Acoustic black holes in BECs with an extended sonic region

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Mathematical Framework

N-body (bosonic) system

$$\hat{H} = \int \mathrm{d}^3x \left(\hat{\Psi}^\dagger \left(-\frac{\hbar}{2m}\nabla^2 + V_{ext} \right) \hat{\Psi} + \frac{g}{2} \hat{\Psi}^\dagger \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right)$$

$$\mathrm{i}\hbar \frac{\partial \hat{\Psi}}{\partial t} = \left[\hat{\Psi}, \hat{H} \right]$$

$$\hat{\Psi} = \Psi_0 (1 + \hat{\Phi}) \mathrm{e}^{-\mathrm{i}\mu t/\hbar}$$

Bogoliubov approximation
(classical part + quantum part)

Gross-Pitaevskii

$$\mathrm{i}\hbar \frac{\partial \Psi_0}{\partial t} = \left(-\frac{\hbar}{2m}\nabla^2 + V_{ext} + g|\Psi_0|^2 \right) \Psi_0$$

Bogoliubov-de Gennes

$$\mathrm{i}\hbar \frac{\partial \hat{\Phi}}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2 + \frac{\hbar^2}{m}\frac{\nabla \Psi_0}{\Psi_0}\nabla \right) \hat{\Phi} + ng(\hat{\Phi} + \hat{\Phi}^\dagger)$$

$n = |\Psi_0|^2$

Mathematical Framework

Bogoliubov-de Gennes

$$i\hbar\frac{\partial\hat{\Phi}}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2 + \frac{\hbar^2}{m}\frac{\nabla\Psi_0}{\Psi_0}\nabla\right)\hat{\Phi} + ng(\hat{\Phi} + \hat{\Phi}^\dagger)$$

Mode expansion (stationary system)

$$\hat{\Phi}(t,x) = \int_0^\infty \mathrm{d}\omega\,\left(\hat{a}(\omega)\phi_\omega(t,x) + \hat{a}^\dagger(\omega)\varphi^*_\omega(t,x)\right)$$

Normalisation $\hbar n \int_{-\infty}^{\infty} \mathrm{d}x\,(\phi_\omega\phi^*_{\tilde{\omega}} - \varphi_\omega\varphi^*_{\tilde{\omega}}) = \pm\delta(\omega - \tilde{\omega})$

Mathematical Framework

$$\hat{\Phi}(t,x) = \int_0^\infty \mathrm{d}\omega \left(\hat{a}(\omega)\phi_\omega(t,x) + \hat{a}^\dagger(\omega)\varphi_\omega^*(t,x)\right)$$

$$\phi_\omega = D(\omega)\mathrm{e}^{-\mathrm{i}(\omega t - k(\omega)x)} \qquad \varphi_\omega = E(\omega)\mathrm{e}^{-\mathrm{i}(\omega t - k(\omega)x)}$$

Dispersion Relation

$$(\omega - vk)^2 = c^2\left(k^2 + \frac{\xi^2 k^4}{4}\right)$$

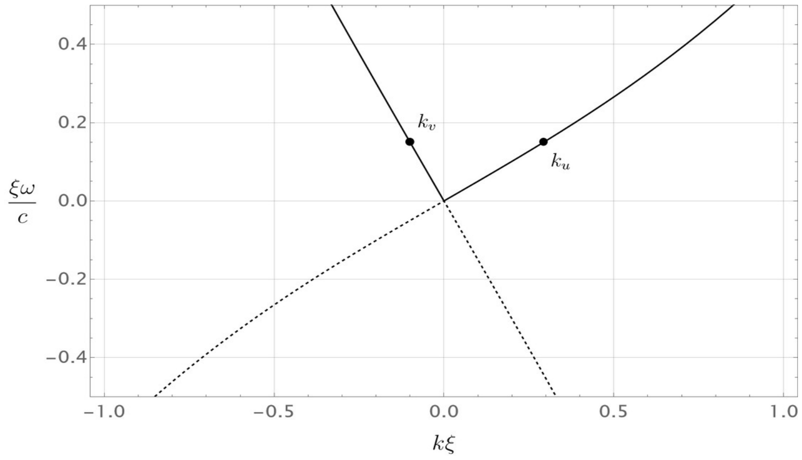
$$v = -\frac{\mathrm{i}\hbar\nabla\Psi_0}{m\Psi_0}$$

$$c = \sqrt{\frac{ng}{m}}$$

$$\xi = \frac{\hbar}{mc}$$

Analyzing the dispersion relation (homogeneous BEC)

SUBSONIC ($c > |v|$)

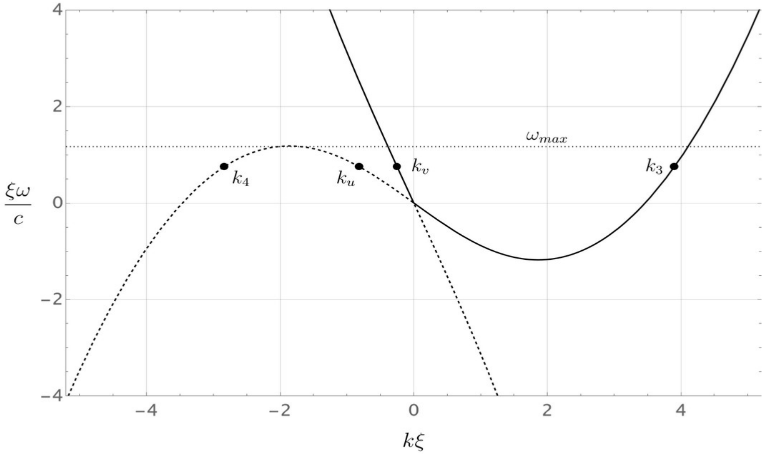


$$k_v = \frac{\omega}{v-c} + \mathcal{O}(\omega^3)$$

$$k_u = \frac{\omega}{v+c} + \mathcal{O}(\omega^3)$$

$$k_\pm = \pm 2\mathrm{i}\frac{\sqrt{c^2-v^2}}{c\xi} + \frac{v\omega}{c^2-v^2} + \mathcal{O}(\omega^2)$$

SUPERSONIC ($c < |v|$)



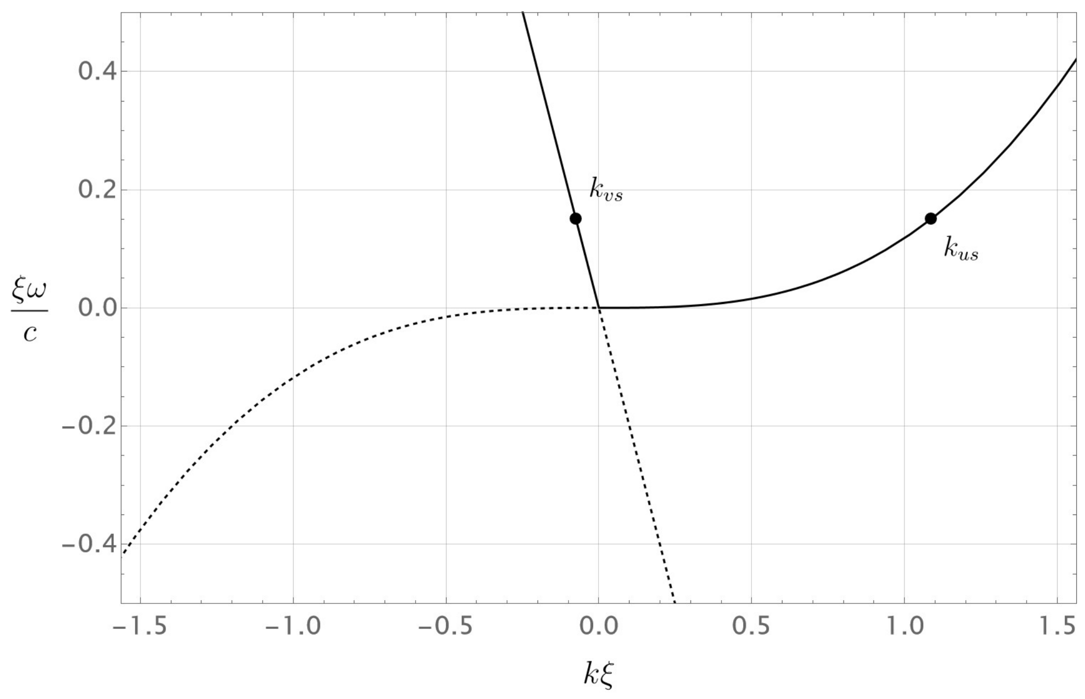
$$k_v = \frac{\omega}{v-c} + \mathcal{O}(\omega^3)$$

$$k_u = \frac{\omega}{v+c} + \mathcal{O}(\omega^3)$$

$$k_{3,4} = \pm 2\frac{\sqrt{c^2-v^2}}{c\xi} + \frac{v\omega}{c^2-v^2} + \mathcal{O}(\omega^2)$$

Sonic Flow

$$\text{SONIC } (c = |v|)$$



$$k_v = -\frac{\omega}{2c} + \mathcal{O}(\omega^3)$$

$$k_u = \frac{2}{\xi}\left(\frac{\xi\omega}{c}\right)^{1/3} + \mathcal{O}(\omega)$$

$$k_\pm = \frac{-1 \pm \mathrm{i}\sqrt{3}}{\xi}\left(\frac{\xi\omega}{c}\right)^{1/3} + \mathcal{O}(\omega)$$

Sonic Region (hydrodynamic approximation)

$$(\omega - vk)^2 = c^2\left(k^2 + \frac{\xi^2 k^4}{4}\right) \xrightarrow{\xi k \ll 1} (\omega - vk)^2 = c^2 k^2$$

$$k_v = \frac{\omega}{v - c}$$

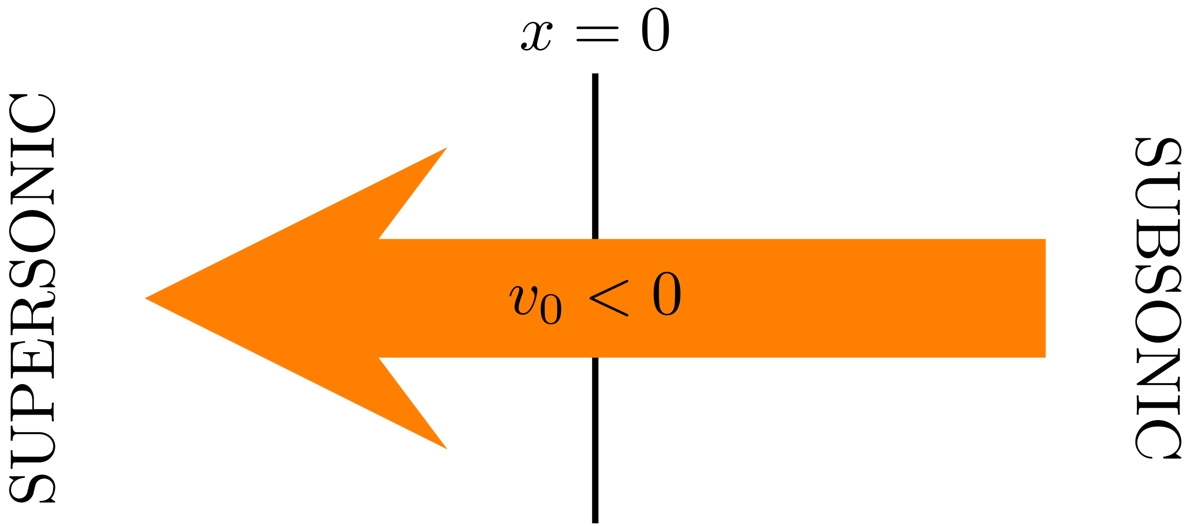
$$k_u = \frac{\omega}{v + c}$$

$$\xrightarrow{c \to |v|}$$

$$k_v = \frac{\omega}{-2c}$$

$$\nexists\ k_u$$

Black Holes in Stepwise BECs



$$n, v = const.$$

$$c = \begin{cases} c_R & (x > 0) \\ c_L & (x > 0) \end{cases}$$

$$\xi = \begin{cases} \xi_R & (x > 0) \\ \xi_L & (x < 0) \end{cases}$$

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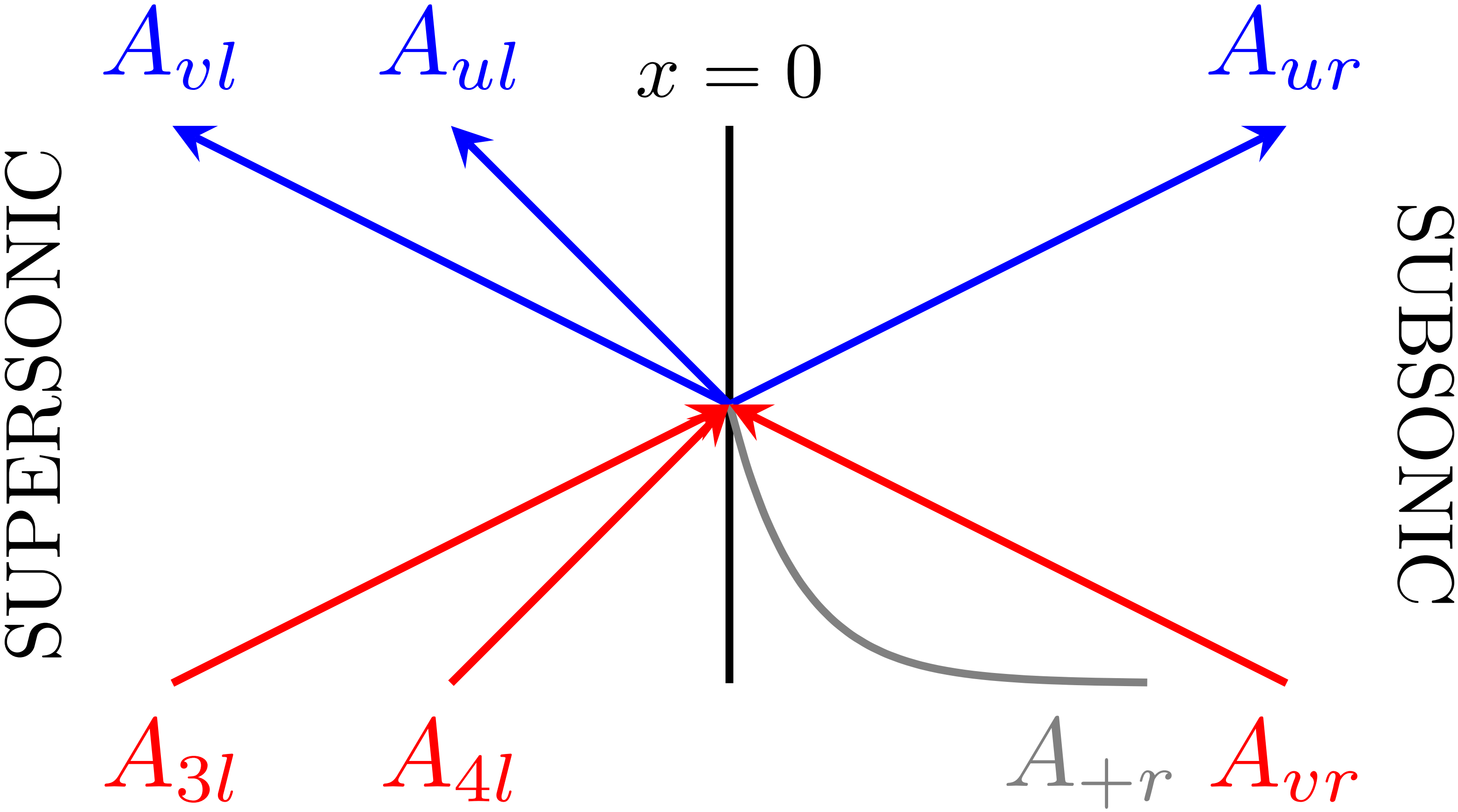
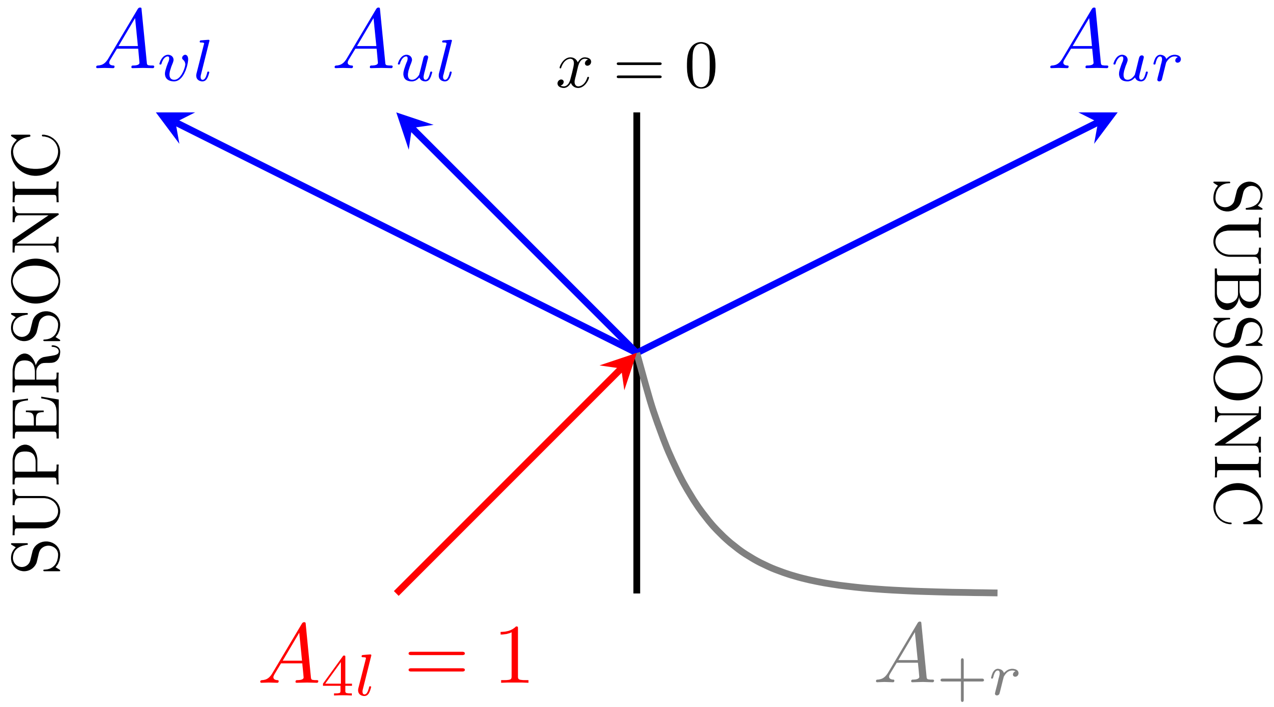
Black Holes in Stepwise BECs



Scattering basis



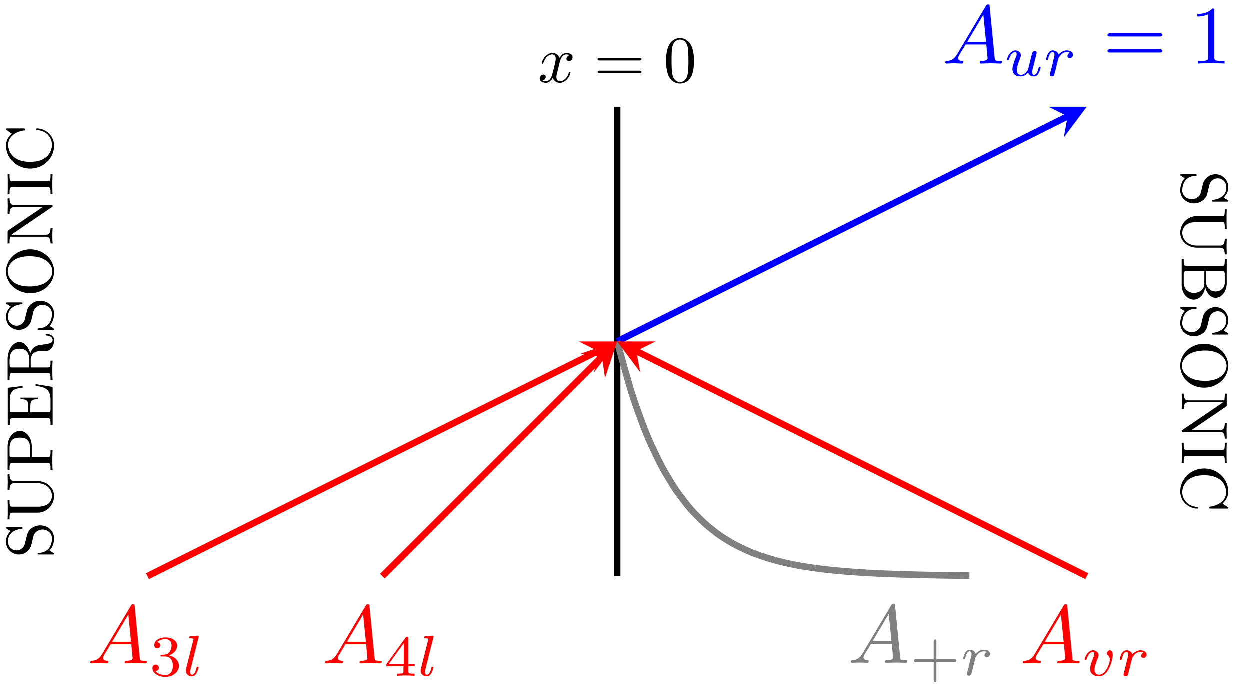
$$\phi_j^{in}, \varphi_j^{in} \in C^1(\mathbb{R})$$

$$(\varphi : D \to E)$$

$$|A_{ur}|^2 + |A_{vl}|^2 - |A_{ul}|^2 = -|A_{4l}|^2$$

$$\phi_{4l}^{in} = \left(D_{4l}\mathrm{e}^{\mathrm{i}k_{4l}x} + A_{vl}D_{vl}\mathrm{e}^{\mathrm{i}k_{vl}x} + A_{ul}D_{ul}\mathrm{e}^{\mathrm{i}k_{ul}x}\right)\theta(-x)$$
$$+ \left(A_{ur}D_{ur}\mathrm{e}^{\mathrm{i}k_{ur}x} + A_{+r}D_{+r}\mathrm{e}^{\mathrm{i}k_{+r}x}\right)\theta(x)$$

Scattering basis



$$\phi_j^{in}, \varphi_j^{in} \in C^1(\mathbb{R})$$

$$(\varphi : D \to E)$$

$$|A_{vr}|^2 + |A_{3l}|^2 - |A_{4l}|^2 = |A_{ur}|^2$$

$$\phi_{ur}^{out} = \left(D_{ur}\mathrm{e}^{\mathrm{i}k_{ur}x} + A_{3l}D_{3l}\mathrm{e}^{\mathrm{i}k_{4l}x} + A_{4l}D_{4l}\mathrm{e}^{\mathrm{i}k_{4l}x}\right)\theta(-x)$$
$$+ \left(A_{vr}D_{ur}\mathrm{e}^{\mathrm{i}k_{vr}x} + A_{+r}D_{+r}\mathrm{e}^{\mathrm{i}k_{+r}x}\right)\theta(x)$$

Bogoliubov transformation

$$\hat{\Phi} = \int d\omega \left(\phi^{in}_{vR} \hat{a}^{in}_{vR} + \phi^{in}_{3L} \hat{a}^{in}_{3L} + \phi^{in}_{4L} \hat{a}^{in\dagger}_{4L} + c.c. \right)$$

$$\hat{\Phi} = \int d\omega \left(\phi^{out}_{uR} \hat{a}^{out}_{uR} + \phi^{out}_{vL} \hat{a}^{out}_{vL} + \phi^{out}_{uL} \hat{a}^{out\dagger}_{uL} + c.c. \right)$$

$$\begin{bmatrix} \phi^{in}_{vR} \\ \phi^{in}_{3L} \\ \phi^{in}_{4L} \end{bmatrix} = S \begin{bmatrix} \phi^{out}_{uR} \\ \phi^{out}_{vL} \\ \phi^{out}_{uL} \end{bmatrix} \qquad \begin{bmatrix} \hat{a}^{out}_{uR} \\ \hat{a}^{out}_{vL} \\ \hat{a}^{out\dagger}_{uL} \end{bmatrix} = S \begin{bmatrix} \hat{a}^{in}_{vR} \\ \hat{a}^{in}_{3L} \\ \hat{a}^{in\dagger}_{4L} \end{bmatrix}$$

Bogoliubov transformation

$$a_j^{in}\,|0, in\rangle = 0$$

$$\begin{bmatrix} \hat{a}_{uR}^{out} \\ \hat{a}_{vL}^{out} \\ \hat{a}_{uL}^{out\dagger} \end{bmatrix} = S \begin{bmatrix} \hat{a}_{vR}^{in} \\ \hat{a}_{3L}^{in} \\ \hat{a}_{4L}^{in\dagger} \end{bmatrix}$$

$$n_\omega^{ur} = \langle 0, in|\hat{a}_{uR}^{out\dagger}\hat{a}_{uR}^{out}|0, in\rangle = |S_{uR,4L}|^2$$

$$n_\omega^{vl} = \langle 0, in|\hat{a}_{vL}^{out\dagger}\hat{a}_{vL}^{out}|0, in\rangle = |S_{vL,4L}|^2$$

$$n_\omega^{ul} = \langle 0, in|\hat{a}_{uL}^{out\dagger}\hat{a}_{uL}^{out}|0, in\rangle = |S_{uL,vR}|^2 + |S_{uL,3L}|^2$$

$$n_\omega^{ul} = n_\omega^{ur} + n_\omega^{vl}$$

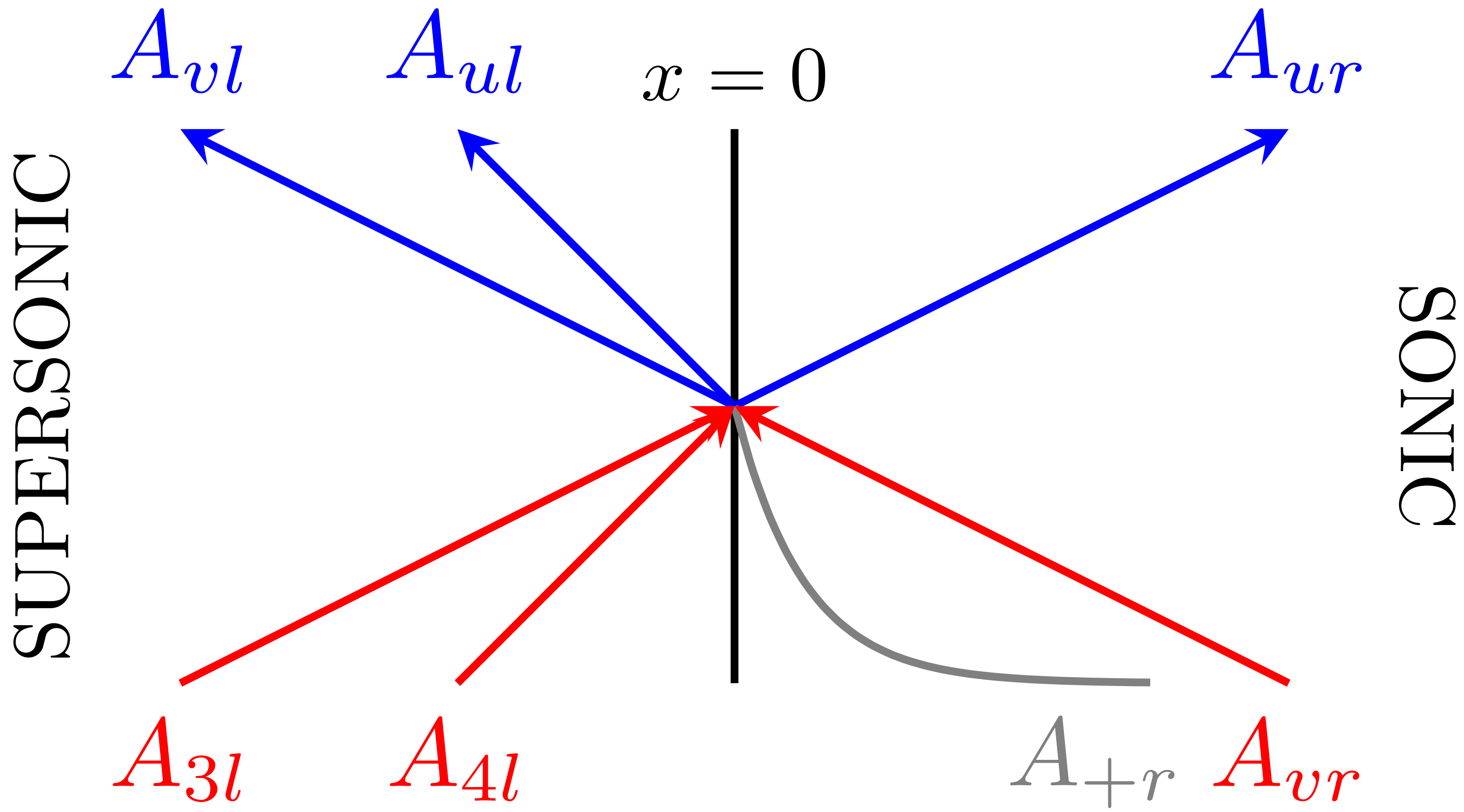
Thermal particle production (at small ω)

$$|S_{uR,4L}|^2 = \frac{c_R - |v|}{c_R + |v|} \frac{(v^2 - c_L^2)^{3/2}}{c_R^2 - c_L^2} \frac{2mc_R}{\hbar\omega} + \dots$$

$$\frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \simeq \frac{k_B T}{\hbar\omega} + \dots$$

$$T = \frac{c_R - |v|}{c_R + |v|} \frac{(v^2 - c_L^2)^{3/2}}{c_R^2 - c_L^2} \frac{2mc_R}{k_B}$$

Two region model with a sonic region

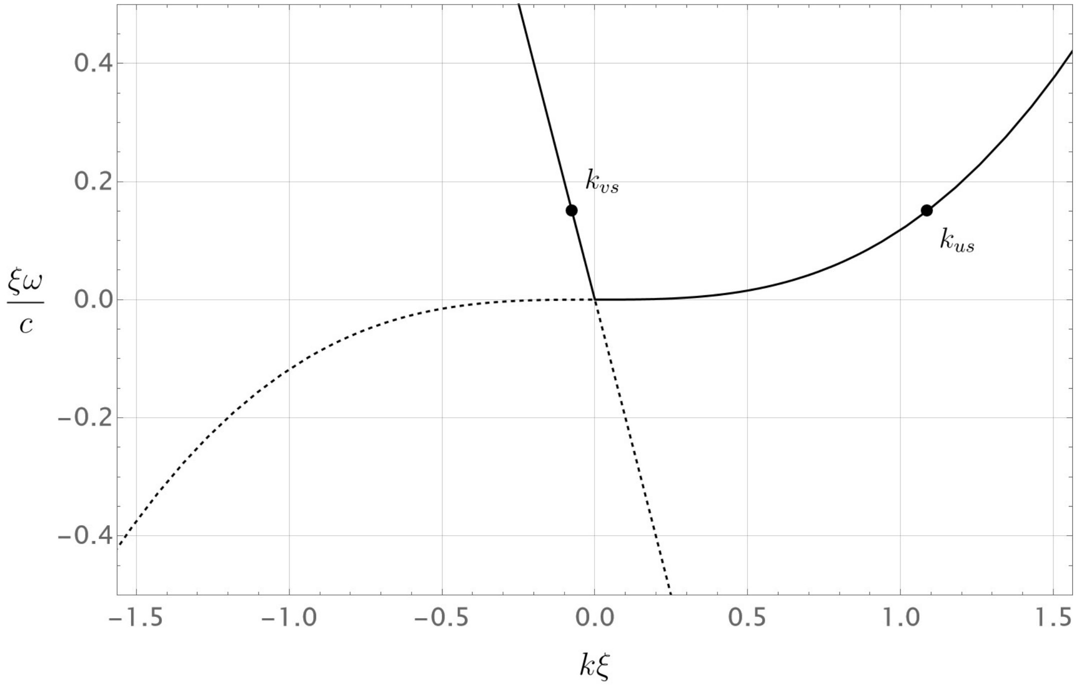


Non thermal particle production

$$|S_{uR,4L}|^2 = \frac{\sqrt{c_R^2 - c_L^2}}{2} \left(\frac{m}{\hbar \omega c_R} \right)^{1/3}$$

Non thermal particle production!

SONIC ($c = |v|$)

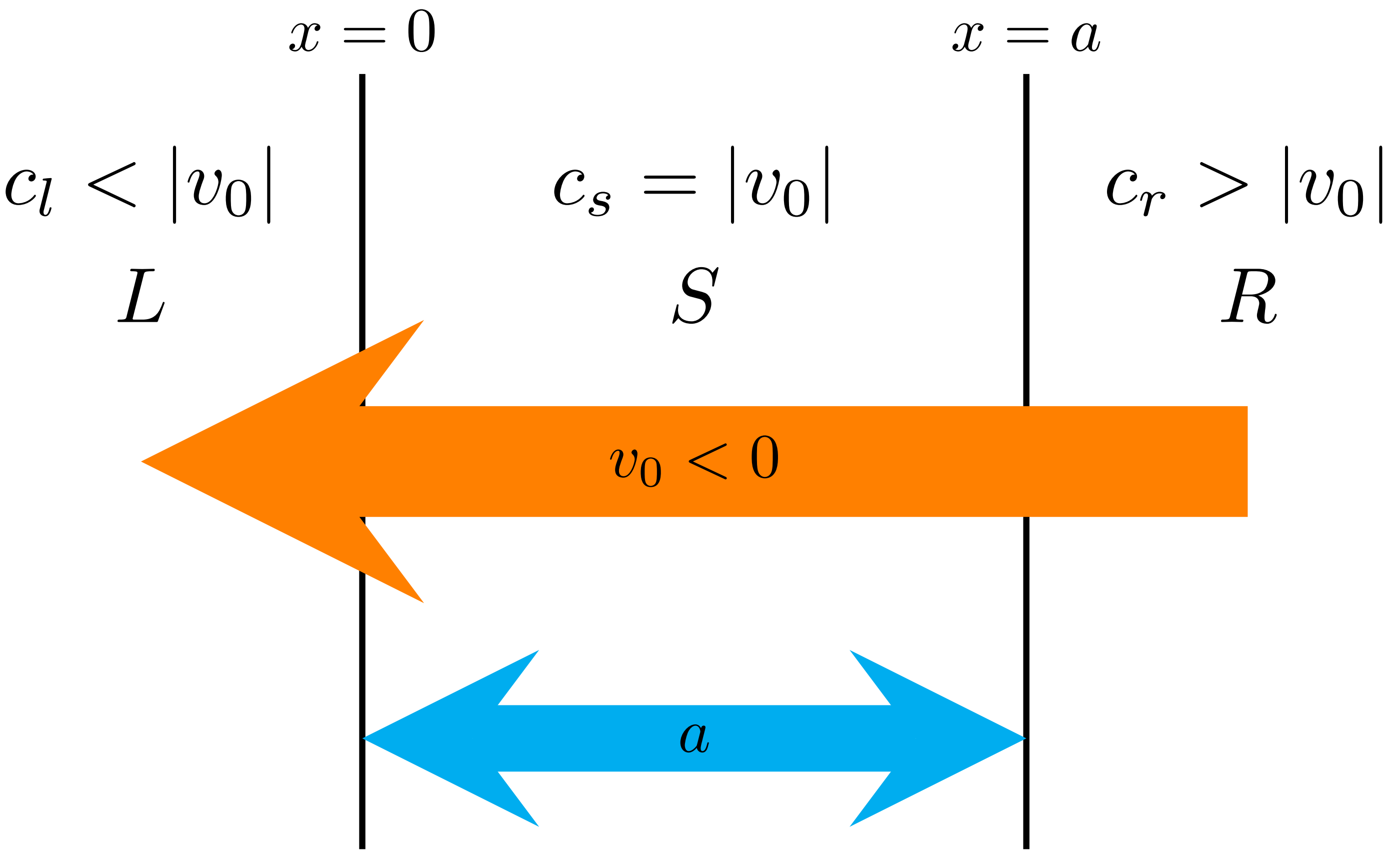


$$k_v = -\frac{\omega}{2c} + \mathcal{O}(\omega^3)$$

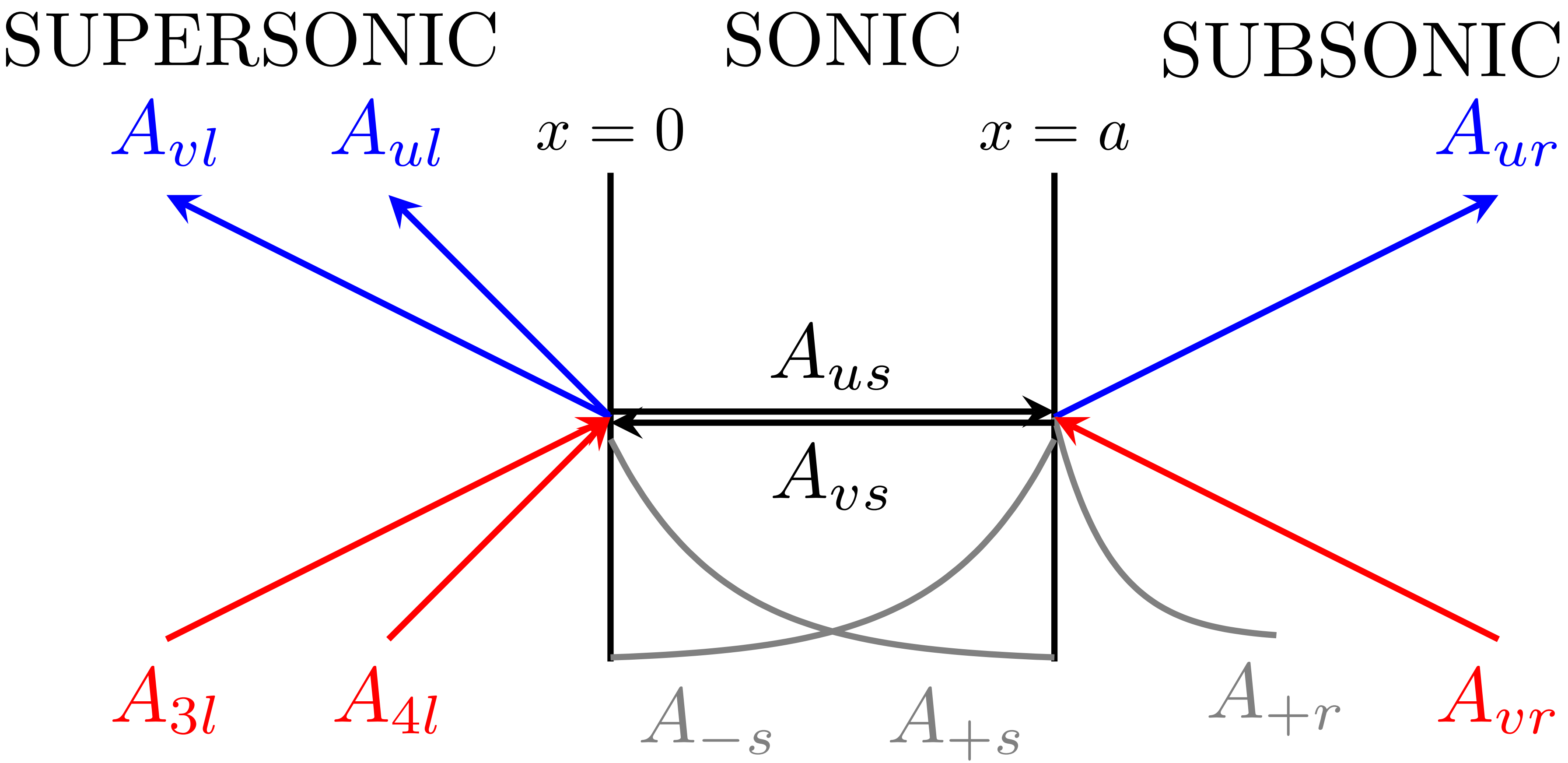
$$k_u = \frac{2}{\xi}\left(\frac{\xi\omega}{c}\right)^{1/3} + \mathcal{O}(\omega)$$

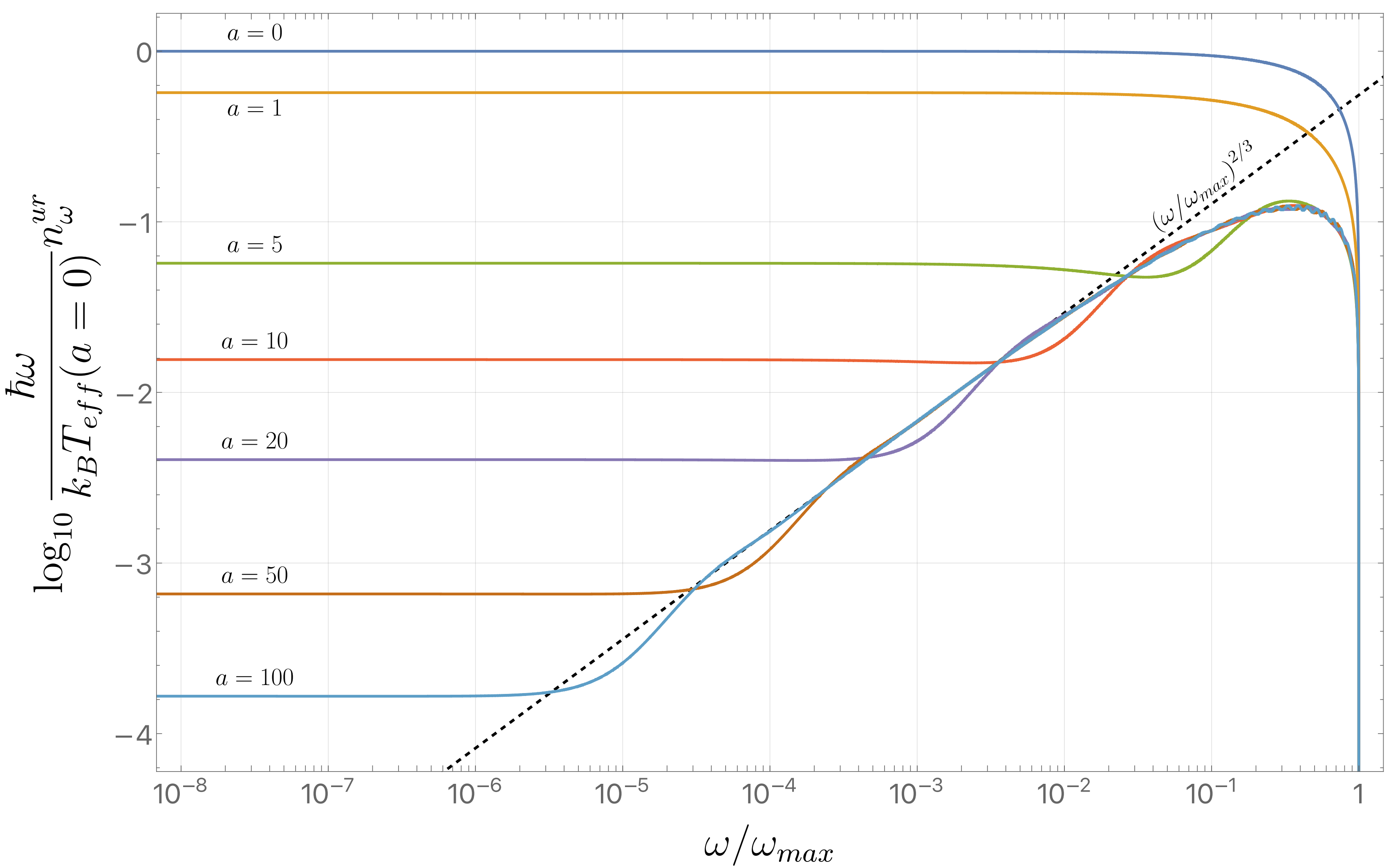
$$k_\pm = \frac{-1 \pm \mathrm{i}\sqrt{3}}{\xi}\left(\frac{\xi\omega}{c}\right)^{1/3} + \mathcal{O}(\omega)$$

The thick horizon model

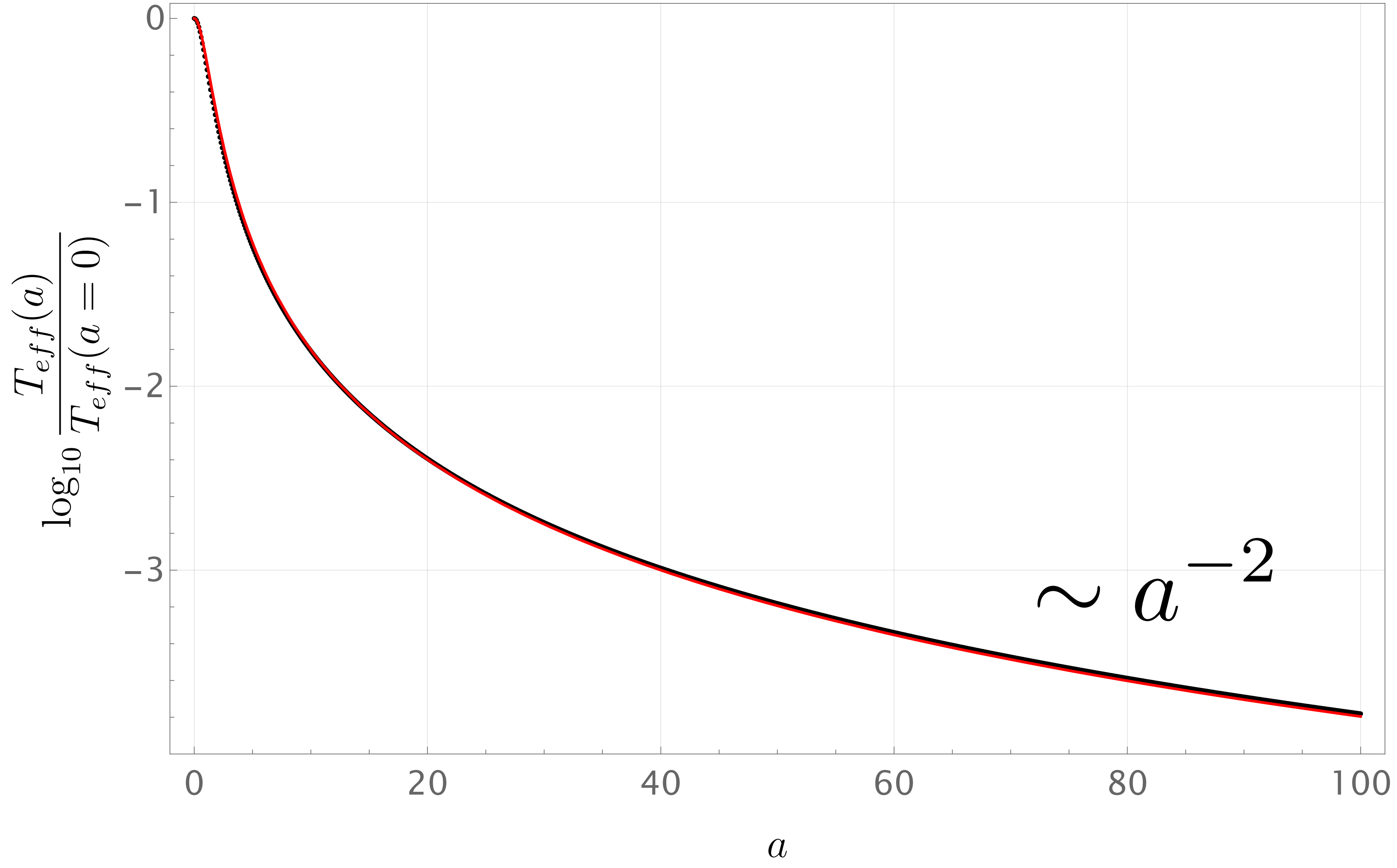


The thick horizon model





$a = 0$
$a = 1$
$a = 5$
$a = 10$
$a = 20$
$a = 50$
$a = 100$
$(\omega/\omega_{max})^{2/3}$
0
−1
−2
−3
−4
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
$\log_{10} \frac{\hbar\omega}{k_B T_{eff}(a=0)} n_\omega^{ur}$
ω/ω_{max}


0
−1
−2
−3
$\log_{10} \frac{T_{eff}(a)}{T_{eff}(a=0)}$
$\sim a^{-2}$
0
20
40
60
80
100
a

Conclusions

- Sonic regions are qualitatively similar to subsonic regions (in the nature of the modes.)
- Sonic regions are quantitatively different to both subsonic and supersonic regions (fractional powers of frequency)
  → **dispersion is crucial!!**
- If we replace subsonic region with a sonic one, the radiation is no longer thermal.
- A thick sonic horizon introduces a gray body factor in the Hawking temperature dependent on a^{-2} when a is sufficiently large.