Worldline instantons for non-perturbative particle production in metrics that depend on both space and time

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Worldline instantons

- Based on the worldline path-integral representation for the propagator.
- Originally for calculating the effective action in QED [1].
- Later extended to calculate momentum-resolved probabilities in QED [2].
- We have recently extended the momentum-resolved approach to curved spacetime [3].

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[1] [Affleck, et.al Nucl. Phys. B 197 (1982)], [Dunne & Schubert, PRD 72 (2005)], [Dunne & Wang PRD 74 (2006)], ...
[2] [Degli Esposti & Torgrimsson, PRD 107, (2023)], [Degli Esposti & Torgrimsson, PRD 109, (2024)], ...
[3] [Semrén & Torgrimsson, preprint [arXiv:2508.01901 [hep-th]]]

Standard tunneling

Standard tunneling and complex path analysis
(Parikh & Wilczek [1], Srinivasan & Padmanabhan [2], Vanzo et. al [3])

- WKB approximation: $\phi \sim e^{iS/\hbar}$
- Hamilton-Jacobi equation:

$$g^{\mu\nu}\partial_\mu S \partial_\nu S + m^2 = 0$$

- Emission rate $\Gamma \sim e^{-2\text{Im}S}$
- Solving for S involves integrals with poles/branch points.

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[1] Phys. Rev. Lett. **85** (2000), 5042-5045
[2] Phys. Rev. D **60** (1999), 024007
[3] Class. Quant. Grav. **28** (2011), 183001

Worldline instantons

Feynman propagator:

$$G(x_+, x_-) = \int_0^\infty \mathrm{d}T \int_{q(0)=x_-}^{q(1)=x_+} \mathcal{D}q \ldots e^{-iS}$$

with action

$$S = \int_0^1 \mathrm{d}\tau \left(\frac{g_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu}{2T} + A_\mu(q)\dot{q}^\mu + \frac{T}{2}\left[m^2 + \xi R(q)\right] \right)$$

Worldline instantons

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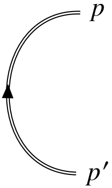
Amplitude using LSZ amputation:



$$M = \lim_{t_\pm \to \infty} \int \mathrm{d}^3x_+ \mathrm{d}^3x_- \, e^{ipx_+ + ip'x_-} \ldots G(x_+, x_-) \ldots$$

$$P(p, p') \sim |M|^2$$

Worldline instantons

Worldline instanton (saddle-point) from

$$\ddot{q}^\mu + \Gamma^\mu_{\nu\sigma}\dot{q}^\nu\dot{q}^\sigma = TF^\mu{}_\nu\dot{q}^\nu$$

$$T^2 = \int \mathrm{d}\tau\, g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu = q_{\mu\nu}\dot{q}^\mu\dot{q}^\nu$$

$$\dot{q}^\mu(1) = Tp^\mu \qquad \dot{q}^\mu(0) = -Tp'^\mu \ .$$

Set $A_\mu = 0$ and rescale $u = T(\tau - \tau_0)$.

$$P(p,p') \sim |M|^2 \sim e^{-\mathcal{A}}, \quad \mathcal{A} = \text{Im}\int_{u_0}^{u_1} \mathrm{d}u\, q^\mu g_{\nu\sigma,\mu}\dot{q}^\nu\dot{q}^\sigma$$

Can take $u_{0,1} \to \mp\infty$.

Worldline instantons

- Need to (in general) determine the instanton $q(u)$ with boundary conditions $\dot{q}(\infty) = p, \quad \dot{q}(-\infty) = -p'$.
- In practice:
  - *Guess* $q(0), \dot{q}(0)$.
  - Find a suitable contour in the complex u plane.
  - Iterate until BCs at $\pm\infty$ satisfied.

Cosmological particle production

Cosmology-inspired:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2A(t,x)}\mathrm{d}x^2$$

Example (H, ω, k constant):

$$A = \frac{H}{\omega}\tanh(\omega t)\,\mathrm{sech}^2(kx)$$

Reduce number of non-trivial parameters by rescaling

$$q \to \frac{q}{\omega}, \quad u \to \frac{u}{\omega} \implies A = \alpha\tanh(t)\,\mathrm{sech}^2(\beta x), \quad \alpha = \frac{H}{\omega}, \quad \beta = \frac{k}{\omega}$$

Cosmological particle production

For $\beta = 0$

$$\dot{t} = \sqrt{1 + p^2 e^{-2A(t)}} \qquad \dot{x} = p e^{-2A(t)}$$

$$P \sim e^{-\mathcal{A}}$$

$$\mathcal{A} = \frac{2}{\omega} \mathrm{Im} \int_{\infty}^{\infty} \mathrm{d}t \sqrt{1 + p^2 e^{-2A(t)}}$$

Go around the branch point:

$$t_B = \operatorname{arctanh}\left(\frac{1}{2\alpha}\left[i\pi + \ln p^2\right]\right)$$

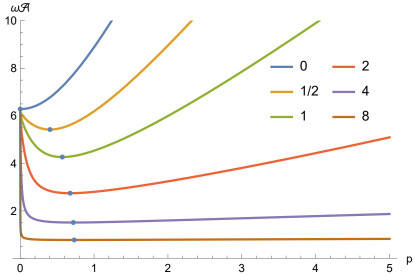


Figure: $\beta = 0$, $\alpha = 0, 1/2, 1, 2, 4, 8$.

$$\alpha \gg 1 \qquad \mathcal{A} \to \frac{2\pi}{\omega\alpha} = \frac{2\pi}{H}$$

$$\alpha \ll 1 \qquad \mathcal{A} \to \frac{2\pi\sqrt{1 + p^2}}{\omega}$$

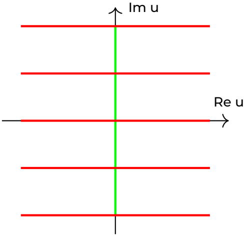
Cosmological particle production

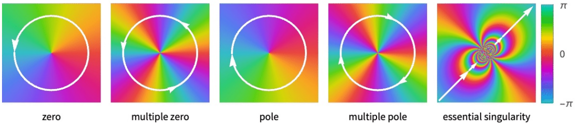
Could find $\mathcal{A}$ without explicit instanton. Will not be possible for $\beta \neq 0$. First consider the instanton for $\beta = 0$.

Initial conditions?

- $t(0) = t_B \implies \dot{t}(0) = 0, \quad \dot{x}(0) = -1/p.$
- Set $x(0) = 0$ (does not matter when $\beta = 0$).
- Let $p = p_s$ be the value of p that minimizes $\mathcal{A}$.

Generate plots over complex u-plane:





Cosmological particle production

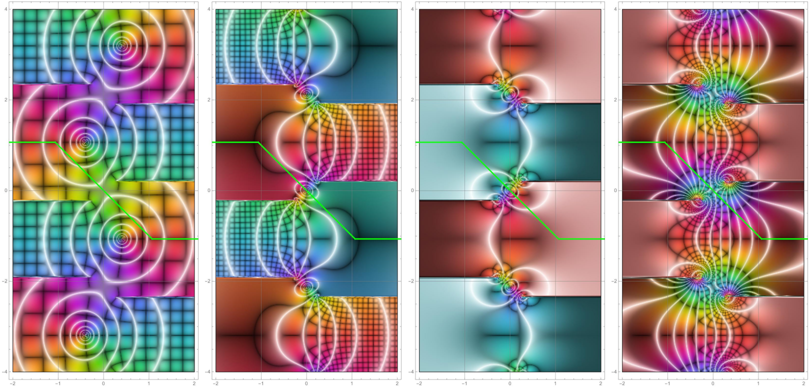
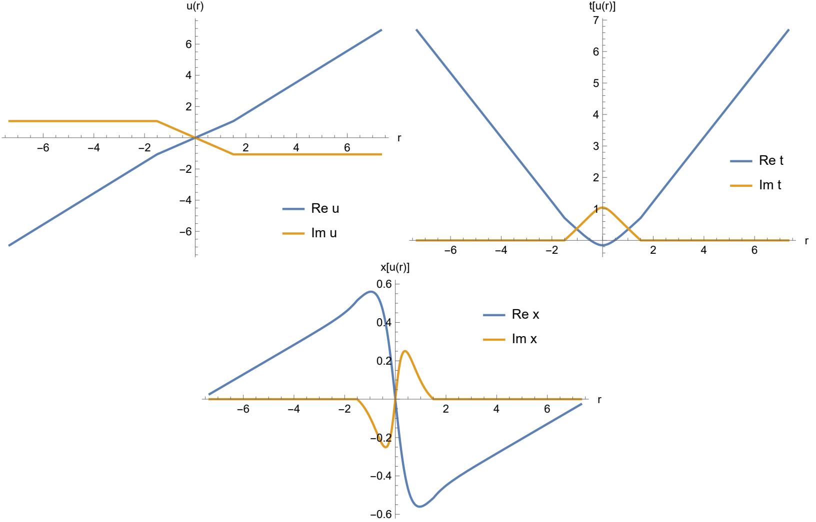


Figure: $\beta = 0$. $t(u)$, $x(u)$, $\dot{t}(u)$ and $\dot{x}(u)$ in the complex proper-time plane, $-2 < \text{Re}\, u < 2$, $-4 < \text{Im}\, u < 4$. $x(0) = \dot{t}(0) = 0$, $t(0) = t_B$, $\alpha = 1$, $p = p_s$.

Cosmological particle production



Cosmological particle production

Numerically continue to $\beta > 0$.

1. Pick a small $\beta > 0$.
2. Use $q(0)$, $\dot{q}(0)$ for $\beta = 0$ as initial guess.
3. Integrate over a similar contour as for $\beta = 0$.
4. Newton-Raphson iterate over $q(0)$, $\dot{q}(0)$ until BCs are matched.
5. Repeat process for larger β, but with the new $q(0)$, $\dot{q}(0)$ as initial guess.

Cosmological particle production

Have to adjust contour as branch-points move around:

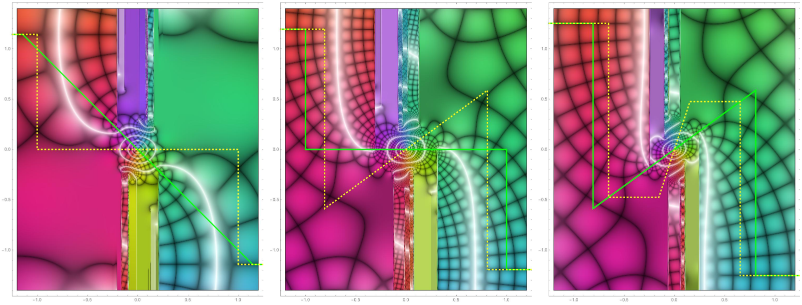


Figure: $x(u)$ for $\beta = \{3, 8, 21.8\}$.

Cosmological particle production

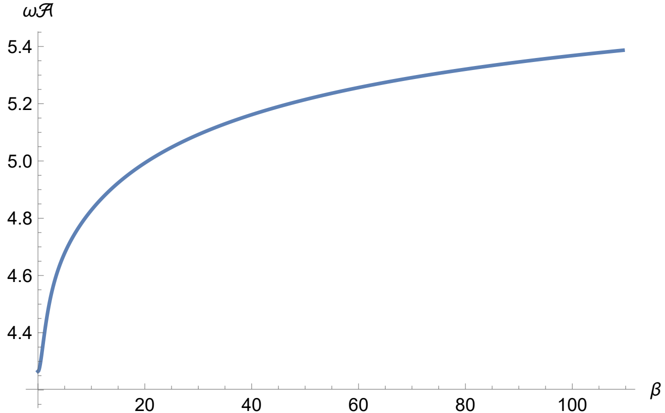


Figure: $\alpha = 1$ and saddle-point values of the momenta.

Hawking Radiation

Gullstrand-Painlevé form:

$$\mathrm{d}s^2 = [1 - v^2(t,x)]\mathrm{d}t^2 + 2v(t,x)\mathrm{d}t\mathrm{d}x - \mathrm{d}x^2$$

If no dependence on time t:

$$\dot{t} = \frac{\sqrt{1+p^2} - v\epsilon\sqrt{p^2+v^2}}{1-v^2} \qquad \dot{x} = \epsilon\sqrt{p^2+v^2(x)} \qquad \epsilon = \pm 1$$

$$\mathcal{A} = 2\mathrm{Im}\int \mathrm{d}x \frac{\sqrt{1+p^2}v - \epsilon\sqrt{p^2+v^2}}{1-v^2}$$

Semicircle in complex x-plane around horizon at $x = x_H$

$$\mathcal{A} = \frac{2\pi p_0}{\kappa}, \quad \kappa = v'(x_H), \quad p_0 = \sqrt{1+p^2} \quad \text{for } \epsilon = 1$$

Hawking Radiation

$$v(t,x) = -\frac{\alpha}{(1+[kx]^2)^2}\,\mathrm{sech}^2(\omega t)$$

Instanton?

- Start with $\omega = 0$.
- For $\alpha > 1$, we have two horizons $v(x = \pm x_H) = -1$.
- Initial conditions:

$$x(0) = 0, \qquad t(0) = 0,$$

$$\dot{x}(0) = \sqrt{p^2 + v^2(0)}, \qquad \dot{t}(0) = \frac{\sqrt{1+p^2} - v\sqrt{p^2 + v(0)^2}}{1 - v(0)^2}$$

- Assume that we only observe one of the particles: $p \neq p_s$, $p' = p'_s$.
- Integrate EoM numerically and generate complex plots:

Hawking Radiation

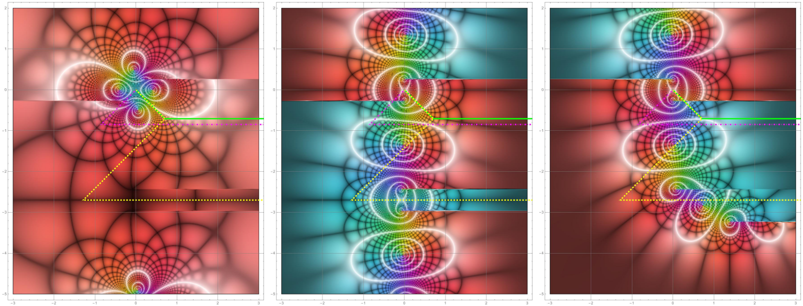


Figure: $\dot{t}(u)$ (first plot) and $\dot{x}(u)$ (second and third plots) for $\alpha = 2$, $k = 1$, $\omega = 0$ and $p = 0.5$.

Hawking Radiation

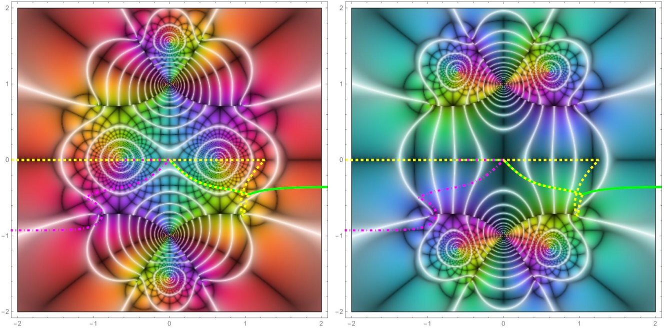


Figure: The integrand in $\mathscr{A}$ in the complex x plane. The two plots show the two parts of the Riemann surface.

Hawking Radiation

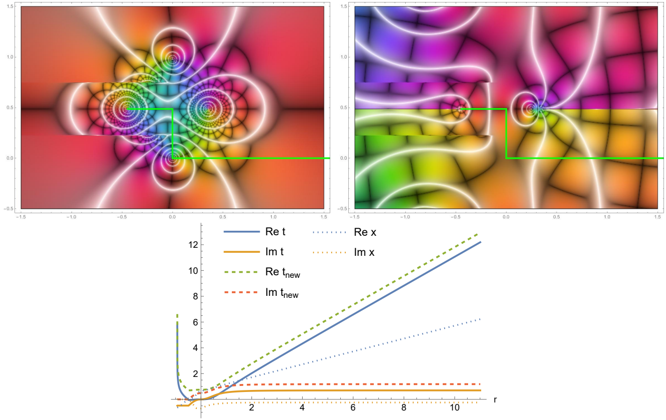


Figure: $\dot{t}(u) = \dot{t}_{\mathrm{new}}(u)$ and $t_{\mathrm{new}}(u)$

Hawking Radiation

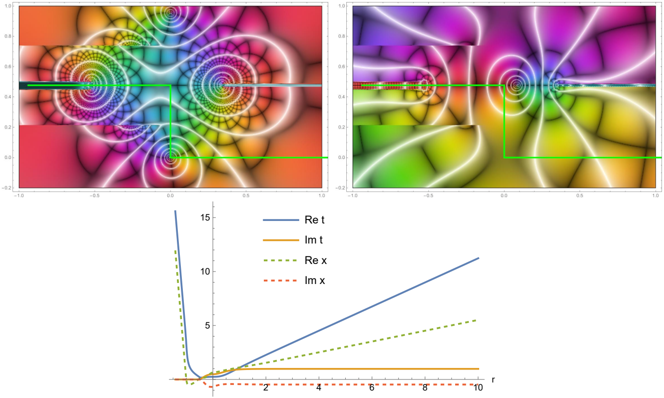


Figure: $\dot{t}(u)$ and $t(u)$ but with $\omega = 0.4$.

Hawking Radiation

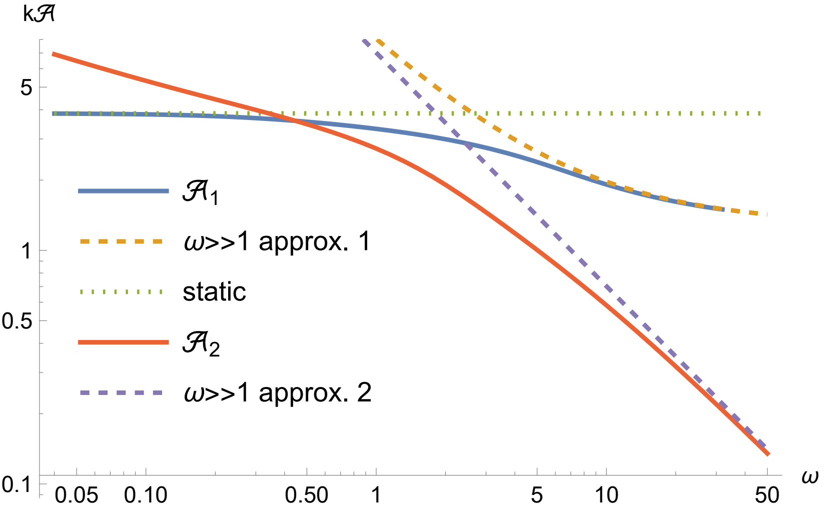


Figure: $k = 1$, $\alpha = 2$, $p^1 = 0.5$ and $p' = p'_s$ (or $p'^1 = 0.5$ and $p = p_s$).

Conclusion

- We have extended a worldline instanton technique from QED to curved spacetime [1].
- Have shown how to calculate the exponential scaling for Hawking radiation and cosmological particle production.
- Current/future work:
  - Determine the prefactor.
  - Consider more geometries and processes (e.g. stimulated emission).
- Main benefits:
  - Can be applied straightforwardly to general metrics.
  - Efficiently determines momentum-resolved probabilities.
  - Can quickly sweep over field parameters using numerical continuation.

[1] P. Semrén & G. Torgrimsson, [arXiv:2508.01901]