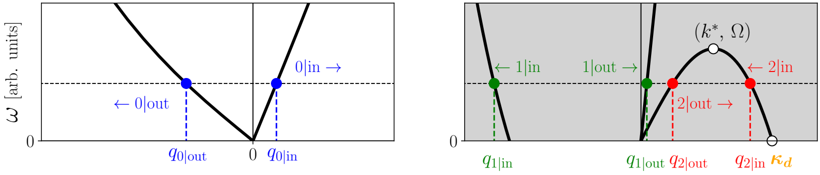
$$-\frac{\hbar^2}{4m}\partial_x^2 \delta\rho + m\left[(c_{GP}^\alpha)^2 - (V_{GP}^\alpha)^2\right]\delta\rho = S_\alpha, \qquad (3)$$

where S_α is a constant source term. Solutions are of the form

$$\delta\rho(x) = \begin{cases} \delta\rho_u + \mathcal{A}_u \exp(\kappa_u x) & \text{when } x \to -\infty, \\ \delta\rho_d + \mathcal{A}_d \sin(\kappa_d x + \phi) & \text{when } x \to +\infty, \end{cases} \qquad (4)$$

where

$$\delta\rho_\alpha = \frac{S_\alpha}{m\left[(c_{GP}^\alpha)^2 - (V_{GP}^\alpha)^2\right]}, \quad \textit{and} \quad \kappa_\alpha = \frac{2m}{\hbar}\left|(c_{GP}^\alpha)^2 - (V_{GP}^\alpha)^2\right|^{1/2}. \qquad (5)$$



Conclusion

BECs offer interesting prospects to observe analogous Hawking radiation

general perspective : **quantum effects** with **nonlinear** matter **waves**

BEC experiments (Steinhauer) provides indisputable evidences

- of the occurrence of a sonic horizon.
- of the associated acoustic Hawking radiation.

The thermality of analogous Hawking radiation is still a matter of debate.

- the quantum nature of the Hawking process
  - ☺ → can be directly addressed by present time experimental setups
  - ☹ → no indisputable result yet

- Gondret *et al.* PRL 2025 : proof of entanglement in pairs produced *via* dynamical Casimir effect

BECs offer interesting prospects to observe analogous Hawking radiation

general perspective : **quantum effects** with **nonlinear** matter **waves**

Quantum backreaction

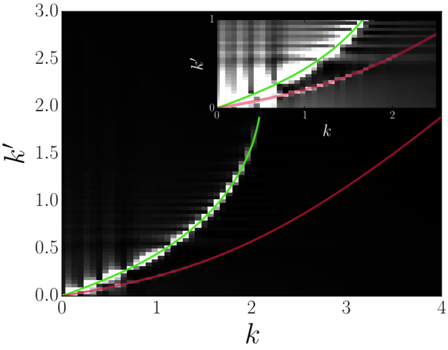
in BEC we have a "theory of everyting": Heisenberg equation for the field operator.
Bogoliubov procedure $\leadsto$ separation of quantum fluctuations from a classical background flow.

- ☺ recent theoretical progresses in 1D
- ☺ downstream undulations of quantum origin (stored information?)
- ☹ technical difficulty : proper inclusion of zero modes
- ☹ + ☺ weak effect

Thank you for your attention

Fourier transform of G_2

$$\textit{compute} \quad G_2(k,k') = \int_{-L_u}^{0} dx \int_0^{L_d} dx'\, e^{-i(kx+k'x')} G_2(x,x')$$

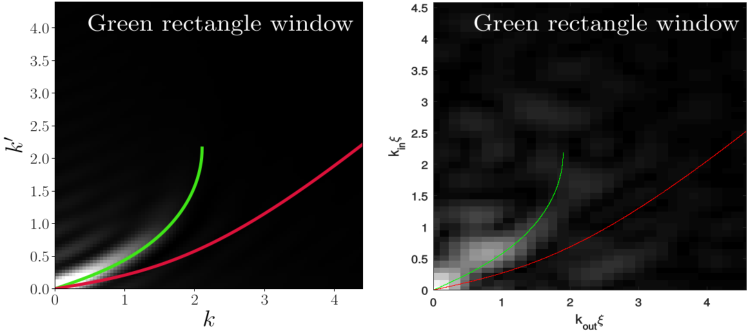


Fourier transform of G_2 computed over the upper left region (L_u and $L_d \to \infty$).

The green curve corresponds to $k = k_0$ and $k' = k_2$ (Hawking-Partner).
The red curve corresponds to $k = k_0$ and $k' = k_1$ (Hawking-Companion).

The inset shows the same plot, but overexposed to enhance the Hawking-Companion correlation signal.

$$\text{compute} \quad G_2(k, k') = \int_{-L_u}^{0} dx \int_{0}^{L_d} dx' \, e^{-i(kx+k'x')} G_2(x, x')$$

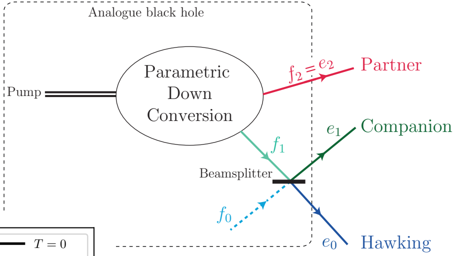


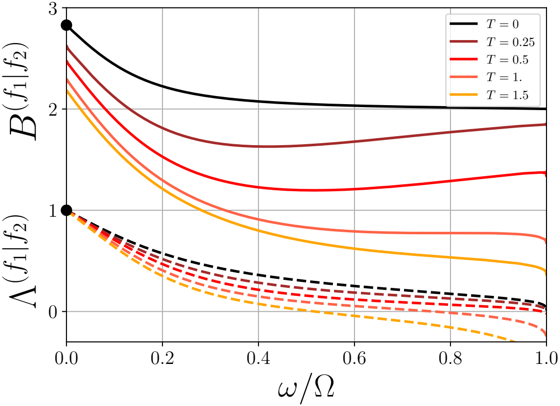
Fourier transform of G_2 computed over the green rectangle. Left: theoretical result, right: experimental result from Muñoz de Nova et al., Nature 2019. The green curve corresponds to $k = k_0$ and $k' = k_2$ (Hawking-Partner). The red curve corresponds to $k = k_0$ and $k' = k_1$ (Hawking-Companion).

$$B^{(f_1|f_2)} = 2\sqrt{1+\frac{4}{\pi^2}\arctan^2[\sinh(2r_2)]}$$

$$\Lambda^{(f_1|f_2)} = 1-\exp(-2r_2)$$

$$r_2(\omega) = \langle \hat{c}_2^\dagger \hat{c}_2 \rangle$$





$$|0\rangle_b = \exp\left\{r_2\left(\hat{f}_1^\dagger \hat{f}_2^\dagger - \hat{f}_1 \hat{f}_2\right)\right\}|0\rangle_c$$

Monogamy inequality : $G^{(i|jk)} \geq G^{(i|j)} + G^{(i|k)}$

☞ $(i|jk)$ pure Gaussian state: squared logarithmic negativity $(\ln ||\rho^{\mathrm{PT}}||)^2$

$$G^{(i|jk)} = 4\, r_i^2 = 4\, \mathrm{arsinh}^2 \left(\sqrt{\langle \hat{c}_i^\dagger \hat{c}_i \rangle} \right)$$

☞ $\rho^{(i|j)}$ mixed (k has been traced out) $G^{(i|j)}$: Gaussian contangle

$$G^{(0|1)} = 0 \qquad G^{(j|2)} = \mathrm{arsinh}^2 \left(\frac{2 |\langle \hat{c}_j \hat{c}_2 \rangle|}{1 + \langle \hat{c}_k^\dagger \hat{c}_k \rangle} \right)$$

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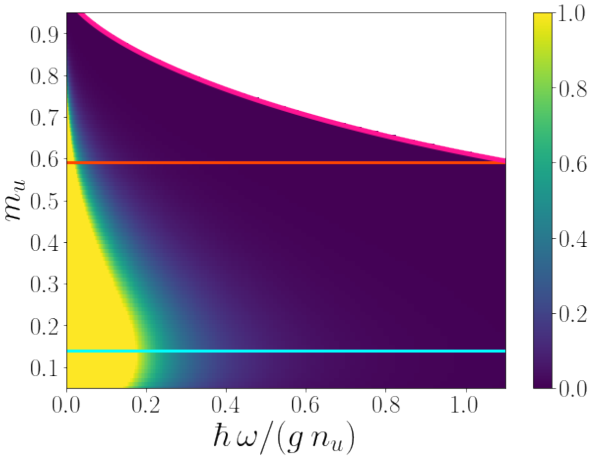
residual contangle:

$$G_{res} = \min \left(G^{(i|jk)} - G^{(i|j)} - G^{(i|k)} \right)$$



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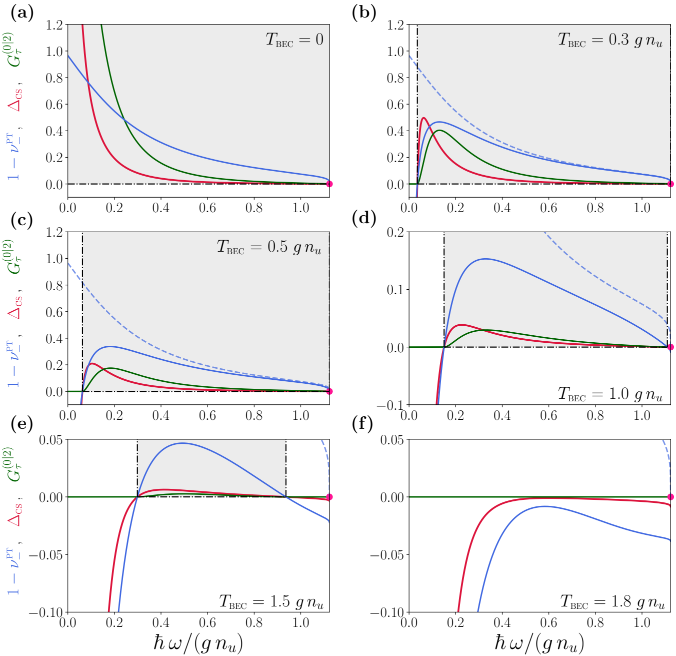
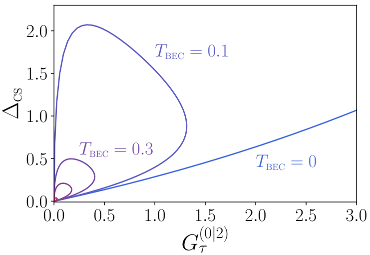
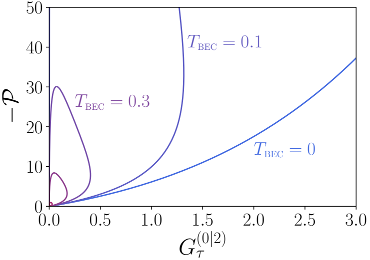
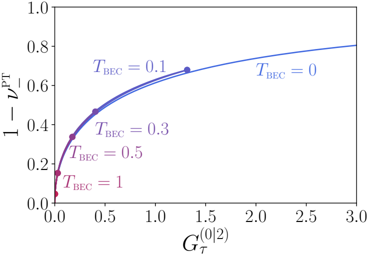
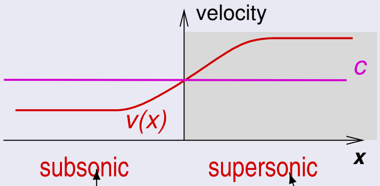

(a)
$T_{\rm BEC} = 0$
(b)
$T_{\rm BEC} = 0.3\ g\,n_u$
(c)
$T_{\rm BEC} = 0.5\ g\,n_u$
(d)
$T_{\rm BEC} = 1.0\ g\,n_u$
(e)
$T_{\rm BEC} = 1.5\ g\,n_u$
(f)
$T_{\rm BEC} = 1.8\ g\,n_u$
$1-\nu_-^{\rm PT}$, $\Delta_{\rm CS}$, $G_\tau^{(0|2)}$
$\hbar\,\omega/(g\,n_u)$


$T_{\text{BEC}} = 0.1$
$T_{\text{BEC}} = 0.3$
$T_{\text{BEC}} = 0$
Δ_{CS}
$G_\tau^{(0|2)}$


$T_{\text{BEC}} = 0.1$
$T_{\text{BEC}} = 0.3$
$T_{\text{BEC}} = 0$
$-\mathcal{P}$
$G_\tau^{(0|2)}$

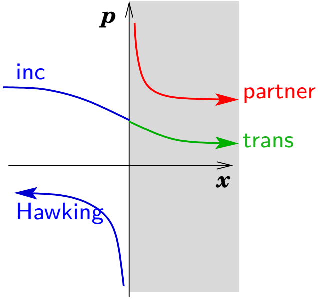

$T_{\text{BEC}} = 0.1$
$T_{\text{BEC}} = 0$
$T_{\text{BEC}} = 0.3$
$T_{\text{BEC}} = 0.5$
$T_{\text{BEC}} = 1$
$1 - \nu_-^{\text{PT}}$
$G_\tau^{(0|2)}$

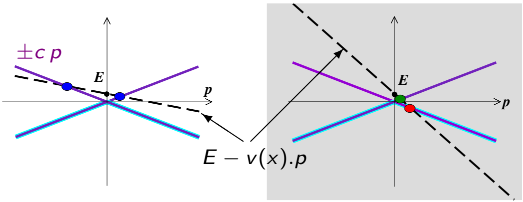
model configuration :



$$E - v(x).p = \pm\, c\, p$$

phase space :

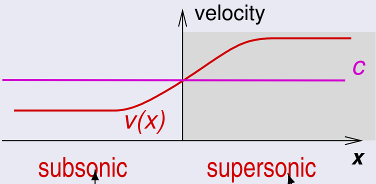


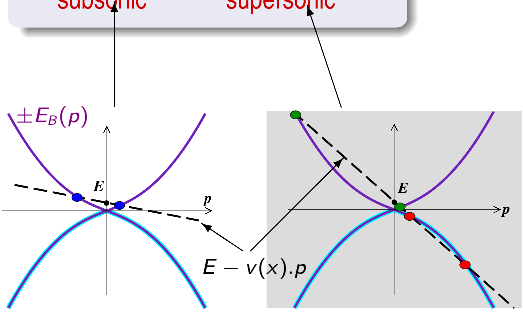


$v(x) < c$ region $x < 0$

$v(x) > c$ region $x > 0$

model configuration :





$v(x) < c$ — region $x < 0$

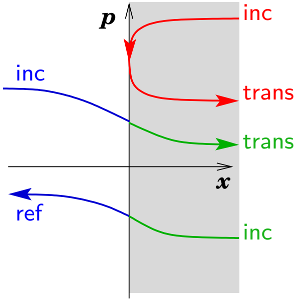
$v(x) > c$ — region $x > 0$

$$E - v(x).p = \pm E_B(p)$$

with

$$E_B(p) = c\,p\,\sqrt{1 + \xi^2 p^2/4}$$

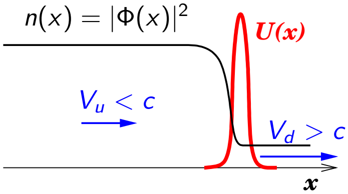
phase space :



How to form a sonic horizon ?

stationnary Gross-Pitaevskii Eq.

$$-\frac{1}{2}\Phi_{xx} + \left(U(x) + g|\Phi|^2\right)\Phi = \mu\,\Phi$$



P. Leboeuf and N. Pavloff, Phys. Rev. A (2001)

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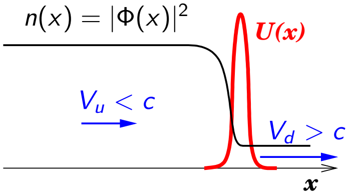

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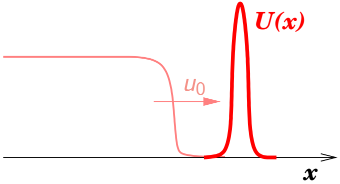

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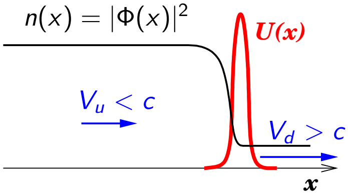
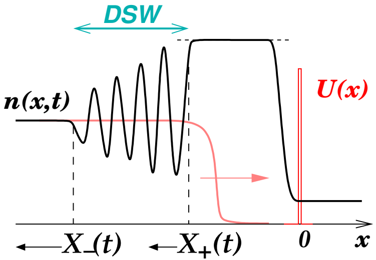
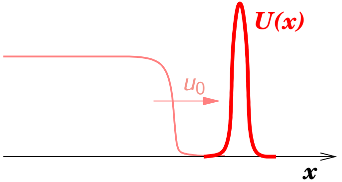
time-dependent Gross-Pitaevskii Eq.

$$-\frac{1}{2}\Phi_{xx} + \left(U(x) + g|\Phi|^2\right)\Phi = i\Phi_t$$



P. Leboeuf and N. Pavloff, Phys. Rev. A (2001)





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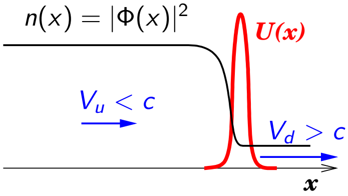

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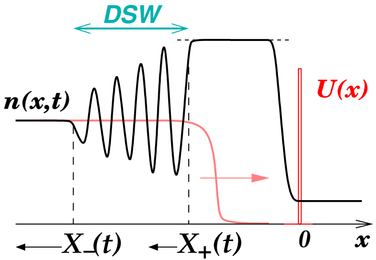
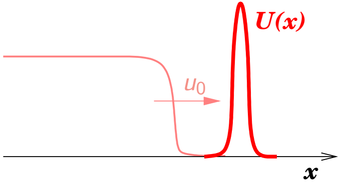
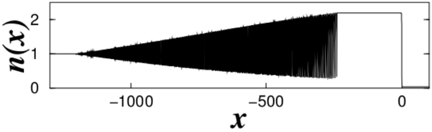
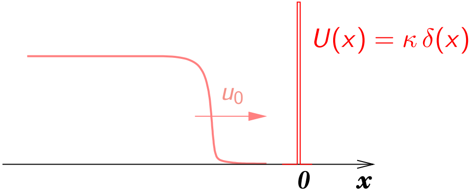
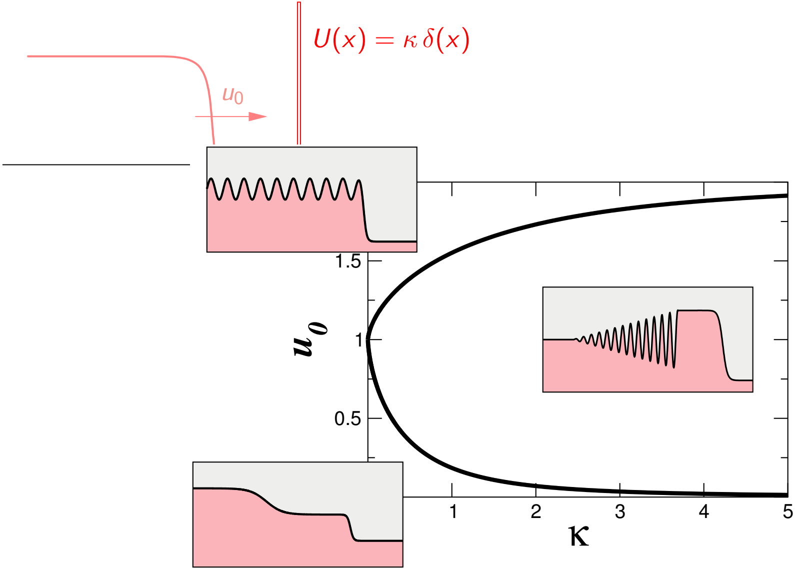






A. Kamchatnov & N. Pavloff, Phys. Rev. A (2012)


$U(x) = \kappa\,\delta(x)$
u_0
0
x


$U(x) = \kappa\,\delta(x)$
u_0
1.5
1
0.5
u_0
1
2
3
4
5
κ