Geometry and Quantized Responses in Floquet Matter: The Středa Approach

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EMMI Workshop: Synthetic Topological Matter
05/09/2025 @ Benasque Science Center, Spain

Outline

- The **Středa Formula (1982)**:

  A thermodynamic relation connecting the bulk and edge of topological quantum matter

- **Generalization to out-of-equilibrium periodically driven (Floquet) systems (2024-2025)**:

→ Can we apply the same Středa approach, *mutatis mutandis*, to derive the bulk-boundary correspondence of Floquet topological matter **from physical principles**?

a) The “mathematical miracle”

b) The physical consequences

c) A geometric description of Floquet topology:
Wannier charge centers and an emergent axion coupling

- Outlook

- **Středa Formula for Floquet Systems:**
  Topological Invariants and Quantized Anomalies from Cesàro Summation

  arXiv:2408.13576 (2024), to appear in PRX **LPG**, G. Usaj, N. Goldman

- **Quantized Chern-Simons Axion Coupling in Anomalous Floquet Systems**

  arXiv:2506.20719 (2025) **LPG**, N. Goldman, G. Usaj

G. Usaj

N. Goldman

The Widom-Středa formula (1982)

$$\frac{\sigma_H}{\sigma_0} = \Phi_0 \frac{\partial n}{\partial B}\bigg|_\mu = \Phi_0 \frac{\partial N}{\partial \Phi}\bigg|_\mu \qquad \text{when gapped}$$

P. Středa, J. Phys. C: Solid State Phys. **15** L717 (1982)
A. Widom, Physics Letters A **90**, 474 (1982)

Modern theory of semiclassical dynamics of Bloch electrons

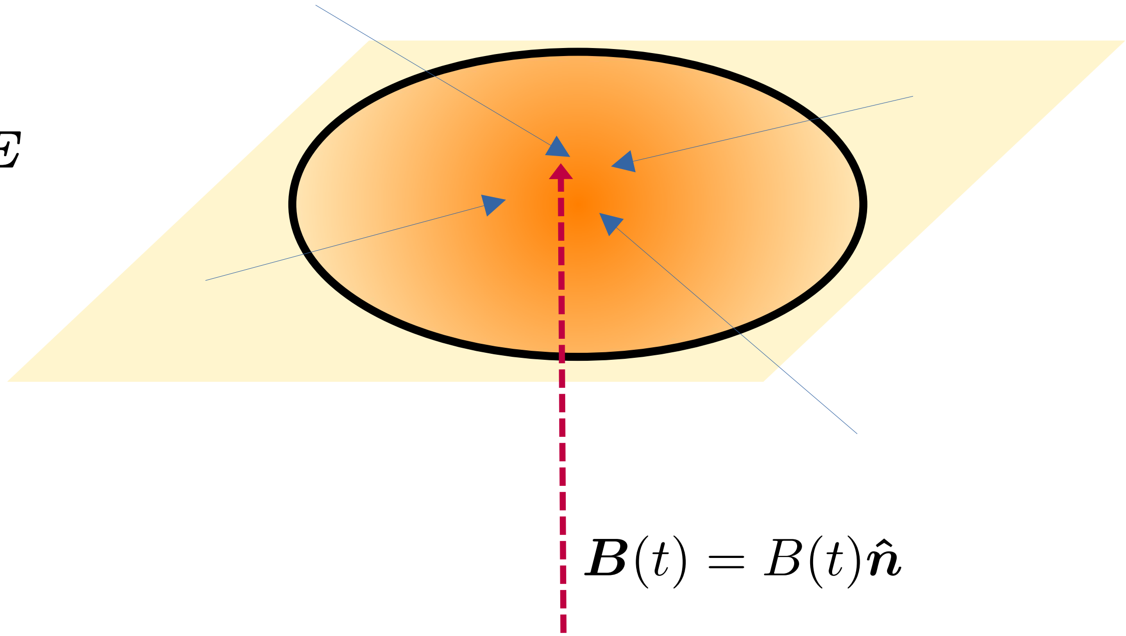
D. Xiao, J. Shi, and Q. Niu, PRL **95**, 137204 (2005)

$$n = \int \frac{d^2k}{(2\pi)^2} \sum_\alpha \left(1 + \frac{2\pi}{\Phi_0} \boldsymbol{B} \cdot \boldsymbol{\mathcal{F}}^\alpha(\boldsymbol{k})\right) \theta(\mu - \varepsilon_{\alpha \boldsymbol{k}})$$

$$\longrightarrow \quad \Phi_0 \frac{\partial n}{\partial B}\bigg|_\mu = \frac{1}{2\pi} \int_{\mathrm{BZ}} \sum_{\alpha \in \mathrm{occ}} d^2k \mathcal{F}^\alpha_{xy} = C$$

An intuitive bulk argument

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{d\boldsymbol{B}}{dt} \qquad \boldsymbol{J} = \sigma_H \hat{\boldsymbol{n}} \times \boldsymbol{E}$$



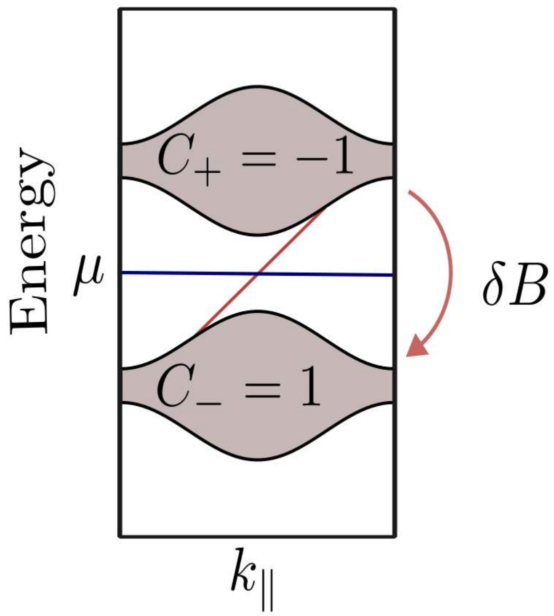
$$-e\frac{\partial n}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}$$

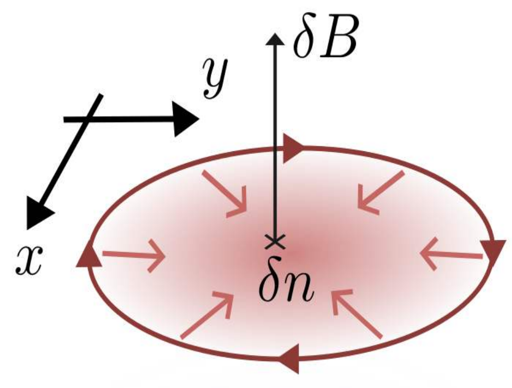
$$\boldsymbol{B}(t) = B(t)\hat{\boldsymbol{n}}$$

$$\frac{\partial n}{\partial t} = \frac{\sigma_H}{ec}\frac{\partial \boldsymbol{B} \cdot \hat{\boldsymbol{n}}}{\partial t} \quad \Rightarrow \quad ec\frac{\Delta n}{\Delta B} = \sigma_H$$

Bulk-boundary correspondence: Spectral flow

$$\frac{\sigma_H}{\sigma_0} = \sum_{\alpha \in \mathrm{occ}} C_\alpha = \#\,\text{chiral edge states}$$

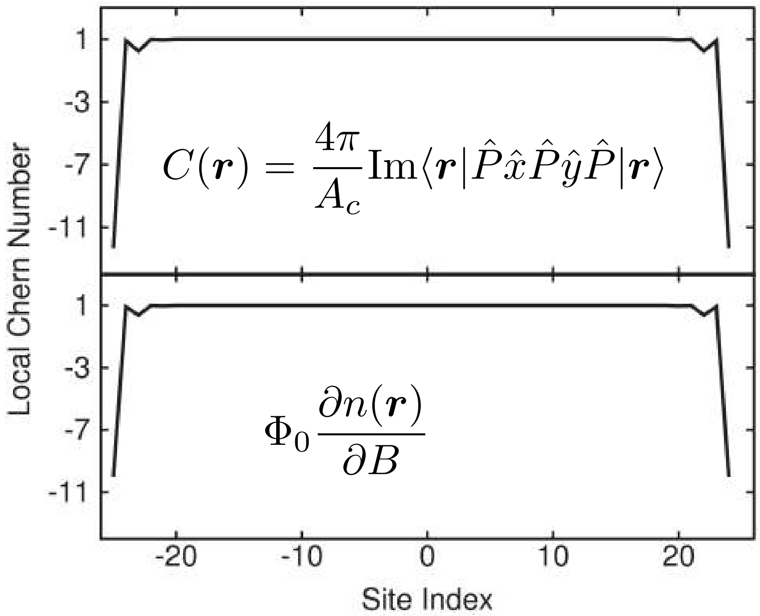




Anomaly inflow / Anomaly cancellation

[C. G. Callan, Jr. and J. A. Harvey, Anomalies and fermion zero modes on strings and domain walls, Nucl. Phys. B 250, 427 (1985)]

Středa formula holds locally: Comparison with local Chern marker



Local Středa marker

$$\widetilde{C}(\boldsymbol{r}) = \Phi_0 \frac{\partial n(\boldsymbol{r})}{\partial B}$$

"Mapping topology in coordinate space"
Bianco & Resta, PRB **84**, 241106(R) (2011)

Bianco & Resta, PRL **110**, 087202 (2013)

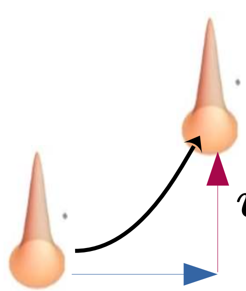
Boundaries act as a particle reservoir (Bulk-boundary)

Magnetic response of the DOS: Energy-resolved Středa response

$$\mathcal{W}(\mu) = \Phi_0 \frac{\partial n}{\partial B}\bigg|_{\mu} = \Phi_0 \int_{-\infty}^{\mu} d\omega \frac{\partial \rho(\omega)}{\partial B}$$

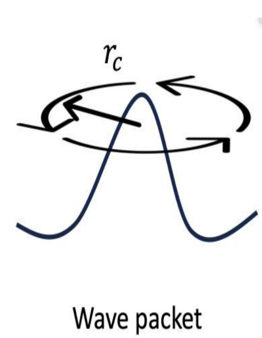
$$\Phi_0 \frac{\partial \rho(\omega)}{\partial B}\bigg|_{B=0} = \int_{\mathrm{BZ}} \frac{d^2 k}{2\pi} \sum_{\alpha} \left[\mathcal{F}_{xy}^{\alpha}(\boldsymbol{k}) \delta(\omega - \varepsilon_{\alpha \boldsymbol{k}}) + \frac{\Phi_0}{2\pi} m_z^{\alpha}(\boldsymbol{k}) \frac{\partial}{\partial \omega} \delta(\omega - \varepsilon_{\alpha \boldsymbol{k}}) \right]$$

Berry curvature



$$\boldsymbol{v}_{\mathrm{anom}} = -\frac{e}{\hbar} \boldsymbol{E} \times \boldsymbol{\mathcal{F}}$$

Intrinsic orbital magnetic moment

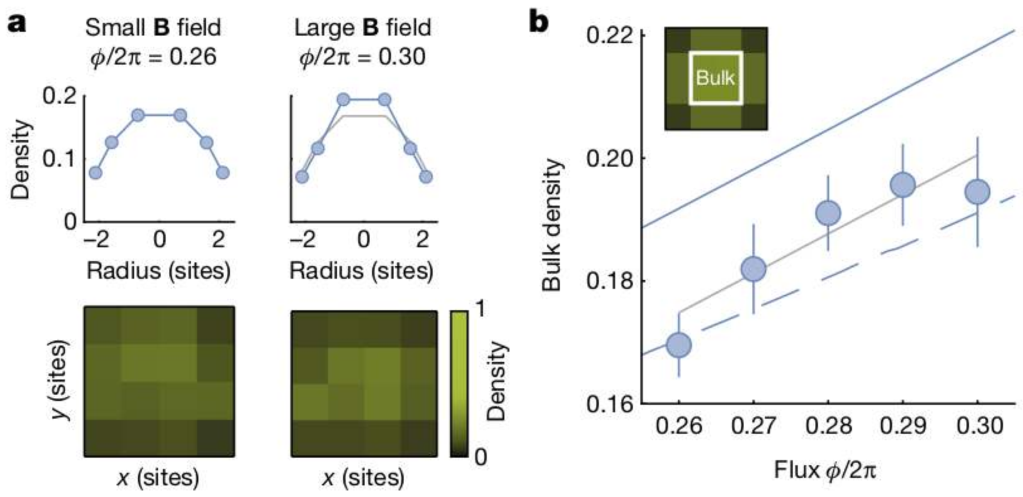


arXiv:2408.13576 (2024), **LPG**, Usaj & Goldman

A probing tool in various contexts

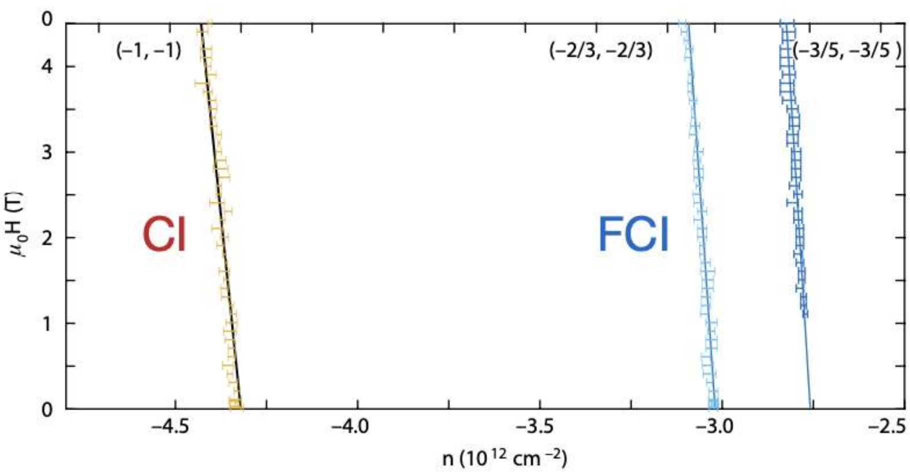
- **Signature of Laughlin state with cold atoms**

J. Léonard et al, Nature **619**, 495–499 (2023)



- **QAH states in twister bilayer materials**

Y. Xie et al, Nature **600**, 439–443 (2021)
Y. Zeng et al, Nature **622**, 69–73 (2023)
J. Cai et al, Nature **622**, 63–68 (2023)



Generalize Středa to out-of-equilibrium? → Floquet engineered topological phases

Floquet Theorem

$$\hat{H}(t) = \hat{H}(t+T) \quad \longrightarrow \quad \exists\, \hat{R}(t)\,,\ \hat{H}_{\text{eff}} = \hat{R}^\dagger(t)\hat{H}(t)\hat{R}(t) - i\hat{R}^\dagger(t)\partial_t\hat{R}(t)$$

$$\hat{U}(t,t') = \hat{R}(t)e^{-i\hat{H}_{\text{eff}}(t-t')}\hat{R}^\dagger(t')$$

Goldman & Dalibard, PRX **4**, 031027 (2014)

Eckardt & Anisimovas, New J. Phys. 17 093039 (2015)

$\hat{H}_{\text{eff}}$: time-independent effective Hamiltonian

$\hat{R}(t)$: unitary micro-motion operator

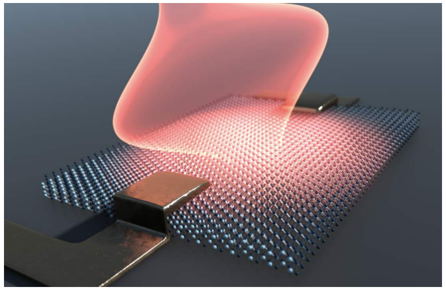
Solutions of the time-dependent Schrödinger equation:

$$\hat{H}(t)|\psi_a(t)\rangle = i\partial_t|\psi_a(t)\rangle$$

ε_a : quasienergies $\left(\hat{H}_{\text{eff}}|u_a^{\text{eff}}\rangle = \varepsilon_a|u_a^{\text{eff}}\rangle\right)$

$$|\psi_a(t)\rangle = e^{-i\varepsilon_a t}|u_a(t)\rangle$$

$|u_a(t)\rangle = R(t)|u_a^{\text{eff}}\rangle$: time-periodic Floquet mode



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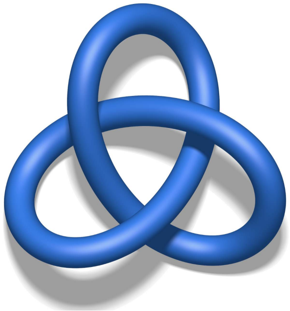


Gauge redundancy and Sambe (Fourier) space

$$|\psi_a(t)\rangle = e^{-i\varepsilon_a t}|u_a(t)\rangle = e^{-i\varepsilon_{as} t}|u_{as}(t)\rangle \quad \begin{cases} \varepsilon_{as} = \varepsilon_a + s\Omega \\ |u_{as}(t)\rangle = e^{is\Omega t}|u_a(t)\rangle \end{cases}$$

First Floquet zone

$$\varepsilon_a \in (-\Omega/2, \Omega/2]$$

$$[\hat{H}(t) - i\partial_t]|u_{as}(t)\rangle = \varepsilon_{as}|u_{as}(t)\rangle \quad \rightarrow$$

Sambe space

$$\hat{\boldsymbol{H}}^F|u_{as}\rangle\rangle = \varepsilon_{as}|u_{as}\rangle\rangle$$

$$|\mathrm{u}_{as}\rangle\rangle = \begin{pmatrix} \vdots \\ |u_{as}^{(1)}\rangle \\ |u_{as}^{(0)}\rangle \\ |u_{as}^{(-1)}\rangle \\ \vdots \end{pmatrix}$$

J. H. Shirley, Phys. Rev. **138**, B979 (1965)
H. Sambe, PRA **7**, 2203 (1973)

Floquet Hamiltonian: $\hat{\boldsymbol{H}}^F = \hat{\boldsymbol{H}} - \Omega\hat{\boldsymbol{N}}$

$$\hat{\boldsymbol{H}}^F = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & \hat{H}_0 - \Omega & \hat{H}_1 & \hat{H}_2 & \dots \\ \dots & \hat{H}_{-1} & \hat{H}_0 & \hat{H}_1 & \dots \\ \dots & \hat{H}_{-2} & \hat{H}_{-1} & \hat{H}_0 + \Omega & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathcal{S} = \mathcal{H}_{\text{sys}} \otimes \text{“Photons”}$$

$$|u_{as}^{(n)}\rangle = \tfrac{1}{T}\int_0^T dt e^{in\Omega t}|u_{as}(t)\rangle$$

$$\hat{\boldsymbol{H}}_{nm} = \frac{1}{T}\int_0^T dt e^{i(n-m)\Omega t}\hat{H}(t)$$

$$\hat{\boldsymbol{N}}_{nm} = n\delta_{nm}$$

Micromotion operator in Sambe space

$$\hat{H}_{\text{eff}} = \hat{R}^\dagger(t)(\hat{H}(t) - i\partial_t)\hat{R}(t)$$

$$\hat{\boldsymbol{H}}^F = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & \hat{H}_0 - \Omega & \hat{H}_1 & \hat{H}_2 & \dots \\ \dots & \hat{H}_{-1} & \hat{H}_0 & \hat{H}_1 & \dots \\ \dots & \hat{H}_{-2} & \hat{H}_{-1} & \hat{H}_0 + \Omega & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

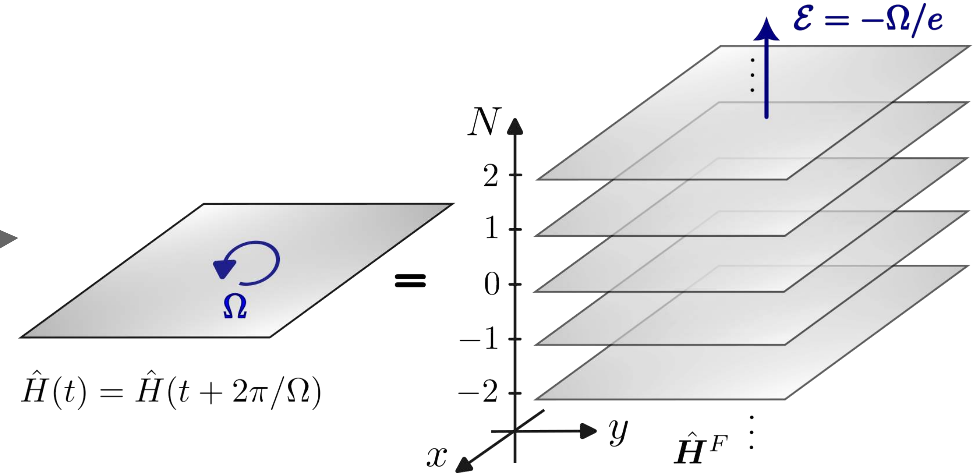
$$\rightarrow \hat{\boldsymbol{R}}^{F\dagger}\hat{\boldsymbol{H}}^F\hat{\boldsymbol{R}}^F = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & \hat{H}_{\text{eff}} - \Omega & 0 & 0 & \dots \\ \dots & 0 & \hat{H}_{\text{eff}} & 0 & \dots \\ \dots & 0 & 0 & \hat{H}_{\text{eff}} + \Omega & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

"Photon" dressing via periodic driving

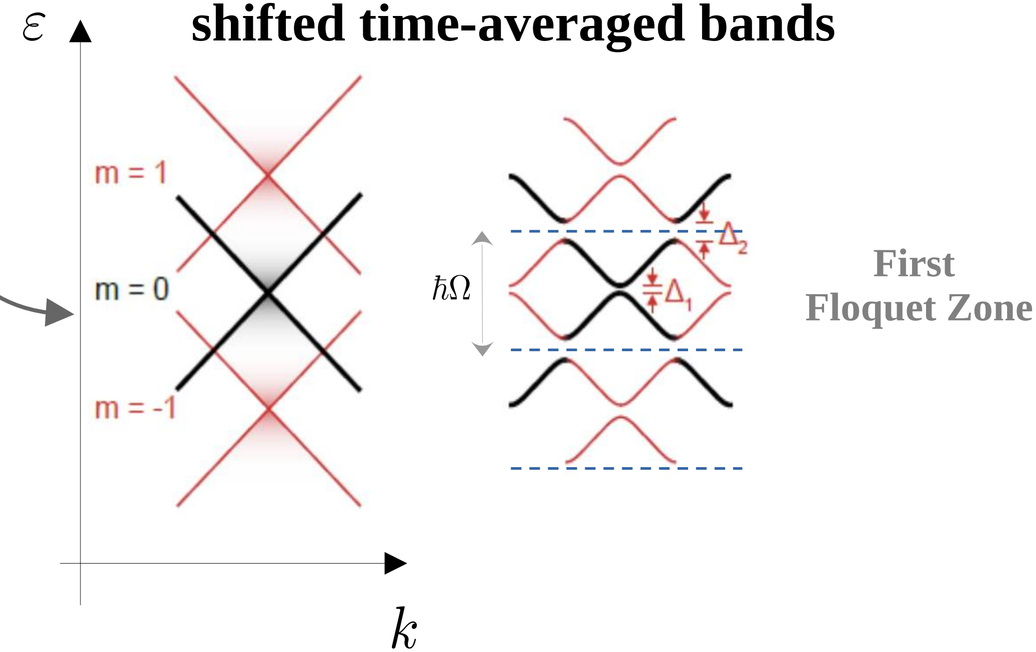
$$\hat{\boldsymbol{H}}^F = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \hat{H}_0 - \Omega & \hat{H}_1 & \hat{H}_2 & \cdots \\ \cdots & \hat{H}_{-1} & \hat{H}_0 & \hat{H}_1 & \cdots \\ \cdots & \hat{H}_{-2} & \hat{H}_{-1} & \hat{H}_0 + \Omega & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathcal{S} = \mathcal{H}_{\text{sys}} \otimes \text{“Photons”}$$

2D + driving = 3D Sambe lattice

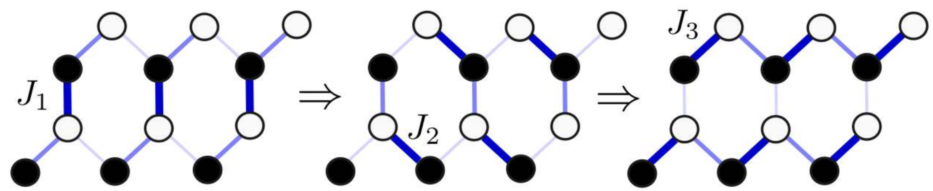


Bloch system: Coupling between shifted time-averaged bands

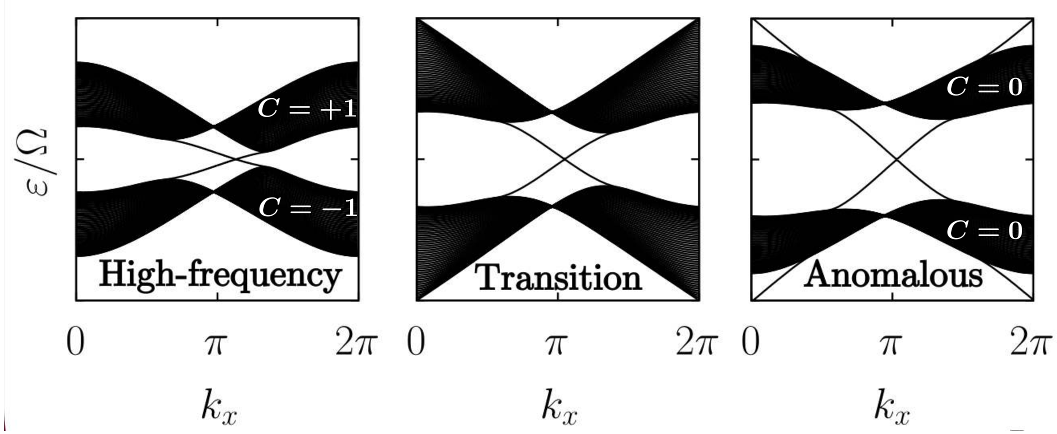


Anomalous Floquet physics: Chern numbers of H_{eff} are not enough

Kitagawa, Berg, Rudner, Demler, PRB **82** (2010)
Rudner, N. Lindner, E. Berg, M. Levin, PRX **3** (2013)



"Purely mathematical considerations suggest a natural guess…" (quote)

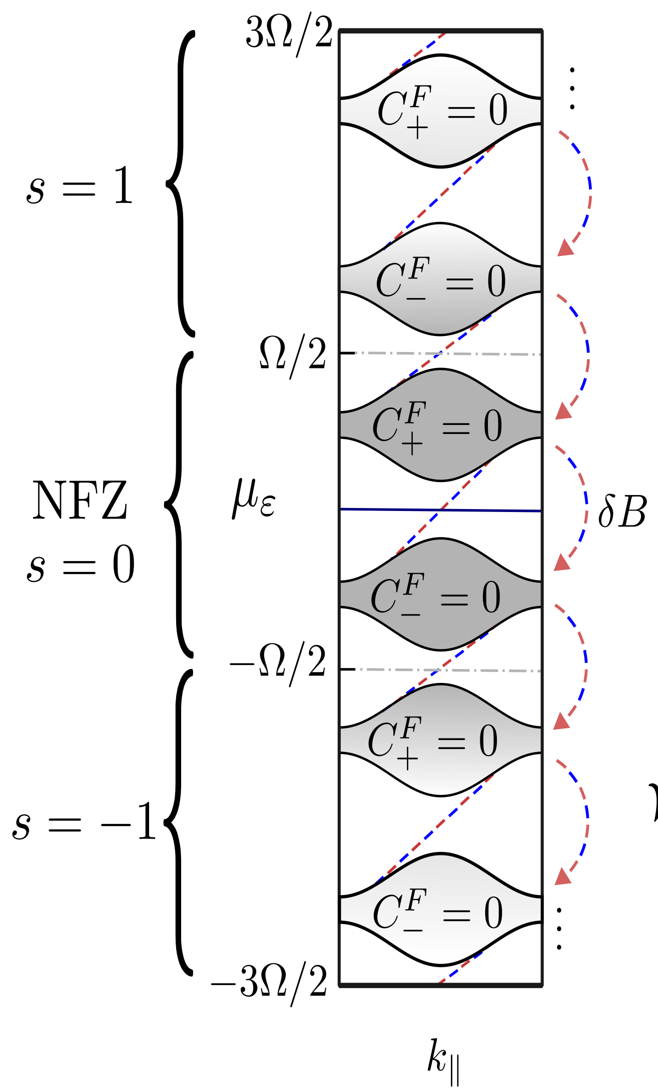


Floquet winding number

$$\mathcal{W}(\mu_\varepsilon) = \sum_{\alpha<\mu_\varepsilon} C_\alpha + N_3[R]$$

$$N_3[R] = \frac{1}{8\pi^2}\int_0^T dt \int_{\mathrm{BZ}} d^2k \mathrm{tr}\left[\hat{R}^\dagger_{\boldsymbol{k}}\frac{\partial \hat{R}_{\boldsymbol{k}}}{\partial t}\hat{R}^\dagger_{\boldsymbol{k}}\frac{\partial \hat{R}_{\boldsymbol{k}}}{\partial k_x}\hat{R}^\dagger_{\boldsymbol{k}}\frac{\partial \hat{R}_{\boldsymbol{k}}}{\partial k_y} - (k_x \leftrightarrow k_y)\right]$$

Spectral flow and Středa response in the extended Floquet picture



Floquet DOS: $\rho^F(\omega) = \frac{1}{A_s} \sum_{a \in \mathrm{NFZ}} \sum_{s=-\infty}^{\infty} \delta(\omega - \varepsilon_a - s\Omega)$

Floquet-Streda response:

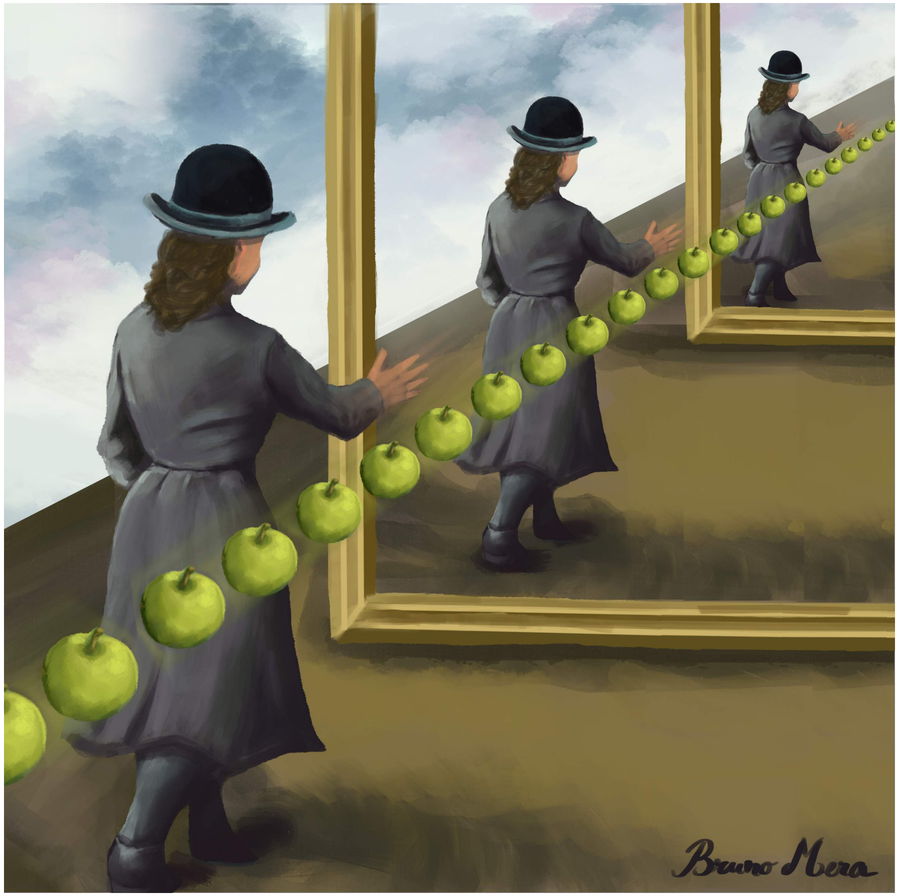
$$\mathcal{W}(\mu_\varepsilon) = \Phi_0 \int_{-\infty}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B}$$

$$\mathcal{W}(\mu_\varepsilon) = \underbrace{\Phi_0 \int_{-\Omega/2}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B}}_{\mathcal{W}^N(\mu_\varepsilon)} + \underbrace{\Phi_0 \int_{-\infty}^{-\Omega/2} d\omega \frac{\partial \rho^F(\omega)}{\partial B}}_{\mathcal{W}^A}$$

$\mathcal{W}^N(\mu_\varepsilon)$ || Chern number

$\mathcal{W}^A$: spectral flow zone edge || #anomalous edge states

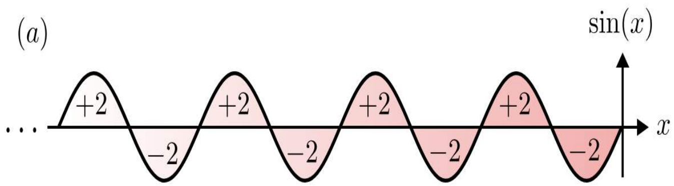
The Miracle



How to regularize the anomalous flow → Cesàro summation

$$\mathcal{W}^A = \Phi_0 \int_{-\infty}^{-\Omega/2} d\omega \frac{\partial \rho^F(\omega)}{\partial B}$$

$$\mathcal{W}^A \sim \int_{-\infty}^{0} \sin(x) dx = (-2+2) + (-2+2) + \ldots = ?$$
$$= 0 + 0 + \ldots + 0 + \ldots = ?$$



Cesàro Regularization

$$\int_{-\infty}^{0} \sin(x) dx \overset{(C,1)}{=} \lim_{\lambda \to \infty} \int_{-\lambda}^{0} \sin(x) \left(1 + \frac{x}{\lambda}\right) dx = -1$$

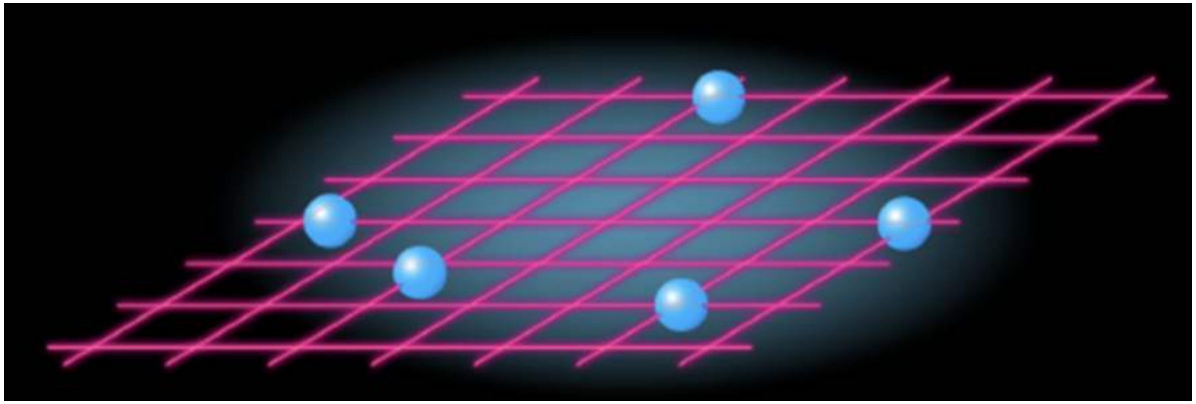
$$\sum_n (-1)^n = ?$$

Luigi G. Grandi
1671-1742

$$\mathcal{W}^A \overset{(C,1)}{=} \frac{\Phi_0}{\Omega} \int_{\mathrm{NFZ}} d\omega \left(\frac{\partial \rho^F(\omega)}{\partial B} \omega \right)$$

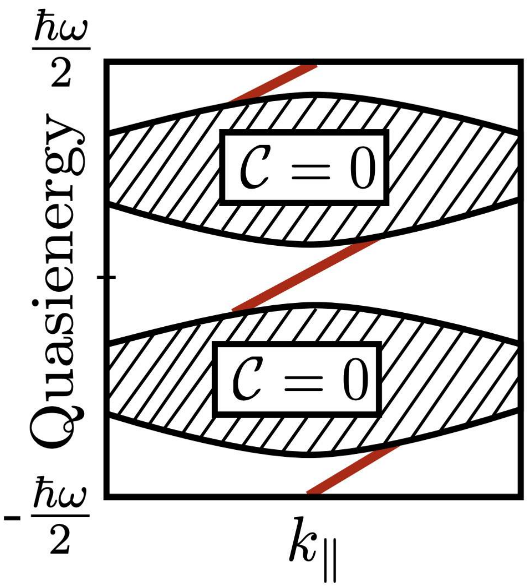
$$\Rightarrow \quad \mathcal{W}^A : \text{flow of energy}$$

The Physical Consequences



Quantized anomaly → Quantized orbital magnetization density

$$\mathcal{W}(\mu_\varepsilon) = \mathcal{W}^N(\mu_\varepsilon) + \mathcal{W}^A \left\{ \begin{array}{l} \mathcal{W}^N(\mu_\varepsilon) = \Phi_0 \int_{-\Omega/2}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B} = \Phi_0 \frac{\partial}{\partial \Phi} N_{\text{eff}}(\mu_\varepsilon) \\ \\ \mathcal{W}^A = \frac{\Phi_0}{\Omega} \int_{\text{NFZ}} d\omega \left(\frac{\partial \rho^F(\omega)}{\partial B} \omega \right) = \frac{\Phi_0}{\Omega} \frac{\partial}{\partial \Phi} \text{Tr}[\hat{H}_{\text{eff}}] = -\frac{\Phi_0}{\Omega} \mathcal{M}_T^F \end{array} \right.$$



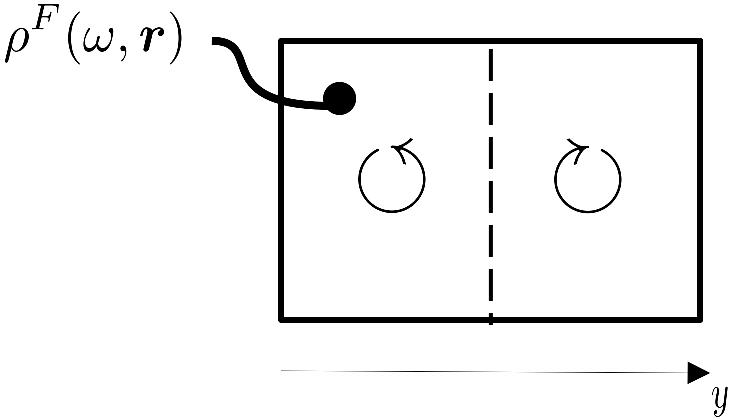
1) Topology can be extracted from stroboscopic dynamics!

2) $\mathcal{W}^A \propto \text{Tr}[\hat{L}_{\text{eff}}] \neq 0$ **Quantized orbital magnetization density: Anomaly in H_{eff}**

F. Nathan, M. Rudner, N. Lindner, E. Berg, G. Refael, PRL **119** (2017)
P. Glorioso, A. Gromov, S. Ryu, PRResearch **3**, 013117 (2021)
LPG, G. Usaj & N. Goldman, arXiv:2408.13576 (2024)

Local Floquet winding numbers from magnetic response of the local DOS

$$\mathcal{W}(\mu_\varepsilon, \boldsymbol{r}) = \Phi_0 \int_{-\Omega/2}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega, \boldsymbol{r})}{\partial B} + \frac{\Phi_0}{\Omega} \int_{\mathrm{NFZ}} d\omega \left(\frac{\partial \rho^F(\omega, \boldsymbol{r})}{\partial B} \omega \right)$$



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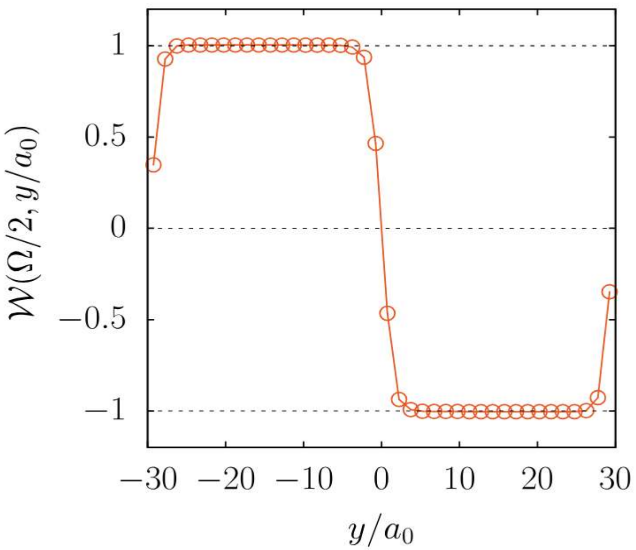
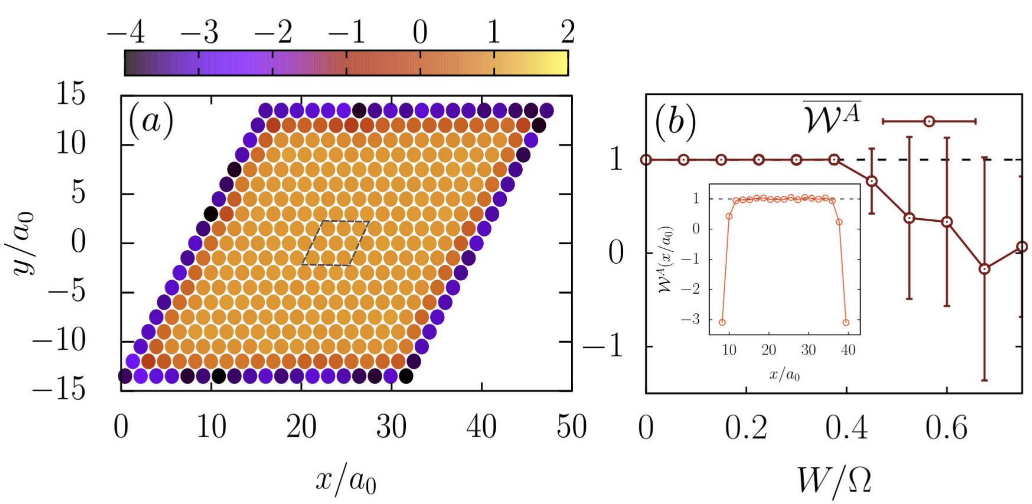


$$\hat{V}_{\mathrm{disorder}} = \sum_{\boldsymbol{R}} w_{\boldsymbol{R}} \hat{n}_{\boldsymbol{R}}$$

$$w_{\boldsymbol{R}} \in [-W, W]$$

$\rightarrow$



Topology in terms of simple properties of Floquet-Bloch bands

$$\mathcal{W}(\mu_\varepsilon) = \mathcal{W}^N(\mu_\varepsilon) + \mathcal{W}^A$$

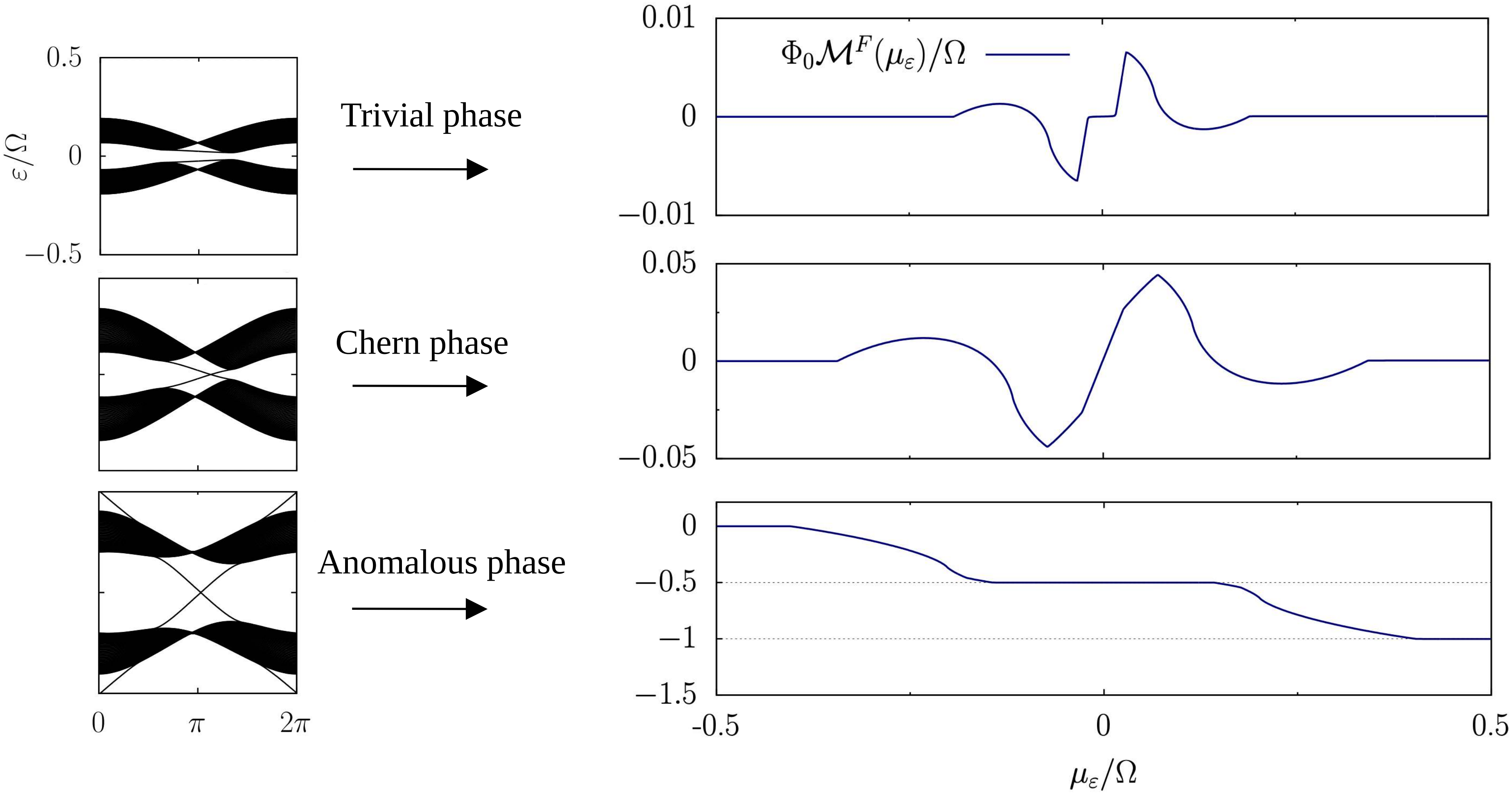
με in a gap: $\mathcal{W}^N(\mu_\varepsilon) = \sum_{\alpha<\mu_\varepsilon} C_\alpha$

$$\mathcal{W}^A = \sum_\alpha \int_{\mathrm{BZ}} \frac{d^2k}{2\pi\Omega} \left[\mathcal{F}^\alpha_{xy}(\boldsymbol{k})\varepsilon_{\alpha\boldsymbol{k}} - \frac{\Phi_0}{2\pi} m^\alpha_z(\boldsymbol{k})\right]$$

Fully quantifies the number of anomalous edge modes and represents a new formulation of Rudner's invariant

$$\mathcal{W}^A = N_3[R] = -\frac{\Phi_0}{\Omega}\mathcal{M}^F_T$$

Bulk orbital magnetization density as a function of filling



Generalized Floquet-Středa response: Exchange of “photons”

$$\mathrm{Tr}[\hat{H}_{\mathrm{eff}}] = \frac{1}{T}\int_0^T dt\mathrm{Tr}[\hat{H}(t)] + \Omega N_1[R]$$

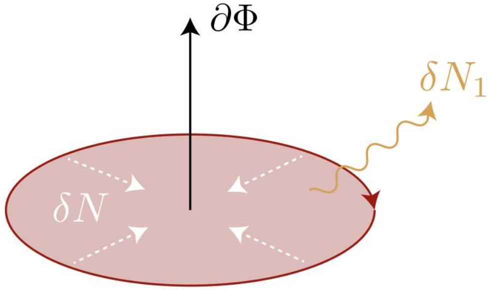
$$\mathcal{W}^A = \frac{\Phi_0}{\Omega}\frac{\partial}{\partial\Phi}\mathrm{Tr}[\hat{H}_{\mathrm{eff}}] = \Phi_0\frac{\partial N_1[R]}{\partial\Phi} = N_3[R]$$

$$N_1[R] = -\frac{i}{2\pi}\int_0^T dt\mathrm{Tr}[\hat{R}^\dagger(t)\partial_t\hat{R}(t)]$$

$$\in \mathbb{Z}$$

First order winding number of R(t)

$$\mathcal{W}(\mu_\varepsilon) = \mathcal{W}^N(\mu_\varepsilon) + \mathcal{W}^A = \Phi_0\left(\frac{\partial N_{\mathrm{eff}}(\mu_\varepsilon)}{\partial\Phi} + \frac{\partial N_1}{\partial\Phi}\right)$$



- **Normal flow:** quantized flow of dressed particles between the edge and the bulk
- **Anomalous flow:** quantized energy flow (exchange of “photons” -- N1) between the system and the driving field

Summary of relations

$$\mathcal{W}(\mu_\varepsilon) \overset{(C,1)}{=} \Phi_0 \int_{-\Omega/2}^{\mu_\varepsilon} d\omega \frac{\partial \rho^F(\omega)}{\partial B} + \Phi_0 \int_{\text{NFZ}} d\omega \frac{\partial \rho^F(\omega)}{\partial B} \frac{\omega}{\Omega}$$

↓ ↓

$$\Phi_0 \frac{\partial N_{\text{eff}}(\mu_\varepsilon)}{\partial \Phi} \quad + \quad \mathcal{W}^A = \frac{\Phi_0}{\Omega} \frac{\partial}{\partial \Phi} \mathrm{Tr}[\hat{H}_{\text{eff}}] = -\frac{\Phi_0}{\Omega} \mathcal{M}_T^F$$

↓ ↓

$$\sum_{\alpha < \mu_\varepsilon} C_\alpha \quad + \quad \mathcal{W}^A = \Phi_0 \frac{\partial N_1[R]}{\partial \Phi} = N_3[R] \quad \bigstar$$

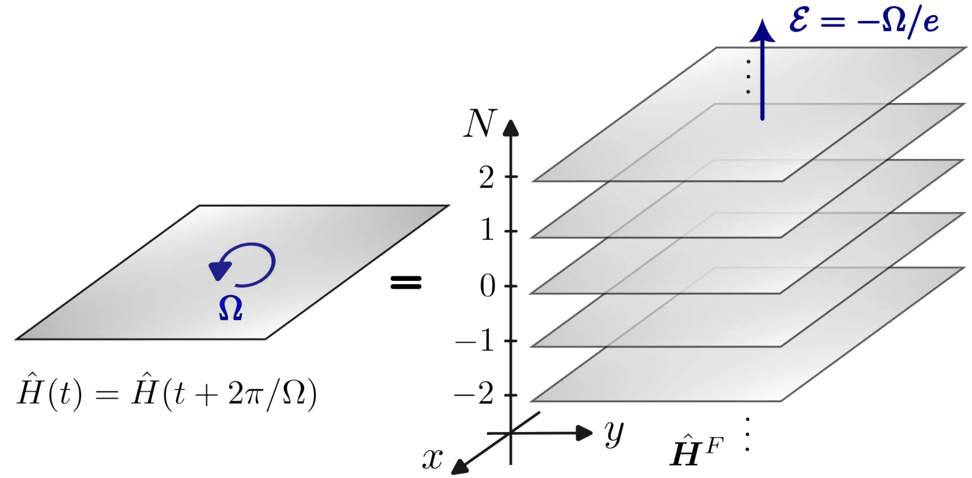
A New Interpretation

Physical meaning of N_1 : Polarization along the "photon-axis"

First order winding number of R(t)

$$N_1[R] = -\frac{i}{2\pi}\int_0^T dt\,\mathrm{Tr}[\hat{R}^\dagger(t)\partial_t \hat{R}(t)] \in \mathbb{Z}$$

$$N_1[R] = -\frac{i}{2\pi}\sum_a \int_0^T dt\langle u_a(t)|\partial_t u_a(t)\rangle$$



Orbital polarization of a 3D crystal along 'z':

$$P_z = -e\sum_{\alpha\in\mathrm{occ}}\int_{\mathrm{BZ}}\frac{d^2k\,dk_z}{(2\pi)^3}\langle u_{\alpha\boldsymbol{k}}(k_z)|i\partial_{k_z}u_{\alpha\boldsymbol{k}}(k_z)\rangle$$

$$P_z = -e\sum_{\alpha\in\mathrm{occ}}\int_{\mathrm{BZ}}\frac{d^2k}{(2\pi)^2}\frac{\overline{z}_\alpha(\boldsymbol{k})}{a_z}$$

$$\overline{z}_\alpha(\boldsymbol{k}) = \int dz\, z|w_{\alpha 0}(\boldsymbol{k},z)|^2 = \langle w_{\alpha 0}|\hat{z}|w_{\alpha 0}\rangle$$

Polarization of Sambe lattice along 'N':

$$P_N = \frac{N_1[R]}{A_s} = -\sum_{\alpha\in\mathrm{NFZ}}\int_{\mathrm{BZ}}\frac{d^2k\,dt}{(2\pi)^3}\langle u_{\alpha\boldsymbol{k}}(t)|i\partial_t u_{\alpha\boldsymbol{k}}(t)\rangle$$

$$P_N = -\sum_{a\in\mathrm{NFZ}}\int_{\mathrm{BZ}}\frac{d^2k}{(2\pi)^2}\overline{N}_\alpha(\boldsymbol{k})$$

$$\overline{N}_\alpha(\boldsymbol{k}) = \sum_n n\langle u^{(n)}_{\alpha\boldsymbol{k}}|u^{(n)}_{\alpha\boldsymbol{k}}\rangle = \langle\langle u_{\alpha\boldsymbol{k}}|\hat{\boldsymbol{N}}|u_{\alpha\boldsymbol{k}}\rangle\rangle$$

Berry Phases in Electronic Structure Theory, D. Vanderbilt

Wannier representation of Floquet topological states, M. Nakagawa, R-J.Slager, S. Higashikawa, T. Oka, PRB **101** (2020)

N_3 as a quantized magneto-polarizability in the Sambe lattice

Magneto-electric response in 3D crystals

$$\frac{\partial P_z}{\partial B_z} = \frac{\partial M_z}{\partial E_z} = \frac{\theta_{\mathrm{CS}} e^2}{4\pi\hbar c}$$

$$\mathcal{L} = \theta_{\mathrm{CS}}(e^2/2\pi hc)\boldsymbol{E}\cdot\boldsymbol{B}$$

3D Chern-Simons term

$$\theta_{\mathrm{CS}} = -\frac{1}{4\pi}\int d^3k\epsilon^{ijl}\mathrm{tr}\left[\mathcal{A}_i\partial_j\mathcal{A}_l - i\frac{2}{3}\mathcal{A}_i\mathcal{A}_j\mathcal{A}_l\right]$$

$$\mathcal{A}_j^{\alpha\beta} = i\langle u_{\alpha\boldsymbol{k}}(k_z)|\partial_{k_j}u_{\beta\boldsymbol{k}}(k_z)\rangle \qquad k = (\boldsymbol{k}, k_z)$$

L. Qi, T. L. Hughes, and S.-C. Zhang, PRB **78**, 195424 (2008)
A. M. Essin, J. E. Moore, and D. Vanderbilt, PRL **102**, 146805 (2009)

Magneto-polarizability in the Sambe lattice

$$e\frac{\partial P_N}{\partial B} = \frac{N_3[R]e^2}{2\pi\hbar c}$$

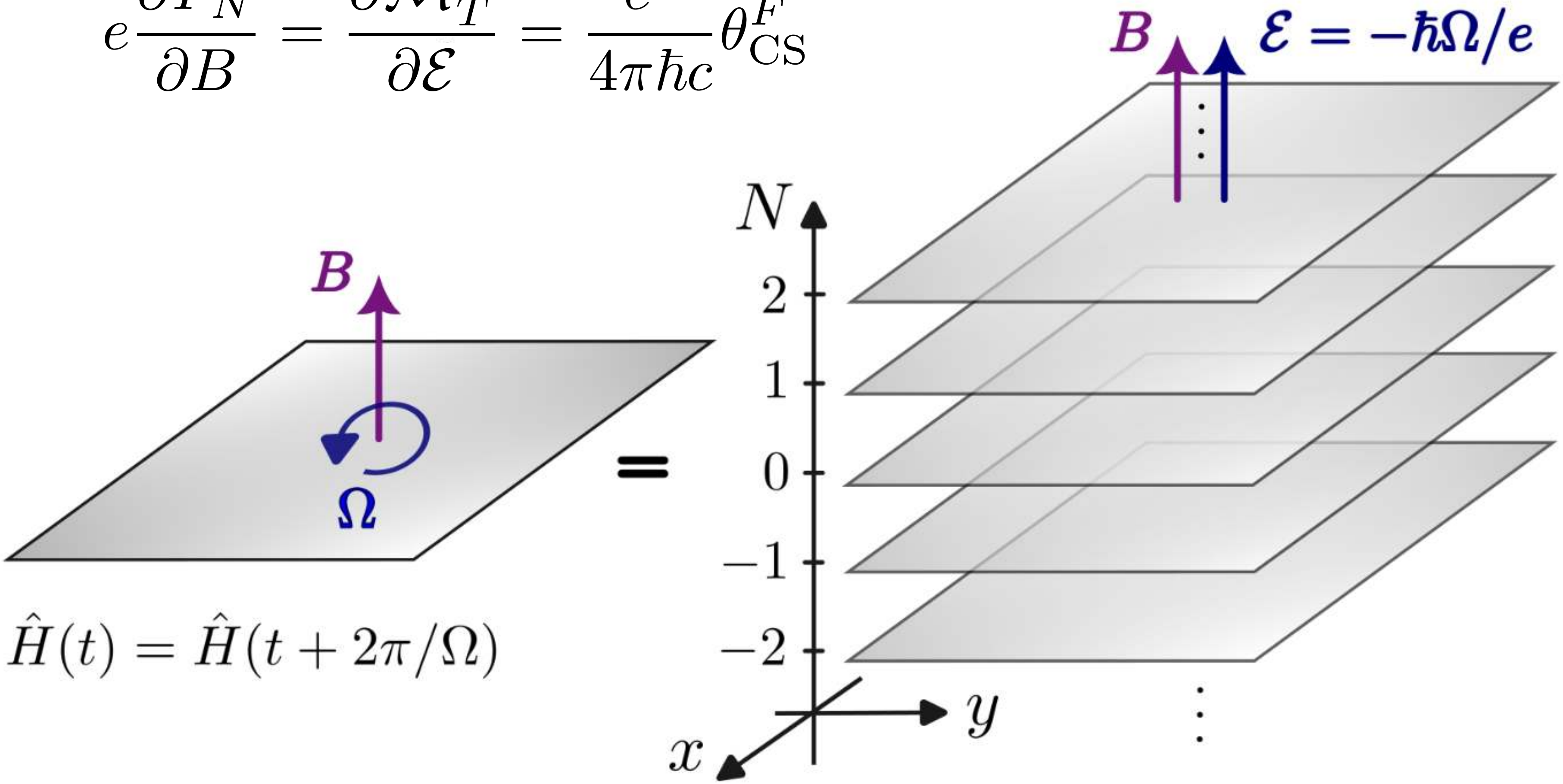
$$\rightarrow N_3[R] = \frac{\theta^F_{\mathrm{CS}}}{2\pi}$$

Floquet Chern-Simons term

$$\theta^F_{\mathrm{CS}} = -\frac{1}{4\pi}\int d^3k\epsilon^{ijl}\mathrm{tr}\left[\mathcal{A}_i\partial_j\mathcal{A}_l - i\frac{2}{3}\mathcal{A}_i\mathcal{A}_j\mathcal{A}_l\right]$$

$$\mathcal{A}_j^{\alpha\beta} = i\langle u_{\alpha\boldsymbol{k}}(t)|\partial_{k_j}u_{\beta\boldsymbol{k}}(t)\rangle \qquad k = (\boldsymbol{k}, t)$$

$$e\frac{\partial P_N}{\partial B} = \frac{\partial \mathcal{M}_T^F}{\partial \mathcal{E}} = \frac{e^2}{4\pi\hbar c}\theta_{\text{CS}}^F$$



Quantized Chern-Simons Axion Coupling in Anomalous Floquet Systems, arXiv:2506.20719 (2025) **LPG**, N. Goldman, G. Usaj

$N_3[R]$ in purely geometrical terms

$$\theta_{\mathrm{CS}}^F = -\frac{1}{4\pi}\int d^3k\epsilon^{ijl}\mathrm{tr}\left[\mathcal{A}_i\partial_j\mathcal{A}_l - i\frac{2}{3}\mathcal{A}_i\mathcal{A}_j\mathcal{A}_l\right] = -\frac{1}{2\pi}\int_{\mathrm{BZ}} d^2k\int_0^T dt\,\mathrm{tr}\left(\mathcal{A}_y\partial_t\mathcal{A}_x + \mathcal{A}_N F_{xy}\right)$$

$$F_{xy} = \mathcal{F}_{xy} - i[\mathcal{A}_x, \mathcal{A}_y]$$

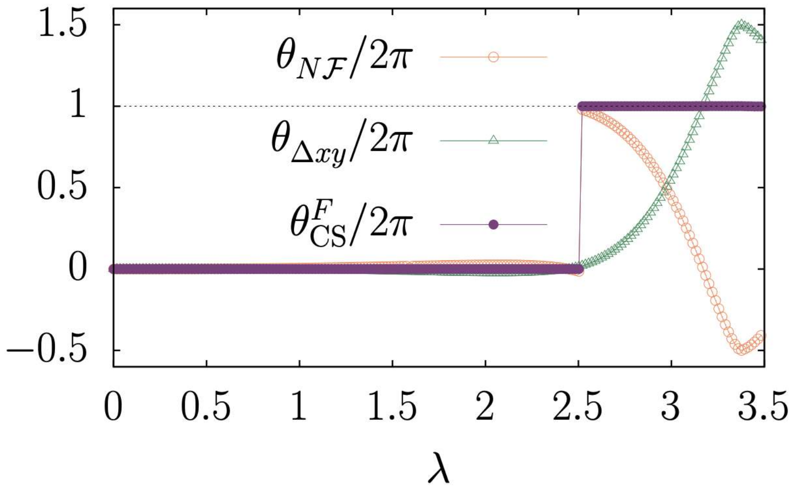
Considering the fully filled NFZ: $F_{xy} = 0$

Representation in terms of hybrid Wannier charge centers in the Sambe lattice

$$\theta_{\mathrm{CS}}^F = -\frac{1}{2\pi}\int_{\mathrm{BZ}} d^2k\int_0^T dt\,\mathrm{tr}\left(\mathcal{A}_y\partial_t\mathcal{A}_x\right) = \theta_{N\mathcal{F}} + \theta_{\Delta xy}$$

$$\theta_{N\mathcal{F}} = -\sum_\alpha\int_{\mathrm{BZ}} d^2k\overline{N}_\alpha(\boldsymbol{k})\mathcal{F}_{xy}^{\alpha 0;\alpha 0}(\boldsymbol{k})$$

$$\theta_{\Delta xy} = i\sum_{\alpha\beta s}\int_{\mathrm{BZ}} d^2k(\overline{N}_\alpha(\boldsymbol{k}) - \overline{N}_{\beta s}(\boldsymbol{k}))\mathcal{A}_x^{\alpha 0;\beta s}\mathcal{A}_y^{\beta s;\alpha 0}$$



$$\mathcal{A}_j^{\alpha 0;\beta s} = \langle\langle u_{\alpha\boldsymbol{k}}|i\partial k_j u_{\beta s\boldsymbol{k}}\rangle\rangle$$

$$\mathcal{F}_{xy}^{\alpha 0;\alpha 0}(\boldsymbol{k}) = \partial_{k_x}\mathcal{A}_y^{\alpha 0;\alpha 0} - \partial_{k_y}\mathcal{A}_x^{\alpha 0;\alpha 0}$$

PRL 114, 096401 (2015) PHYSICAL REVIEW LETTERS week ending 6 MARCH 2015

Adiabatic Pumping of Chern-Simons Axion Coupling

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(Received 7 November 2014; published 4 March 2015)

$$\theta = -\frac{1}{4\pi}\int d^3k\epsilon^{ijk}\mathrm{Tr}\left[A_i\partial_j A_k - i\frac{2}{3}A_iA_jA_k\right]$$

$$\theta = \theta_{z\Omega} + \theta_{\Delta xy}$$

$$\theta_{z\Omega} = -\frac{1}{c}\int d^2k\sum_n \bar{z}_n\Omega_{xy,0n,0n},$$

$$\theta_{\Delta xy} = \frac{i}{c}\int d^2k\sum_{lmn}(\bar{z}_{lm} - \bar{z}_{0n})A_{x,0n,lm}A_{y,lm,0n},$$

In correspondence with our decomposition!

Outlook

- Cavity quantum materials:

  The driving field is a quantum degree of freedom, is there a Středa-type back-action in the photon-field?

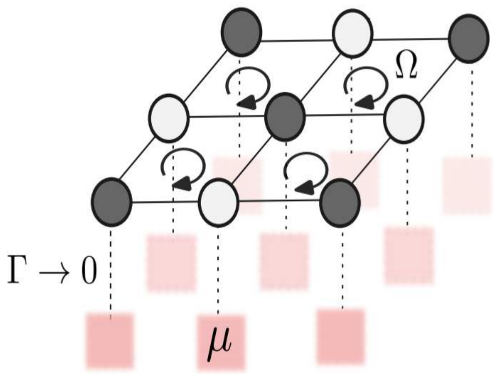
- Effective field theory describing the coupling between the synthetic electric field and the external magnetic field in the Sambe lattice → axion-like electrodynamics?

Refs.:

- Středa Formula for Floquet Systems: Topological Invariants and Quantized Anomalies from Cesaro Summation, arXiv:2408.13576 (2024), to appear in PRX, **LPG**, G. Usaj, N. Goldman

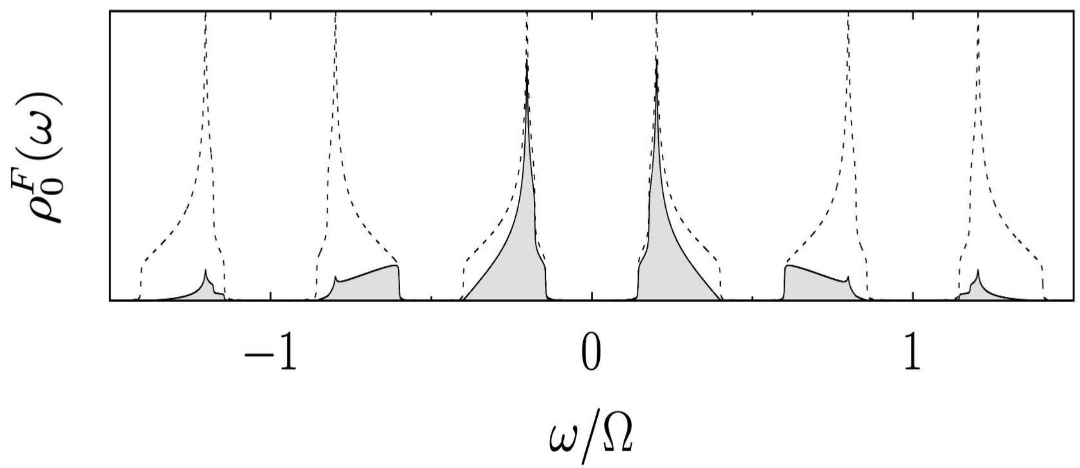
  Quantized Chern-Simons Axion Coupling in Anomalous Floquet Systems, arXiv:2506.20719 (2025) **LPG**, N. Goldman, G. Usaj

Sum-rule protocol: coupling to Buttiker-type reservoirs



$$\overline{n}|_\mu = \int_{-\infty}^{\infty} d\omega f(\omega)\rho_0^F(\omega)$$

"Time-averaged" density of states: $\rho_0^F(\omega) = \sum_{a,l} \langle u_a^{(l)} | u_a^{(l)} \rangle \delta(\omega - (\varepsilon_a + l\Omega))$



Total DOS from the projected one:

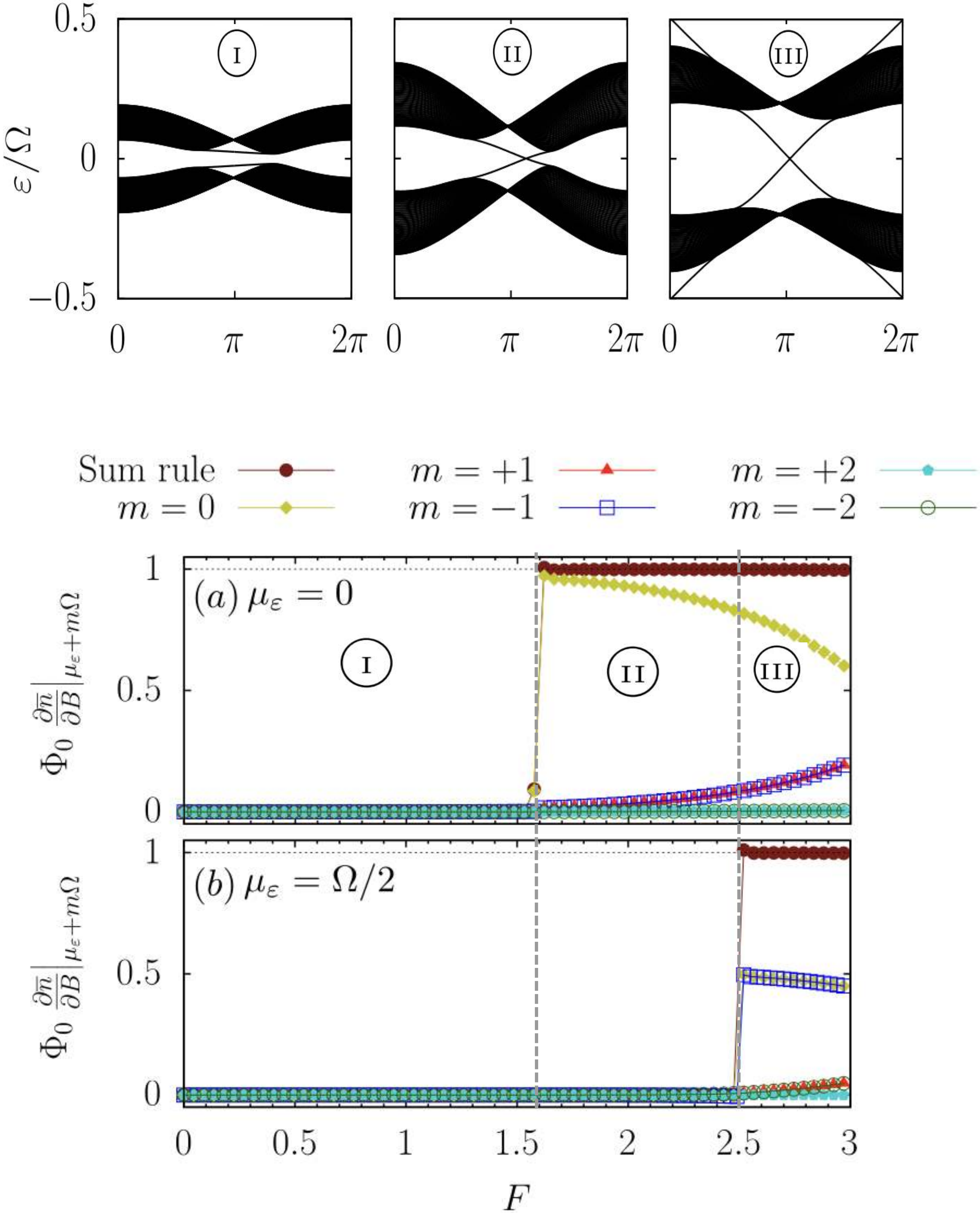
$$\rho^F(\omega) = \sum_{m=-\infty}^{\infty} \rho_0^F(\omega + m\Omega)$$

$$\mathcal{W}(\mu_\varepsilon) = \Phi_0 \int_{-\infty}^{\mu_\varepsilon} \sum_{m=-\infty}^{\infty} d\omega \frac{\partial \rho_0^F(\omega + m\Omega)}{\partial B}$$

$$\stackrel{!}{=} \Phi_0 \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\mu_\varepsilon + m\Omega} d\omega \frac{\partial \rho_0^F(\omega)}{\partial B}$$

Floquet-Streda sum rule recovers quantization

$$\mathcal{W}(\mu_\varepsilon) = \Phi_0 \sum_{m=-\infty}^{\infty} \frac{\partial \overline{n}}{\partial B}\bigg|_{\mu = \mu_\varepsilon + m\Omega}$$



0.5
0
−0.5
ε/Ω
I
II
III
0
π
2π
Sum rule
m = 0
m = +1
m = −1
m = +2
m = −2
(a) μ_ε = 0
(b) μ_ε = Ω/2
1
0.5
0
Φ0 ∂n̄/∂B|με+mΩ
0
0.5
1
1.5
2
2.5
3
F

Wannier sheets and their Berry curvature distribution