normal diffusion with anomalous fluctuations

sarang gopalakrishnan (princeton)

de nardis, sg, vasseur, ware, pnas **119**, e2202823119 (2022)
sg, morningstar, vasseur, khemani, prb **109**, 024417 (2024)
sg, mcculloch, vasseur, pnas **121**, e2403327121 (2024)
review article: sg + vasseur, rep. prog. phys. **86**, 036502 (2023)

a simple but nontrivial model

- General Hamiltonian

$$H = \sum_i (X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}) \xrightarrow{\Delta \to \infty} H_{\mathrm{XXZ}} = \Pi(\sigma_i^+ \sigma_{i+1}^- + \mathrm{h.c.})\Pi$$

allows up-spins (particles) to hop

kills processes that change number of domain walls

- Ballistic energy transport due to integrability
- Separate conservation of charge and DW's

a simple but nontrivial model

- General Hamiltonian

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allows up-spins (particles) to hop

kills processes that change number of domain walls

- Separate conservation of charge and DW's
- $\ldots \downarrow \downarrow \uparrow \uparrow \uparrow \downarrow \downarrow \ldots$ cannot grow or shrink or break — so it's stuck
- But a single flipped spin like $\ldots \downarrow \downarrow \uparrow \downarrow \downarrow \ldots$ can move around freely
- Model is integrable:
  - Magnons move ballistically even at finite density
  - Magnons and frozen domains are separately conserved

quasiparticle picture of integrable systems

Two variables (velocities), two constraints (momentum + k.e.)

If particles have equal mass:

$$v_1^f = v_2^i, v_2^f = v_1^i$$

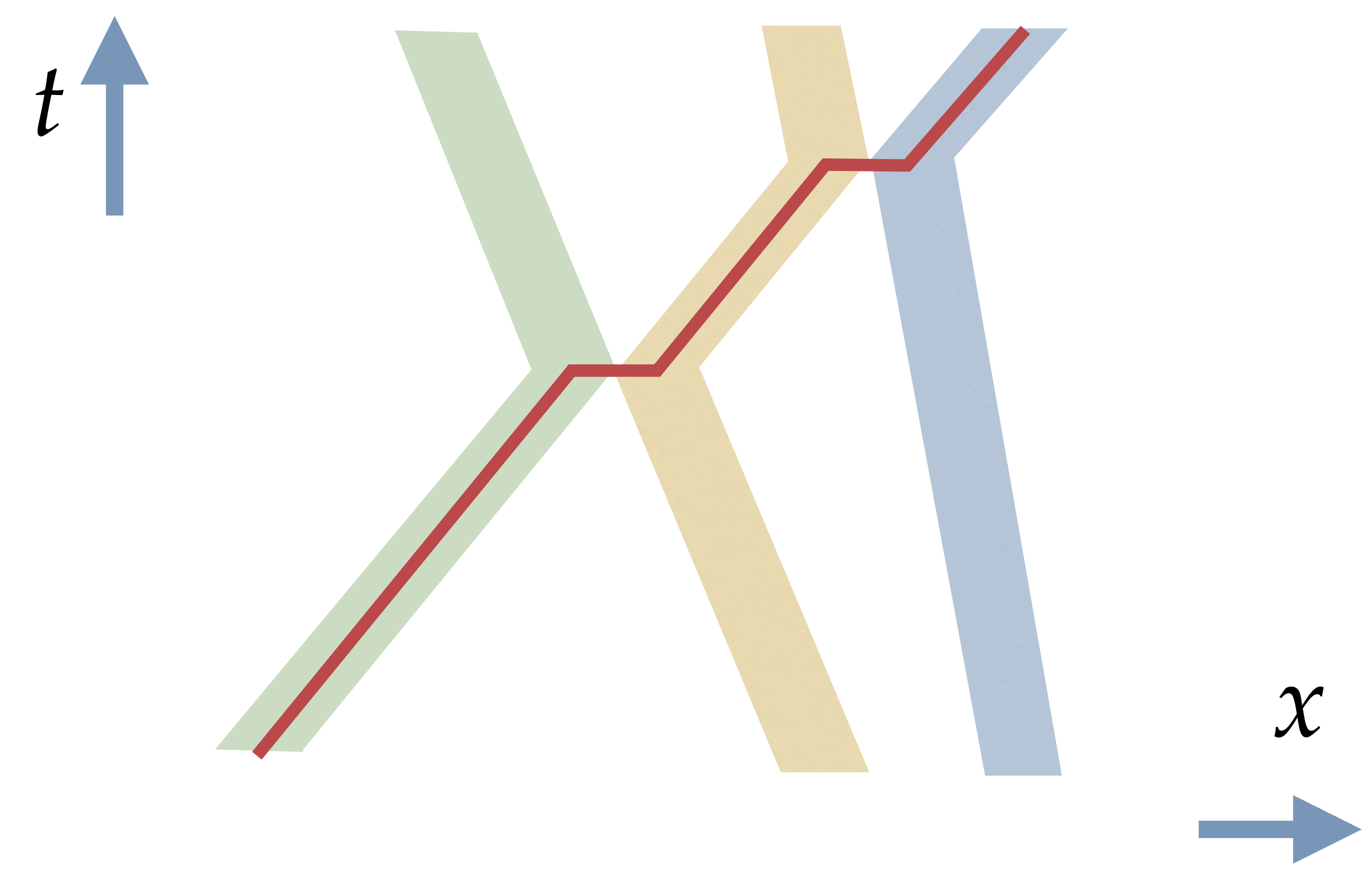
Set of velocities $\{v\}$ preserved

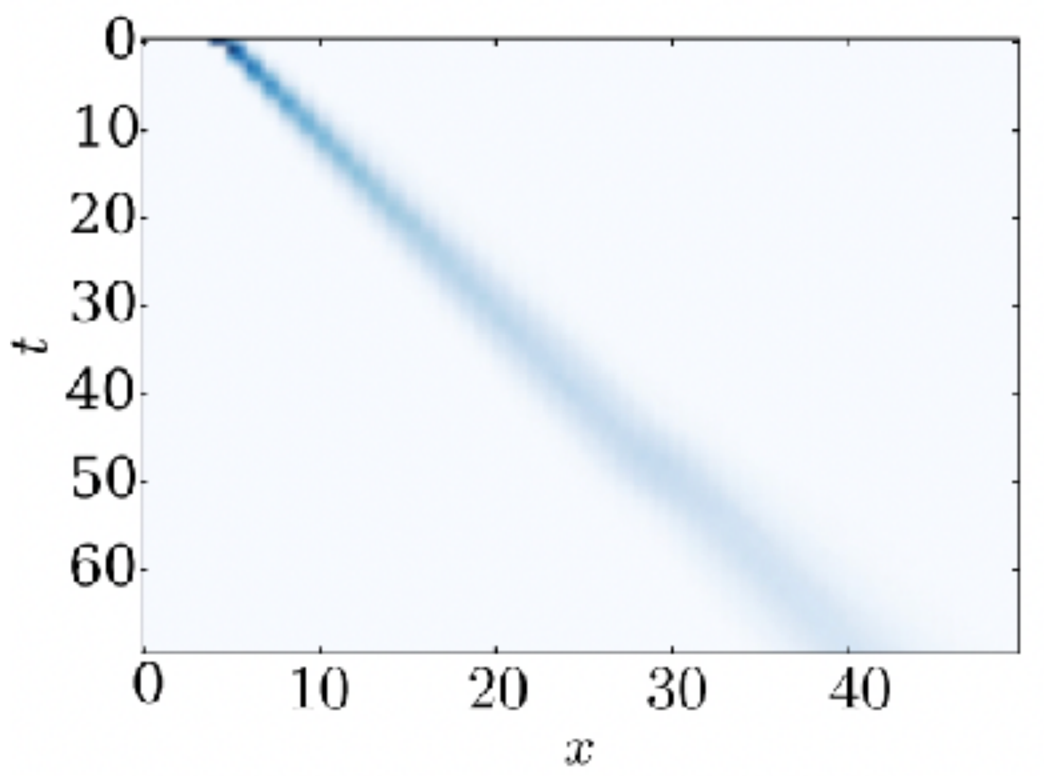
Three-body collisions relax $\{v\}$ *unless* they factorize (Hubbard/Heisenberg)

In integrable systems a picture of colliding trolleys can be made exact

Each trolley moves at a renormalized velocity that depends on the density of other trolleys that are in the way

So why isn't everything always ballistic?





SG, Huse, Khemani, Vasseur, PRB (2018)

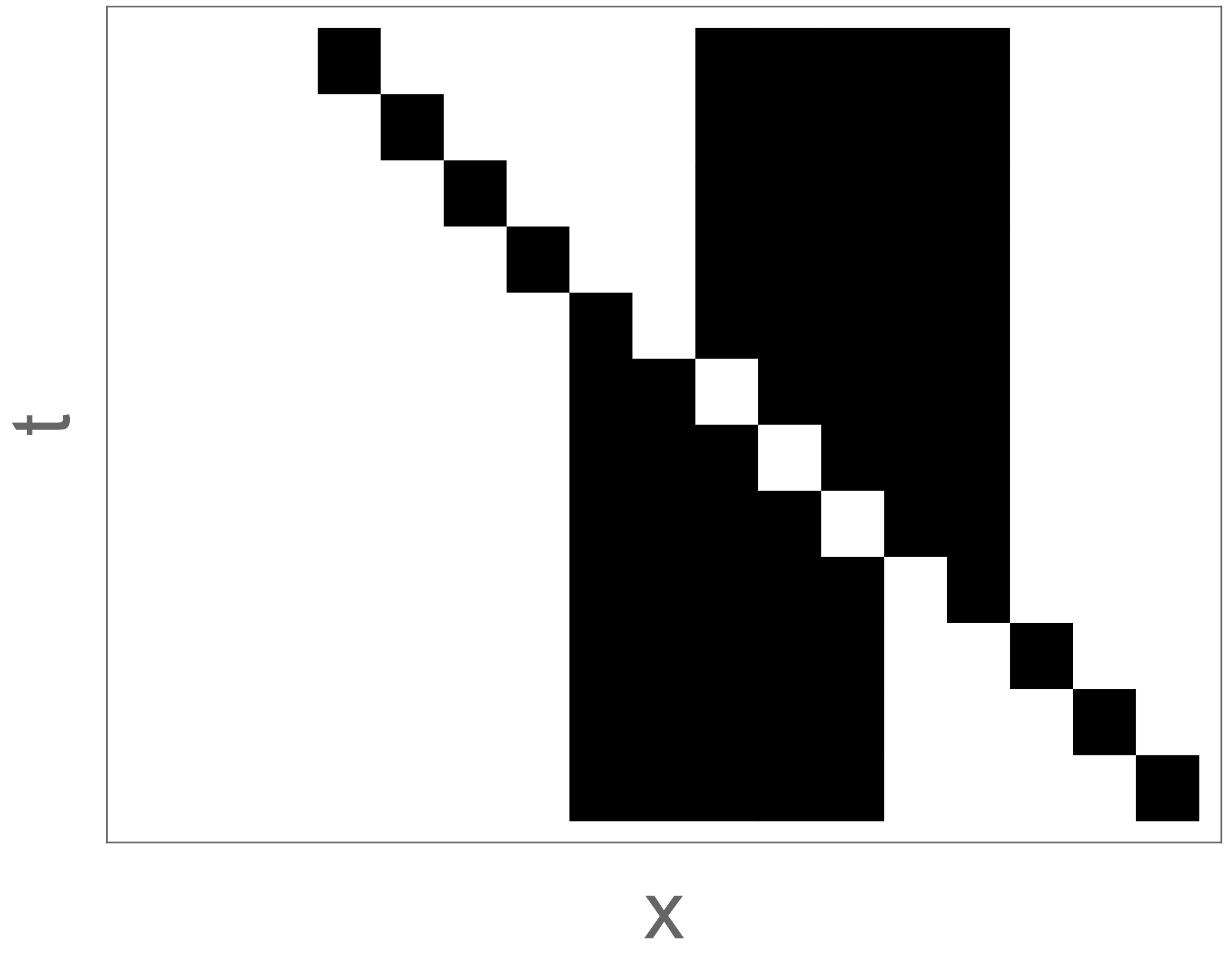
outline

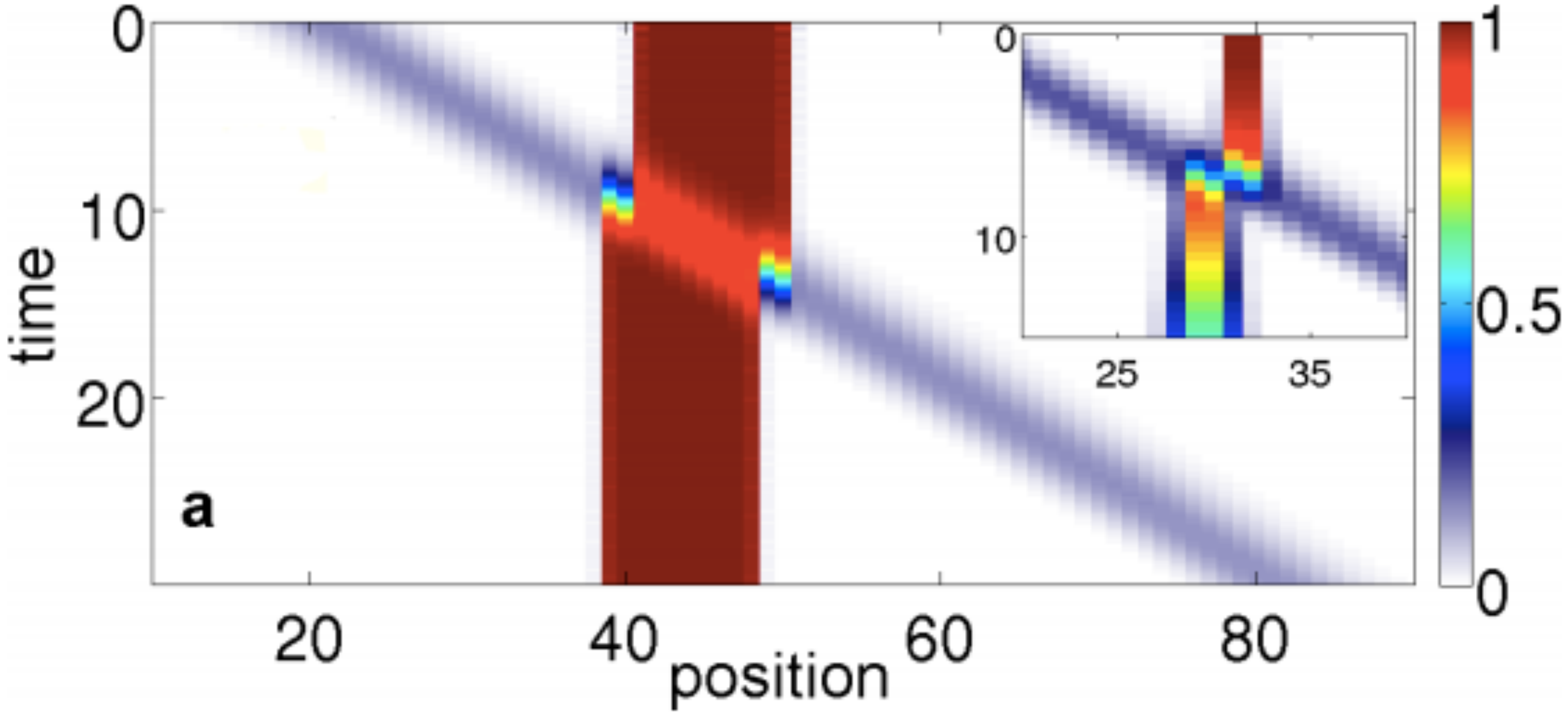
- Diffusive spin transport
- Effect of breaking integrability
- Full counting statistics (FCS)
- General hydrodynamic picture of FCS

why is spin transport diffusive?

why is there no ballistic spin transport?

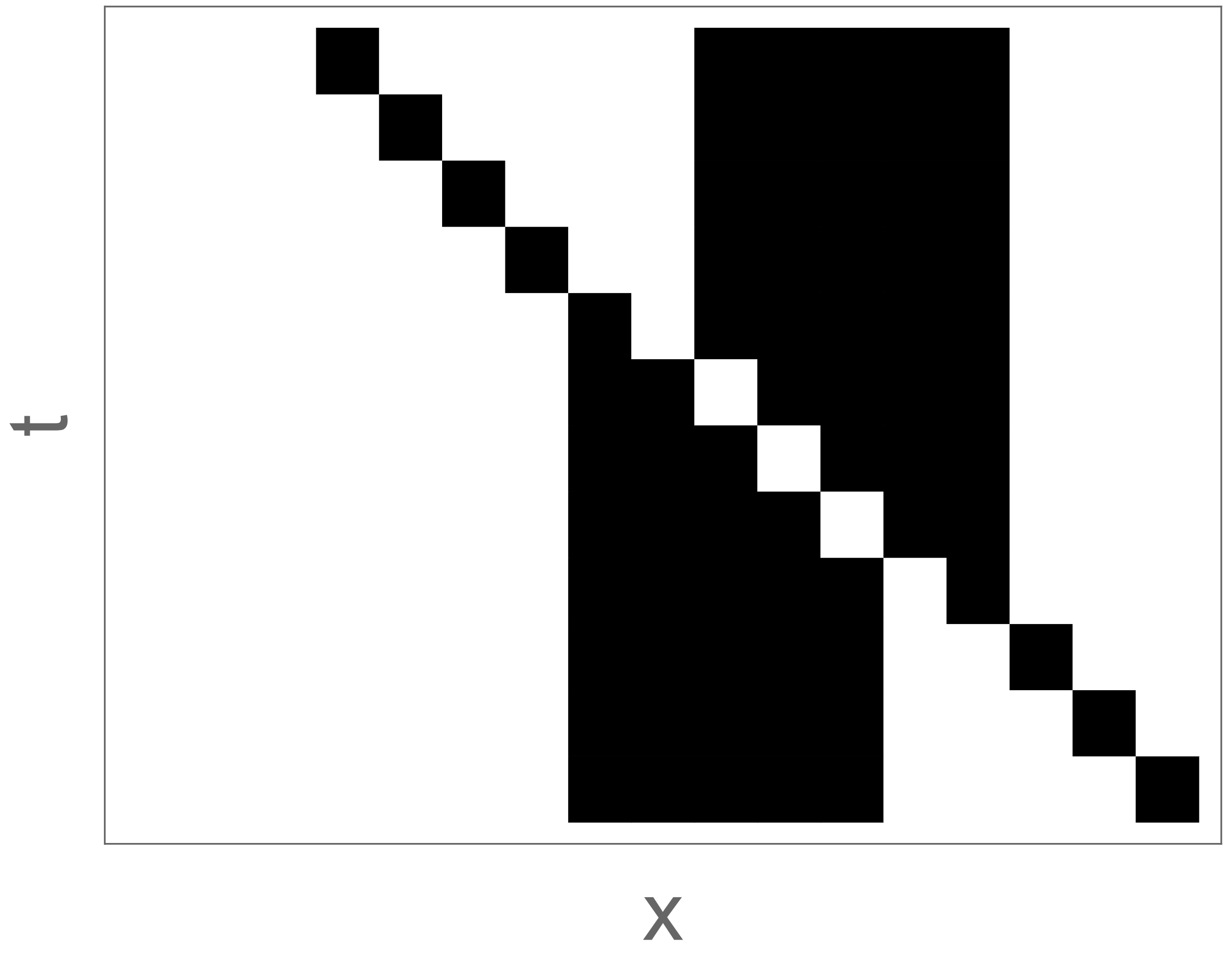
Ganahl et al., 2013

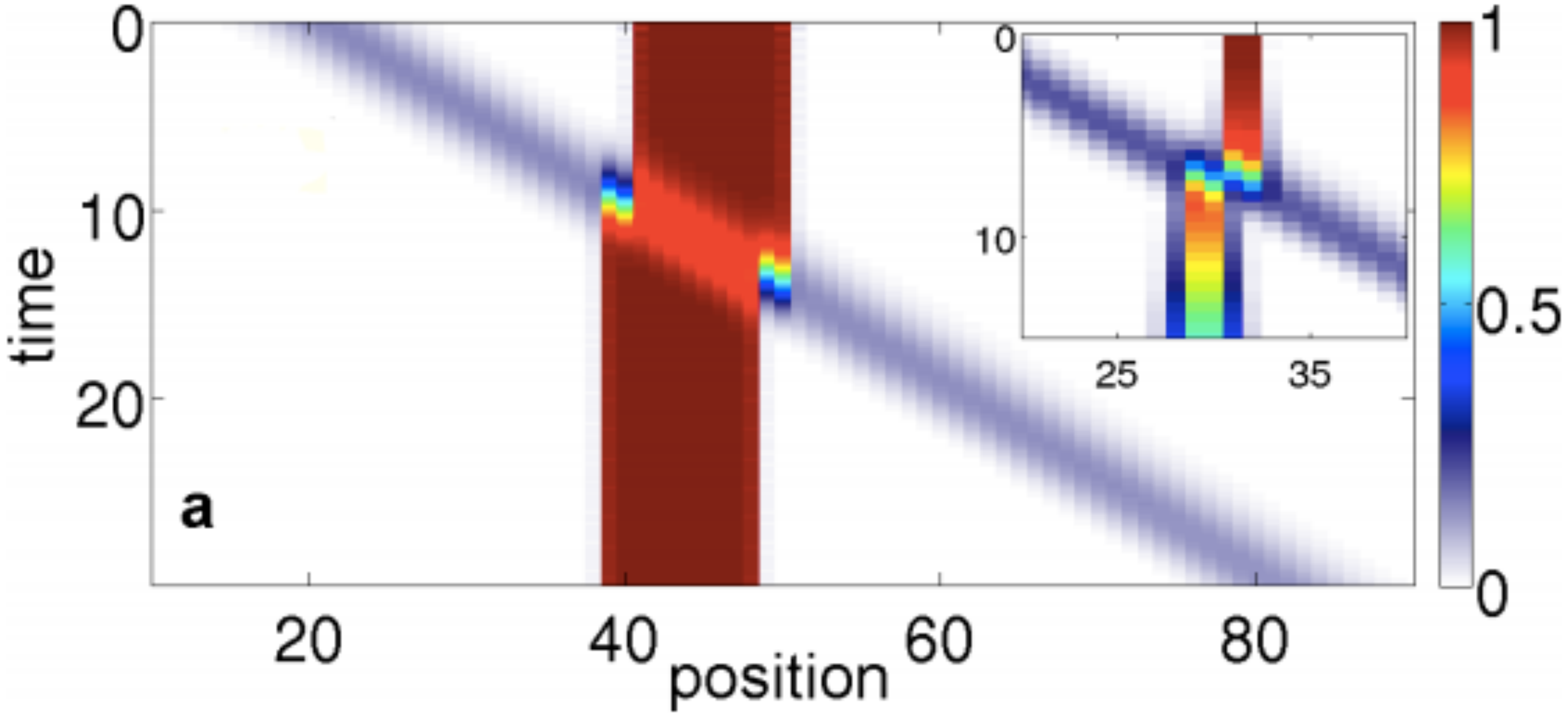




- What happens when a small mobile domain hits a large immobile domain?
- Small domain is stripped of spin, hence no ballistic spin transport (but still ballistic energy transport)
- Large domain undergoes Brownian motion from repeated collisions

intuitive argument for diffusion





- In time t a magnon "sees" a system of size $x \sim t$
- Positive and negative domains in this finite-size region cancel only up to a factor $\sim 1/\sqrt{t}$
- Residual magnetization carried by quasiparticle: $m^{\rm dr} \sim 1/\sqrt{t}$
- Amount of magnetization transported:

$$\langle \delta (m^{\rm dr} x)^2 \rangle \sim \frac{v^2 t^2}{vt} \sim vt$$

structure factor at general filling

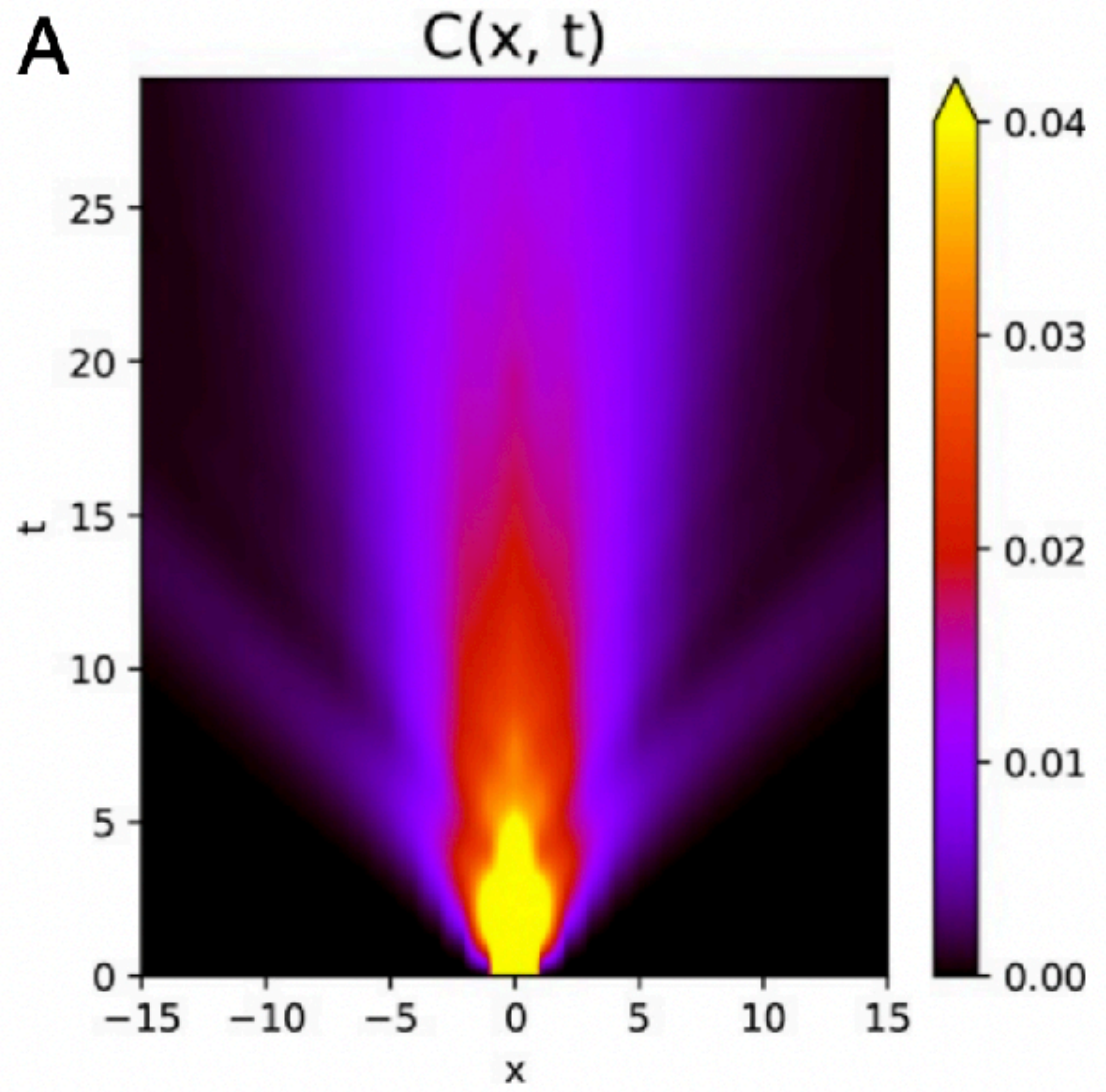
- Contributions to correlation function $S(x,t) \equiv \langle \sigma^z(x,t)\sigma^z(0,0)\rangle - \langle \sigma^z \rangle^2$:
  - Magnon moves from one point to the other
  - Frozen pattern of domain walls diffuses from one point to the other
- Magnon carries "charge" $\sim -\langle \sigma^z \rangle$
- Leading magnon contribution to structure factor:

$$\langle \sigma^z \rangle^2 \sum_v P(v)\delta(x - vt)$$

- Frozen pattern undergoes Brownian motion, gives contribution:

$$(1 - \langle \sigma^z \rangle^2)\exp(-x^2/(Dt))$$

where D is some O(1) number set by magnon density



integrability-breaking

breaking integrability with noise

- Time-dependent Hamiltonian:

  $$H = H_0 + H(t) = \sum_i (X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}) + h_i(t) Z_i$$

  where $\langle h_i(t) \rangle = 0, \langle h_i(t) h_i(t') \rangle = \gamma f(t - t')$

- Respects constraint but breaks integrability

- Why is this a good idea? Average over noise, get a Lindblad master equation

  $$\partial_t \rho = \mathscr{L}(\rho) = -i[H_0, \rho] + \gamma \sum_i (Z_i \rho Z_i - \rho)$$

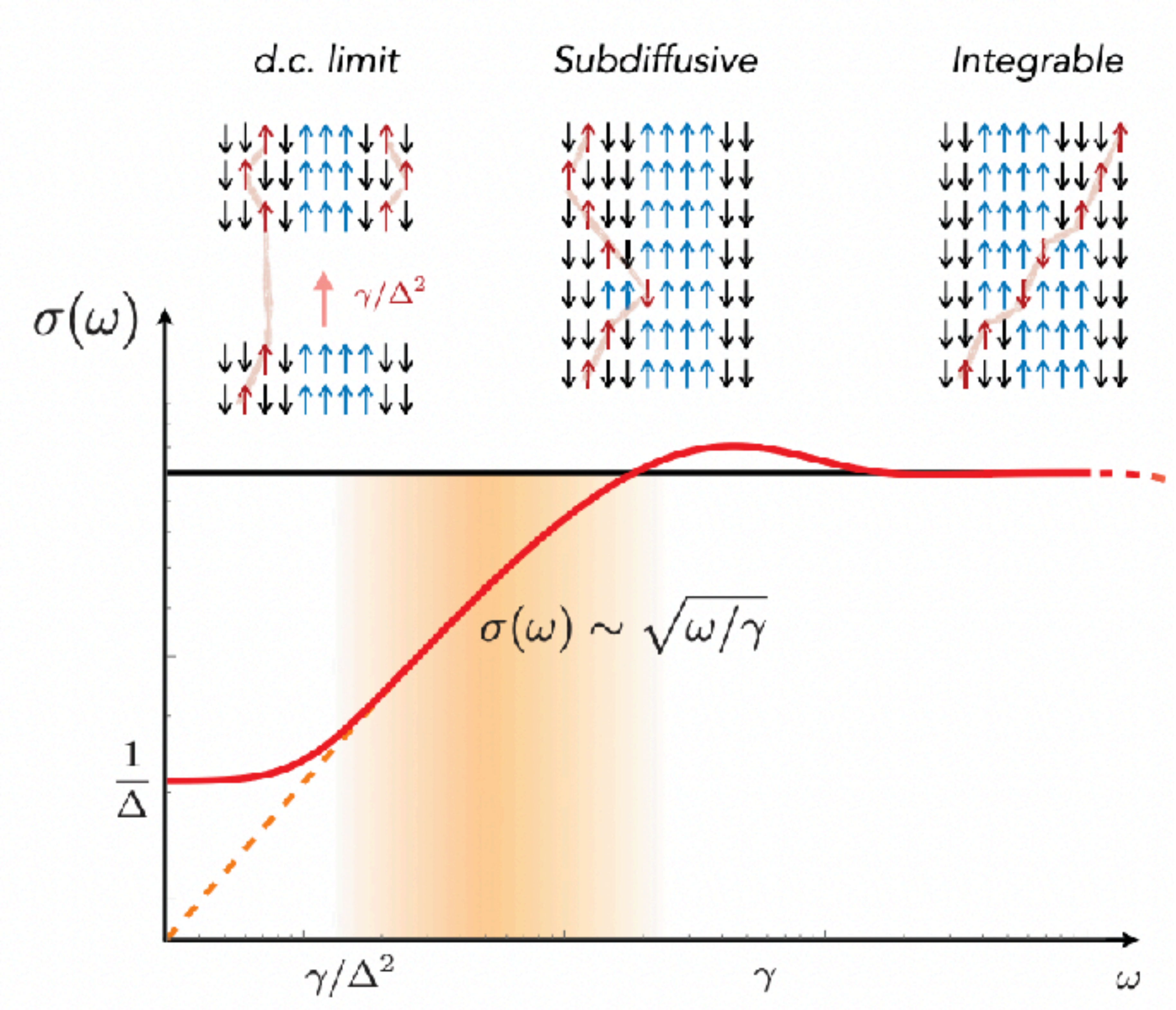
- This Lindblad master equation has two useful features:

  - Immediately restores diffusion for a single quasiparticle (otherwise, you would need collisions for diffusion)

  - Permits efficient numerical simulations because noise kills entanglement

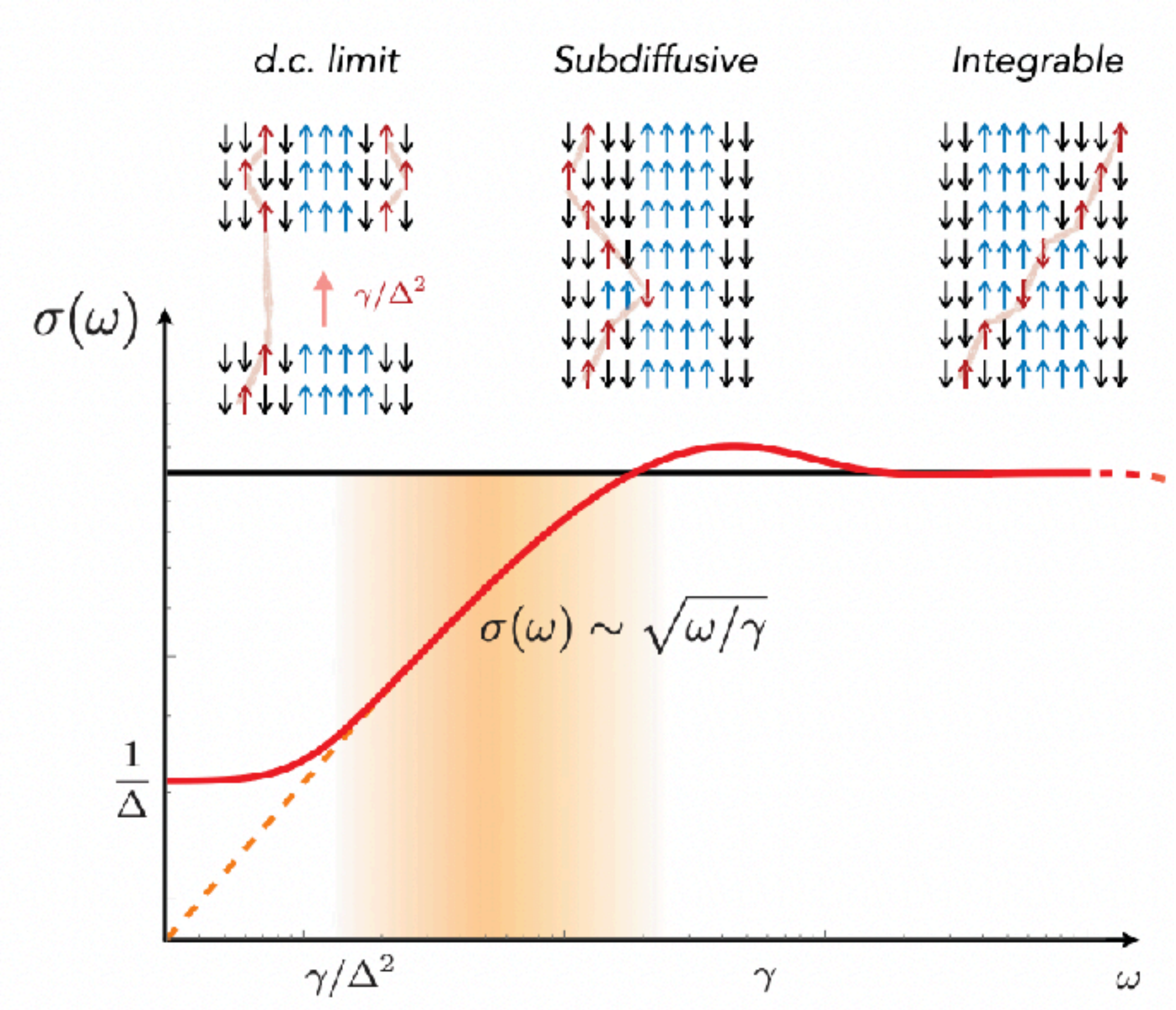
so what does it do?

- In time t a magnon sees a region of size $\sqrt{t}$
- So the magnetization it sees is $t^{-1/4}$
- Transported spin: $t^{1/2} \times t^{-1/4} \sim t^{1/4}$: subdiffusion!
- Energy is transported diffusively
- What is the subdiffusion rate?
    - Mean free time, mean free path, energy diffusivity $\sim 1/\gamma$
    - Spin transport $\sim (t/\gamma)^{1/4}$
- How to see this from the domain wall picture? Same magnon repeatedly interacts with domain wall, leads to anticorrelations in the Brownian motion



so what does it do?

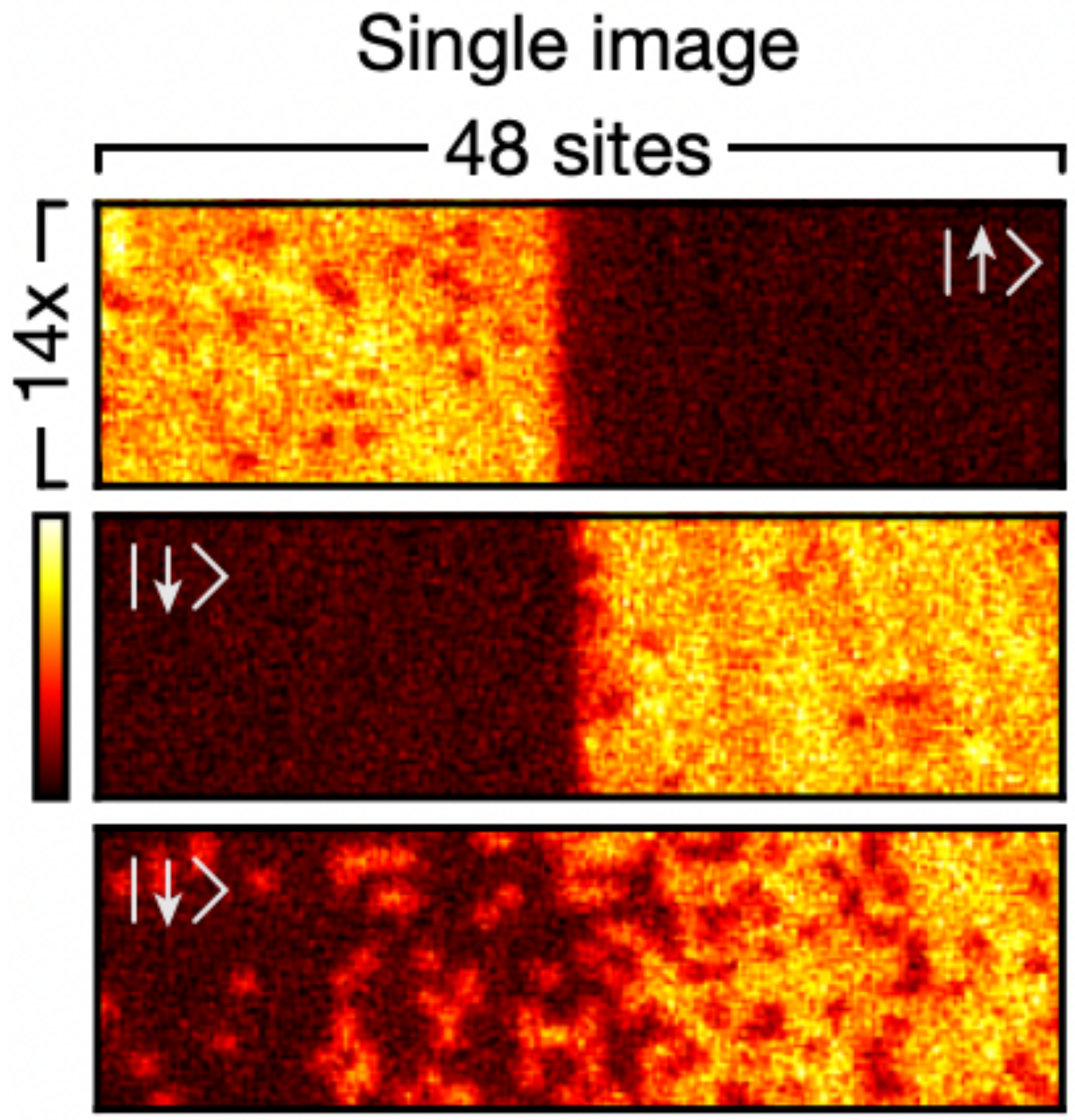
- Spin transport $(t/\gamma)^{1/4} \sim \sqrt{D(t)t}, \quad D(t) \sim (\gamma t)^{-1/2}$
- Spin conductivity $\sim \sqrt{\omega/\gamma}$ at low frequencies
- Integrability-breaking can be detected at frequencies $\leq 1/\gamma, \Rightarrow \omega^* \sim \gamma$
- Away from the infinite-Δ limit, eventually get diffusion at very low frequencies, but the diffusion constant is discontinuous from the integrable limit
- Numerical methods that break integrability weakly can converge to the wrong diffusion constant

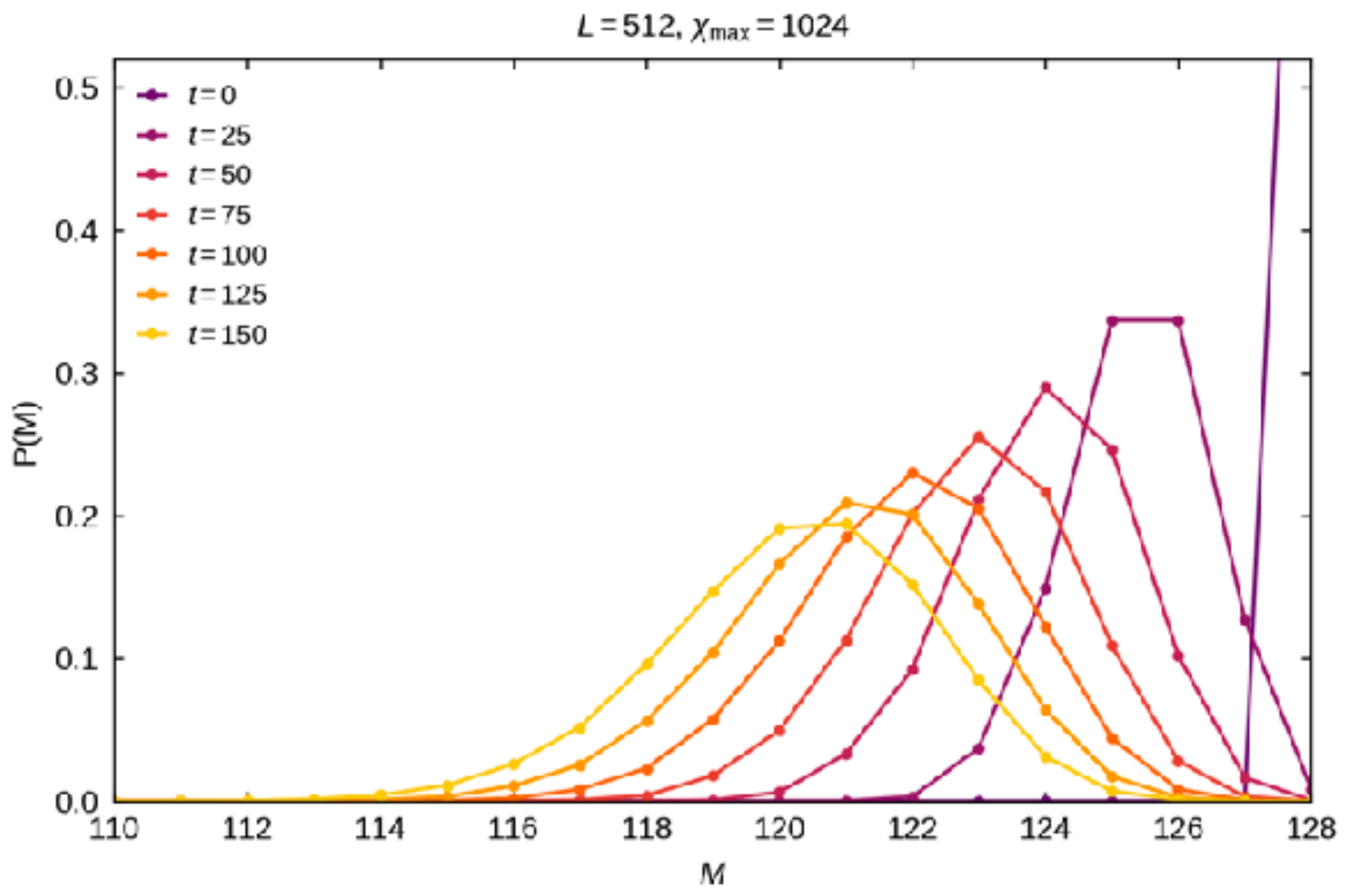


full counting statistics

full counting statistics (fcs)

- Single-site resolved projective measurement of all atoms
- Lots of data, need good summary statistics going beyond exp. vals.
- One way to organize the data: full counting statistics (fcs)

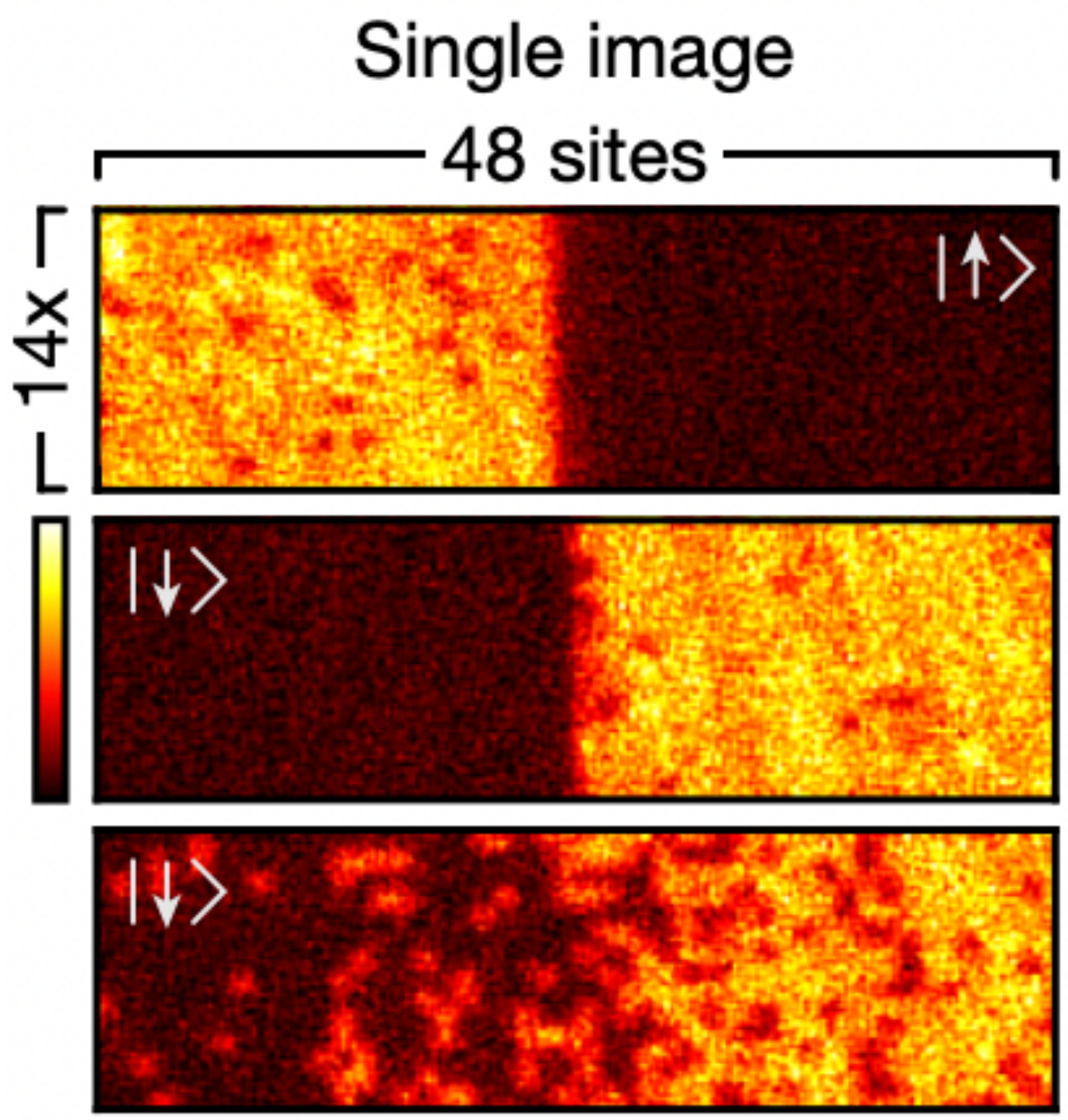


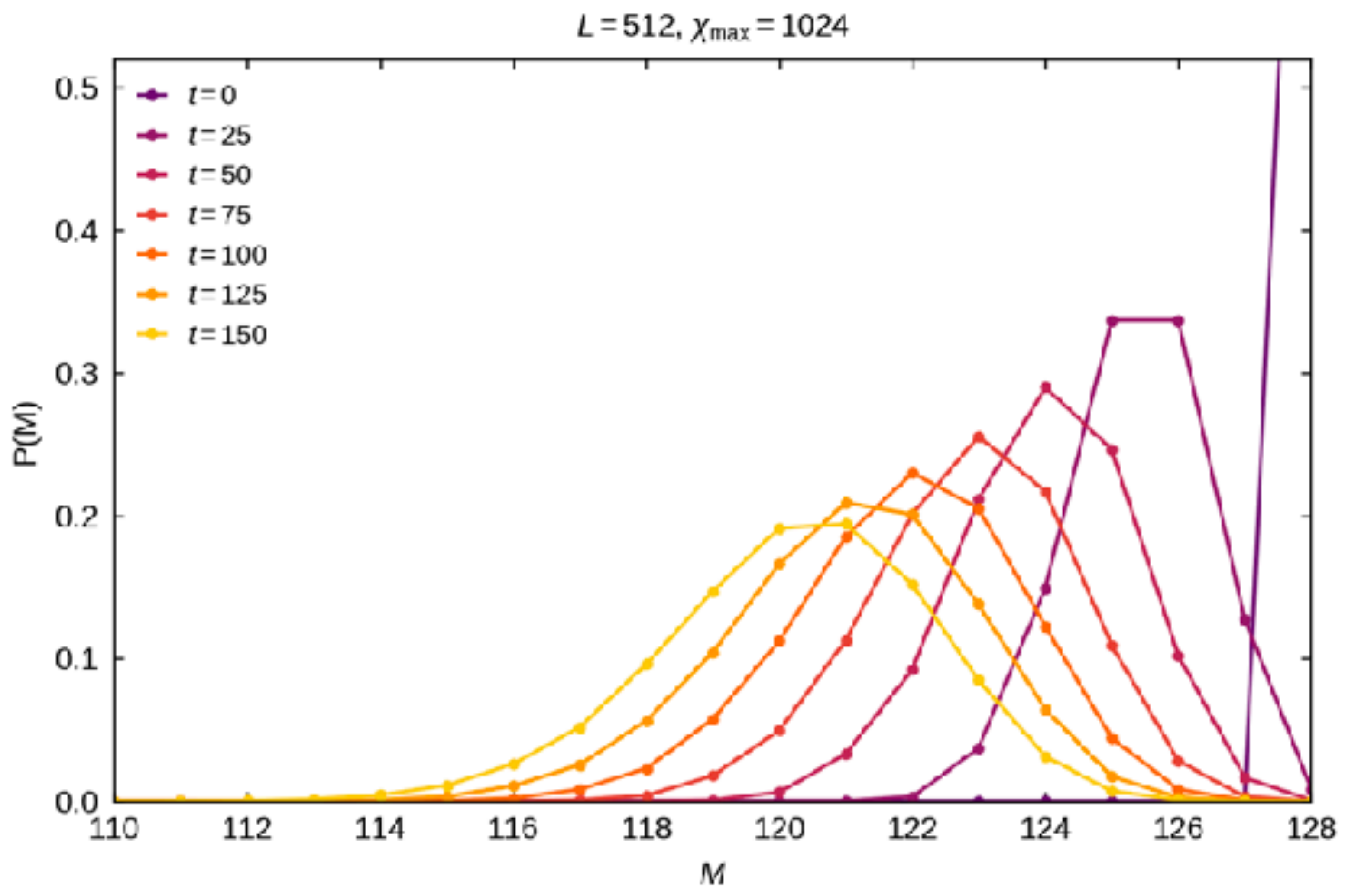


Wei et al. 2022

full counting statistics (fcs)

- Single-site resolved projective measurement of all atoms
- Lots of data, need good summary statistics going beyond exp. vals.
- One way to organize the data: full counting statistics (fcs)
- Experimental protocol for fcs:
    - Initialize two half-systems separated by a barrier, at (sharp) particle numbers Q_L^0, Q_R^0
    - Lower barrier and run the dynamics to time t, measure all particle positions
    - This gives conditional distribution $P(Q_R^t, Q_L^t \,|\, Q_R^0, Q_L^0)$
    - Compute *particle transfer* as $P(Q_R^t - Q_L^t - (Q_R^0 - Q_L^0))$





Wei et al. 2022

“standard” fcs

fcs for conventional diffusion

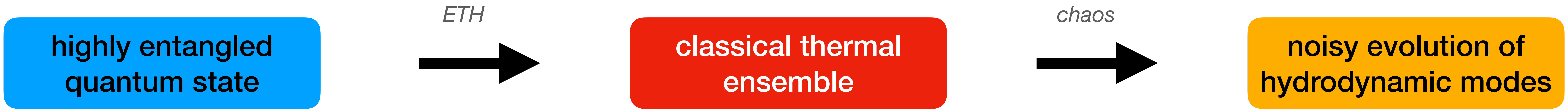
- All cumulants of the charge transfer scale as $\sqrt{t}$
  - From nonequilibrium initial condition, mean $\sim \sqrt{t}$
  - Standard deviation $\sim t^{1/4}$: equilibration over scale $\sqrt{t}$, fluctuations $\sqrt{\sqrt{t}}$
- Full distribution function follows from solving the fluctuating hydro equations with white noise

$$\partial_t n = \partial_x \left(D(n)\partial_x n + \sqrt{D(n)\chi(n)}\, \xi \right)$$

- Only cares about density-dependent transport/thermodynamic coefficients

why would this apply to quantum systems?

	classical noisy	**classical det.**	**quantum noisy**	**quantum det.**
random state	noise + ensemble	ensemble	noise + ensemble + projection	ensemble + projection
deterministic state	noise	none	noise + projection	projection
integrable vs. chaotic	*chaotic*	*either*	*chaotic*	*either*



explicit calculation for random circuits (McCulloch, De Nardis, SG, Vasseur, PRL (2022))

how could this go wrong?

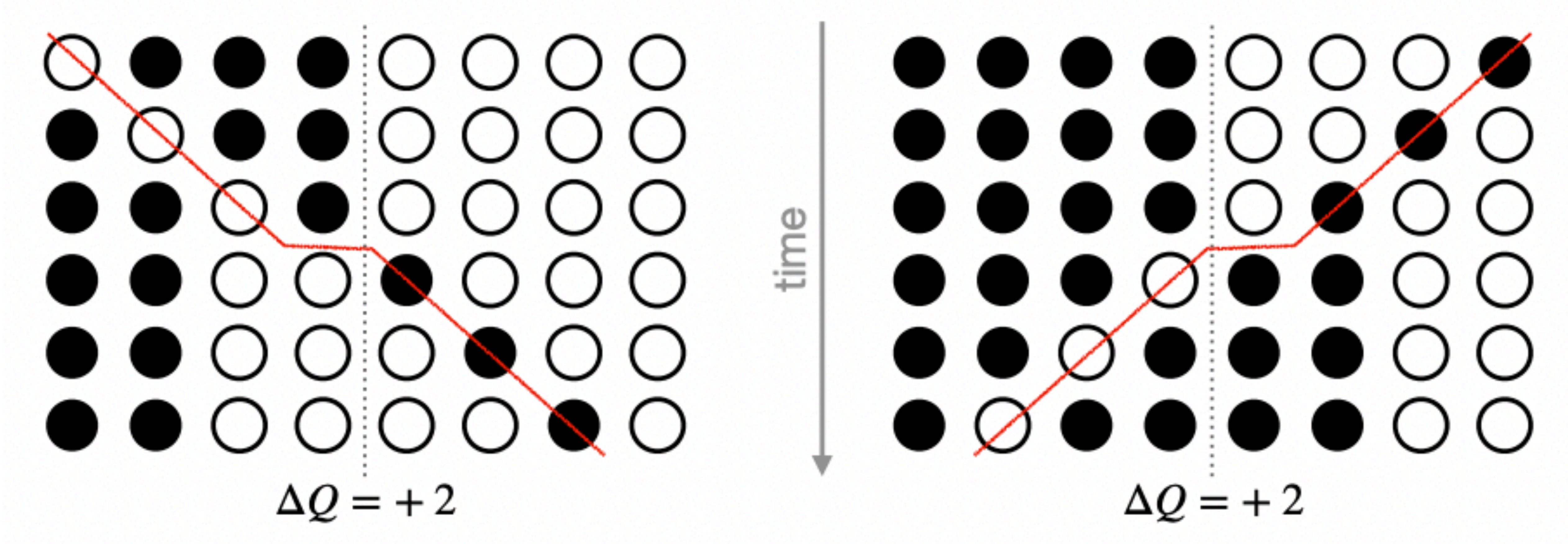
- Basic MFT thesis is that all fluctuations are set by density-dependent diffusion constant:

$$\partial_t n = \partial_x \left(D(n)\partial_x n + \sqrt{D(n)\chi(n)}\, \xi \right)$$

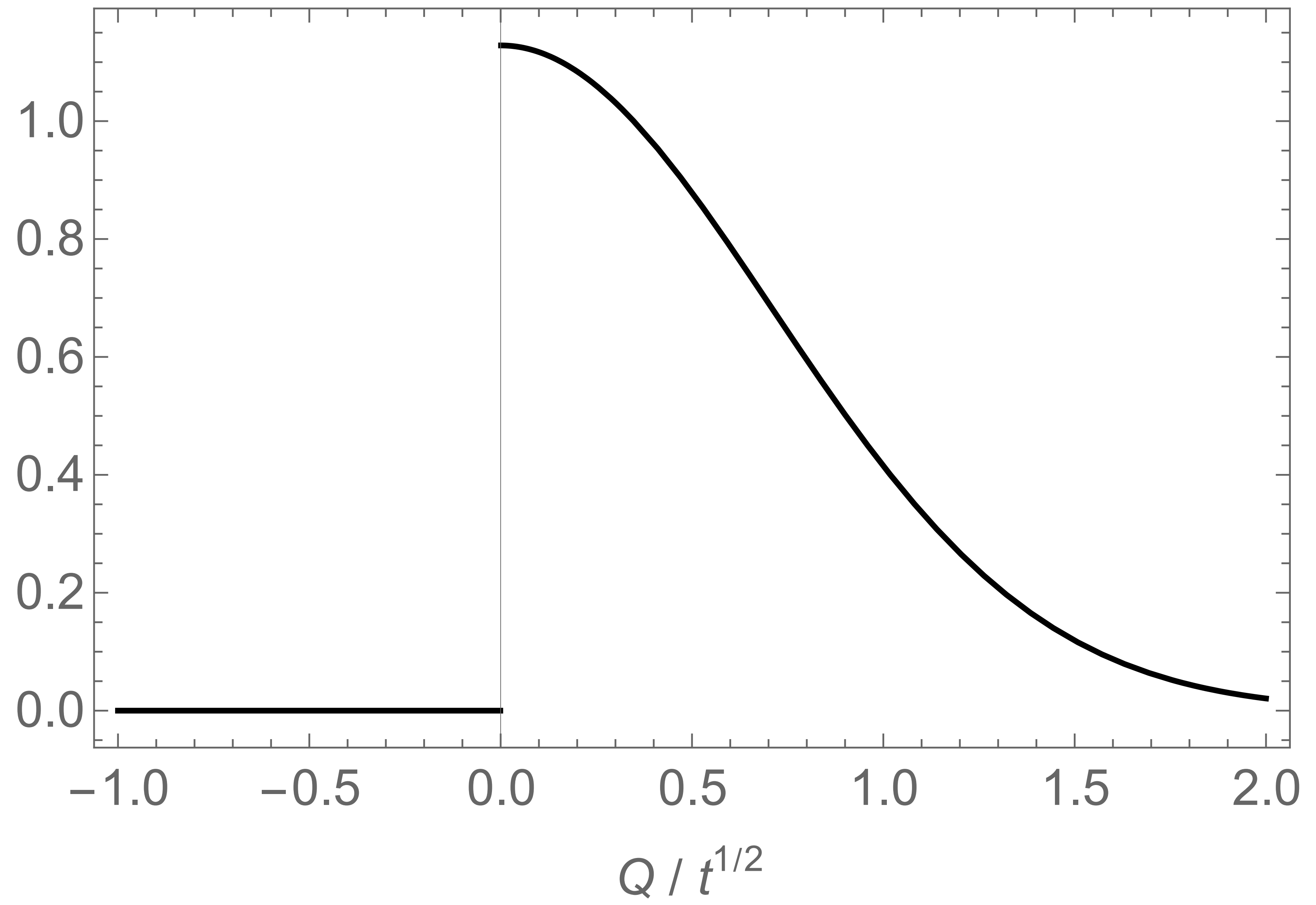
- Much of MFT is unchanged if $D(n)$ is a smooth function of density
- Why would this ever fail?
- In the XXZ model, the diffusion constant is *infinite* away from half filling
- Also nonintegrable models with this feature (e.g., graphene at charge neutrality)

fcs in the xxz spin chain

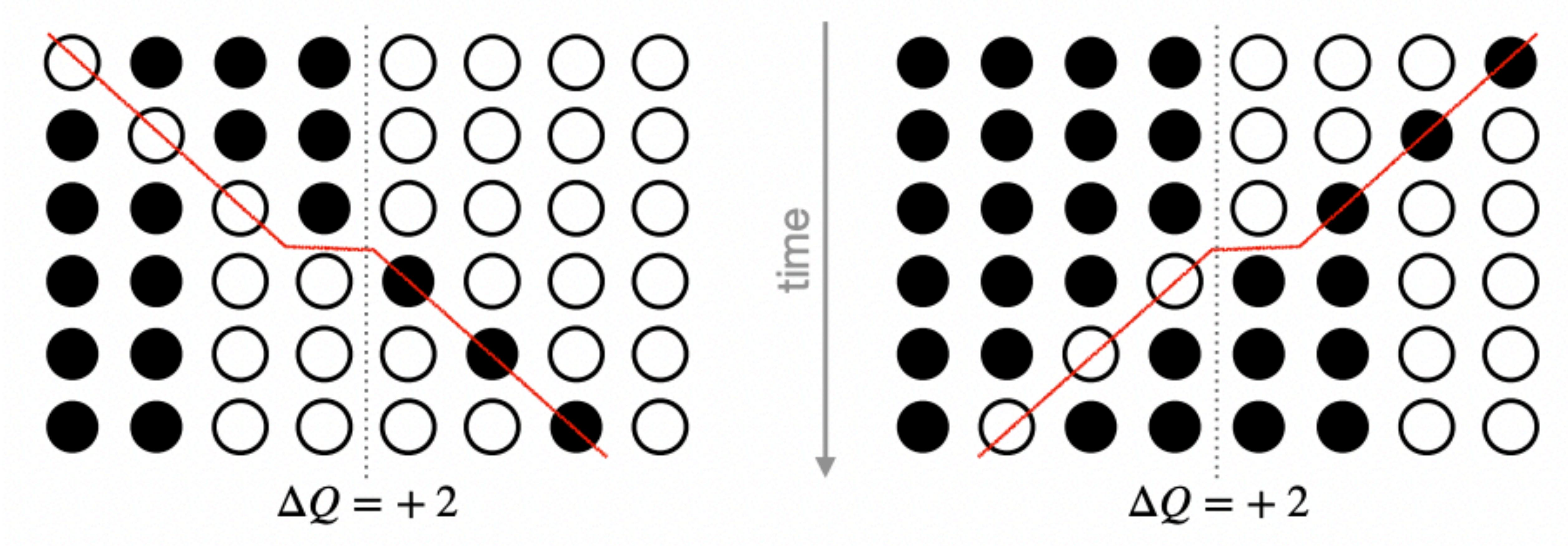
xxz at large polarization



- Big immobile domain has intermittent collisions with magnons
- Domain edge undergoes random walk due to collisions
- Relation between domain motion and Q(t):
  - When the domain wall moves *toward* the origin, Q decreases
  - When the domain wall moves *away from* the origin, Q increases
- Domain wall starts at the origin, so $Q(t) = |x(t)|$ – absolute value of displacement of a random walker



xxz at large polarization



- Domain wall starts at the origin, so probability distribution of magnetization transfer is the absolute value of a random walk

- Two implications:
  - Strongly skewed distribution
  - Mean and *standard deviation* scale the same way as $\sqrt{t}$: mean and variance scale with different powers
  - Main reason: not independent walkers, just one giant walker

cf. standard diffusive systems

particles equilibrate across a distance $\sqrt{t}$

so distribution width is $\sqrt{\sqrt{t}}$

general case: three-mode hydrodynamics

hydro with a conserved energy current

- Continuity equation for energy: $\partial_t e + \partial_i \phi_i = 0$
- Continuity equation for energy current: $\partial_t \phi_i + \partial_j q_{ji} = 0$
- Constitutive relation for q_{ji}: $q_{ji} = Be\delta_{ji} + \ldots$
- Continuity equation for energy current, at Euler scale: $\partial_t \phi_i + B\partial_i e = \ldots$

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- Continuity equation for energy current, at Euler scale: $\partial_t \phi_i + B\partial_i e = \ldots$
- These two equations describe sound waves; how do the sound waves couple to charge?

hydrodynamics of charge

- Previously: $\partial_t e + \partial_i \phi_i = 0$, $\partial_t \phi_i + B \partial_i e = 0$ (at Euler scale)
- Continuity equation for charge: $\partial_t n + \partial_i j_i = 0$
- Only Euler-scale term allowed in $j_i = n\phi_i$
- Away from states with particle-hole symmetry, $\delta j_i \sim n_0 \delta\phi_i$: particles carry energy + charge
- At charge neutrality this coupling is absent, so linearized hydro of charge is purely diffusive:

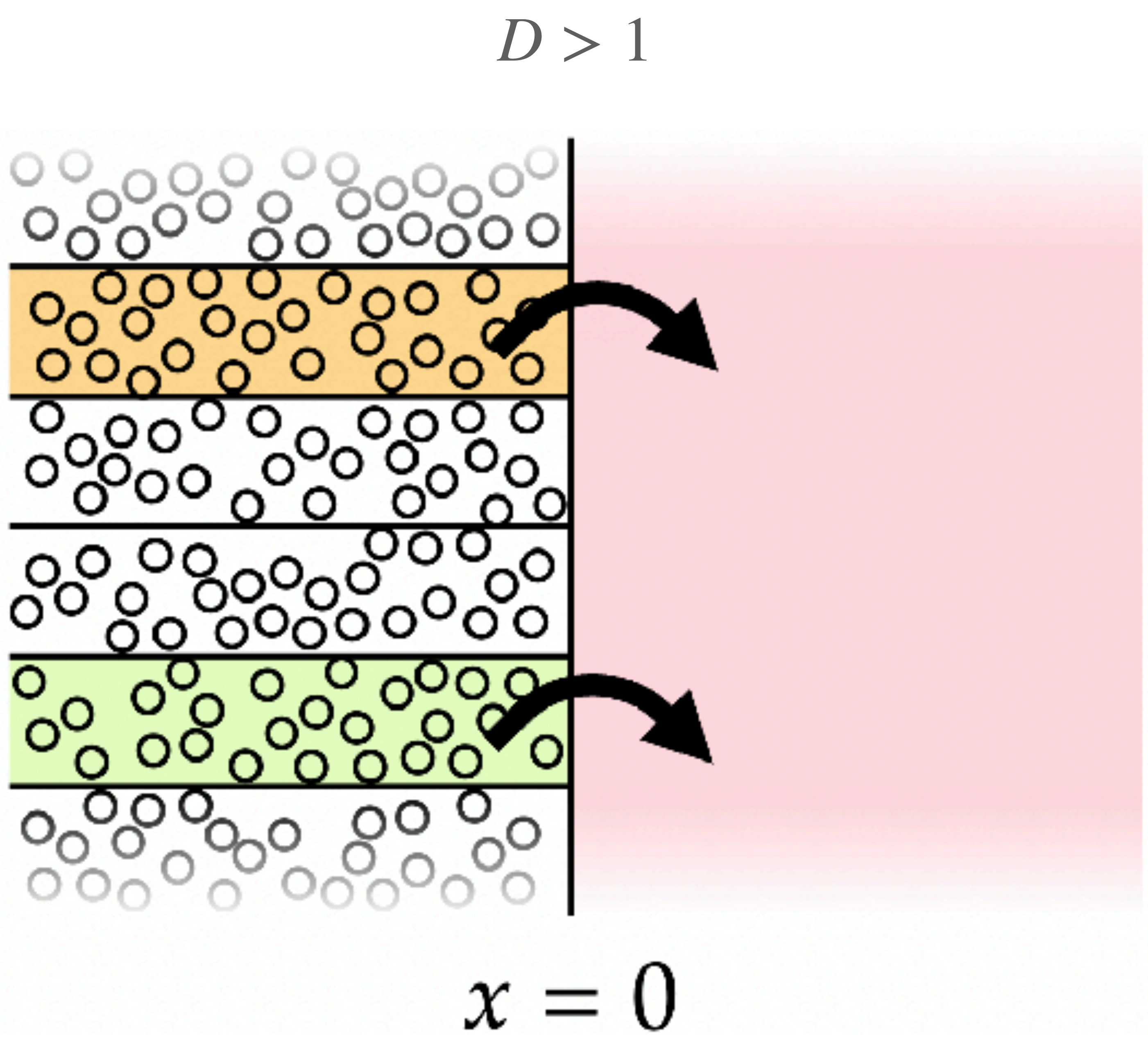
$$j_i \sim -D\partial_i n + \xi_i + \ldots$$

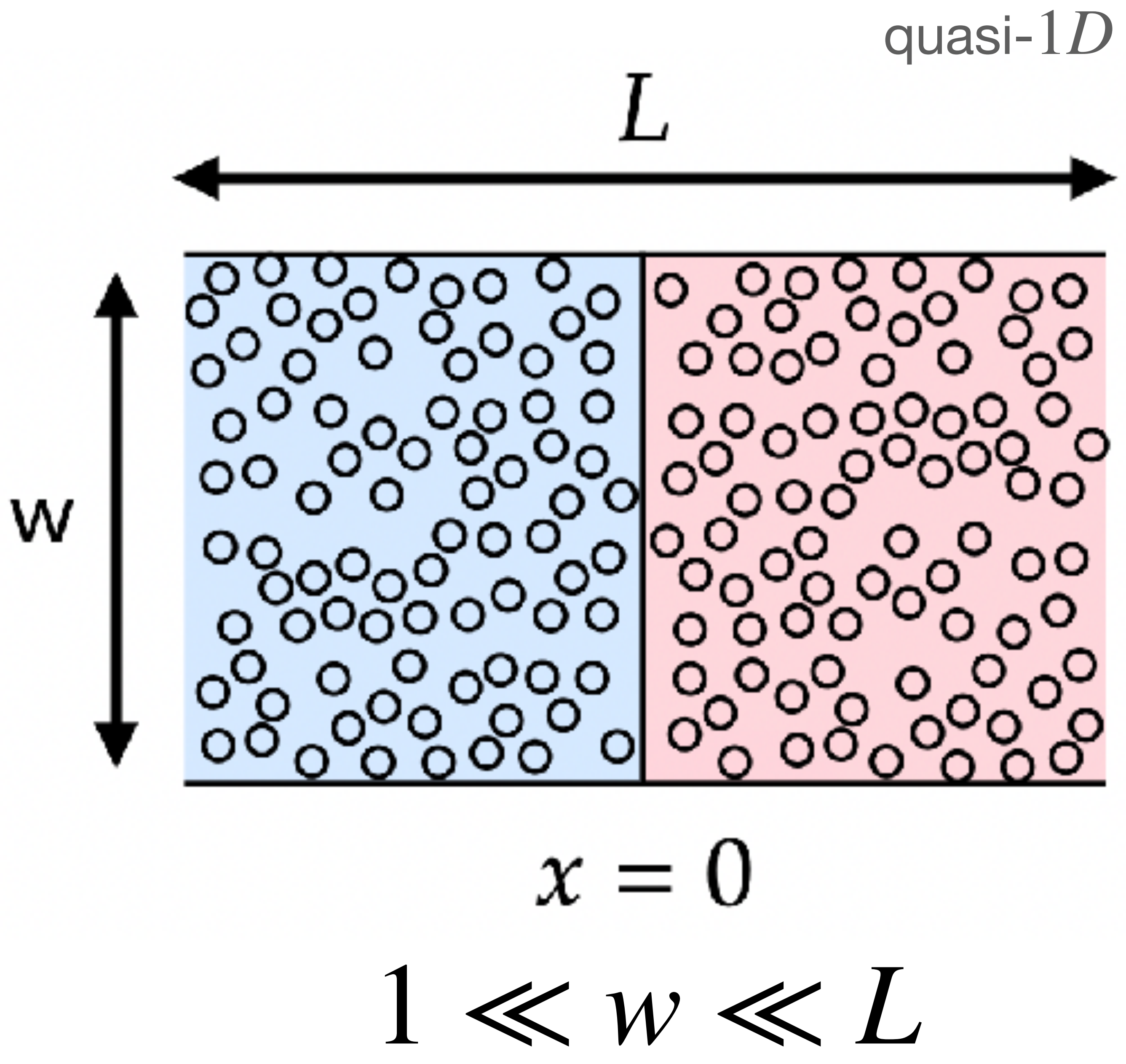
- But *nonlinear fluctuations* matter, so general hydro is

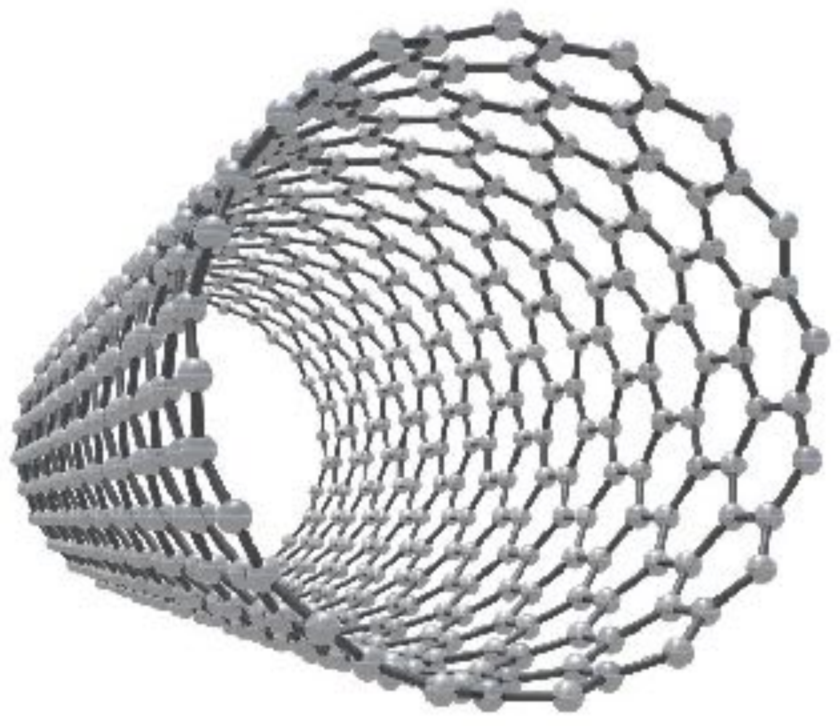
$$\partial_t n + \partial_i (n\phi_i) = D\partial_i \partial_i n + \xi_i$$

why quasi-$1D$?

- In higher dimensions, macroscopic number of independent regions contribute to charge transfer — sum over which is asymptotically Gaussian.







Use graphene bulk dispersion

In $D > 1$, sound waves propagate in a continuum of directions, only charge along the path of a wave feel same convective force

In $1D$, every parcel of charge feels the effect of every sound wave.

hydrodynamic decoupling

- Effect of "Brownian coupling":

  $$\partial_t n + \partial_i (n\phi_i) = 0$$

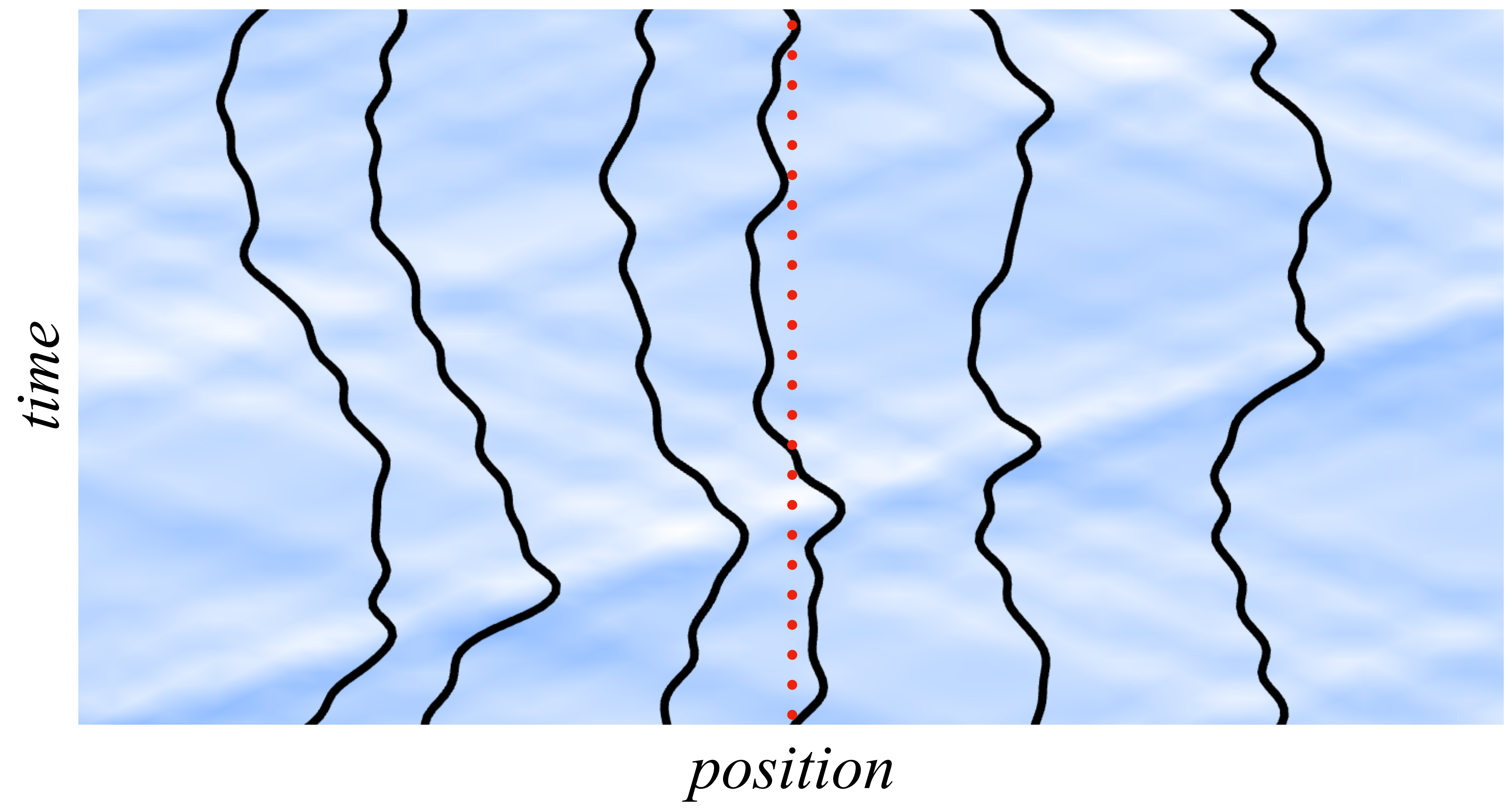
- Fluctuations in ϕ_i are rapidly moving sound waves that impart random kicks to n

- Because n is slowly fluctuating these kicks on a particular fluid element are effectively uncorrelated in time: Brownian motion

- Nonlinearity reduces to tackling multiplicative noise with ballistic correlations

  $$\langle \phi(x,t)\phi(0)\rangle \sim \delta(x - vt)$$

- **These correlations matter for FCS since all the particles are feeling the same noise**



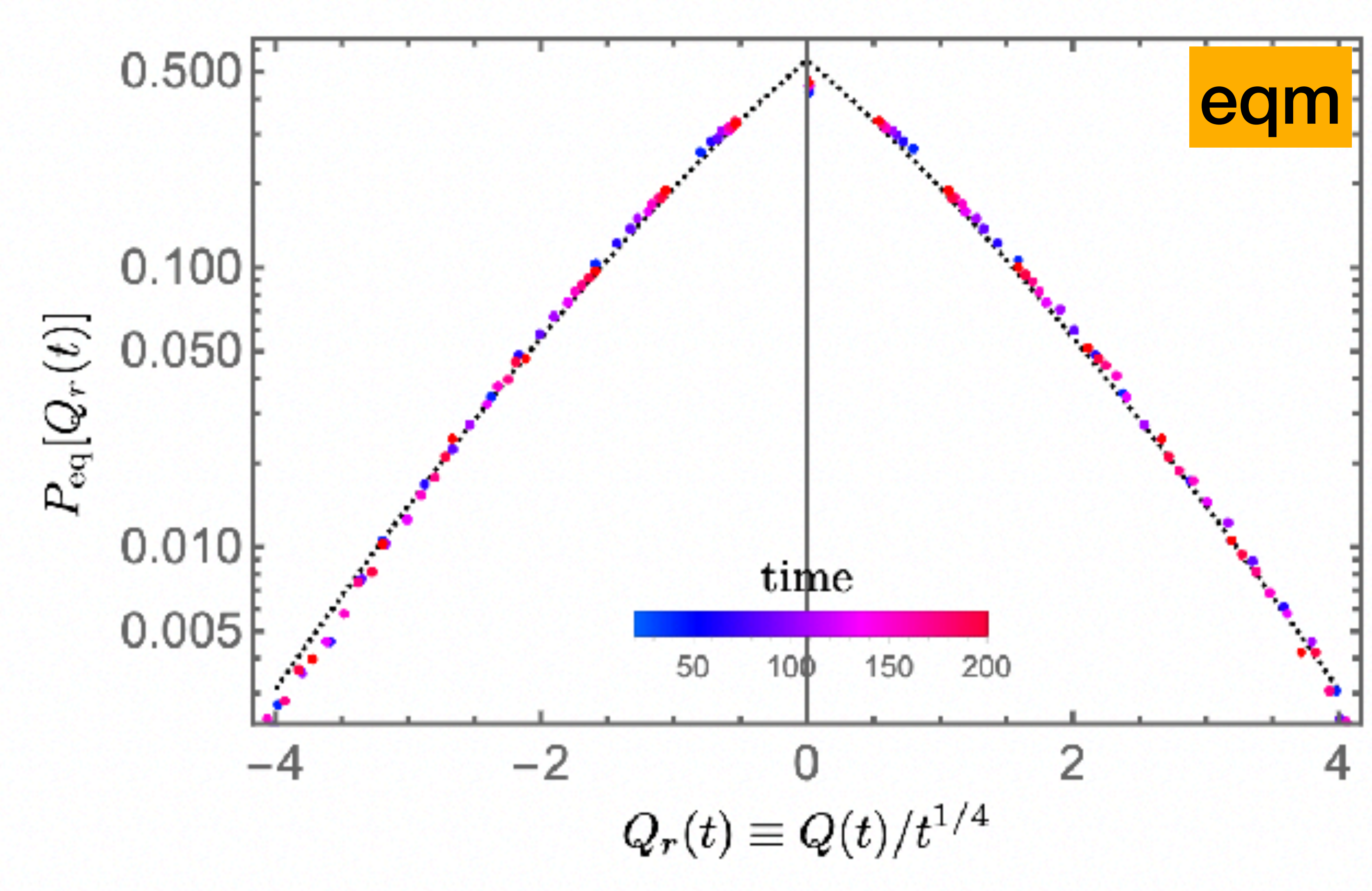
solution by characteristics

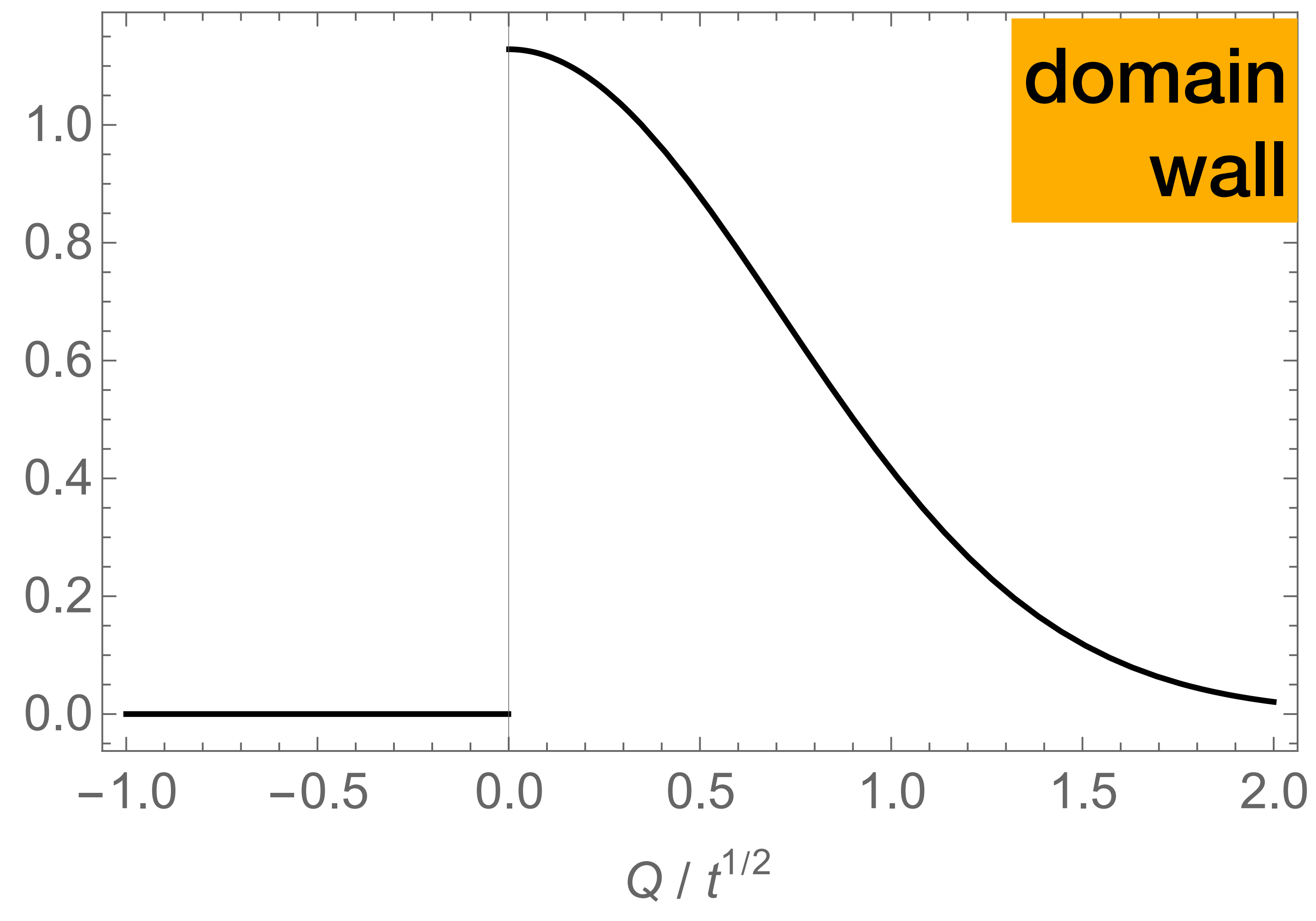
- $\partial_t n + \partial_x(n\phi) = 0$, $\langle \phi(x,t)\phi(0)\rangle \sim \delta(x - vt)$
- Formal solution:

$$n(x,t) \approx n_0\left(x - \int_0^t dt'\phi(x,t')\right)$$

where $n_0(x) = n(x,0)$

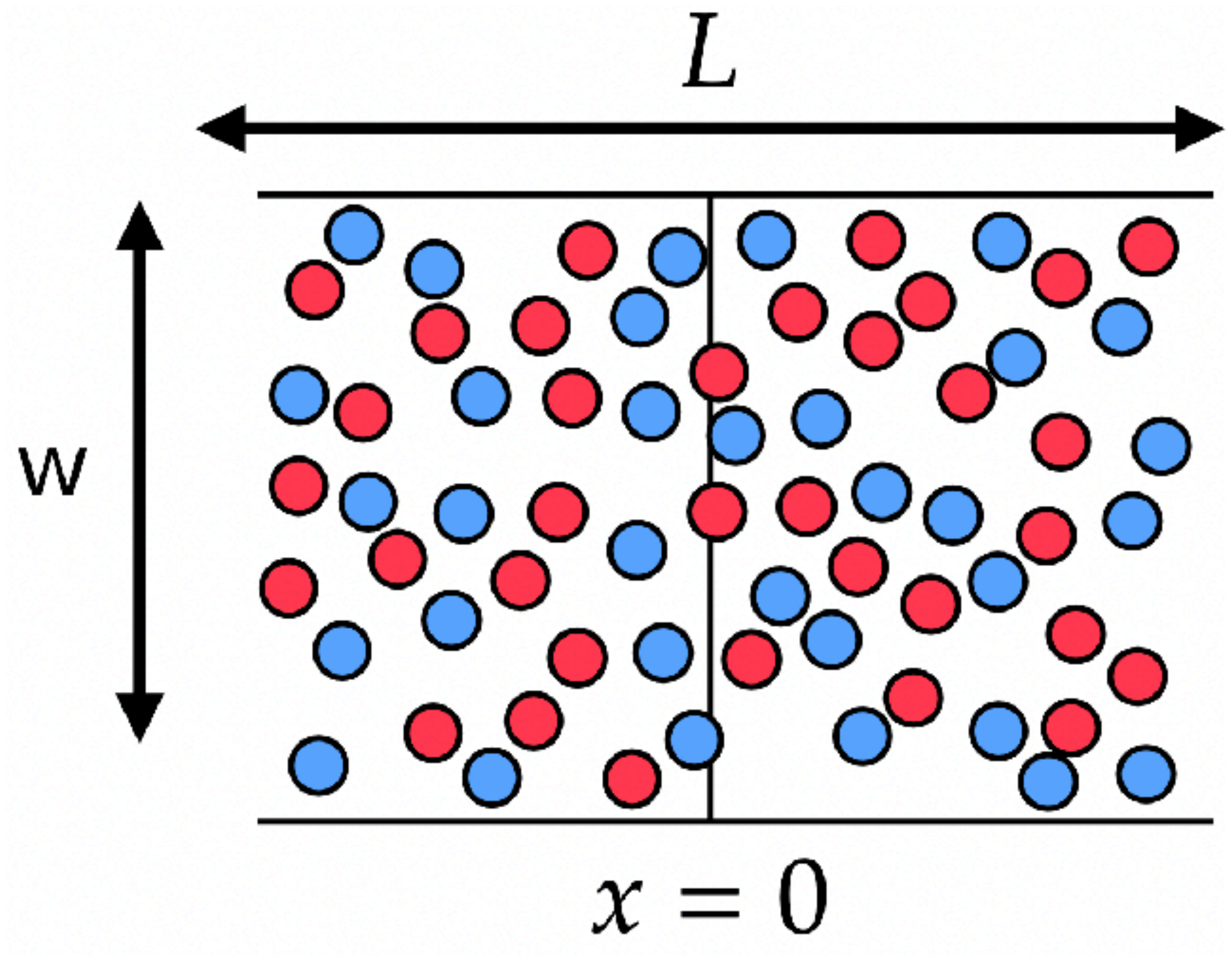
- Distribution reaches a nongaussian limit shape which is different for eq'm and biased states
- Cumulants scale as $t^{n/2}$ rather than $t^{1/2}$ as in standard diffusive systems
- Matches recent results for integrable XXZ (sg et al, 2022; krajnik et al., 2022)





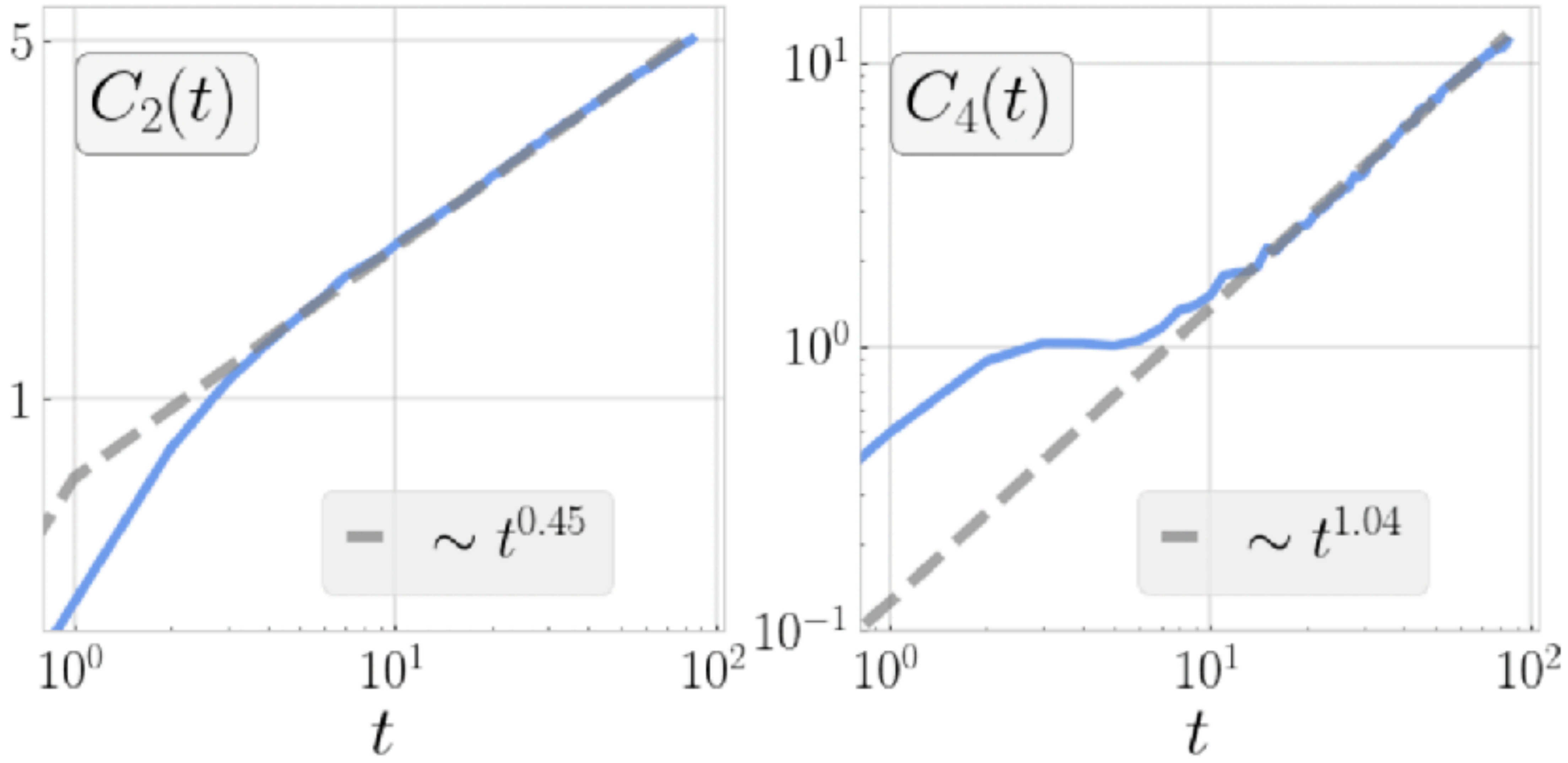
two-color fluid

- Galilean (rather than Lorentz) invariance
- Conserved quantities: total number, imbalance, momentum, energy
- At half filling, imbalance does not overlap with total momentum, and is transported diffusively [cf. Sachdev-Damle]
- Very similar to Dirac fluid, but easy to simulate
- Quasi-1D geometry: wide enough to be chaotic, narrow enough to not average out FCS

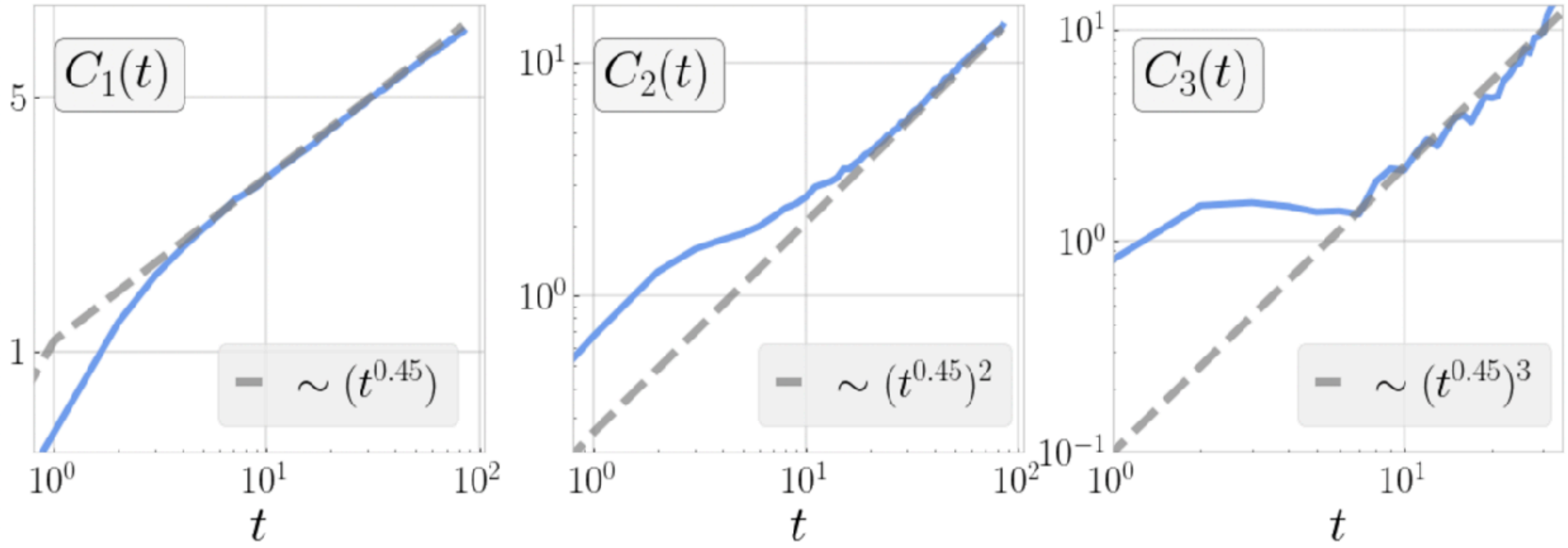


two-color fluid

Equilibrium:
cumulants $C_n \sim t^{n/4}$



Domain wall:
cumulants $C_n \sim t^{n/2}$



summary

- Transport can be diffusive at special points in systems with ballistic modes or systems with a nonequilibrium drive (e.g., asymmetric exclusion)
- This "diffusive" transport has strongly enhanced, nongaussian fluctuations
- Effect is general in nonequilibrium stochastic systems / "active matter" [mcculloch, vasseur, sg, *in prep*]
- Also the basis of fluctuations in integrable systems [sg, huse, khemani, vasseur (2018); medenjak et al. (2019); doyon (2020)]
- What if the ballistic modes are not strictly ballistic but just long-lived? What is the crossover to regular MFT?
- What about disorder—e.g., smoothly varying chemical potentials?
- What is the best way to probe this physics in higher-dimensional settings? (nonlinear response? multipoint noise correlations?)
- Does quantum mechanics play any part here?