emergence of hydrodynamics / classical simulations of quantum transport

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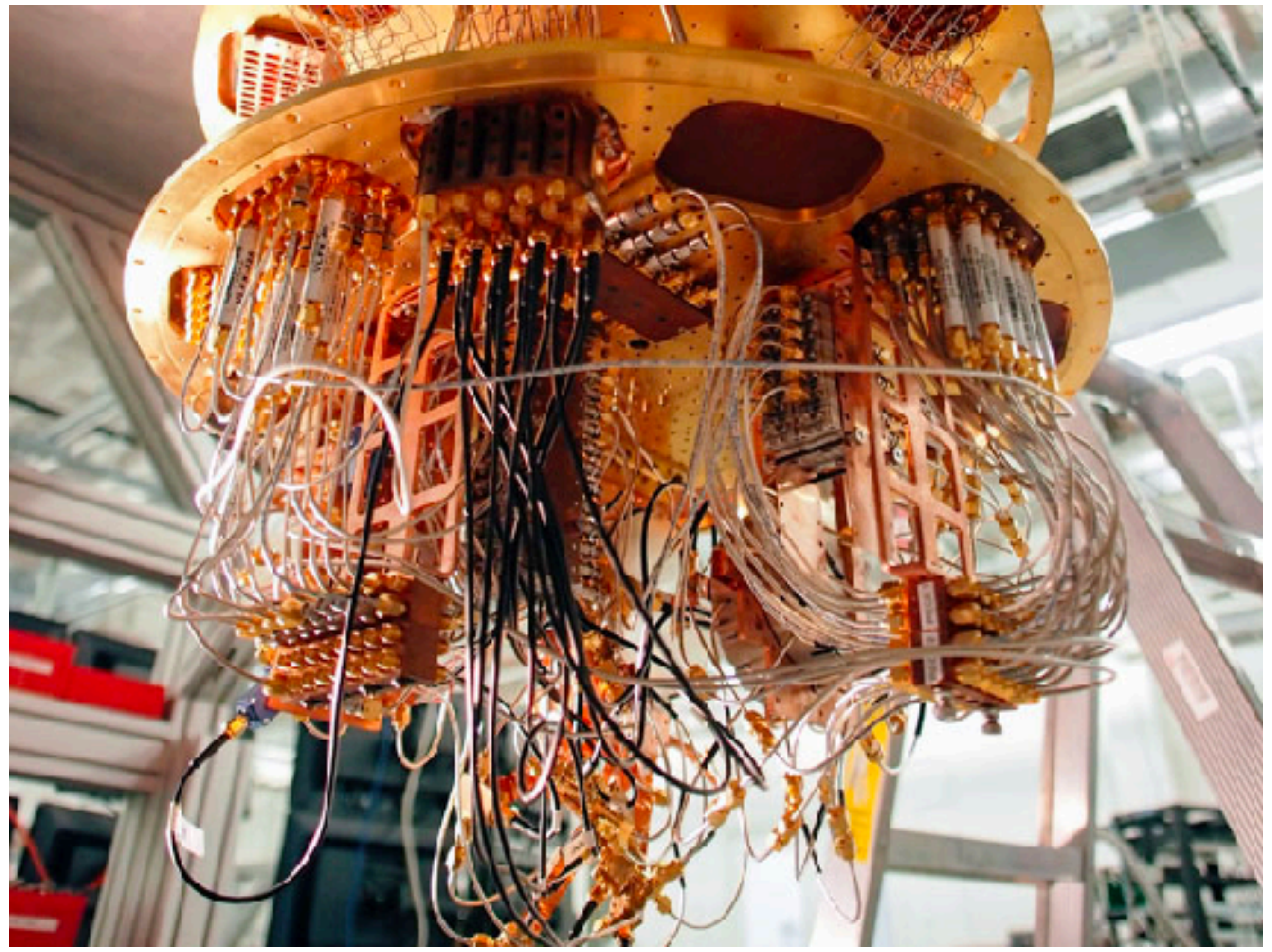
limitations of phenomenological hydro

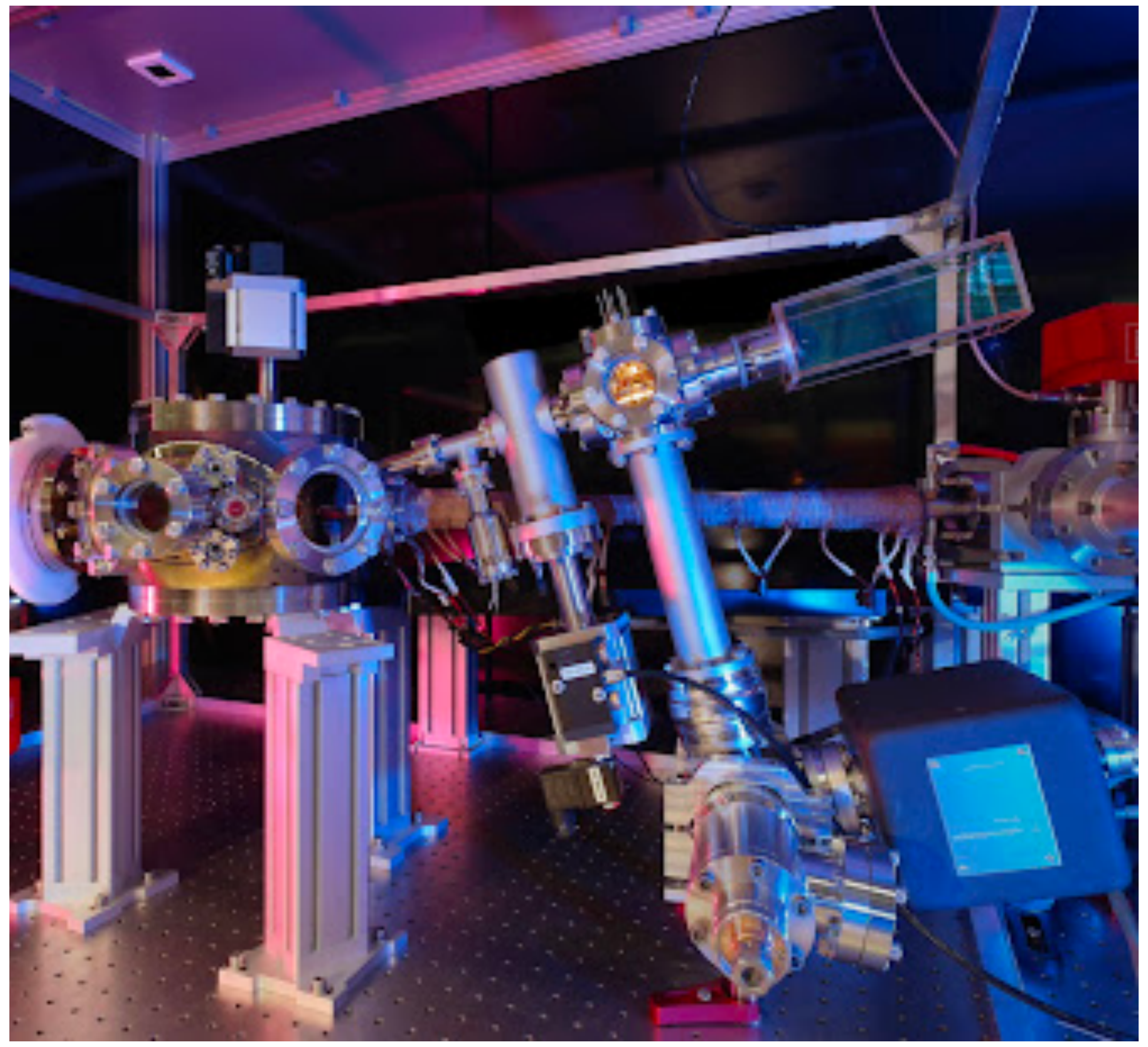
- “Standard” hydro philosophy:
  - Start with symmetries, identify conserved charges and Goldstone modes
  - Write down all terms in gradient expansion, with general coefficients
  - Gives dynamical implications of a particular symmetry structure, could allow one to infer symmetries from experiment
- Basic limitation: disconnected from microscopics
  - Attempts to derive hydro from microscopics: Boltzmann kinetic theory, etc., controlled in specific cases (e.g., weakly interacting systems)
  - But we care about generic strongly interacting systems
- Closely related question: how hard is it to classically predict quantum dynamics?

worst-case scenario for numerics?

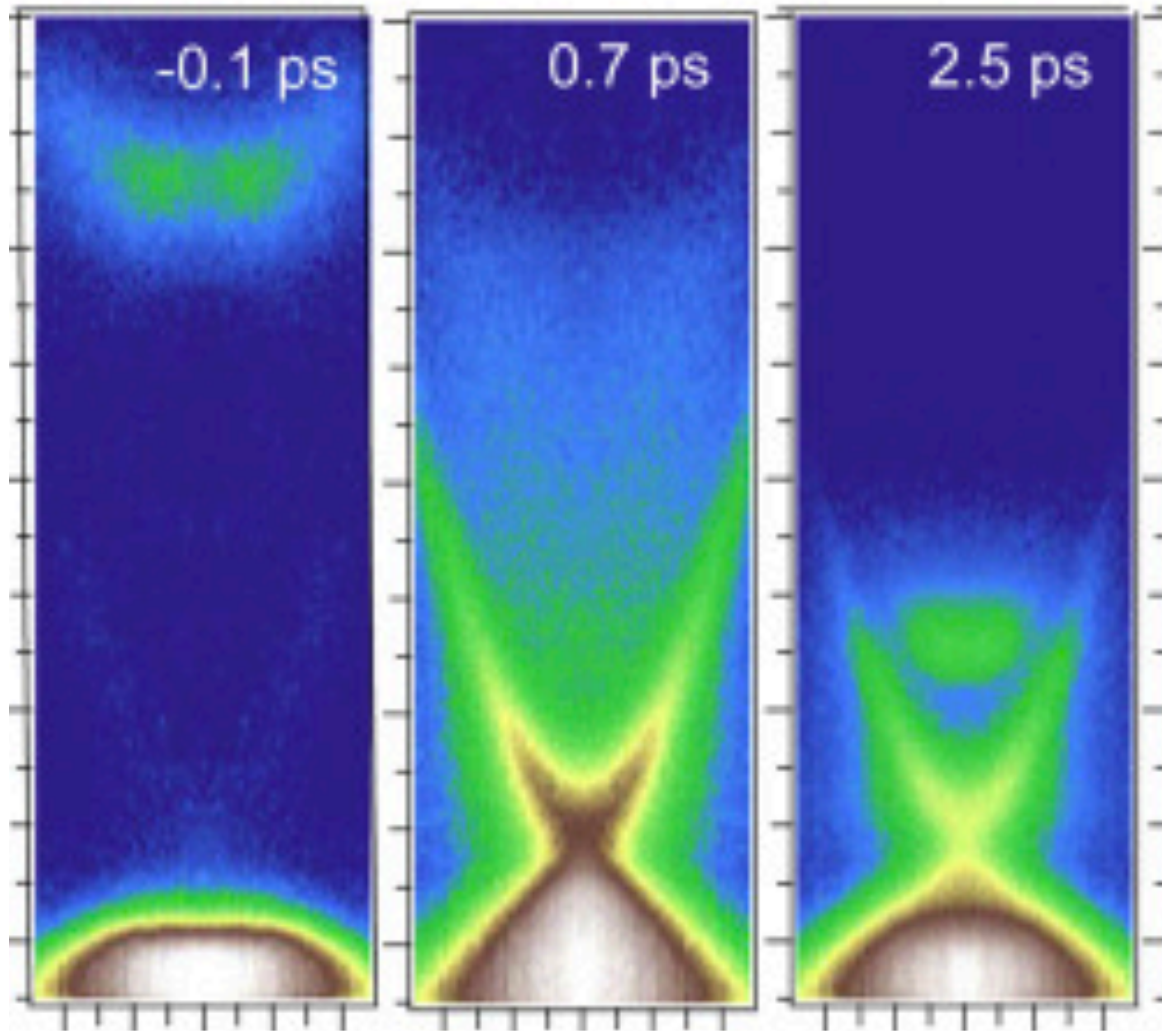
- Tensor-network methods work well when entanglement is low
- Dynamics starting from states that are high-energy-density (far from ground state) generate a lot of entanglement (exponentially hard in time to simulate the state exactly)
- Are we stuck?



superconducting qubit arrays



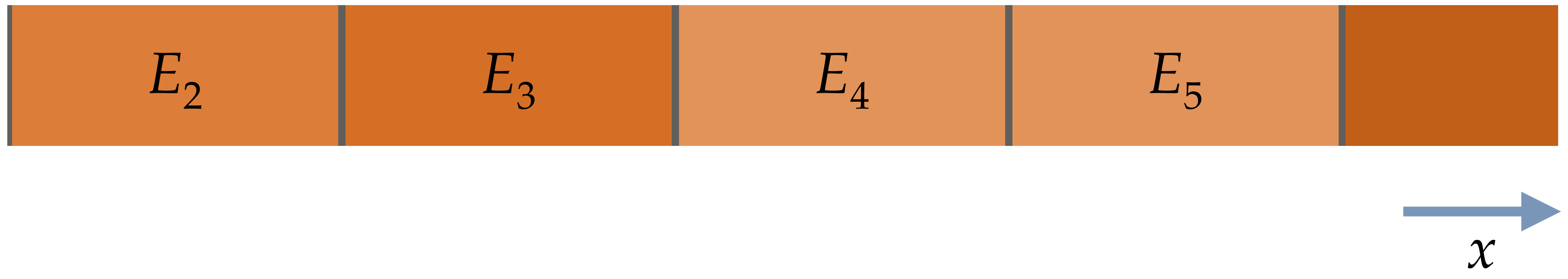
ultracold atoms



ultrafast probes of materials

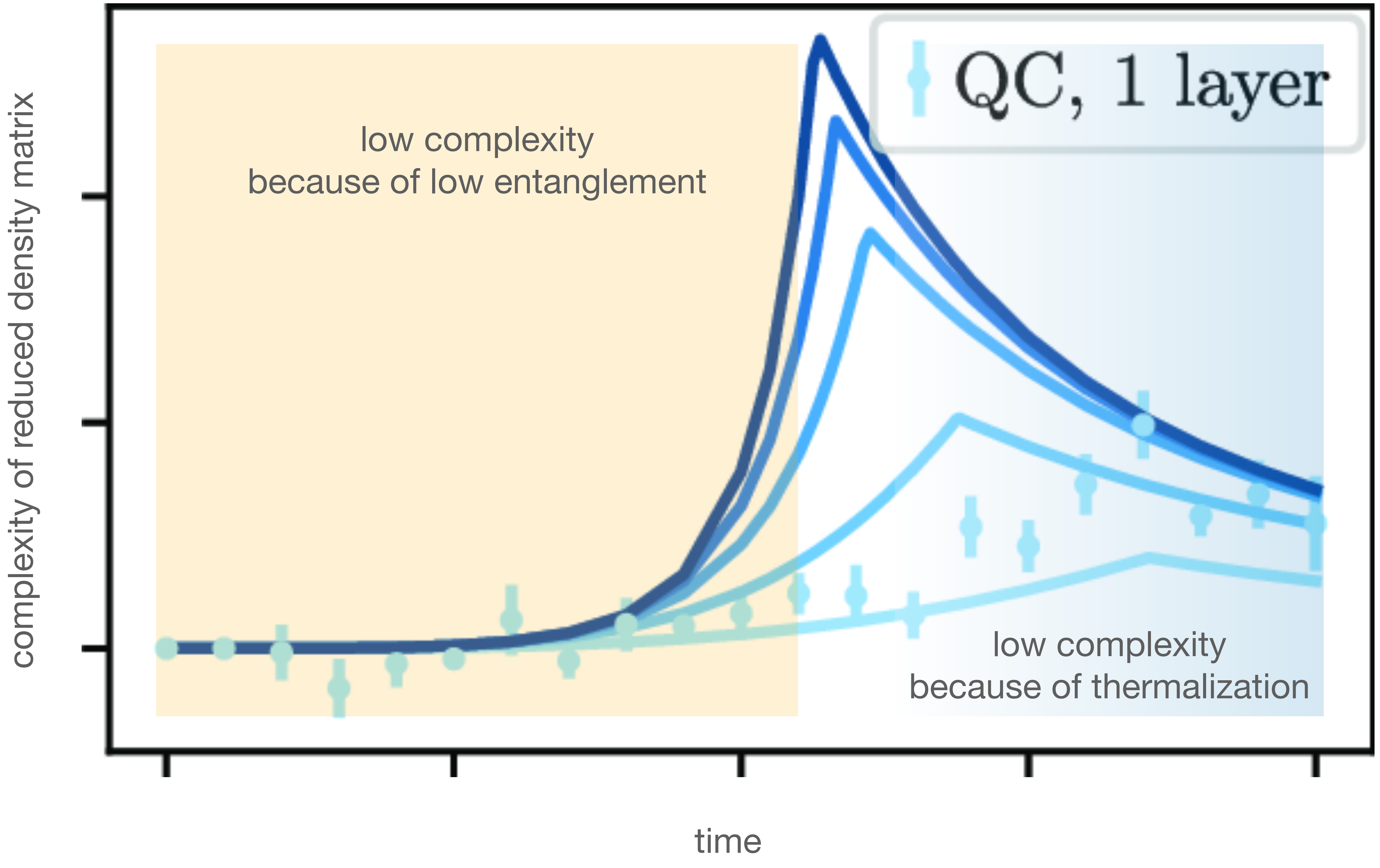
thermalization and hydrodynamics

- High temperature dynamics is complex, chaotic
- Chaos leads to effective randomization of state
    - Local subsystems have thermal density matrices, full system does not but we only care about local properties
- System goes to maximum entropy state subject to conservation laws (“thermalization”)
- Hydrodynamics:
    - Assume system is *locally* in some thermal state (described by local values of conserved variables)
    - Write down equations of motion for conserved variables by gradient expansion (assume that variations are smooth)



entanglement barrier

- Entanglement grows linearly in time, complexity is exponential in entanglement
- High-temperature thermal states like $\exp(-\beta H)$ have low complexity
- Thermalization: reduced density matrix for subsystem can be replaced by thermal state
- Problem: how to bridge between the early and late time physics? ("Crossing the entanglement barrier")
- Equivalent to asking: how do we derive hydrodynamics?



why do we want to do this?

- Intellectual satisfaction
- Might want precise estimates of diffusion constant, etc. (Trivedi et al., 2024)
- We are interested in specific models, we want to be open to discovering new structures beyond what hydro puts in (e.g., quantum scars, strong zero modes, many-body localization)

Article | Published: 30 November 2017

Probing many-body dynamics on a 51-atom quantum simulator

Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Pichler, Soonwon Choi, Alexander S. Zibrov, Manuel Endres, Markus Greiner, Vladan Vuletić & Mikhail D. Lukin

Nature **551**, 579–584 (2017) | Cite this article

finding slow operators

- Picture behind hydro:
  - Most operators relax fast and can be treated as white noise
  - Remaining operators form a “slow operator Hilbert space”
  - Want to perform “adiabatic elimination”/Schrieffer-Wolff to eliminate the fast operators
- Naive idea: work with Liouville superoperator:

$$\mathscr{L}(O) = i[H, O]$$

- Annoyingly, the eigenmodes of $\mathscr{L}$ are just $|E_i\rangle\langle E_j|$, eigenstates of the true evolution; too nonlocal to be helpful

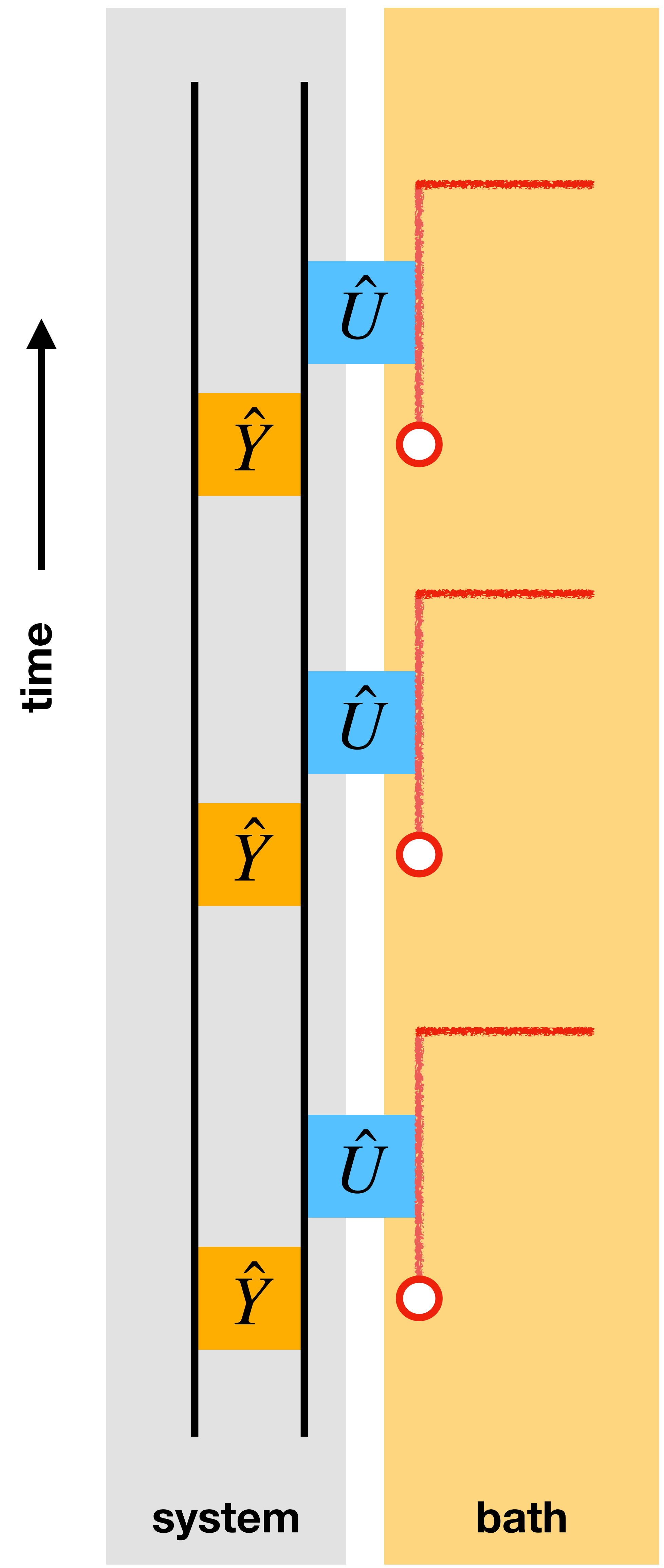
the rest of this talk

- Noise as a way to cut off the entanglement barrier
- Case with no conservation laws: Liouvillian gap and Ruelle resonances
- Case with charge conservation; hydrodynamic projections
- Note: for simplicity we will be working with discrete-time/Floquet dynamics
- Key refs:

Prosen, J Phys A 35, L737 (2002)
Von Keyserlingk, Pollmann, Rakovszky, arXiv:2111.09904
Nahum, Roy, Vijay, Zhou, arXiv:2205.11544
Mori, arXiv:2311.10304
Jacoby, Huse, SG, arXiv:2409.17238
Zhang, Nie, Von Keyserlingk, arXiv:2409.17251

weak dissipation + no conserved quantities

markovian open quantum systems



quantum channel $\mathscr{E}$
(discrete-time version
of Lindblad master eq)

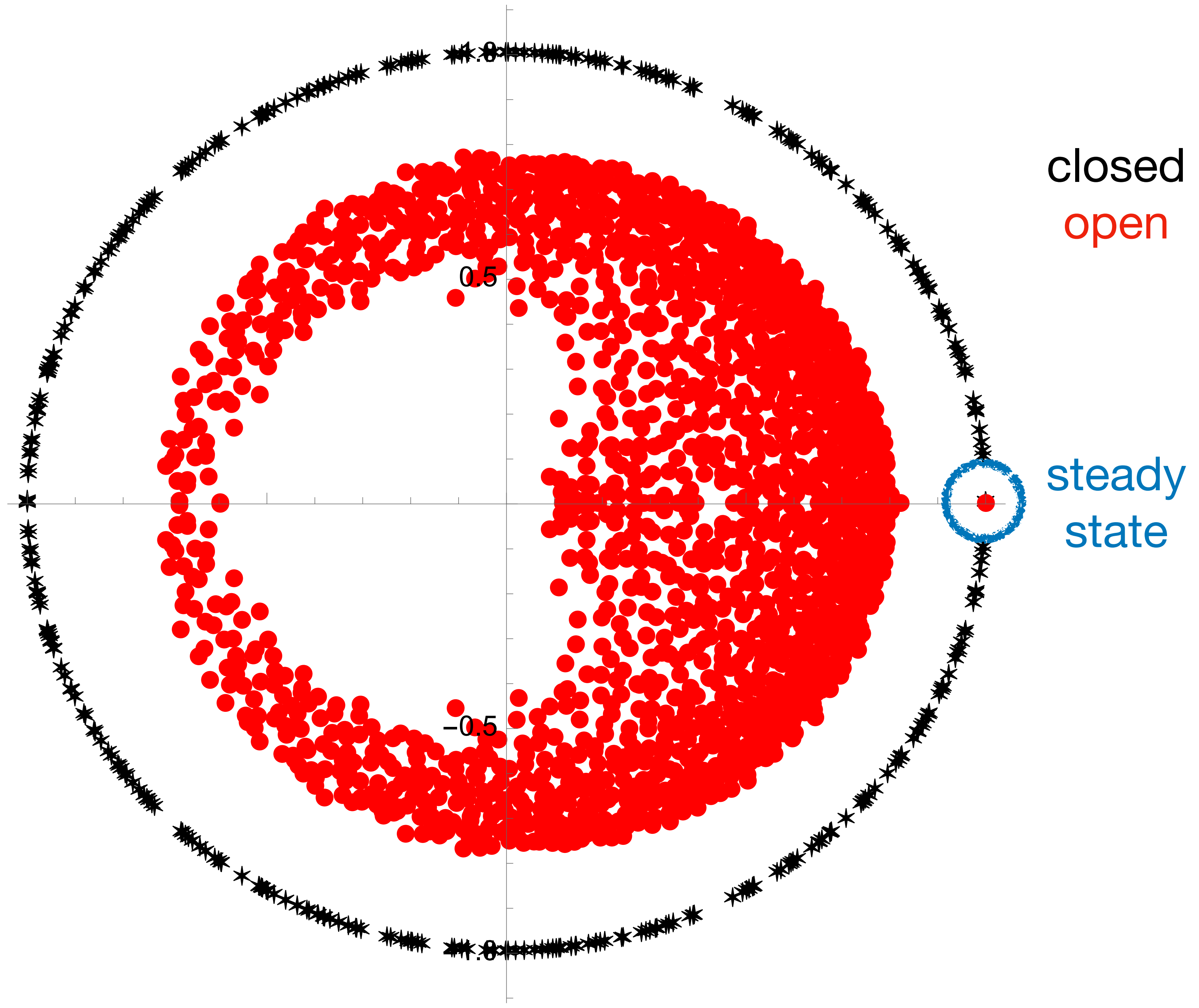
- Q: how does the spectrum of $\mathscr{E}$ evolve in the weak-dissipation limit?

mori's result

- Specifically, consider the following Lindblad master equation with $H(t+\tau) = H(t)$:

$$\partial_t \rho = -i[H(t), \rho] + \gamma \sum_i \left(\sigma_i^z \rho \sigma_i^z - \rho \right)$$

- Integrating this equation for a time τ gives one step of evolution under $\mathscr{E}$



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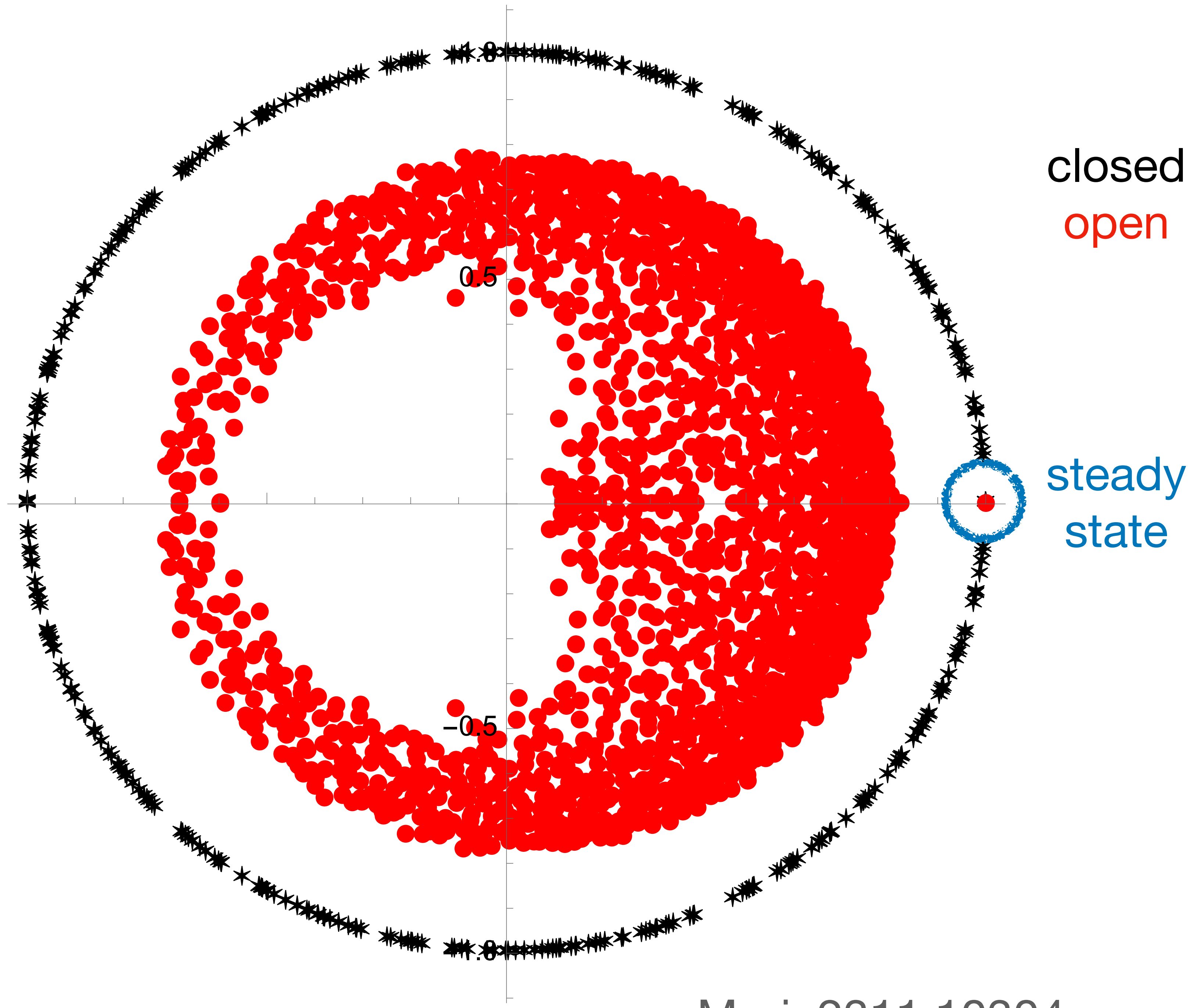
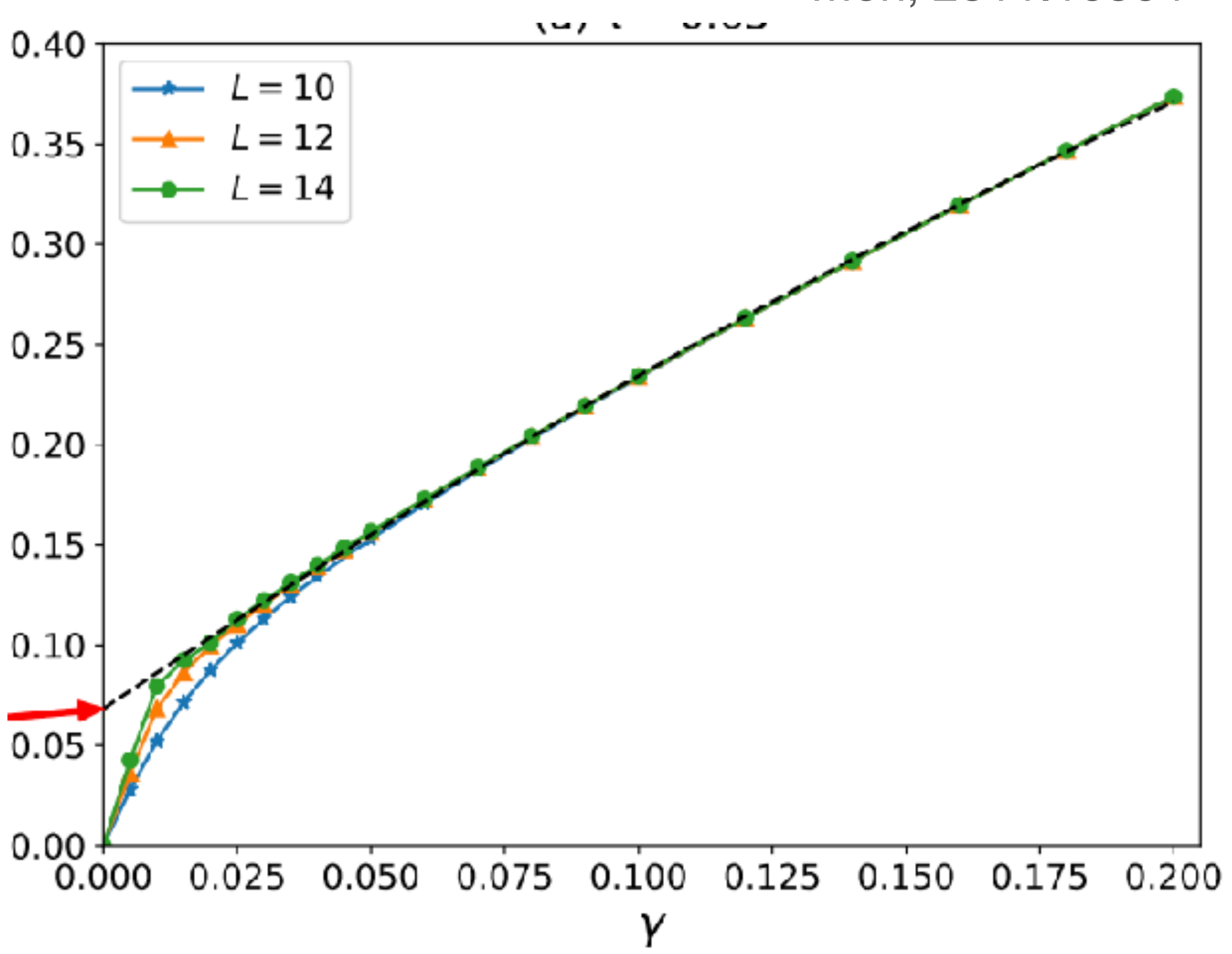
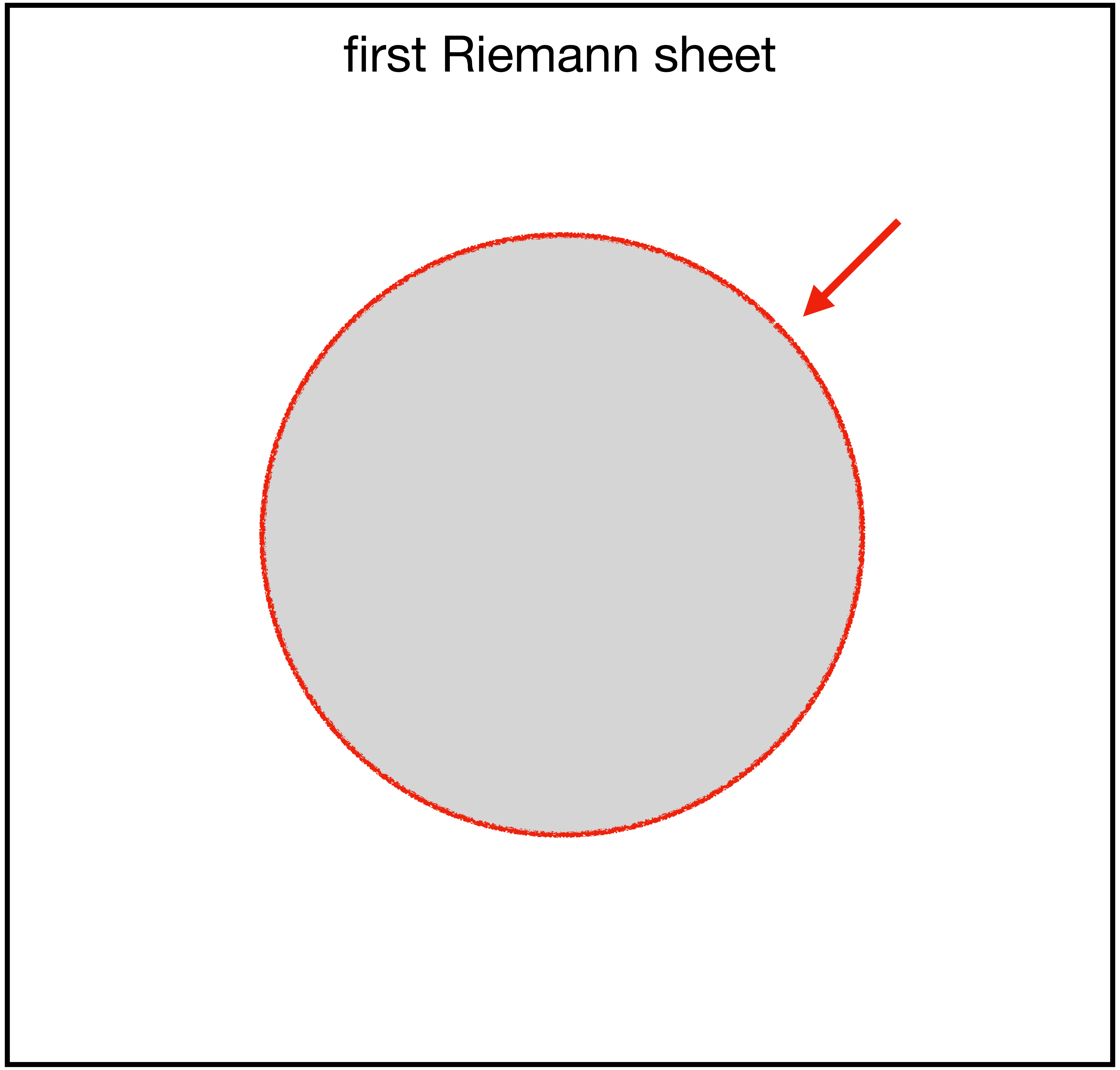
- Integrating this equation for a time τ gives one step of evolution under $\mathscr{E}$
- Why time-periodic? Otherwise, for $\gamma \to 0$, H itself becomes a zero mode
- Mori's finding:
  - Gap stays open and O(1) as $\gamma \to 0$
  - Gap of $\mathscr{E}$ related to decay of local correlation functions

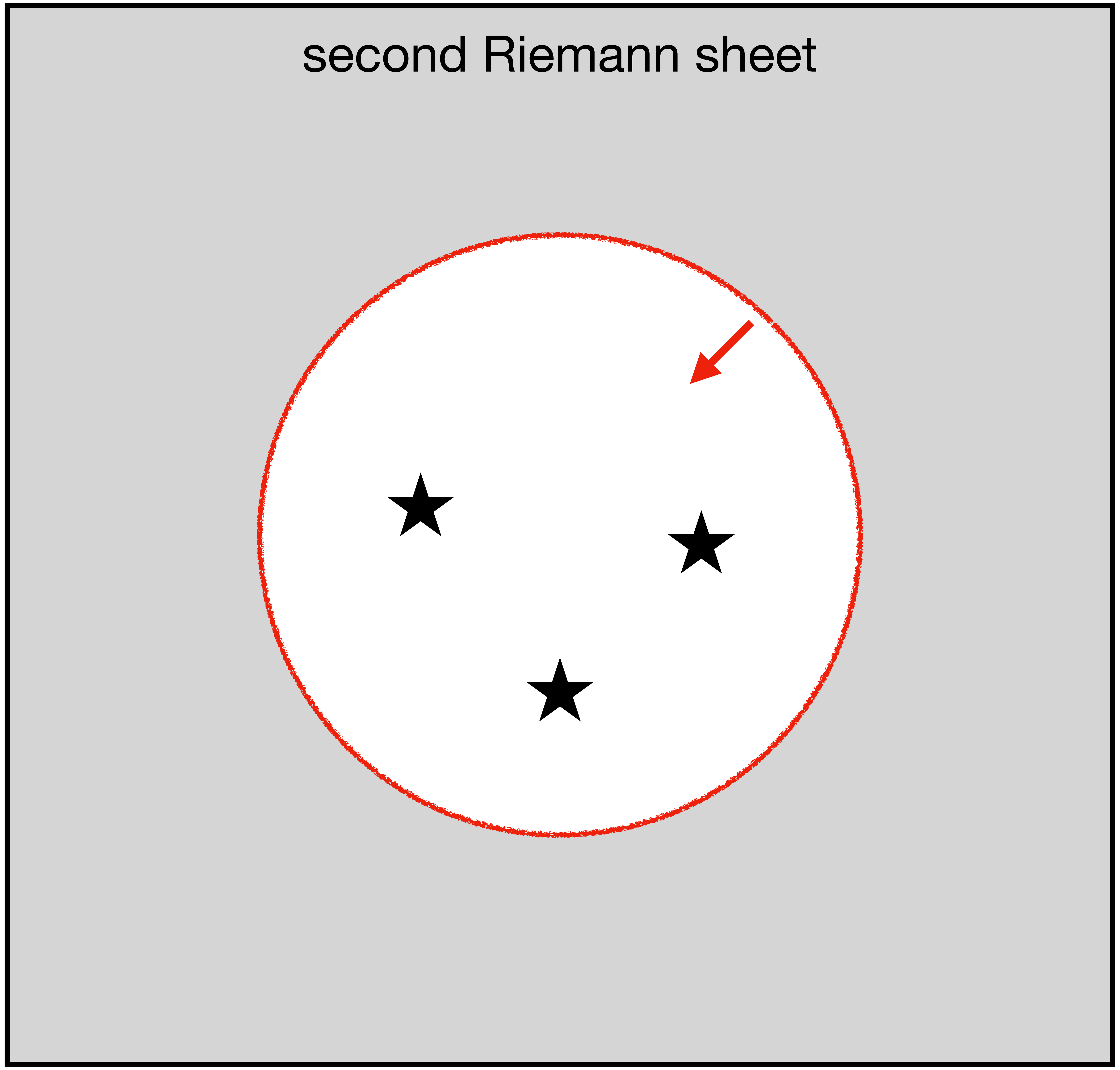


Mori, 2311.10304



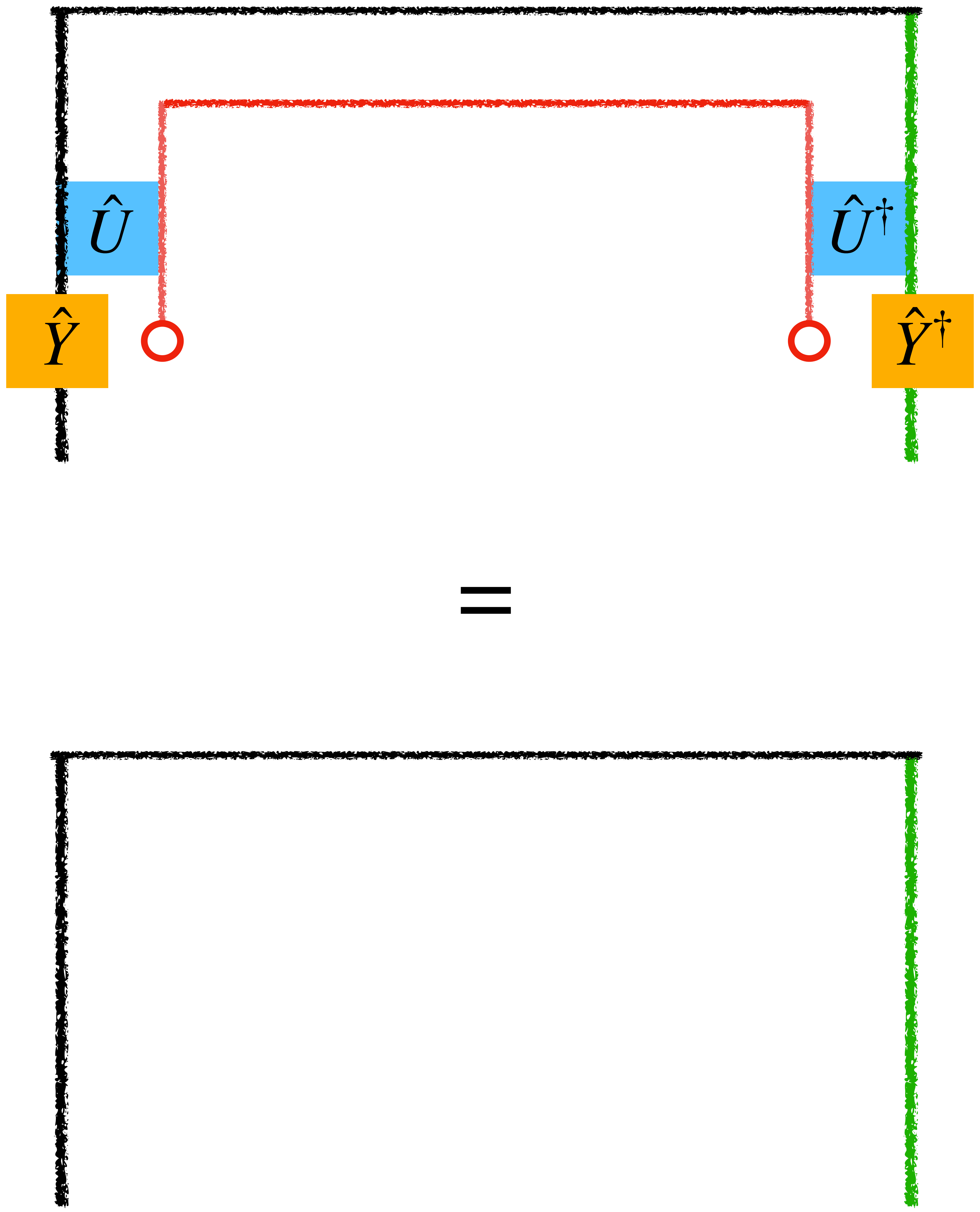
ruelle resonances: analytic structure of $(z - \mathscr{E})^{-1}$





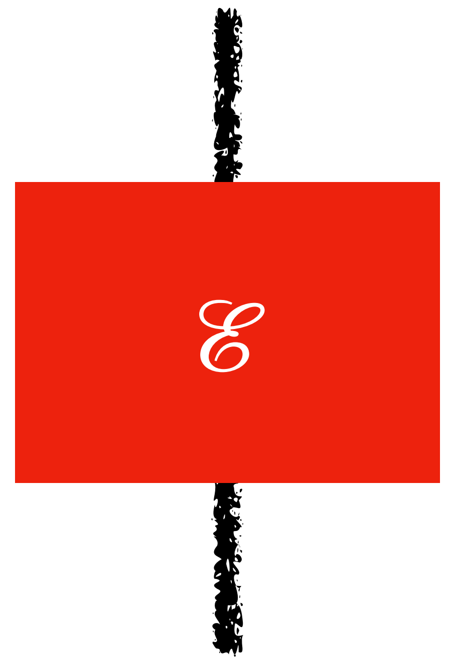
quantum channels in the heisenberg picture

- Density matrix evolves under CPTP map, $\rho \mapsto \mathscr{E}(\rho)$
- We want to compute exp val of some observable,
$\langle O(t) \rangle = \mathrm{Tr}(O\mathscr{E}_t(\rho))$
- Can define a Heisenberg picture for CPTP maps,
$\mathrm{Tr}(\mathscr{E}^A(O)\rho) \equiv \mathrm{Tr}(O\mathscr{E}(\rho))$
- Helpful picture for superoperators:
  - States are column vectors (“superkets”) $|\rho)$
  - Operators are “superbras” $(O|$
  - Bracket $(O|\rho) \equiv \mathrm{Tr}(O^\dagger \rho)$
  - Expectation value looks like $(O|\hat{\mathscr{E}}|\rho)$
  - Left and right action have the same eigenvalues (but different eigenvectors)
- Trace preservation implies $(\mathbb{I}|\hat{\mathscr{E}}|\rho) = (\mathbb{I}|\rho) \equiv \mathrm{Tr}(\rho)$, so
$(\mathbb{I}|\hat{\mathscr{E}} = (\mathbb{I}|$, identity is always a left eigenvector of the adjoint map
(“unital property”)

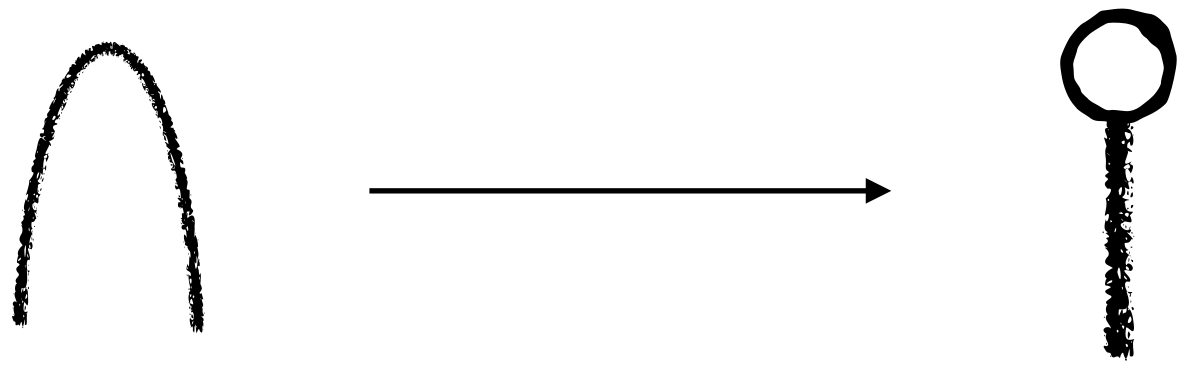


fat-line notation

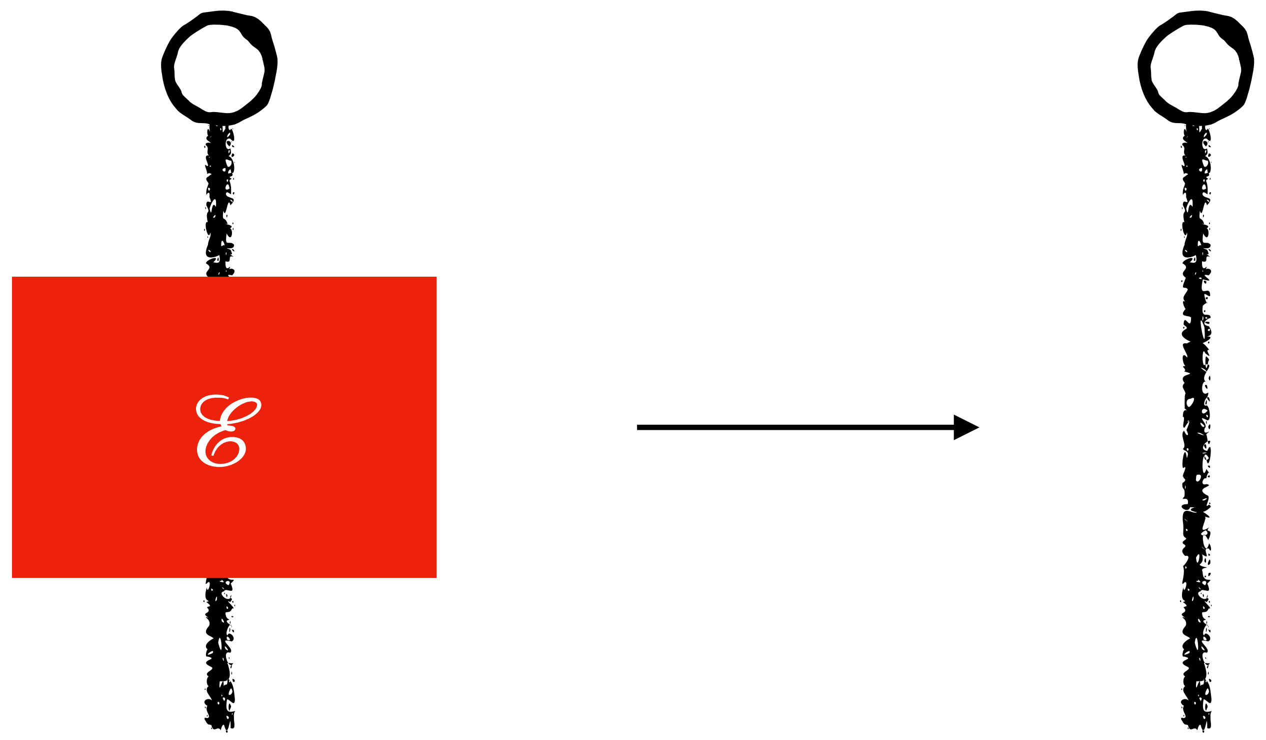
- Fat lines carry density matrices,
  a channel is a superoperator on density matrices



- Observables are vectors acting from the top, identity

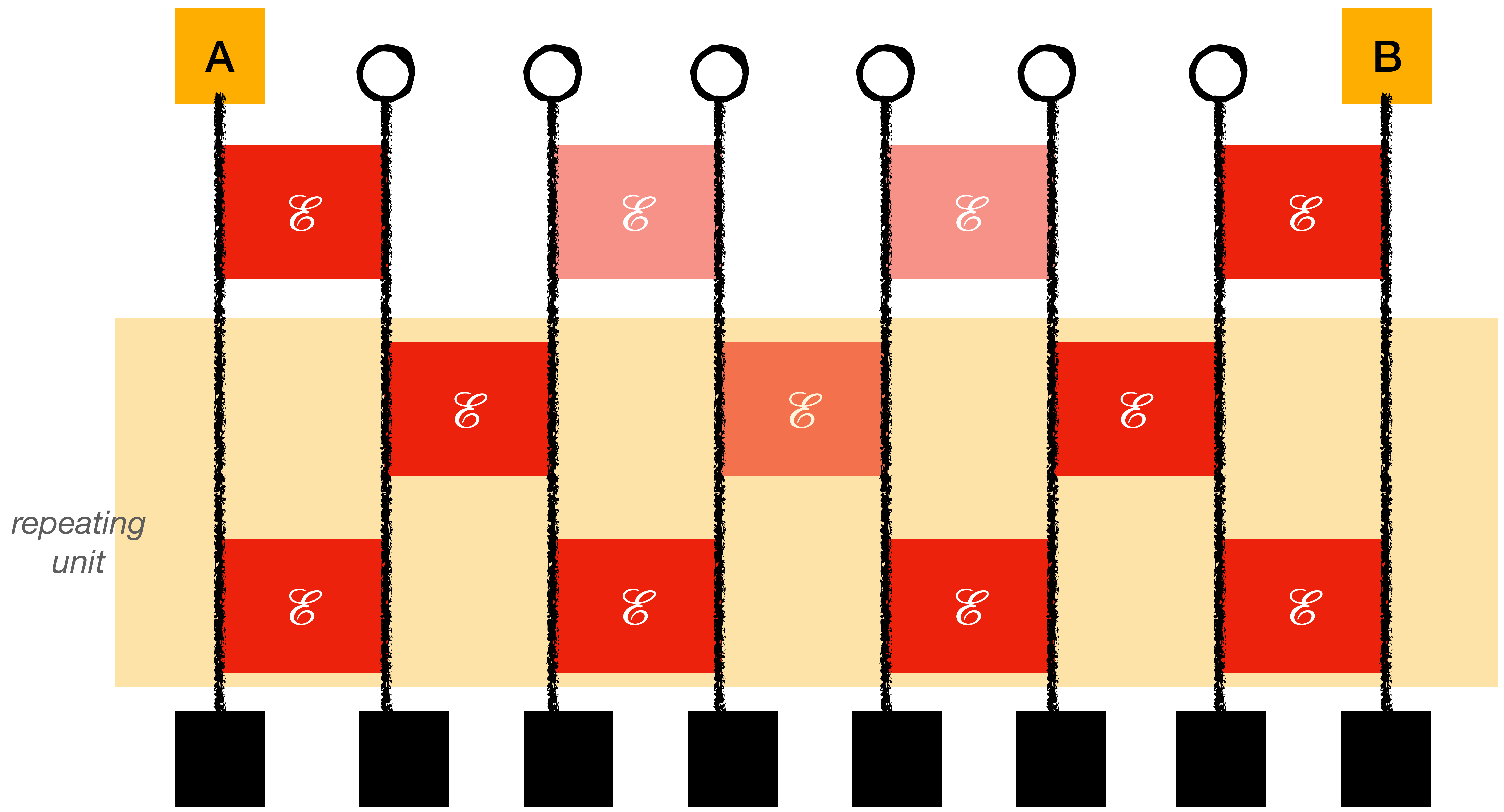


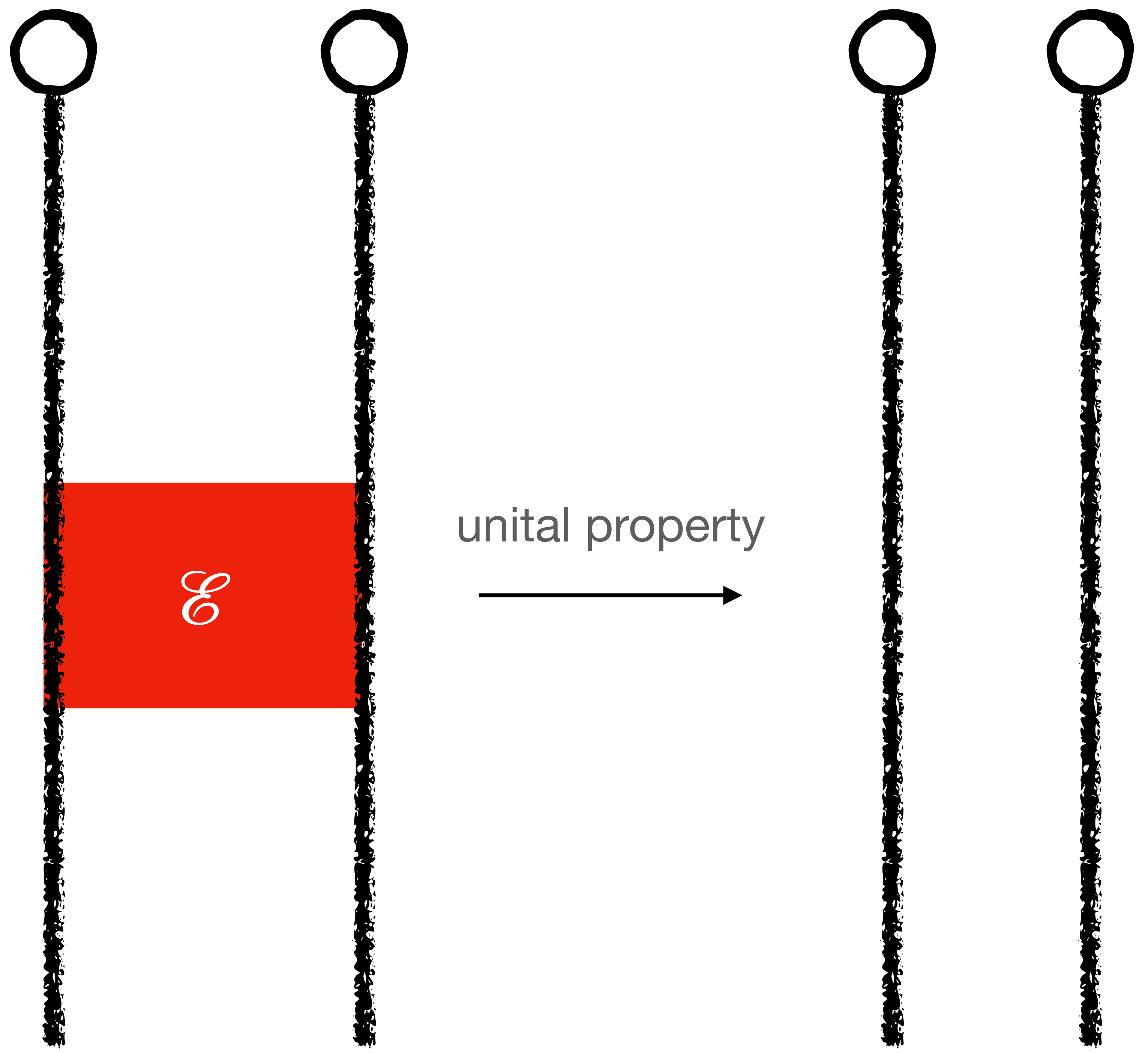
- Unital property:



circuits composed of quantum channels

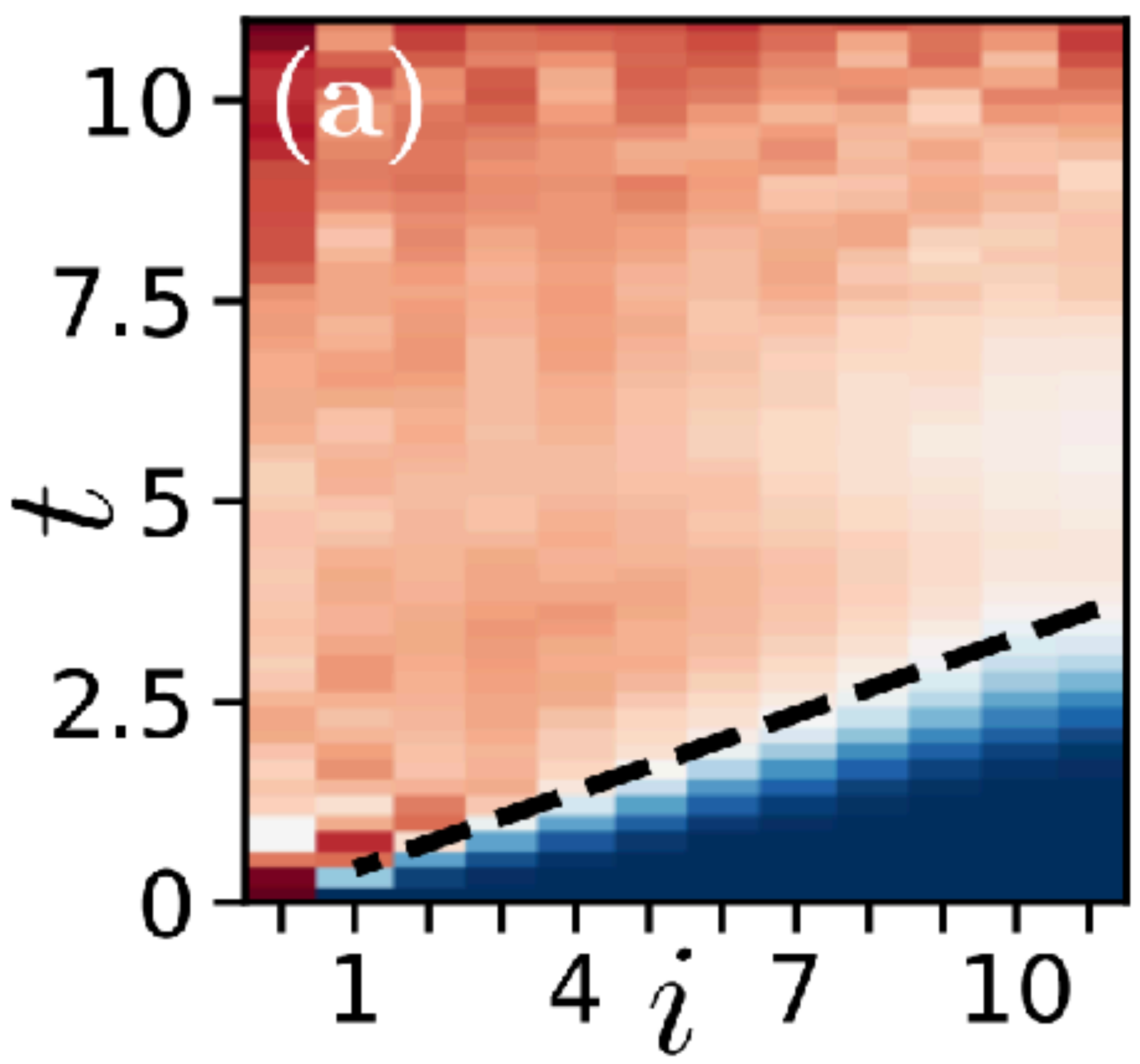
- Lieb-Robinson arguments carry over more or less directly from unitary systems

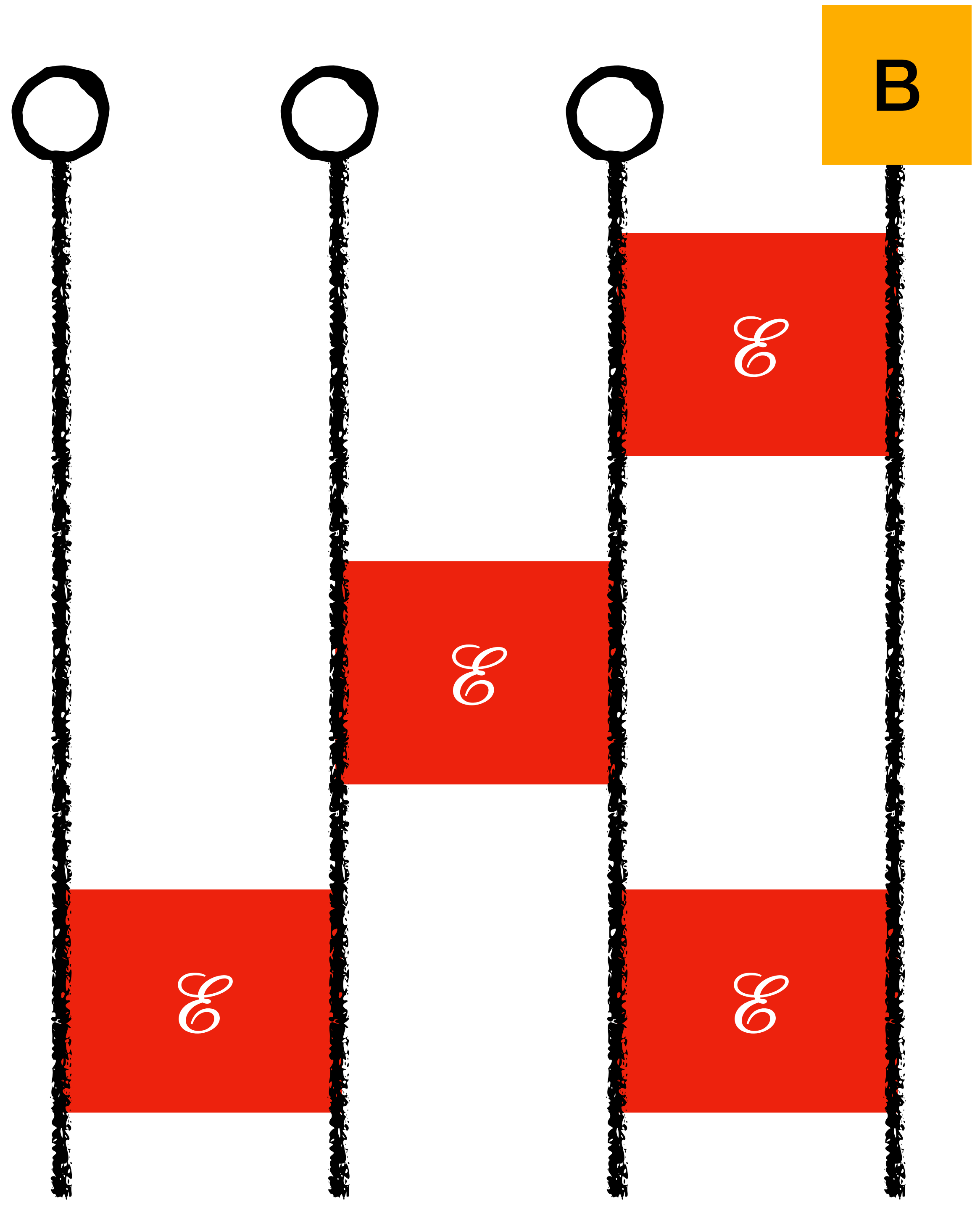




operator growth

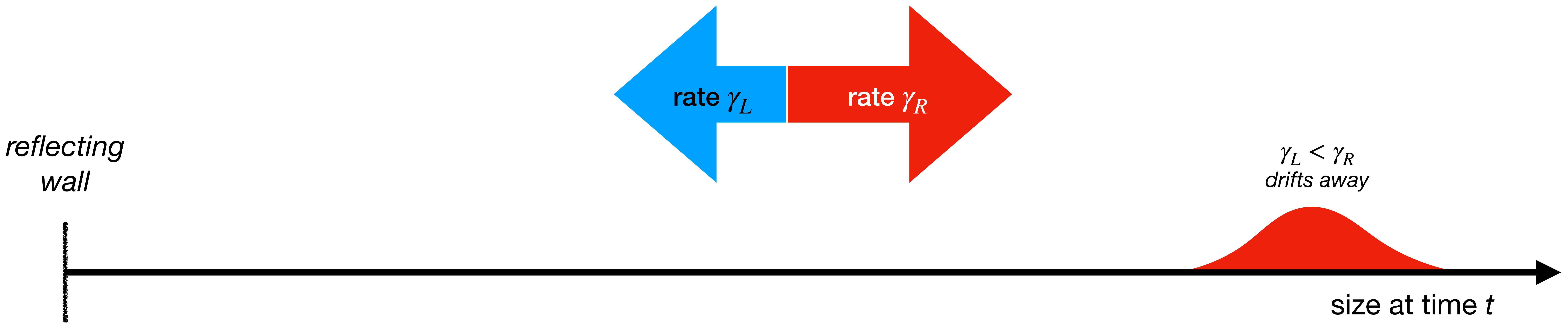
- Over time an initially local operator “grows” to fill in its light cone
- Basis for operator space: Pauli strings
- Support of an operator: set of sites with Paulis that are not the identity
- There are four Paulis and only one is the identity, so operator growth is *entropic* in origin
- Can solve explicitly for random quantum circuits





Schuster, Yao, 2208.12272

markov process for operator growth



- Operator growth in chaotic systems is a biased random walk (toward larger size)
- Effect of dissipation increases linearly as one goes to larger size, effective process:

$$\dot{\rho}_x = w_+\rho_{x-1} + w_-\rho_{x+1} - (w_- + w_+ + \gamma x)\rho_x$$

- What are the spectrum and eigenstates of the non-Hermitian operator $M_{x,x'}$?

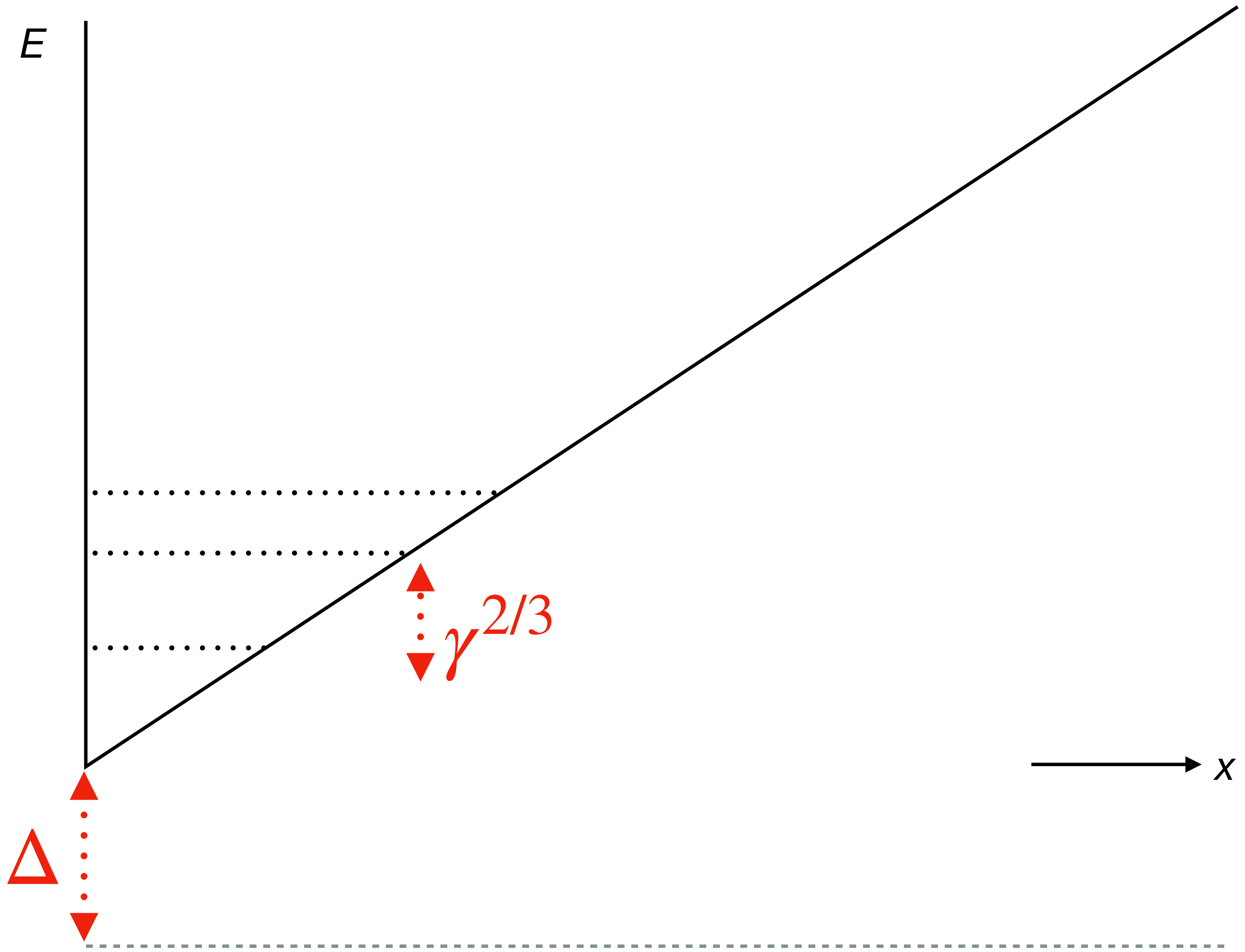
“hermitian” frame

- $\dot{\rho}_x = w_+\rho_{x-1} + w_-\rho_{x+1} - (w_- + w_+ + \gamma x)\rho_x$
- Perform a similarity transform $\tilde{M} = T^{-1}MT$, where $T_{x,x'} = e^{ax}\delta_{x,x'}$ (leaves spectrum unchanged)
- For appropriate a, $\tilde{M}$ is a Hermitian matrix
- Continuum limit of eigenvalue problem:

$$(\lambda - \Delta)\psi(x) = -w\psi''(x) + \gamma x\psi(x)\,.$$

$$w = \sqrt{w_+w_-},\ \Delta = \left(\sqrt{w_+} - \sqrt{w_-}\right)^2$$

- Gap arises from coordinate transformation
- Eigenstates delocalized (in this frame) on a length-scale $\gamma^{-1/3}$



low-lying eigenstates

- Eigenstates are orthogonal in Hermitian frame, but not after transformation $T|\psi_n^H\rangle = |\phi_n^{lab}\rangle$

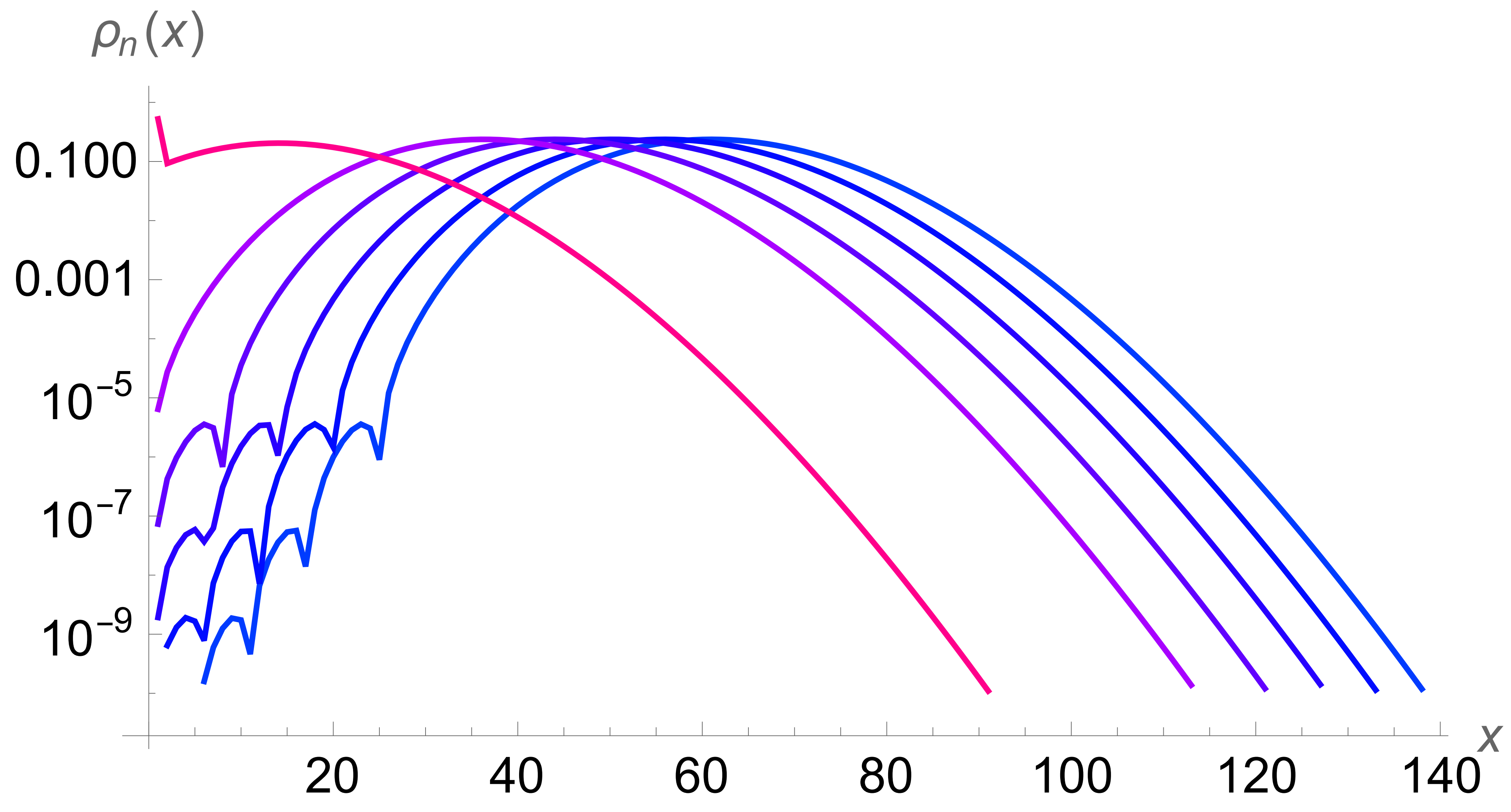
- Location of maximum is set by balancing operator growth e^{ax} vs. Airy function decay

- Peak at $1/\gamma$

- Why is the decay rate O(1)? Intuitively: eigenmodes have characteristic operator size $1/\gamma$, their decay rate is $\gamma \times 1/\gamma = O(1)$

- Eigenstates are increasingly non-orthogonal as $\gamma \to 0$

- Bottom line: in the absence of conservation laws, the evolution operator is **gapped**!

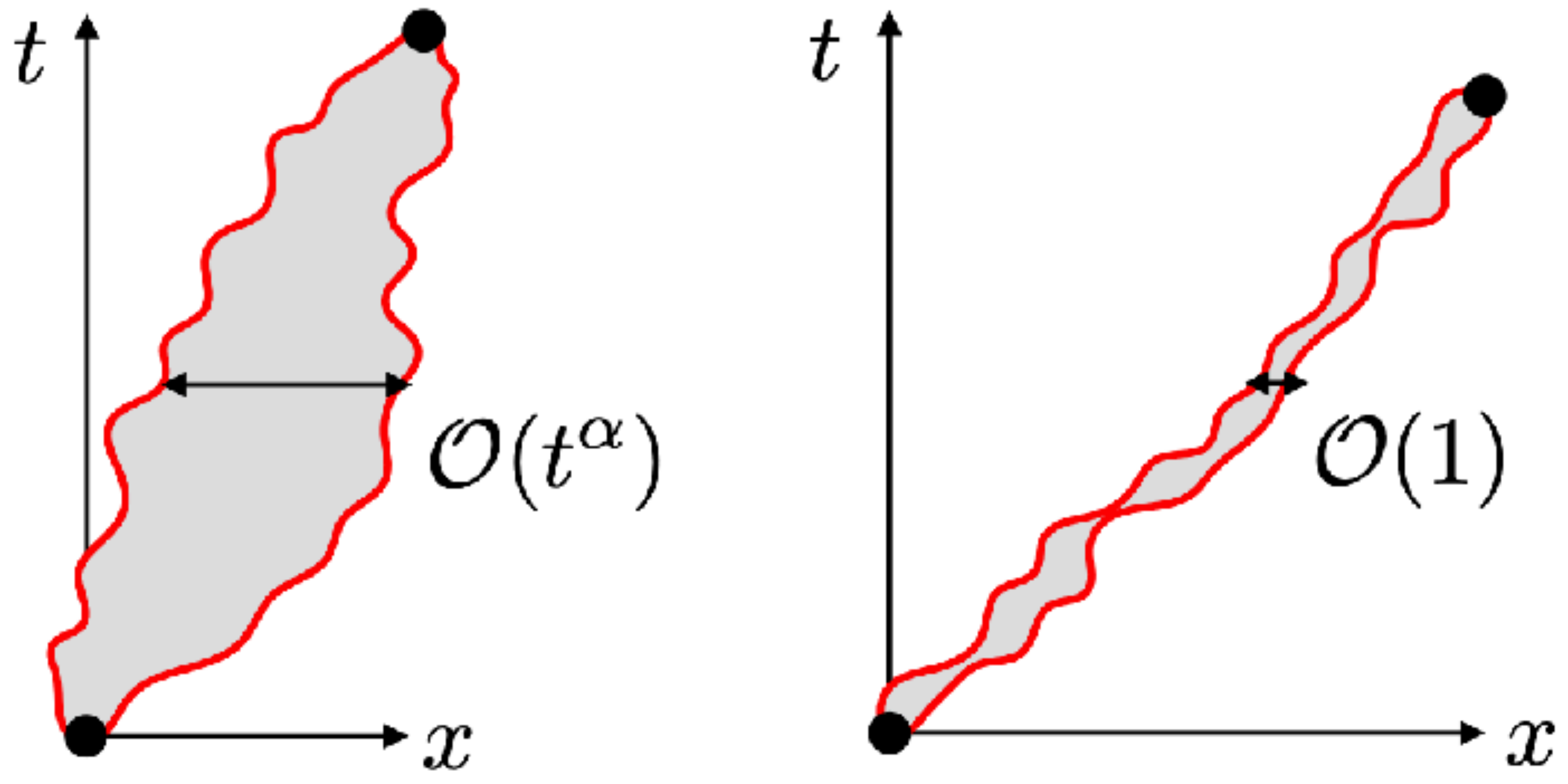


correlation functions

- Return probability for an operator to start and end at size 1
- Recall, generator $M = T\tilde{M}T^{-1}$

  where $\tilde{M} = \Delta \mathbb{1} +$ Schroedinger eq

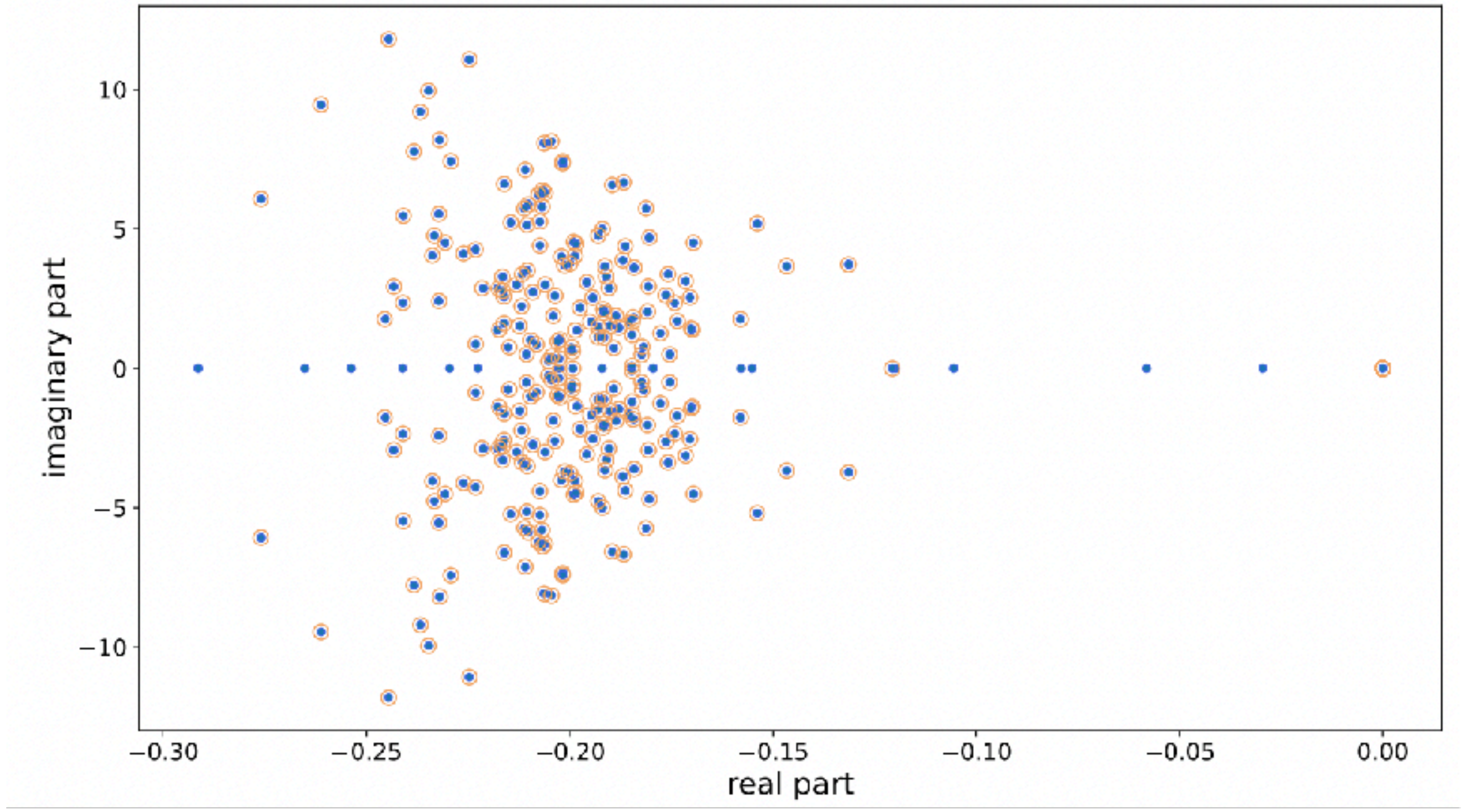
  and $T = \sum_{xx'} e^{ax}\delta_{xx'}$
- Correlation function is $\langle 1 \,|\, M \,|\, 1 \rangle \approx \langle 1 \,|\, \tilde{M} \,|\, 1 \rangle$
- Because it concerns return probabilities it does not “feel” the change of coordinates
- Bound state vs. lack of bound state ~ Feynman trajectories dominating correlation function (Nahum et al., 2022)



adding conservation laws (speculative)

what we hope to see

- Operator Hilbert space separates into hydro and non-hydro subspaces
- Adiabatic elimination of non-hydro subspace: to leading order, simply project it out
- To get the long-time dynamics of a general operator, compute $P_{\text{hydro}} O P_{\text{hydro}}$, which lives in a much smaller space
- Extra slow operators show up as slow modes of $\mathscr{E}$
- Many practical challenges to making this work…



toy application: mazur bounds

- Suppose we have some number of exactly conserved quantities Q_i
- Then the late-time limit of any autocorrelation function is bounded by the Mazur bound

  $$\lim_{t\to\infty}\langle O(t)O(0)\rangle \geq \sum_i \langle OQ_i\rangle (M^{-1})_{ij}\langle Q_j O\rangle, M_{ij} = \langle Q_i Q_j\rangle$$

- Can construct time-dependent versions (wang, ren, sg, vasseur, 2025): from the positivity of Lehmann representation, we have that

  $$\int_0^t dt \langle J(t)J(0)\rangle \geq \langle J_s^2\rangle t - O(\epsilon)$$

  where $||[J_s, H]|| \leq \epsilon$

- So far, useful mostly in cases where we already know the structure of the problem
- Similar approach has been used to find slow operators: Banuls et al. (2015, 2018)

wrapping up

- Even if exact simulations of quantum states are hard, many of the questions we are interested in might not be
- Hydrodynamics is a successful theory of some observables in some systems
- Where are its boundaries? Can we quantify non-hydrodynamic effects and compute their timescales? (Recall: bit-string distribution has specifically quantum fluctuations)
- Are there universal effects in transport beyond hydrodynamics?
- How do we turn the “hydrodynamic projection” into an efficient numerical scheme?