measuring and thinking about transport

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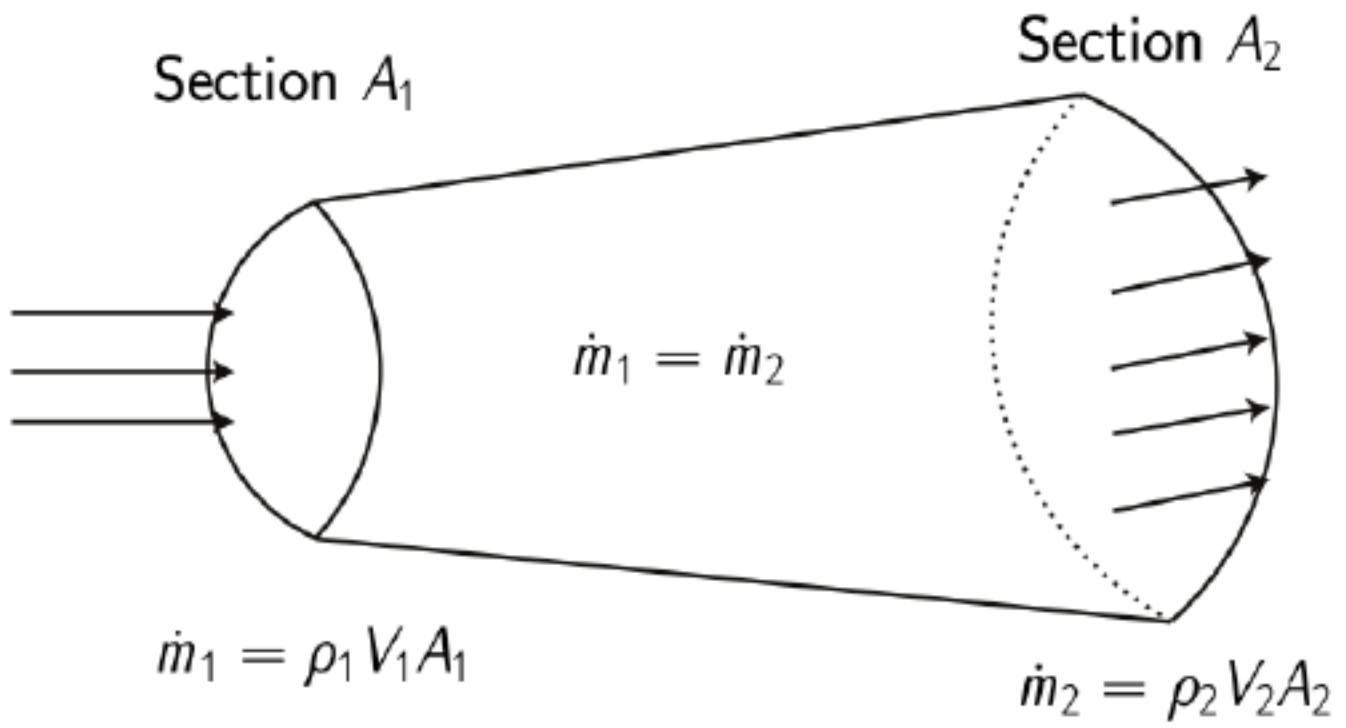
what is transport and why should you care?

- Charge, energy, spin conductivity
- Boring practical reasons:
    - One of the few things you can consistently measure in the solid state
    - Technologically important
- More interesting formal reasons:
    - Involves **large-scale** collective behavior
    - One of the simplest examples of scale-invariant behavior, e.g., heat equation, $\rho(x,t) \sim t^{-1/2} \exp(-x^2/(Dt))$
    - Transport has multiple universality classes (with dynamical phase transitions between them)

basics: continuity equation

- Intuition: if the charge inside a region changes, it must have left through the boundary

- Relies on *locality* of dynamics



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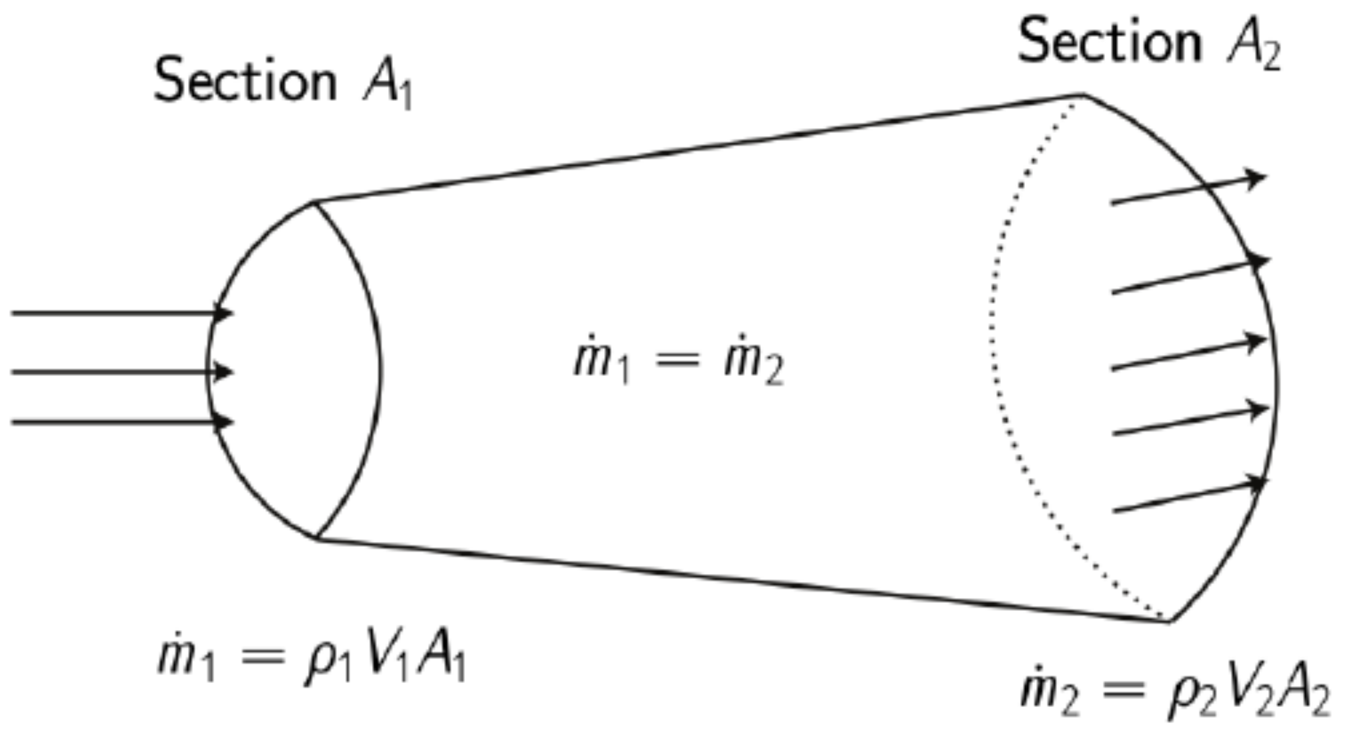


More formally:

- Conserved local charge $Q = \sum_{\mathbf{x}} q(\mathbf{x})$, sum of local terms
- Conservation law: $\partial_t Q = 0 \Rightarrow \partial_t q_i =$ total derivative
- Continuity equation $\partial_t q(\mathbf{x}, t) + \nabla \cdot \mathbf{j}(\mathbf{x}, t) = 0$ *defines* "current"
- We will be interested in the dynamics of q, **j**

correlation and response functions

- Response: **perturb** at time 0, measure at time t
- $R_{AB}(t) = \mathrm{Tr}\left(BU\exp(-i\epsilon A)\rho_{\mathrm{eq}}\exp(i\epsilon A)U^\dagger\right) = \mathrm{Tr}\left(U^\dagger BU(-i\epsilon A\rho_{\mathrm{eq}} + i\epsilon\rho_{\mathrm{eq}}A)\right) = i\epsilon\langle[A, B(t)]\rangle$
- Correlations: **measure** at time 0, measure at time t
- Kraus operators $M_\pm = \sqrt{1/2}\sqrt{1 \pm \epsilon A}, \qquad M_+^\dagger M_+ + M_-^\dagger M_- = \mathbb{I}$
- $C_{AB}(t) = \mathrm{Tr}(BUM_+\rho M_+^\dagger U^\dagger) = \epsilon\langle\{A, B(t)\}\rangle$
- In equilibrium, related by fluctuation-dissipation theorem,
  $C_{AB}(\omega) = (n_B(\omega) + 1)R_{AB}(\omega)$
- At low frequencies and finite temperatures, $C_{AB}(\omega) \simeq (T/\omega)R_{AB}(\omega)$
- Response vanishes at infinite temperature but fluctuations survive

kubo formula

- Linear response a.c. conductivity

$$\sigma(\omega) = \omega^{-1} R_{JJ}(\omega) \simeq T^{-1} C_{JJ}(\omega), \quad C_{JJ}(t) = L^{-1} \langle J(t) J(0) \rangle$$

- Probes intrinsic fluctuations of current
- Lehmann representation,

$$C_{JJ}(\omega) \propto \sum_{mn} e^{-\beta E_m} |\langle m | J | n \rangle|^2 \delta(\omega - (E_m - E_n))$$

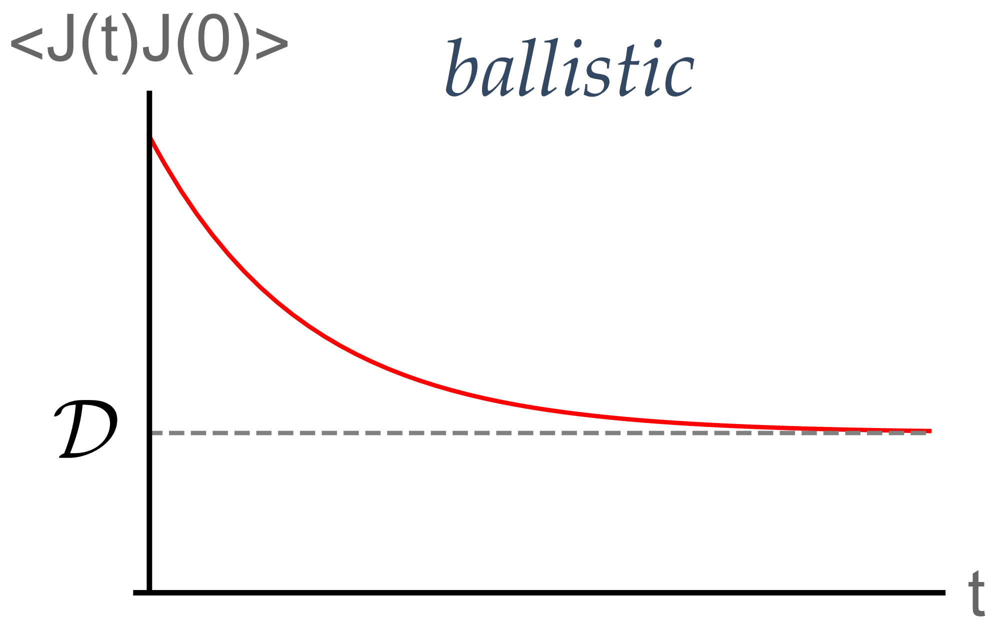
- Other basic object of interest, $S(x, t) = \langle q(\mathbf{x}, t) q(0,0) \rangle$
- These quantities are related by continuity equation: for example, Einstein relation, in diffusive systems

$$\lim_{t\to\infty} t^{-1} \int dx x^2 S(x, t) = D, \quad D = \int_0^\infty C_{JJ}(t)$$

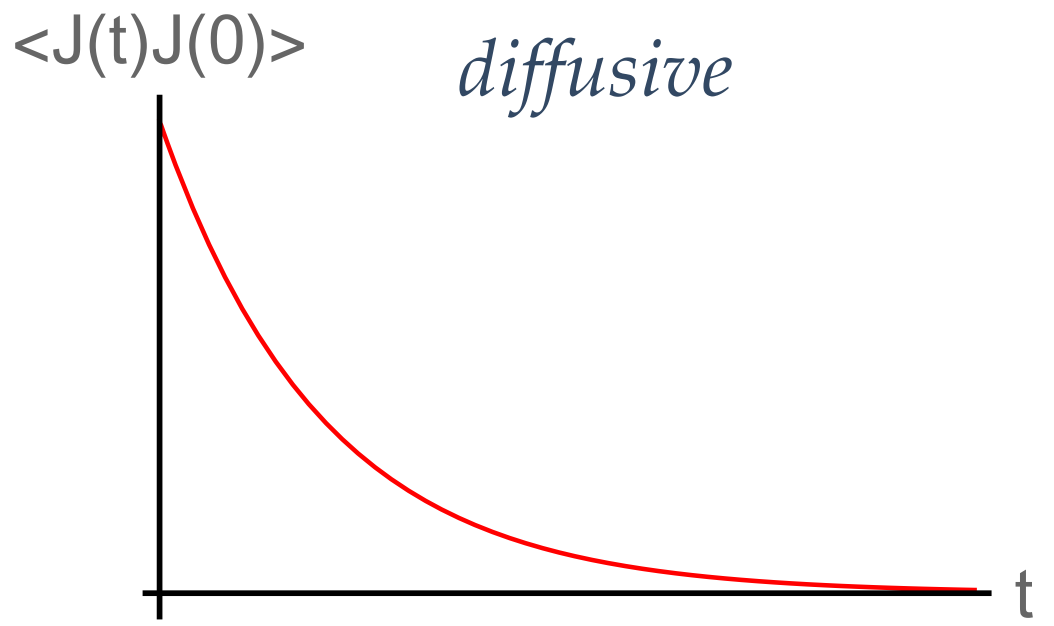
- You will be seeing lots of limits here…

hydrodynamics for linear response

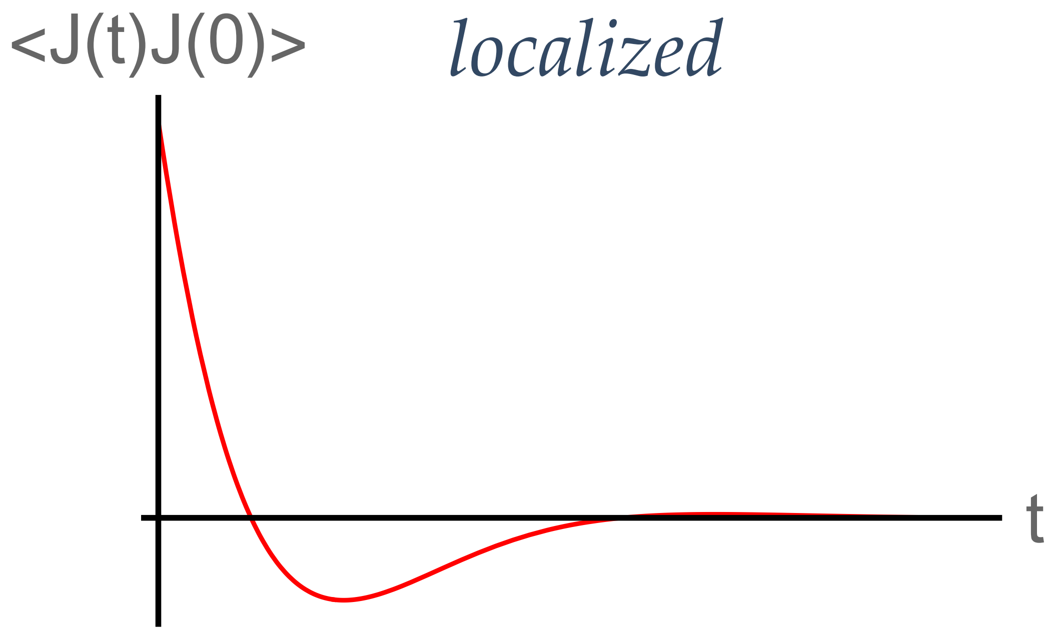
types of linear-response transport



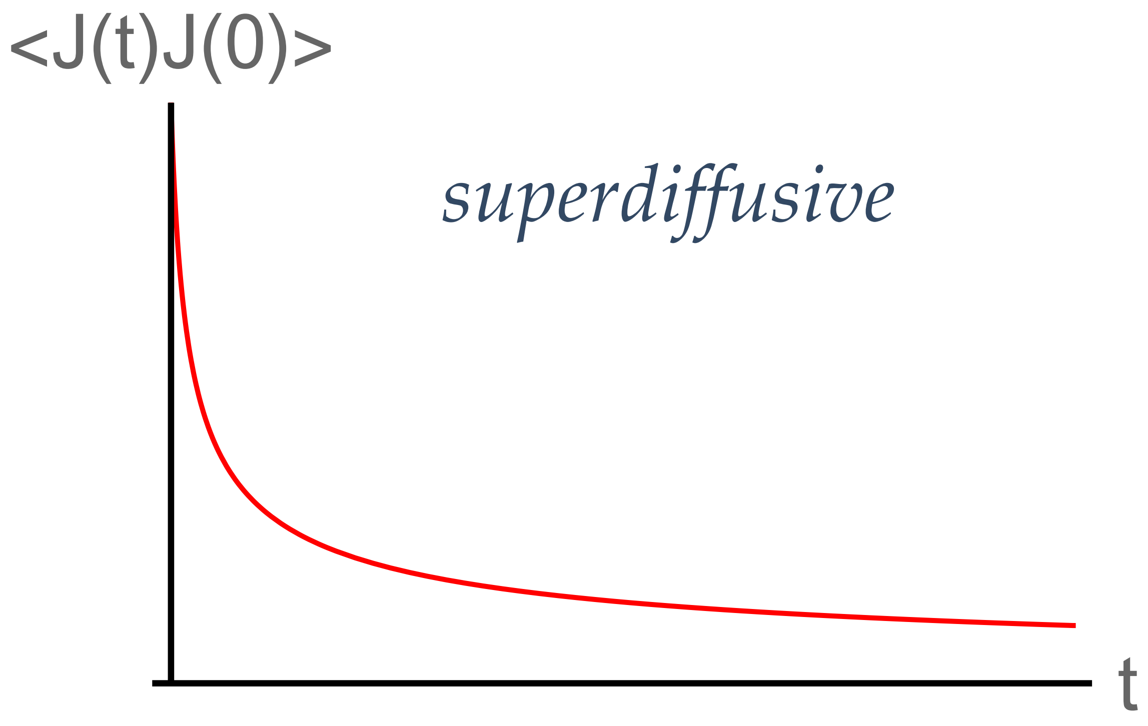
$$\sigma(\omega) = \mathcal{D}\delta(\omega) + \sigma^{\text{reg}}(\omega)$$



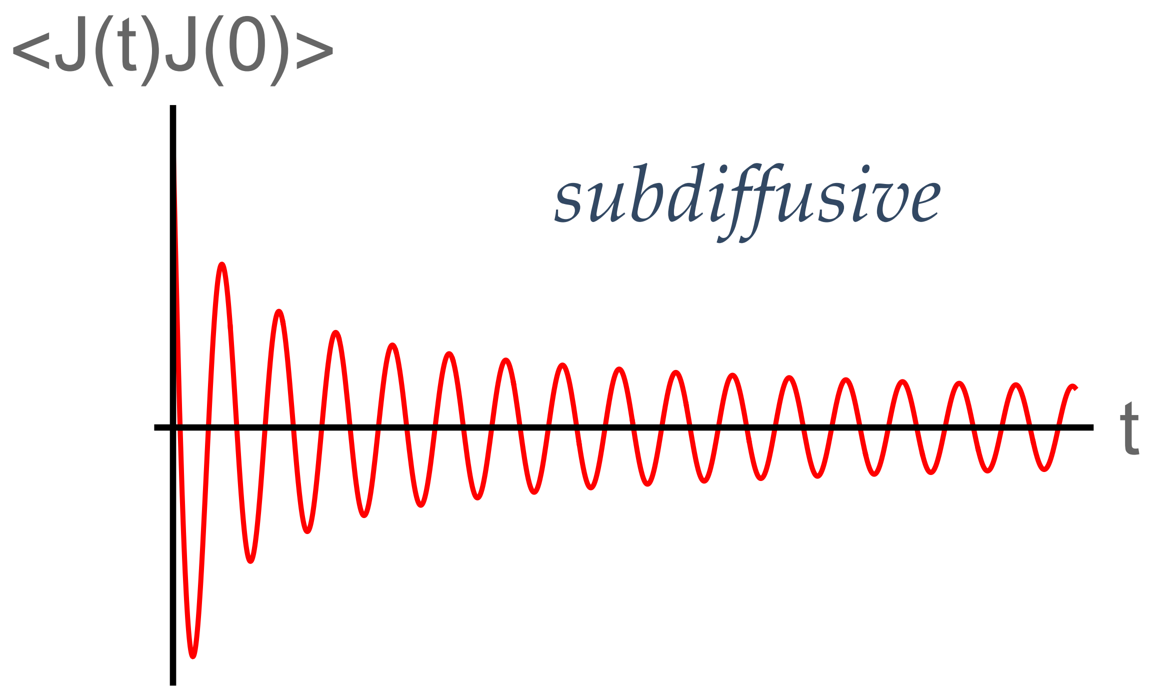
$$\sigma(\omega) = \sigma_0/[1 + (\omega\tau)^2] + \ldots$$



$$\sigma(\omega) \sim \omega^\alpha, \alpha \geq 1$$



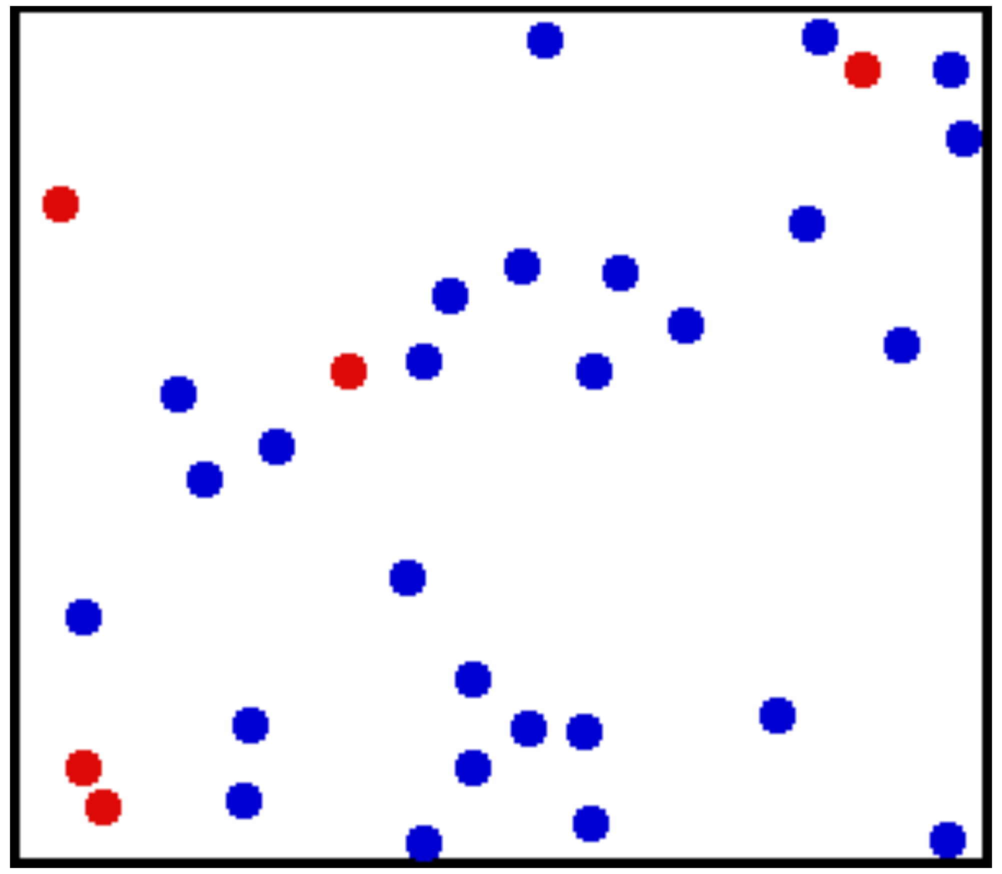
$$\sigma(\omega) \sim \omega^\alpha, -1 < \alpha < 0$$



$$\sigma(\omega) \sim \omega^\alpha, 0 < \alpha < 1$$

high temperatures: hydrodynamics

- High temperature dynamics is complex, chaotic
- Chaos leads to effective randomization of state
  - How quantum mechanics interacts with chaos is a bit subtle…
- System goes to maximum entropy state subject to conservation laws ("thermalization")



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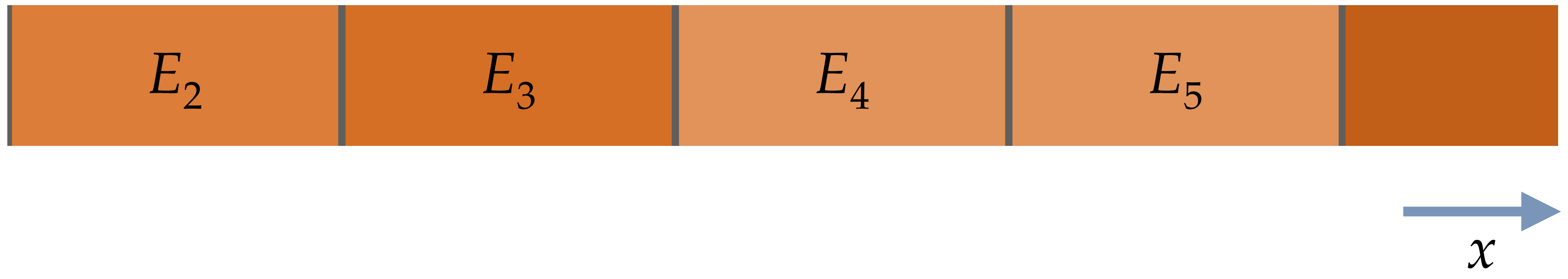
- System goes to maximum entropy state subject to conservation laws (“thermalization”)
- Hydrodynamics:
  - Assume system is *locally* in some thermal state (described by local values of conserved variables)
  - Write down equations of motion for conserved variables by gradient expansion (assume that variations are smooth)



standard diffusive hydrodynamics

- Work in 1D, suppose the only conserved quantity is charge $N = \sum_x n(x)$
- Continuity equation: $\partial_t n + \partial_x j = 0$
- Write current out in gradients of density:
  - Current cannot be proportional to density: opposite symmetries under time reversal and parity
  - Next order in gradients, $j = D\partial_x n$, fixes parity (and D fixes an arrow of time)
- Deterministic diffusion equation: $\partial_t n = \partial_x (D \partial_x j)$
  - "Too good" at smoothing out fluctuations, does not preserve equilibrium local density fluctuations
  - Fix by adding noise: $j = D\partial_x n + \sqrt{D\chi}\xi$ where noise strength is fixed by thermodynamic susceptibility $\chi(n)$
- Move away from linearized theory by letting all coefficients depend on densities:

$$\partial_t n = \partial_x \left(D(n)\partial_x n + \sqrt{D(n)\chi(n)}\ \xi \right)$$

main equation of "nonlinear fluctuating hydrodynamics" in the diffusive case

1d case with translation invariance

- Two conserved quantities, charge and momentum (resp. even/odd under parity/time reversal)
- $\partial_t q + \partial_x j_q = 0, \quad \partial_t p + \partial_x j_p = 0$
- By symmetry these terms are allowed in constitutive relation: $j_q = c_1 p + c_3 p^3 + \ldots, j_p = b_1 q + b_2 q^2 + \ldots$
- At linear order, you get wave equation: $\partial_t^2 q - b_1 c_1 \partial_x^2 q = 0$
- Write down left-moving wave: $(\partial_t + v\partial_x)q = \ldots$
- Velocity can depend on density, and diffusion term is allowed, so we have

  $\partial_t q + v\partial_x q + v' q \partial_x q = D\partial_x^2 q + \partial_x \xi$
- Transform into co-moving frame, get Burgers equation

  $\partial_t q + v' q \partial_x q = D\partial_x^2 q + \partial_x \xi$
- Nonlinear term **not** allowed by symmetry for lattice diffusion; this model “remembers” about ballistic modes

burgers and superdiffusion

- $\partial_t q + v' q \partial_x q = D\partial_x^2 q + \partial_x \xi$
- Ignore nonlinear term, get diffusion equation: but nonlinear term is "RG relevant"
- Simple argument for what happens:
  - Noise preserves thermal fluctuations in stationary state
  - Consider motion over a length-scale ℓ
  - Over this scale, $q \sim 1/\sqrt{\ell}$
  - Charge moves with velocity $v'/\sqrt{\ell}$
  - Time taken to traverse region: $\ell/(v'/\sqrt{\ell}) \sim \ell^{3/2}$

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Fluctuating hydrodynamics for a discrete Gross-Pitaevskii equation: Mapping onto the Kardar-Parisi-Zhang universality class

Manas Kulkarni,[1] David A. Huse,[2] and Herbert Spohn[3,4]

Breakdown of Diffusion on Chiral Edges

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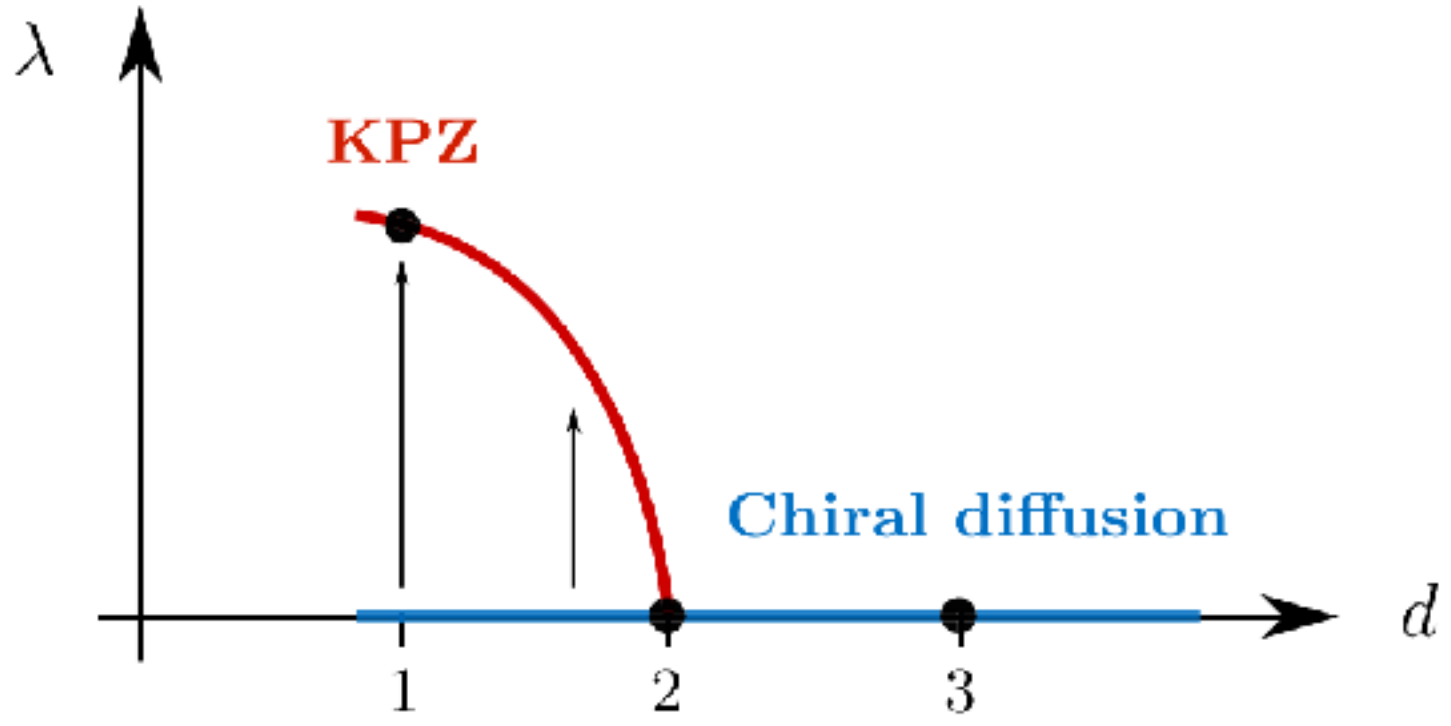
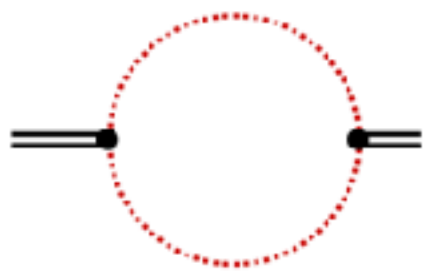


long-time tails

- Consider the relaxation of a density wave initialized at momentum k
- Linear diffusion equation: $\partial_t q_k = -Dk^2 q_k$
- However, diffusion constant can depend on density, allowing for real-space terms like $\partial_t q = \partial_x((D + D'q)\partial_x q + \xi)$
- These terms lead to "three-wave mixing" between different k modes

$\langle \mathcal{O}\mathcal{O} \rangle (t,k) =$  $+$  $+ \cdots$

$$\sim g(t,k) + k^d g(t, \tfrac{k}{2})^2 + \cdots$$

$$\sim e^{-Dk^2 t} + k^d e^{-Dk^2 t/2} + \cdots$$

- Current belief: at very late times, density wave relaxation is $\sim e^{-\sqrt{Dk^2 t}}$

general framework

- Philosophy of hydro: identify slow modes (typically charges or Goldstone modes)
- Write down all symmetry-allowed terms in the equations of motion for these slow modes, working to lowest possible order in fluctuations and gradients
- This approach already leads to surprising nonperturbative predictions
- How can we deal with/discover new hydrodynamic modes? (See third lecture)

linear response vs. experiment

singular limits

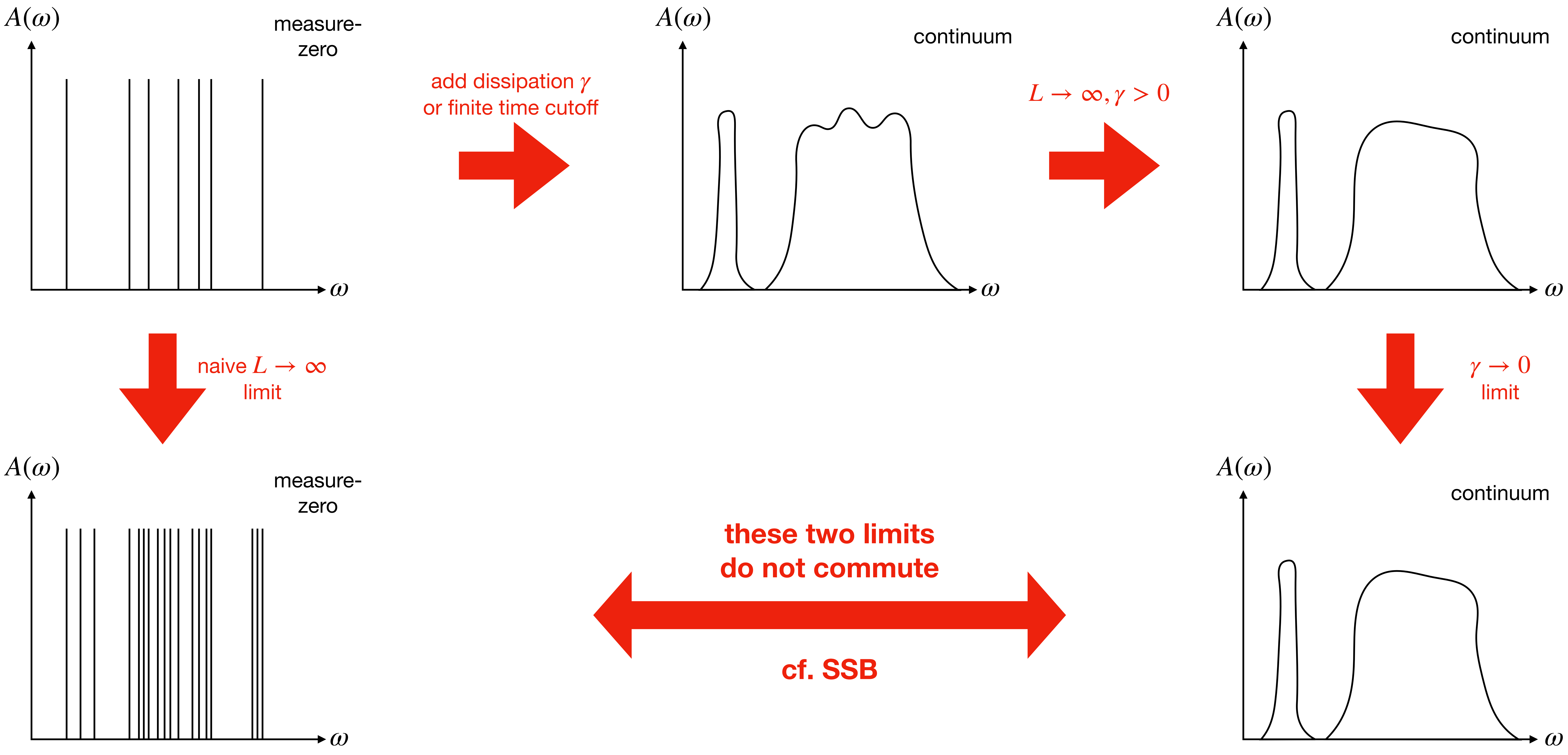
"Biting into an apple and finding a maggot is bad enough, but finding half a maggot is worse. Discovering one-third of a maggot would be more distressing still: The less you find, the more you must have eaten. Extrapolating to the limit, an encounter with no maggot at all should be the ultimate bad-apple experience. This remorseless logic fails, however, because the limit is singular: A very small maggot fraction (f approaching 0) is qualitatively different from no maggot (f = 0)."

michael berry

kubo order of limits

- Standard hydrodynamic order of limits
- First, take linear response limit: $\epsilon \to 0$
- Second, take thermodynamic limit: $L \to \infty$
- Third, take late-time limit: $t \to \infty$
- **These limits might not commute!**
- But you can interchange linear response and TDL in local systems, because of locality — at a finite time, a system does not “see” regions that are very far away in space

getting to a continuum



protocol 1: quenches

- Idea: create an initial state

  $\rho \propto \prod_x e^{-\mu \,\mathrm{sign}(x)\sigma^z(x)}$

- Measure $\langle \sigma^z(x,t) \rangle$

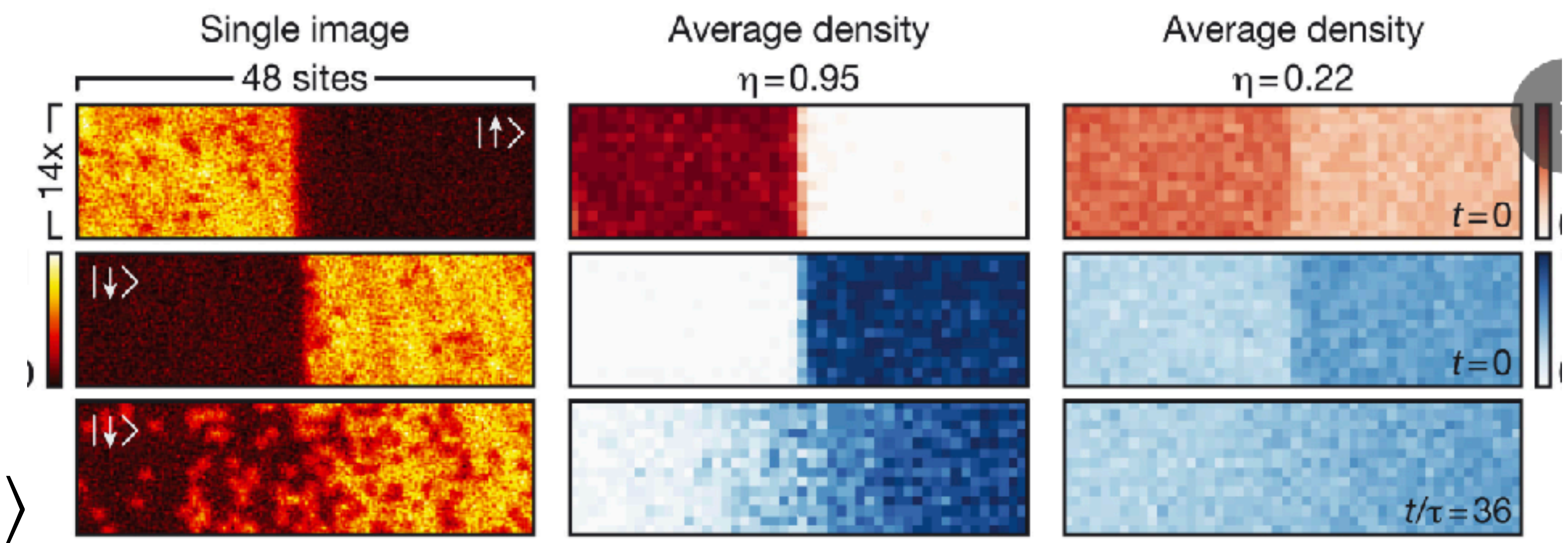
- $\langle \sigma^z(x,t) \rangle \simeq \mu \sum_y \mathrm{sign}(y) \langle \sigma^z(x,t)\sigma^z(y,0) \rangle$

- Related to linear response if you take $\mu \to 0$ at finite time

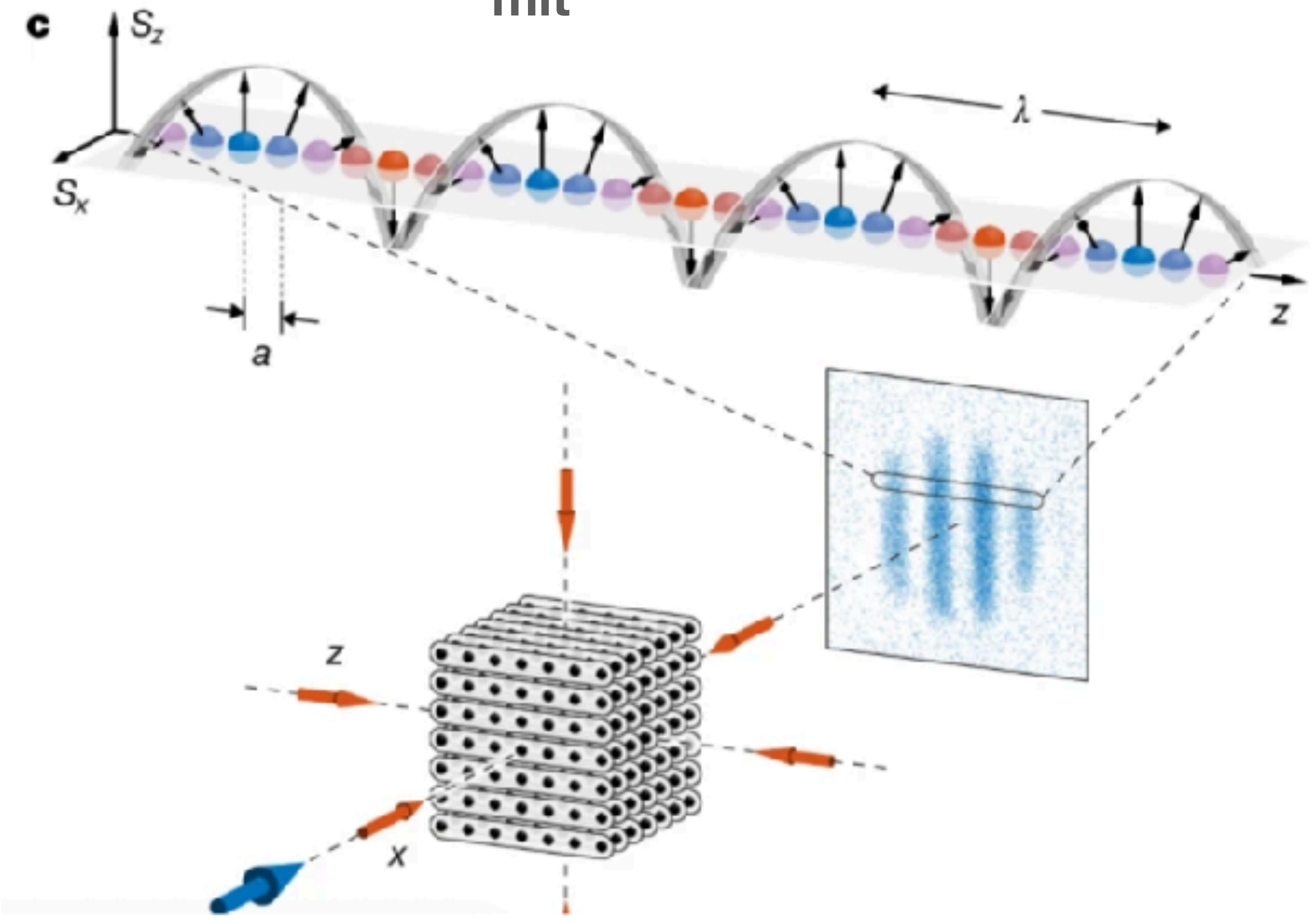
- Fundamental tradeoff between signal to noise and staying in linear response

- Does this matter in practice?

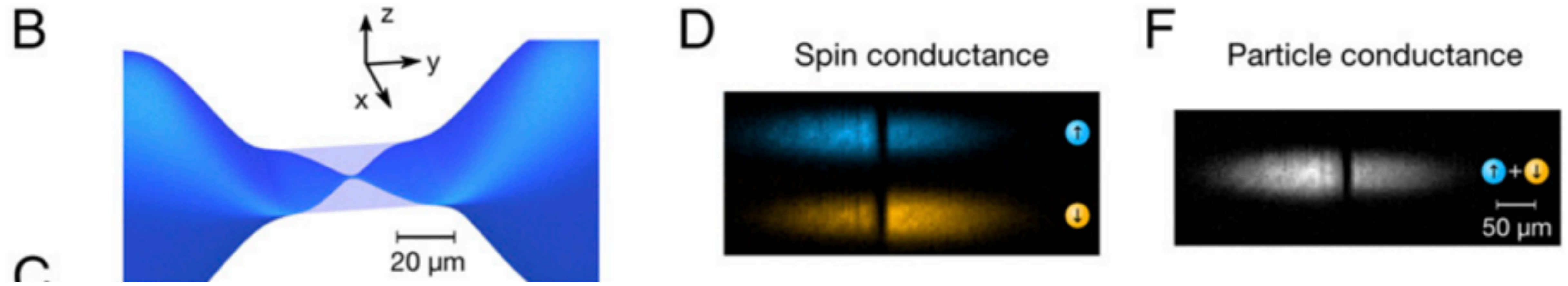
mpq



mit



protocol 2: lindblad/landauer



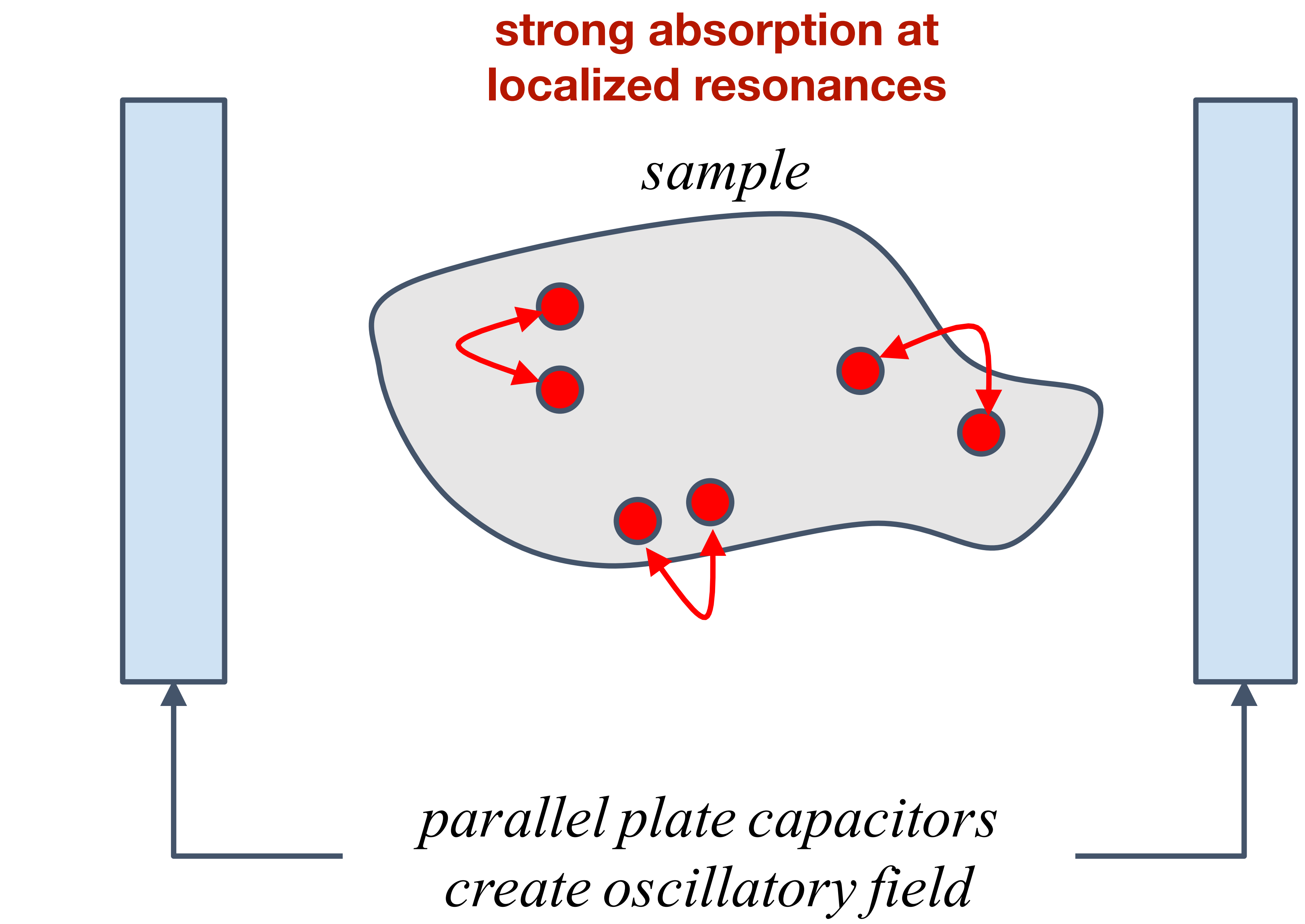
- Take finite system coupled to infinite leads with finite coupling γ
- Take leads to be infinite size, replace with free particle / markovian bath approx
- Find steady state at finite system size L: *interchanges* order of thermodynamic and dc limits
- Does this matter? Yes, especially for integrable systems: boundary breaks integrability

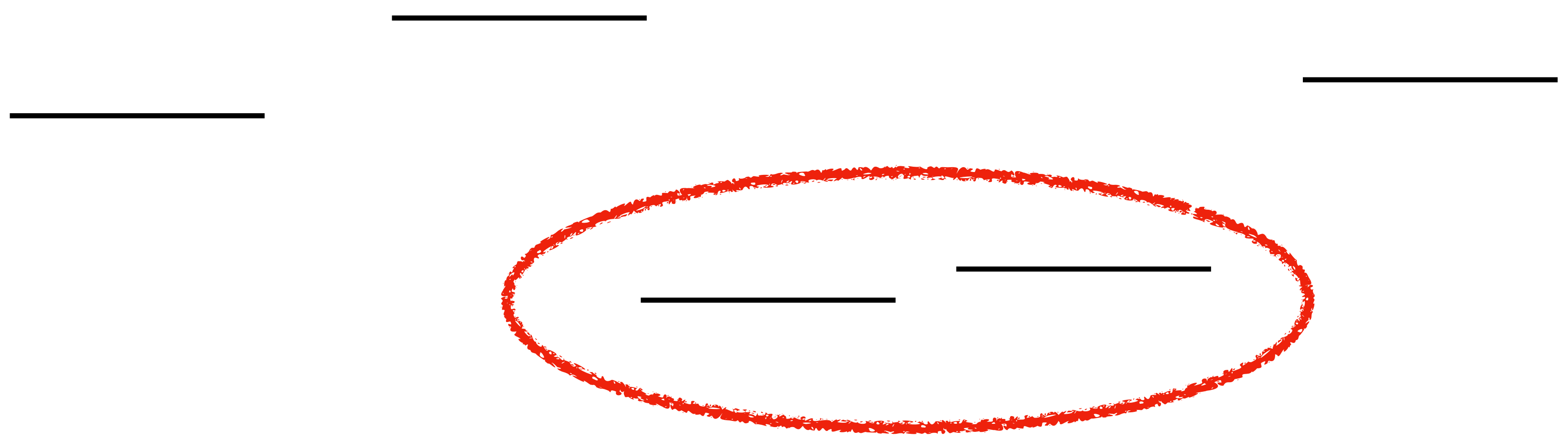
strange case 1: heisenberg chain

- Heisenberg chain in one dimension, $H = \sum_i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$
- Linear-response transport is superdiffusive at nonzero temp: $S(x,t) \sim t^{-2/3} f(x/t^{2/3})$
- However, for any finite μ, this physics is cut off on a length-scale $\ell \sim 1/\mu^2$
- More general scaling form for domain wall state,

  $S(x,t,\mu) = \mu^2 C(x/\mu^2, t/\mu^3)$
- When the arguments are large, $S(x,t) \sim (\mu t)^{-1/2} f(x/\sqrt{t})$
- **Experiments with fixed contrast going to late times will see diffusion, not superdiffusion**

strange case 2: driven insulators





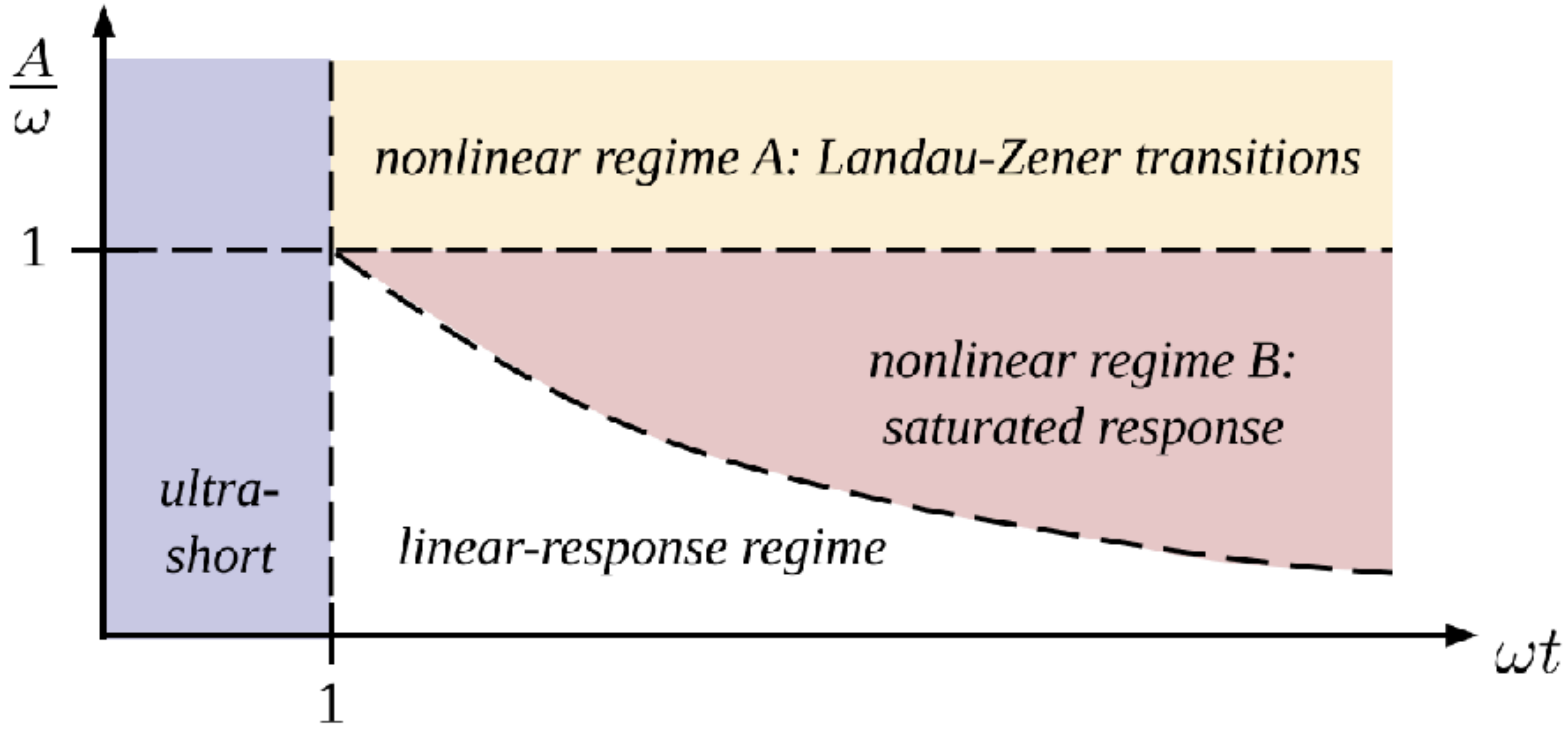
Mostly isolated two-level systems (TLS)

When you drive a TLS, two possible behaviors:

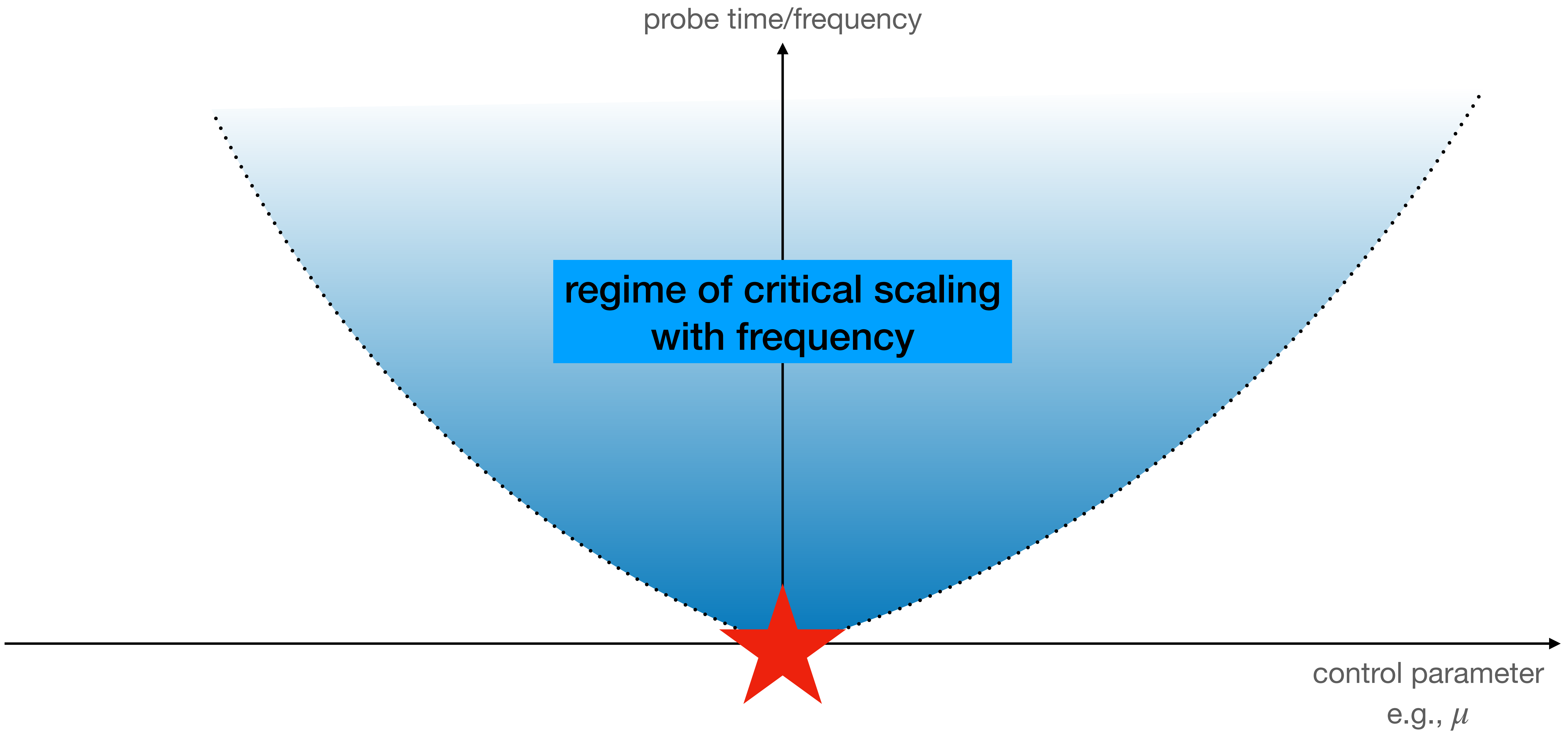
amplitude $\ll \omega$, rotating-wave approx is good

Excitation rate given by Fermi Golden Rule

amplitude $\gg \omega$, Landau-Zener transitions, excitation rate is a nontrivial power of amplitude



more generally: critical response



some general thoughts

- Challenges of hydrodynamics in the “interesting” regime:
  - Noncommuting limits make it hard to interpret experimental data
  - Dynamics is “interesting” precisely because it is not purely dictated by obvious symmetries
  - Is there a treatment that connects to microscopics and allows us to “discover” new slow modes?
- Plan for the next two lectures:
  - Lecture 2: XXZ spin chain as a case study
  - Lecture 3: “modern” microscopic approaches; quantum advantage (?) in transport