# Introduction to Quantum Optics

## From Maxwell's Equations to Multi-Mode Fields

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# Outline

- Quantization of the electromagnetic field
- Different quantum states for the field
- Correlation functions of the electromagnetic field
- Young's double slit from a quantum optical perspective

# Maxwell's Equations in Free Space

In vacuum the Maxwell equations read:

$$\nabla \cdot \mathbf{E} = 0 \qquad \text{(Gauss's Law)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No magnetic monopoles)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère-Maxwell Law)}$$

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Electromagnetic potentials

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

# Gauge Freedom and the Coulomb Gauge

The **electromagnetic potentials** $\phi$ and $\mathbf{A}$ are not uniquely defined; their redundancy allows a gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \quad \phi' = \phi - \frac{\partial \chi}{\partial t}$$

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$$\phi = 0$$

In the **Coulomb gauge** we also impose:

$$\nabla \cdot \mathbf{A} = 0$$

so that the electric field reduces to:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$$

# Wave Equation for the Vector Potential

**Ampère-Maxwell law**

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Using the **vector identity**

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

$$\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} \quad (\text{since } \nabla \cdot \mathbf{A} = 0)$$

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we obtain the **wave equation** for the vector potential:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

# Separation Ansatz and the Helmholtz Equation

Assume a separable solution of the form:

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$$\nabla^2 \mathbf{a}(\mathbf{r}) + k^2 \mathbf{a}(\mathbf{r}) = 0$$

and the time-dependent part satisfies harmonic oscillator ODE

$$T''(t) + c^2 k^2 T(t) = 0$$

# Separation of Variables and Mode Expansion

Mode expansion

$$\mathbf{A}(\mathbf{r},t) = \sum_m \sqrt{\frac{\hbar}{2\epsilon_0\omega_m}}\left[a_m(t)\,\boldsymbol{u}_m(\mathbf{r}) + a_m^\dagger(t)\boldsymbol{u}(\mathbf{r})\right]$$

Here $a_m(t)$ and $a_m^\dagger(t)$ are complex numbers.

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Inserting this into wave equation yields:

$$\nabla^2 \mathbf{u}_m(\mathbf{r}) + \frac{\omega_m}{c} \mathbf{u}_m(\mathbf{r}) = 0$$

$$\frac{\partial^2 a_m(t)}{\partial t^2} + \omega_m^2 a_m(t) = 0$$

# Solutions for Separation Ansatz

Solution for time-dependent coefficients:

$$a_m(t) = a_m e^{-i\omega_m t}$$
$$a_m^\dagger(t) = a_m^\dagger e^{+i\omega_m t}$$

$a_m(t)$, $a_m^\dagger(t) \in \mathbb{C}$ as before.

# Solutions for Separation Ansatz

Use **cuboid** with volume V for the domain of the solution.
Assume **periodic boundary** conditions

$$\mathbf{A}(\mathbf{r} + \mathbf{k}_m, t) = \mathbf{A}(\mathbf{r}, t)$$

with

$$\mathbf{k}_m = m_1 \mathbf{a} + m_2 \mathbf{b} + m_3 \mathbf{c}, \quad m_1, m_2, m_3 \in \mathbb{Z}$$

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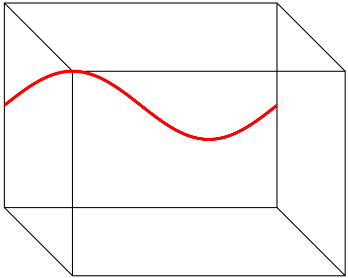
Solution of Helmholtz equation:

$$\mathbf{u}_m(\mathbf{r}) = \frac{1}{\sqrt{V}} \mathbf{e}_m \exp(i \mathbf{k}_m \cdot \mathbf{r})$$

# Sketch of EM Mode in a 3D Box



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Orthogonality of modes

$$\int_V \mathbf{u}_m^*(\mathbf{r})\mathbf{u}_n(\mathbf{r})dV = \delta_{m,n}$$

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Allowed are two transversal polarizations in the plane perpendicular to the propagation direction $\mathbf{k}_m$.

# Explicit form of solutions

**Vector potential**

$$\mathbf{A}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \boldsymbol{\epsilon}_{\mathbf{k},\lambda} \left[ a_{\mathbf{k},\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + a^{\dagger}_{\mathbf{k},\lambda}\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right]$$

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**Electric field**

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \\ &= i \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \boldsymbol{\epsilon}_{\mathbf{k},\lambda} \left[ a_{\mathbf{k},\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - a^\dagger_{\mathbf{k},\lambda}\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \end{aligned}$$

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**Magnetic field**

$$\begin{aligned} \mathbf{B}(\mathbf{r},t) &= \nabla \times \mathbf{A}(\mathbf{r},t) \\ &= -\frac{i}{c} \sum_{\mathbf{k},\lambda} \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} \boldsymbol{\epsilon}_{\mathbf{k},\lambda} \times \mathbf{k} \left[ a_{\mathbf{k},\lambda}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - a^\dagger_{\mathbf{k},\lambda}\, e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right] \end{aligned}$$

# Explicit form of solutions

Energy of the (classical) Multi-Mode field

$$
\begin{aligned}
H &= \frac{1}{2}\int_V \epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_2}\mathbf{B}^2 dV \\
&= \frac{1}{2}\int_V \epsilon_0 \left(-\frac{\partial A}{\partial t}\right)^2 + \frac{1}{\mu_2}(\nabla \times \mathbf{A})^2 dV \\
&= \frac{1}{2}\sum_m \hbar\omega_m (a_m a_m^\dagger + a_m^\dagger a_m)
\end{aligned}
$$

Up to here, we still have $a_m$, $a_m^\dagger \in \mathbb{C}$ as before.

# Quantization of the Multi-Mode field

Quantization of the Multi-Mode field: Bosonic commutation relations

$$[\hat{a}_n, \hat{a}_m] = 0, \quad [\hat{a}_n^\dagger, \hat{a}_m^\dagger] = 0, \quad [\hat{a}_n, \hat{a}_m^\dagger] = i\hbar\,\delta_{nm}$$

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Reminder: Harmonic Oscillator (for a single mode)

$$H_n = \frac{\hat{p}_n^2}{2} + \frac{1}{2}\omega_n^2 \hat{q}_n^2, \quad [\hat{q}_n, \hat{p}_n] = i\hbar$$

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Creation and annihilation operators:

$$\hat{a}_n = (2\hbar\omega_2)^{-1/2}(\omega_n\hat{q}_n + i\hat{p}_n), \quad \hat{q}_n = (\hbar/2\omega_n)^{1/2}(\hat{a}_n + \hat{a}_n^\dagger)$$
$$\hat{a}_n^\dagger = (2\hbar\omega_2)^{-1/2}(\omega_n\hat{q}_n - i\hat{p}_n), \quad \hat{p}_n = i(\hbar/2\omega_n)^{1/2}(\hat{a}_n - \hat{a}_n^\dagger)$$

# Hamilton of the Multi-Mode field

Classical Multi-Mode Hamiltonian

$$H = \frac{1}{2} \sum_m \hbar\omega_m (a_m a_m^\dagger + a_m^\dagger a_m)$$

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$$a_m \longrightarrow \hat{a}_m$$
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Quantized Hamilton of the Multi-Mode field

$$\hat{H} = \frac{1}{2} \sum_m \hbar\omega_m (\hat{a}_m \hat{a}_m^\dagger + \hat{a}_m^\dagger \hat{a}_m)$$
$$= \sum_m \hbar\omega_m (\hat{a}_m^\dagger \hat{a}_m + 1/2)$$

The free EM field corresponds in quantized form to a sum of free quantum harmonic oscillators.

# Fock number states

Single mode

$$\hat{a}_m^\dagger \hat{a}_m \left| n \right\rangle = n \left| n \right\rangle$$

n photons in state $\left| n \right\rangle$.

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**Fock number states** are energy eigenstates of the Oscillator.

$$\begin{aligned} \hat{H} \left|n\right\rangle &= \hbar\omega(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}) \left|n\right\rangle \\ &= \hbar\omega(n + \frac{1}{2}) \left|n\right\rangle \\ &= E_n \left|n\right\rangle \end{aligned}$$

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Raising and lowering

$$\begin{aligned} \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\ \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \end{aligned}$$

# Multi-mode field

EM field state vector for M oscillators

$$|n_1\rangle \otimes |n_2\rangle \otimes ... \otimes |n_M\rangle = |n_1 n_2 ... n_m\rangle$$

Ground state of the EM field

$$|0\rangle \otimes |0\rangle \otimes ... \otimes |0\rangle = |0, 0, ...., 0\rangle = |0\rangle$$

Energy expectation value of ground state

$$\begin{aligned} \langle 0| \hat{H} |0\rangle &= \langle 0| \sum_m \hbar\omega_m (\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}) |0\rangle \\ &= \frac{1}{2} \sum_{m=1}^{M} \hbar\omega_m \end{aligned}$$

# Multi-mode field

Vacuum energy diverges

$$\lim_{M\to\infty} \frac{1}{2} \sum_{m=1}^{M} \hbar\omega_m \to \infty$$

Not a problem in practice, since only energy differences can be measured.

**Normal ordering**
By imposing

$$\langle 0| : \hat{a}\hat{a}^\dagger : |0\rangle = 0$$

the vaccum expectation value of the Multi-mode Hamiltonian can be set to zero.

# Multi-mode wavefunction in position space

$$\langle x_1, x_2, \ldots, x_m | n_1, n_2, \ldots, n_m \rangle =$$

$$\frac{1}{\sqrt{n_1!\, n_2! \cdots n_m!}} \begin{vmatrix} \phi_{n_1}(x_1) & \phi_{n_2}(x_1) & \cdots & \phi_{n_m}(x_1) \\ \phi_{n_1}(x_2) & \phi_{n_2}(x_2) & \cdots & \phi_{n_m}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n_1}(x_m) & \phi_{n_2}(x_m) & \cdots & \phi_{n_m}(x_m) \end{vmatrix}_+$$

The notation $|\cdot|_+$ is used to indicate that the sum over permutations is taken with only positive signs – that is, this object is the permanent, rather than the determinant.

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The orbitals are the usual oscillator states

$$\phi_n(x) = \left(\frac{1}{2^n\, n!}\right)^{\frac{1}{2}} \left(\frac{1}{\pi^{1/2}\, a}\right)^{\frac{1}{2}} H_n\!\left(\frac{x}{a}\right) \exp\!\left(-\frac{x^2}{2a^2}\right),$$

where $H_n$ is the nth Hermite polynomial and $a = \sqrt{\hbar/\omega_n}$ is the oscillator length.

# Coherent states / Glauber states - Eigenvalue equation

The coherent state $|\alpha\rangle$ is defined by the eigenvalue equation of the annihilation operator $a$

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The coherent state can be expressed as an infinite superposition of number states:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

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In a similar way, the bra is written as:

$$\langle\alpha| = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\,\langle n|$$

# Coherent states - Check of the Eigenvalue Equation

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Change the summation index via $m = n-1$:

$$a\,|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^{m+1}}{\sqrt{(m+1)!}} \sqrt{m+1}\;|m\rangle = \alpha\, e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}}\;|m\rangle$$

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Hence, we recover

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

# Coherent states - Position and Momentum

The position and momentum operators for the harmonic oscillator are written as:

$$\hat{x} = \sqrt{\frac{\hbar}{2\omega}}\,(a + a^\dagger), \quad \hat{p} = i\sqrt{\frac{\hbar\omega}{2}}\,(a^\dagger - a).$$

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The expectation values in the coherent state $|\alpha\rangle$ are:

$$\langle x \rangle = \sqrt{\frac{\hbar}{2\omega}}(\alpha + \alpha^*),$$

$$\langle p \rangle = i\sqrt{\frac{\hbar\omega}{2}}(\alpha^* - \alpha).$$

## Coherent states - Variances / Uncertainty Product

For coherent states, the fluctuations are independent of $\alpha$, yielding:

$$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\hbar}{2\omega},$$

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Using the variances,

$$\Delta x^2 \, \Delta p^2 = \sqrt{\frac{\hbar}{2\omega} \cdot \frac{\hbar\omega}{2}} = \frac{\hbar}{2},$$

which is the minimum allowed by the Heisenberg uncertainty principle.

# Coherent states - Relation to number operator and states

For the number operator $\hat{N} = a^\dagger a$, the expectation value in a coherent state is:

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The overlap (or expansion coefficient) is provided directly by the expansion:

$$\langle n | \alpha \rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}.$$

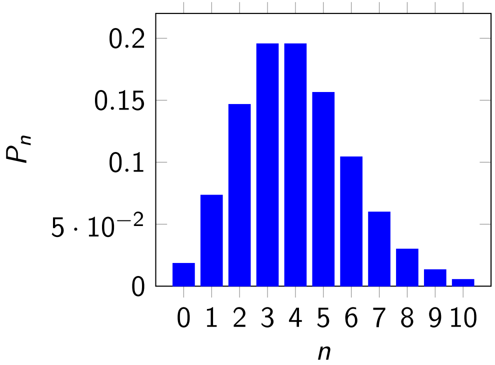
This shows that the probability of finding $n$ quanta in the state is a **Poisson distribution**.

# Coherent States - Poissonian Distribution

Example: $|\alpha|^2 = 4$, the probability is

$$P_n = e^{-4}\,\frac{4^n}{n!},$$

for $n = 0, 1, 2, \ldots$.



# Coherent states - Position Representation

A common form for the coherent state in position space is:

$$\langle x|\alpha\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{\omega}{2\hbar}(x-x_0)^2 + \frac{i}{\hbar} p_0 x\right],$$

where the displacement parameters relate to $\alpha$ as

$$x_0 = \sqrt{\frac{2\hbar}{\omega}}\,\Re\alpha, \qquad p_0 = \sqrt{2\hbar\omega}\,\Im\alpha.$$

An overall phase factor may be present but does not affect observables.

# Glauber model for ideal photon detection

The electric field operator is decomposed into positive and negative frequency components:

$$\hat{E}(\mathbf{r}, t) = \hat{E}^{(+)}(\mathbf{r}, t) + \hat{E}^{(-)}(\mathbf{r}, t)$$

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Positive and negative components are adjoint to each other

$$\hat{E}^{(-)}(\mathbf{r},t) = \left(\hat{E}^{(+)}(\mathbf{r},t)\right)^{\dagger}$$

The positive frequency component describes photon absorption at space time $\mathbf{r}, t$.

# Intensity in Photon Detection

Transition rate between quantum EM field states $|i\rangle$ and $|f\rangle$ states according to Fermi's Golden Rule

$$w_{i\to f} = \frac{2\pi}{\hbar} \left| \langle f | \hat{E}^{(+)}(\mathbf{r}, t) | i \rangle \right|^2 \rho(E_f)$$

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Intensity (summing over all possible final states)

$$\begin{aligned} I(\mathbf{r},t) &\propto \sum_f \left| \langle f| \hat{E}^{(+)}(\mathbf{r},t) |i\rangle \right|^2 \\ &= \sum_f \langle i| \hat{E}^{(-)}(\mathbf{r},t) |f\rangle \langle f| \hat{E}^{(+)}(\mathbf{r},t) |i\rangle \\ &= \langle i| \hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t) |i\rangle \end{aligned}$$

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$$w_{i\rightarrow f} = \frac{2\pi}{\hbar} \left|\langle f|\, \hat{E}^{(+)}(\mathbf{r}, t)\, |i\rangle\right|^2 \rho(E_f)$$

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Similar for mixed states (described by $\hat{\rho} = \sum_j p_j\, |\psi_j\rangle \langle\psi_j|$)

$$I(\mathbf{r}, t) \propto \mathrm{Tr}(\hat{\rho}\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t))$$

# Intensity for a Single Fock State

Example: single mode with frequency $\omega_{\mathbf{k}}$, field in Fock number state $|n_k\rangle$

$$\hat{E}^{(+)}(\mathbf{r}, t) = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 V}}\, \hat{a}_{\mathbf{k}}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}$$

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Intensity

$$\begin{aligned} I(\mathbf{r}, t) &\propto \langle n_k | \hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t) | n_k \rangle \\ &= \frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 V} \langle n_k | \hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}} | n_k \rangle \\ &= \frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 V} n_{\mathbf{k}} \end{aligned}$$

The intensity is proportional to the photon number.

# Correlation Functions of the EM field

$n$-th order correlation function of the EM field

$$G^{(n)}(r_1, t_1 \dots r_n, t_n; r_1', t_1' \dots r_n', t_n') = \\ \langle \hat{E}^{(-)}(r_1, t_1) \cdots \hat{E}^{(-)}(r_n, t_n) \hat{E}^{(+)}(r_n', t_n') \cdots \hat{E}^{(+)}(r_1', t_1') \rangle$$

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Intensity and first order correlation

$$I(\mathbf{r}, t) = G^{(1)}(r, t; r, t)$$

# Correlation Functions of the EM field

$n$-th order correlation function of the EM field

$$G^{(n)}(r_1, t_1 \dots r_n, t_n; r'_1, t'_1 \dots r'_n, t'_n) = \langle \hat{E}^{(-)}(r_1, t_1) \cdots \hat{E}^{(-)}(r_n, t_n) \hat{E}^{(+)}(r'_n, t'_n) \cdots \hat{E}^{(+)}(r'_1, t'_1) \rangle$$

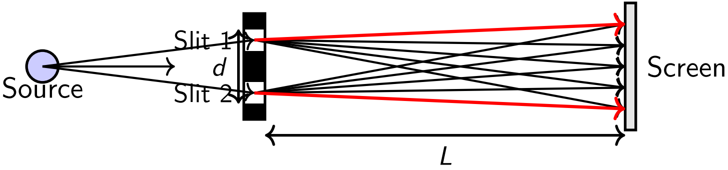
Intensity and first order correlation

$$I(\mathbf{r}, t) = G^{(1)}(r, t; r, t)$$

Positive definiteness

$$G^{(1)}(r, t; r, t) \geq 0$$
$$G^{(n)}(r_1, t_1 \dots r_n, t_n; r_1, t_1 \dots r_n, t_n) \geq 0$$

# Young's Double-Slit Experiment



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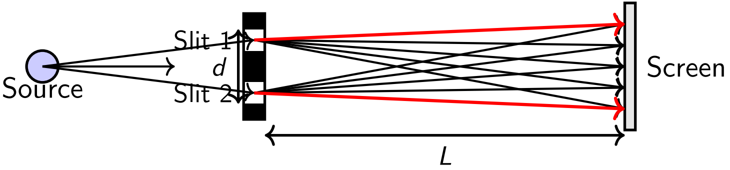


Positive frequency component of the electric field on the observation screen

$$\hat{E}^{(+)}(\mathbf{r},t) = \hat{E}_1^{(+)}(\mathbf{r},t) + \hat{E}_2^{(+)}(\mathbf{r},t)$$

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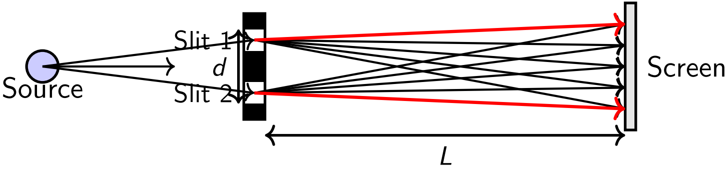


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Spherical waves from slit $j$

$$\hat{E}_j^{(+)}(\mathbf{r}, t) = \underbrace{\hat{E}_j^{(+)}(\mathbf{r}_j, t - s_j/c)}_{\text{field at slit j}} \frac{1}{s_j} e^{ks_j - \omega t}$$

# Young's Double-Slit Experiment

Intensity

$$
\begin{aligned}
I(\mathbf{r}, t) &= G^{(1)}(r, t; r, t) \\
&\approx \frac{1}{R^2}\left(\underbrace{G^{(1)}(x_1, x_1)}_{\text{I at slit 1}} + \underbrace{G^{(1)}(x_2, x_2)}_{\text{I at slit 2}} + \underbrace{2\cos(k(s_1 - s_2))\, G^{(1)}(x_1, x_2)}_{\text{Interference}}\right)
\end{aligned}
$$

where $x_j = (r_j, t - s_j/c)$, $R \approx s_1$, $R \approx s_2$

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Interference maxima

$$\cos(k(s_1 - s_2)) = 1 \longrightarrow k(s_1 - s_2) = 2\pi q, \quad q \in \mathbb{N}$$

# Young's Double-Slit Experiment

Two spherical modes

$$\hat{E}^{(+)} = i\sqrt{\frac{\hbar\omega_{\mathbf{k}}}{2\epsilon_0 V}}\frac{e^{-\omega_{\mathbf{k}} t}}{R}\left(\hat{a}_1\, e^{i\mathbf{k}\cdot\mathbf{s_1}} + \hat{a}_2\, e^{i\mathbf{k}\cdot\mathbf{s_2}}\right)$$

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Intensity

$$I \propto \langle a_1^\dagger a_1\rangle + \langle a_2^\dagger a_2\rangle + 2|\langle a_1^\dagger a_2\rangle|\cos(k(s_1 - s_2))$$

# Young's Double-Slit Experiment

Examples for different field states

1. Fock state

$$|\psi\rangle = |n_1 = 1, n_2 = 1\rangle = |1, 1\rangle = a_1^\dagger a_2^\dagger |0\rangle$$

Gives no interference, since

$$\langle\psi| a_1^\dagger a_2 |\psi\rangle = \langle 0| a_2 a_1 a_1^\dagger a_2 a_1^\dagger a_2^\dagger |0\rangle = 0$$

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2. Coherent state

$$|\psi\rangle = |\alpha, \alpha\rangle = |\alpha\rangle |\alpha\rangle$$

Interference visible, since

$$\begin{aligned} I &\propto \langle a_1^\dagger a_1\rangle + \langle a_2^\dagger a_2\rangle + 2|\langle a_1^\dagger a_2\rangle| \cos(k(s_1 - s_2)) \\ &= 2(1 + \cos(k(s_1 - s_2)))|\alpha|^2 \end{aligned}$$

# Young's Double-Slit Experiment

3. Single photon state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2^\dagger)\,|0\rangle$$

Interference visible, since

$$I \propto \frac{1}{2} + \frac{1}{2} + \cos(k(s_1 - s_2)))|\alpha|^2$$

$\longrightarrow$ the single photon is interfering with itself

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