QFT in curved spacetimes and analogue gravity
Lecture 1
Classical radiation and quantum particle creation

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Analogue Gravity in 2026

Benasque Science Center, January 8-17, 2026

January 8, 2026, Benasque

Contents

Introduction and orientation

- I have been asked to review the foundations of quantum field theory in curved spacetime in the context of analogue gravity
- It is therefore quite natural to approach this objective by focusing our discussion on gravitational particle creation, since it has been the driving force in the development of quantum field theory in curved spacetime:
- In historical reverse order:
  1. Pair creation by black holes (Hawking, 1974)
  2. Pair creation by the expansion of the universe (Parker, 1963-66)

     Since the focus of the workshop is analogue gravity, I will begin by taking you back before those discoveries
  3. Single-particle creation and pair creation in electrodynamics (1951-1963)

Main references

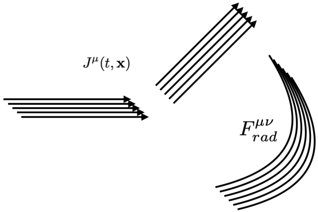
- Lecture 1: QFT, Itzykson and Zuber, McGraw-Hill, 1980
- Lecture 2: QFT in curved spacetime, Parker and Toms, CUP, 2008
- Lecture 3: Modeling black hole evaporation, A. Fabbri and J. N-S, ICP-World Scientific, 2005
- For an overall historical perspective, see:
  A. Ferreiro, J. N-S and S. Pla, "The Birth of Gravitational Particle Creation ... ": [arXiv:2511.13518] To appear in The European Physical Journal H
- In preparation: J. Navarro-Salas, Lectures on Quantum Field Theory and Particle Creation in Flat and Curved Spacetimes

Introduction and motivation

- Quantum field theory was born with quantum electrodynamics

  QED is the prototype of a QFT model

- To better understand quantum particle creation, it is helpful to step back to **classical electrodynamics**
- A central prediction in classical electrodynamics is the phenomena of electromagnetic radiation



Accelerating charges emit electromagnetic waves

Introduction and motivation

- A natural question then arises:
- How is the phenomenon of electromagnetic radiation recovered in the quantized Maxwell theory?
- Answer: particle (photon) creation
- One of the main aims of our first lecture is to recast the concept of radiation in classical field theory within the framework of quantum field theory

Maxwell theory

- Maxwell theory is encapsulated in the action functional

$$S_{Maxwell} = \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu]$$

- In classical physics, electromagnetic radiation arises from accelerating charges and currents, with the fields being determined by the solution of Maxwell's equations subject to causal boundary conditions [Jackson, Classical Electrodynamics]

$$A^\mu_{rad}(x) \;=\; \int d^4x'\, G_{rad}(x-x')\, J^\mu(x')$$

where $G_{rad}(x-x')$ denotes an appropriated (homogeneous) Green's function of the wave operator

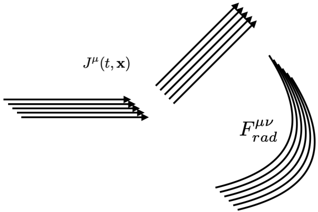
$$\Box_x G(x-x') = 0$$

Classical electromagnetic radiation

- $G_{\rm rad}(x-x')$ isolates the part of the field that propagates freely to infinity, corresponding to electromagnetic radiation

$$A^{\mu}_{\rm rad}(x) = \int d^4x'\, G_{\rm rad}(x-x')\, J^{\mu}(x') \, .$$

$$\begin{aligned} G_{\rm rad}(x-x') &\equiv G_{\rm ret}(x-x') - G_{\rm adv}(x-x') \\ &= \frac{1}{2\pi}\epsilon(x^0 - x'^0)\,\delta\left((x-x')^2\right) \end{aligned}$$



- It is illustrative to evaluate the energy emitted by the source
- Energy-momentum tensor:

$$T^{\mu\nu}_{\rm rad} = F^{\mu\alpha}_{\rm rad} F^{\rm rad \ \nu}_{\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta}_{\rm rad} F^{\rm rad}_{\alpha\beta} \, ,$$

where, we recall, $F^{\mu\nu}_{\rm rad} = \partial^\mu A^\nu_{\rm rad} - \partial^\nu A^\mu_{\rm rad}$.

- The result for the radiated energy is given by

$$E_{\rm rad} = \int d^3x \, T^{00}_{\rm rad} = \int \frac{d^3p}{2(2\pi)^3 \omega_p} \omega_p [|\tilde{J}_1(\mathbf{p})|^2 + |\tilde{J}_2(\mathbf{p})|^2]] \, ,$$

where $\omega_p = |\mathbf{p}|$ and $\tilde{J}_1(\mathbf{p})$ and $\tilde{J}_2(\mathbf{p})$ are the two independent polarization components of the electric current and

$$\tilde{J}^\mu(\mathbf{p}) = \int d^4x \, e^{ipx} J^\mu(x) \, .$$

- The first main aim of this Lecture is to recast the classical concept of the radiation field in electrodynamics, as outlined above, within the framework of quantum field theory

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

$$A_\mu \sim quantum\ field$$

$$J^\mu \sim classical\ (macroscopic)\ source$$

- To focus on the main concepts I assume the simplified case of a scalar field linearly coupled to an external source [we will go back to electrodynamics later on]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \phi J$$

$$\phi \sim quantum\ field$$

$$J \sim classical\ (macroscopic)\ source$$

- This is the scalar version of the electromagnetic coupling $A_\mu J^\mu$
- From now on we will assume that ϕ is a quantized field and $J(x)$ is a prescribed space-time c-number function

Scalar field linearly coupled to an external source

- We now focus on solving the theory of a quantized scalar field linearly coupled to an external scalar source $J(x)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \phi(x)J(x)$$

Field equation:

$$\Box\phi = J \ .$$

- We will also assume that the external source decays in the time and spatial directions

$$J(t,\vec{x}) \sim_{t\to\pm\infty} 0 \qquad\qquad J(t,\vec{x}) \sim_{|\vec{x}|\to+\infty} 0 \ .$$

- This model can be though of as a simplified description of the interaction between light, modeled by the field ϕ (without considering the spin) and matter (modeled by the classical source $J(x)$)
- We will employ standard techniques of QFT in Minkowski spacetime.

  One of the overarching goals of this minicourse is to offer a gentle transition from QFT in flat spacetime to its formulation in curved spacetime.

Interaction Picture

- We will use the Interaction Picture to compute the S operator
- Therefore, since J is a c-number the interaction part of the Hamiltonian in the interaction picture is obtained immediately

$$H^I_{int} = \int d^3x \mathcal{H}^I_{int} = \int d^3x(-\phi_I J) \ .$$

- The field ϕ_I in the interaction picture can be expanded as a free field $[\omega_p = |\vec{p}|]$

$$\phi_I(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx}) \ ,$$

- The operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$, obeying $[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = \delta^3(\mathbf{p} - \mathbf{p}')$, annihilate and create the asymptotic particle states.
- The scattering operator S is defined as(T is the time-ordering operation)

$$S = T\exp\left[-i\int_{-\infty}^{+\infty} dt' H^I_{int}(t')\right] . \qquad (Dyson's \quad formula)$$

- From the above expression one can see that S is formally a unitary operator

$$SS^\dagger = I = S^\dagger S$$

S-Matrix

- In our model we have

$$S = T \exp\left[i \int d^4x \phi_I(x) J(x)\right]$$

- The S operator can then be regarded as a map relating the initial and final Hilbert spaces, both isomorphic to the Fock space of the free field theory
- To evaluate S we will use:
- Note: From now on we will drop the label (I) to refer to the interaction picture.Our simplified notation is $\phi_I \to \phi$, so from now on ϕ denotes a free field.
- Wick's theorem:

$$T(\phi_1\phi_2\cdots\phi_n) =: \phi_1\phi_2\cdots\phi_n : + : \sum_{all \quad posible \quad contractions} \phi_1\phi_2\cdots\phi_n :$$

The general trick to compute a T-ordered product of operators (relating it into a normal-ordered operator) in field theory

Wick's theorem and Feynman propagator

- Let ϕ_1 and ϕ_2 be free fields. The contraction of both fields is defined as

$$\phi_1(x)\phi_2(y) \equiv T(\phi_1(x)\phi_2(y)) - :\phi_1(x)\phi_2(y):$$

For free fields the contraction is a c-number. By taking the vacuum expectation value, one gets

$$\phi_1(x)\phi_2(y) = \langle 0|T(\phi_1(x)\phi_2(y))|0\rangle$$

- This two-point function is called the **Feynman propagator**:

$$\begin{aligned} \langle 0|T(\phi(x)\phi(y))|0\rangle &\equiv D_F(x-y) \\ &= \int \frac{d^3p}{(2\pi)^3 2\omega_p}[e^{-ip(x_1-x_2)}\theta(x_1^0 - x_2^0) + e^{-ip(x_2-x_1)}\theta(x_2^0 - x_1^0) \end{aligned}$$

S-matrix from Wick's algorithm

- We have all the ingredients to compute the S-matrix for our model. The Wick algorithm implies

$$S = I + i\int d^4x_1 : \phi(x_1) : J(x_1) - \frac{1}{2!}\int d^4x_1 d^4x_2 : \phi(x_1)\phi(x_2) : J(x_1)J(x_2)$$
$$-\frac{1}{2!}\int d^4x_1 d^4x_2 D_F(x_1 - x_2) J(x_1)J(x_2) + \mathcal{O}(J^3)$$

- A generic term in the above expansion is of the form

$$\sum_{n,c} \frac{i^n}{n!}\ \frac{n!}{2^c c!(n-2c)!}[\int d^4x_1 d^4x_2 \phi(x_1)\phi(x_2) J(x_1)J(x_2)]^c : [\int d^4y \phi(y)J(y)]^{n-2c} :$$

  The number n refers to the order in perturbation theory (i.e, the power in J), while c indicates the number of contractions.

- The combinatorial factor

$$\frac{n!}{2^c c!(n-2c)!}$$

  counts the number of ways of choosing c pairs (contractions) from n objects (number of fields involved at order n).

S-matrix from Wick's algorithm

- We can arrange the above expression in the form

$$\sum_{u,c} \frac{1}{c!} [\frac{-1}{2} \int d^4x_1 d^4x_2 \phi(x_1)\phi(x_2) J(x_1)J(x_2)]^c \frac{i^u}{u!} : [\int d^4y \phi(y)J(y)]^u : ,$$

where c refers to the number of contractions, u is the number of uncontracted fields and the sum is restricted to $2c + u = n$. Finally, we can encapsulate the result for S in the following more compact expression

$$S = : \exp i \int d^4x \phi(x)J(x) : \exp [-\frac{1}{2} \int d^4x_1 d^4x_2 D_F(x_1 - x_2)J(x_1)J(x_2)] .$$

- To get a more explicit expression we naturally introduce the "on-shell" Fourier transform of $J(x)$

$$\int d^4x e^{ipx} J(x) \equiv \int d^4x e^{i(\omega_p x^0 - \vec{p}\cdot\vec{x})} J(x) \equiv \tilde{J}(\mathbf{p}) ,$$

- And write the free-field operator ϕ as

$$\phi(x) = \int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}} (a_{\mathbf{p}} e^{-ipx} + a^\dagger_{\mathbf{p}} e^{ipx})$$

- Therefore

$$
\begin{aligned}
: e^{i\int d^4x\phi(x)J(x)} : \quad &= \; : e^{i\int d^4x \int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}}(a_{\mathbf{p}}e^{-ipx}+a^\dagger_{\mathbf{p}}e^{ipx})J(x)} : \\
&= \; : e^{i\int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}}(a_{\mathbf{p}}\tilde{J}^*(\mathbf{p})+a^\dagger_{\mathbf{p}}\tilde{J}(\mathbf{p}))} : \\
&= \; e^{i\int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}} a^\dagger_{\mathbf{p}}\tilde{J}(\mathbf{p})} e^{i\int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}} a_{\mathbf{p}}\tilde{J}^*(\mathbf{p})}
\end{aligned}
$$

- Moreover, we can also write [recall that $D_F(x_1 - x_2)$ has both real and imaginary parts]

$$
e^{-\frac{1}{2}\int d^4x_1 d^4x_2 D_F(x_1-x_2)J(x_1)J(x_2)} = e^{-\frac{1}{2}(\bar{n}+i\beta)}
$$

where $\bar{\mathbf{n}}$ is

$$
Re[\int d^4x_1 d^4x_2 D_F(x_1-x_2)J(x_1)J(x_2)] \equiv \bar{\mathbf{n}}
$$

- The imaginary part β appears as a pure phase in the S matrix

- We can evaluate the real number α from the expression for the real part of the propagator

$$Re\ D_F(x_1 - x_2) = Re \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ip(x_1 - x_2)} \ .$$

- We immediately get (exercise: check it)

$$\bar{n} \equiv Re \int d^4x_1 d^4x_2 D_F(x_1 - x_2) J(x_1) J(x_2) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} |\tilde{J}(\mathbf{p})|^2 \geq 0 \ .$$

- The above expression shows that $\bar{n}$ is non-negative real number !!!
- Therefore, our final expression for the operator S is, up to a global phase factor,

$$\begin{aligned} S &= \exp\left[i \int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}} a^\dagger_{\mathbf{p}} \tilde{J}(\mathbf{p})\right] \exp\left[i \int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}} a_{\mathbf{p}} \tilde{J}^*(\mathbf{p})\right] \\ &\times \exp\left[-\frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2\omega_p} |\tilde{J}(\mathbf{p})|^2\right] \ . \end{aligned}$$

Particle creation induced by the external scalar source

- At very early times the vacuum is well-described by the vacuum $|0\rangle$ of the free field theory
- The action of S on $|0\rangle$ will give the particle content at late times, when the source has ceased to operate
- For example, the probability amplitude for creating the 1-particle state $|\mathbf{p}\rangle$ induced by the source is (help: expand the exponentials in S and operate)

$$\langle \mathbf{p}|S|0\rangle = \langle 0|a_{\mathbf{p}} S|0\rangle = \langle 0|S|0\rangle \frac{i\tilde{J}(\mathbf{p})}{\sqrt{2(2\pi)^3\omega_p}}$$

- While for the 2-particle state $|\mathbf{p}_1\mathbf{p}_2\rangle$ the probability amplitude is

$$\langle \mathbf{p}_1\mathbf{p}_2|S|0\rangle = \langle 0|S|0\rangle \frac{i\tilde{J}(\mathbf{p}_1)}{\sqrt{2(2\pi)^3\omega_{p_1}}} \frac{i\tilde{J}(\mathbf{p}_2)}{\sqrt{2(2\pi)^3\omega_{p_2}}} \ .$$

- From these one can obtain the probability of creating 1-particle (independent of momenta)

$$P(1) = \int d^3p |\langle \mathbf{p}|S|0\rangle|^2 = |\langle 0|S|0\rangle|^2 \int \frac{d^3p}{(2\pi)^3 2\omega_p} |\tilde{J}(\mathbf{p})|^2 \ .$$

Particle creation induced by the external source

- Similarly, the probability of producing 2-particles is

$$P(2) = \frac{1}{2!} \int d^3p_1 d^3p_2 |\langle \mathbf{p}_1 \mathbf{p}_2 | S | 0 \rangle|^2 = \frac{1}{2!} \langle 0|S|0\rangle|^2 [\int \frac{d^3p}{(2\pi)^3 2\omega_p} |\tilde{J}(\mathbf{p})|^2]^2 \ .$$

  The factor $1/2!$ is needed to avoid overcounting, since $|\vec{p}_1\vec{p}_2\rangle = |\vec{p}_2\vec{p}_1\rangle$.

- In general, the probability $P(n)$ of finding n particles in the final state is given by

$$P(n) = \langle 0|S|0\rangle|^2 \frac{1}{n!} [\int \frac{d^3p}{(2\pi)^3 2\omega_p} |\tilde{J}(\mathbf{p})|^2 \ ]^n \ .$$

$$P(n) = e^{-\bar{n}} \frac{1}{n!} \bar{n}^n$$

- The expression obtained for $P(n)$ is consistent. Adding up $P(n)$ for any n covers all possible processes

$$\sum_{n=0}^{\infty} P(n) = 1 \ .$$

  This reflects the unitary of the S-matrix.

- The spectrum of the particles created by the source at late-times is a *Poisson distribution*

$$P(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!} ,$$

- The mean number of created particles is just given by

$$\langle N \rangle = \sum_{n=0}^{\infty} n \, P(n) = \bar{n} \, .$$

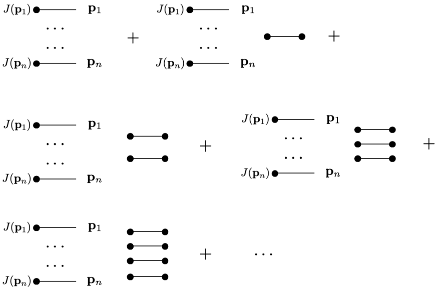
This result provides a justification for the notation $\bar{n}$

- The appearance of the Poisson distribution reflects the statistical independence of the successive creation of particles

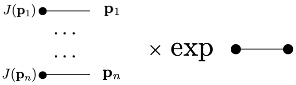
- Remark 1: Particles are created individually, not in pairs. Particles are singly created
- Remark 2: When initial particles are present, particle creation is accompanied by stimulated emission (Bose enhancement).
- Remark 3: Particles can also be annihilated: $\langle 0|S|\mathbf{p}\rangle = \langle 0|S|0\rangle \frac{i\tilde{J}^*(\mathbf{p})}{\sqrt{2(2\pi)^3\omega_p}}$

Feynman Diagrams for the all orders S matrix: individual particle creation

- Graphical representation of the (all orders) particle creation process



- The contributions of the vacuum diagrams can be summed exactly into the exponential of the single **connected** vacuum diagram



Comparison with the prediction of classical field theory

- Furthermore, one can also easily show that the mean value of the energy of the created particles is given by

$$\langle 0|S^{\dagger} H S|0\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\omega_p |\tilde{J}(p)|^2$$

- Very important: this prediction agrees with the analogous prediction within the classical theory !!!

$$E_{\rm rad} = \int d^3x\, T^{00}_{\rm rad} = \int \frac{d^3p}{2(2\pi)^3\omega_p}\omega_p |\tilde{J}(p)|^2 \ ,$$

where $T^{\mu\nu}_{\rm rad}$ is the classical stress-energy tensor of the scalar field evaluated of the classical (scalar) radiation field.

Coherent states

- Underlying reason: the quantum state $S|0\rangle$ is a *COHERENT STATE*

$$S|0\rangle = \exp\left[-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3 2\omega_p}|\tilde{J}(\mathbf{p})|^2\right] \times \exp\left[i\int \frac{d^3p}{\sqrt{2(2\pi)^3\omega_p}} a^\dagger_{\mathbf{p}}\tilde{J}(\mathbf{p})\right]|0\rangle \ .$$

- This means that it is an eigenvector of the annihilation part of the field (usually denoted by $\phi^{(+)}(x)$) [recall coherent states of the harmonic oscillator]

$$\phi^{(+)}(x)S|0\rangle = \phi^{(+)}_{\rm rad}(x)S|0\rangle$$

$$\phi^{(+)}_{\rm rad}(x) = i\int \frac{d^3p}{2(2\pi)^3\omega_p}\tilde{J}(\mathbf{p})e^{-ipx}$$

- The expectation value of the quantized field $\phi(x)$ agrees precisely with the classical radiation field

$$\langle 0|S^{-1}\phi(x)S|0\rangle = \phi_{\rm rad}(x) = i\int \frac{d^3p}{2(2\pi)^3\omega_p}[\tilde{J}(\mathbf{p})e^{-ipx} - \tilde{J}^*(\mathbf{p})e^{ipx}]$$

- Conclusion: when quantum field fluctuations are negligible, the classical results are recovered

Vacuum persistence amplitude and effective action

- Of special relevance is the quantity $P(0)$

$$\langle 0|S|0\rangle \equiv Z[J] = e^{-\frac{1}{2}\int d^4x_1 d^4x_2 D_F(x_1-x_2)J(x_1)J(x_2)} ,$$

known as the vacuum persistence amplitude. It gives the probability amplitude of no particle creation

- It is also denoted, using a slightly different notation, by

$$\langle 0|S|0\rangle \equiv Z[J] \equiv e^{iW[J]} = e^{-\frac{1}{2}\int d^4x_1 d^4x_2 D_F(x_1-x_2)J(x_1)J(x_2)}$$

- $W[J]$ is usually refereed as the **effective action**.
- The imaginary part of $W[J]$ causes the creation of particles out of the vacuum

$$|\langle 0|S|0\rangle|^2 = e^{-2\mathrm{Im}W} = e^{-\bar{n}} \leq 1 \, .$$

Going back to Maxwell theory

- For Maxwell theory

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

we obtain

$$\langle 0|S|0\rangle = \exp -\frac{1}{2} \int d^4x_1 d^4x_2 D^{\text{Feyn}}_{\mu\nu}(x_1 - x_2) J^\mu(x_1) J^\nu(x_2) \ ,$$

where now $D^{\text{Feyn}}_{\mu\nu}$ is the Feynman propagator

- The imaginary part of the effective action becomes ($p_\mu J^\mu = 0$)

$$ImW = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2\omega_p} J^\mu(\mathbf{p}) J^{\mu *}_\mu(\mathbf{p}) \geq 0$$

and hence

$$|\langle 0|S|0\rangle|^2 = \exp\left[-\int \frac{d^3p}{(2\pi)^3 2\omega_p} \sum_{s=1}^{2} |\tilde{J}^s(p)|^2\right] \leq 1 \ .$$

$$\tilde{J}^s(\mathbf{p}) \equiv \epsilon^s_\mu(\mathbf{p}) \tilde{J}^\mu(\mathbf{p})$$

$\epsilon^s_\mu \sim$ corresponds to transverse polarizations

Comparison with the prediction of classical electrodynamics

- Alongside the corresponding result for the scalar-field model, we can also readily show that the mean energy of the created photons is given by

$$\langle 0|S^{\dagger}HS|0\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\omega_p[\sum_{s=1}^{2}|\tilde{J}^s(p)|^2]$$

- This prediction agrees with the analogous prediction from classical electrodynamics

$$E_{\rm rad} = \int d^3x \, T^{00}_{\rm rad} = \int \frac{d^3p}{2(2\pi)^3\omega_p}\omega_p[\sum_{s=1}^{2}|\tilde{J}^s(p)|^2] \ ,$$

  where $T^{\mu\nu}_{\rm rad}$ is now the classical stress-energy tensor of the electromagnetic field evaluated of the classical radiation field.

- Remark: $S|0\rangle$ is also a coherent state

Quantum electrodynamics

- In the previous slide we have considered Maxwell theory

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

- Next, we turn to a more refined framework of electrodynamics in which charged matter is modeled by a quantized Dirac field

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\bar{\Psi}\gamma^\mu\Psi + i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$$

- We now truncate the theory by assuming an external classical electromagnetic field $A_\mu(x)$, quadratically coupled to a quantized (charged) Dirac field Ψ
- A_μ should be regarded as an external source acting on a quantized Dirac field

$$\mathcal{L}_{QED'} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu$$

$$A_\mu \sim classical\ source$$

$$\Psi \sim quantized\ field$$

Quantum electrodynamics: QED'

- Assume QED' with the action (here A_μ is a prescribed classical field)

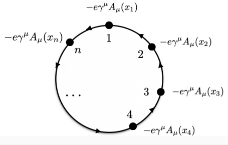
$$\mathcal{L}_{QED'} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi - e\bar{\Psi}\gamma^\mu\Psi A_\mu$$

- Perturbation theory together with the Dyson formula yields

$$\langle 0|S|0\rangle = e^{iW[A^\mu]}$$

$$iW[A^\mu] = -\sum_{n=1}^{\infty}\frac{(-ie)^n}{n}\int d^4x_1\cdots d^4x_n \mathrm{tr}[S_F(x_n-x_1)\not{A}(x_1)S_F(x_1-x_2)\not{A}(x_2) \cdots S_F(x_{n-1}-x_n)\not{A}(x_n)]$$

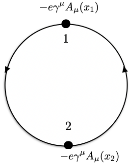
- The diagrammatic representation of the expansion is [$S_F \equiv$ Feynman propagator]



- The overall minus sign is also due to the fermion loop.

Results at finite perturbative order

- At leading order $\mathcal{O}(e^2)$ the contribution of the vacuum polarization diagram



  to ImW requires the Fourier components $F_{\mu\nu}(q)$ to obey

$$q^2 > 4m^2$$

- Also $F_{\mu\nu}F^{\mu\nu} < 0$, i.e., an electric-type field
- The prediction of perturbation theory for electron-positron production for a constant electric field $\vec{E}$ is

$$ImW = 0 + \mathcal{O}(e^{2n}) \ , \quad \forall n$$

Schwinger's non-perturbative calculation for W

- Schwinger was able to manage the all-orders expression for $W[A^\mu]$ introducing many important technical tools:
- Schwinger's proper time formalism

$$\frac{1}{A - i\epsilon} = i \int_0^\infty ds e^{-is(A - i\epsilon)} \ ,$$

- Trace and determinant of operators in field theory

$$Tr[\log O] = \int d^4x \ \langle x | \log O | x \rangle \ .$$

- Thus, we have

$$\begin{aligned} e^{iW[A^\mu]} &= \exp\{\mathrm{Tr}[\sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} ((i\not\partial - m)^{-1}(-e\not A))^n]\} \\ &= \exp\{\mathrm{Tr}[\log(1 + (i\not\partial - m)^{-1}(-e\not A))]\} \end{aligned}$$

$$W[A] = +\frac{i}{2} \int d^4x \int_0^\infty \frac{ds}{s} e^{-i(m^2 - i\epsilon)s} \mathrm{tr}\langle x | e^{-i(D^2 + \frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu})s} | x \rangle \ .$$

Schwinger's Tour de Force for a constant electric field $\vec{E}$

- An explicit prediction can be obtained by assuming a constant electric field $\vec{E} = (0, 0, E)$ and a zero magnetic field.
- Schwinger derived the exact propagator in the presence of a constant electromagnetic field $F_{\mu\nu}$

$$\langle x | e^{-i(D^2 + \frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu})s} | x \rangle$$

- The effective Lagrangian density [$W = \int d^4x \mathcal{L}_{eff}$] for a charged Dirac field in the presence of a constant electric field turns out to be

$$\mathcal{L}_{eff} = 2 \int_0^\infty \frac{ds}{(4\pi)^2 s^3} e^{-i(m^2 - i\epsilon)s} [\frac{eEs}{\sinh eEs} \cosh eEs - 1 - \frac{1}{3}(eEs)^2] \, .$$

- Note 1: We have included the additive constant necessary to make $\mathcal{L}_{eff}$ vanish in the absence of a field.
- Note 2: We have also removed the term (renormalization) that produces the logarithmic divergence.
- Note 3: The real part of $\mathcal{L}_{eff}$ gives, for a generic and constant $F_{\mu\nu}$,the Heisenberg-Euler effective Lagrangian (1936).

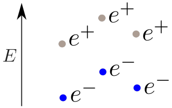
Schwinger's effect for charged fermions

- Due to the poles at $s_n = -i\pi n/|eE|$ the above effective action has a non-zero imaginary part Im$\mathcal{L}_{eff}$.
- The result gives the fermionic pair production rate per unit volume

$$2\text{Im}\mathcal{L}_{eff} = 2\Gamma = 2\frac{e^2E^2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp(-\frac{n\pi m^2}{|eE|}) \ .$$

in the limiting case of a homogeneous and constant electric field [$V \equiv$ tridimensional volume, $T \equiv$ observation time lapse]

$$|\langle 0|S|0\rangle|^2 = e^{-2VT\Gamma_{Schwinger}} \ ,$$



- $e^{-2VT\Gamma} < 1$ measure the probability that no pairs are created over time T and volume V.

- Since $e^{-2VT\Gamma} \approx 1 - 2VT\Gamma$ is a very small number,
- 2Γ(Electric field $\to e^+e^-$ pairs) estimates the probability, per unit time and volume, that any number of pairs (electron-positron) are created
- Since it is a fundamental physical result, it is useful to reinstate the physical constants $\hbar$ and c (we use units where the fine-structure constant is $\alpha \equiv e^2/\hbar c$)

$$\Gamma_{Schwinger} = \frac{e^2E^2}{8\pi^3 c\hbar^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\exp[-\frac{n\pi m^2c^3}{\hbar|eE|}] \, .$$

- It is a non-perturbative effect. Note that the above formula cannot be expanded as a power series in e.
- The rate of pair production is very small. A very strong electric field is needed to reach the critical value $E_{critical} = \frac{m_e^2c^3}{e\hbar} \approx 10^{18} volts/meter$
- Such electric field strengths lie far beyond current technological capabilities, even for the most intense lasers available today.
- The assumption of a constant electric field is not physically realistic; however, it serves as a useful starting point for more refined analyses that incorporate time-dependent electric fields

Summary and Remarks

- Spontaneous particle creation is deeply rooted in the fundamental principles of quantum field theory
- In the case of a linear coupling,
$$A_\mu J^\mu ,$$
it correctly reproduces the familiar results of electromagnetic radiation from classical electrodynamics
- For quadratic couplings,
$$e\,\bar{\Psi}\gamma^\mu \Psi A_\mu ,$$
the phenomenon becomes substantially richer and more intricate
- In particular, it allows for genuinely non-perturbative particle-antiparticle pair creation
- This framework provides a natural first step toward understanding pair creation induced by gravitational fields
- Viewed from the opposite perspective, it also constitutes a prototypical example of *analogue gravity*