Symmetry-Resolved Entanglement Entropy in a Non-Abelian Quantum Hall State

based on forthcoming work with Valentin Crépel, Nicolas Regnault, and Benoit Estienne

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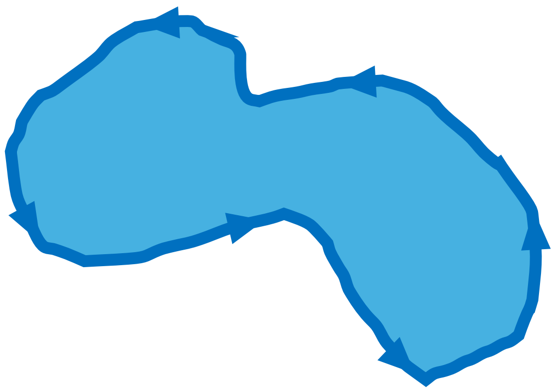
Chiral topological states in (2+1)D

Characterized by a non-zero *chiral central charge*, a property of the bulk (2+1)D Topological Quantum Field Theory, e.g. a (2+1)D Chern-Simons theory

This is reflected in a boundary exhibiting a (1+1)D chiral gapless theory (in particular, a conformal field theory—CFT)



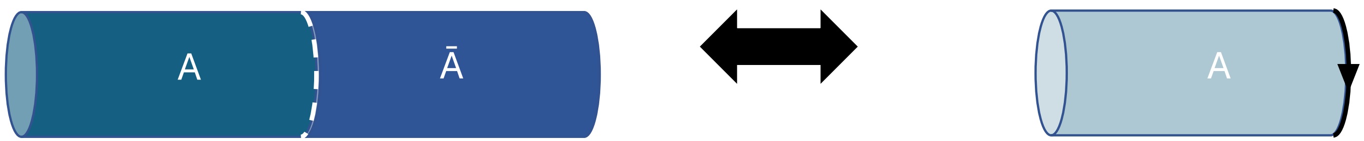
Kitaev, Ann. Phys. 2006
Witten, Comm. Math. Phys. 1989

The Li+Haldane correspondence

Li and Haldane observed a correspondence for chiral topological states between

the low-energy part of the entanglement spectrum (ES)
(spectrum of the entanglement Hamiltonian)

the physical theory on the edge
(chiral CFT)



Reduced density matrix

Entanglement Hamiltonian

$$\rho_A = \mathrm{Tr}_{\bar{A}}(\rho) = \frac{e^{-H_A}}{Z}$$

$$\boxed{H_A \simeq \beta_{\text{effective}} H_{\text{CFT}}}$$

H. Li, F.D.M. Haldane, PRL 2008
(also see e.g. X.-L. Qi, H. Katsura, A.W.W. Ludwig, PRL 2012)

Entanglement and symmetry

Consider a U(1) charge

$$\hat{Q} = \hat{Q}_A + \hat{Q}_{\bar{A}}$$

Reduced density matrix: $\rho_A = \mathrm{Tr}_{\bar{A}}(\rho)$

where $\left[\rho_A, \hat{Q}_A\right] = 0.$

$$\rho_A = \bigoplus_i p_{q_i} \rho_A(q_i) = \begin{pmatrix} p_{q_1}\rho_A(q_1) & & \\ & p_{q_2}\rho_A(q_2) & \\ & & \ddots \end{pmatrix}$$

where the q_i are eigenvalues of $\hat{Q}_A$.

p_q is the full counting statistics (FCS) for the charge, a probability distribution.

$p_q = \mathrm{Tr}(\Pi_q \rho_A)$ (where Π_q is the projector onto the charge sector q)

$\rho_A(q)$ is the symmetry-resolved reduced density matrix:

$$\mathrm{Tr}_A\left[\rho_A(q)\right] = 1$$

Equipartition of entanglement

We can use $\rho_A(q)$ to understand better how entanglement interacts with the U(1) symmetry through looking at the symmetry-resolved entanglement spectrum and quantifying the symmetry-resolved entanglement entropy (SREE).

Rényi SREEs:

$$S_n(q) = \frac{1}{1-n} \log\left(\mathrm{Tr}_A\left[\rho_A^n(q)\right]\right) \xrightarrow{n \to 1}$$

von Neumann SREE:

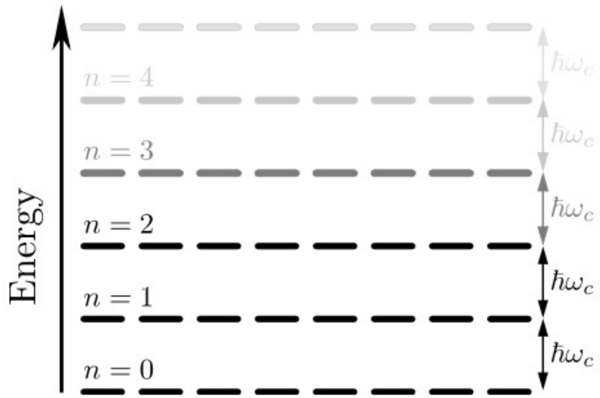
$$S_1(q) = -\mathrm{Tr}[\rho_A(q) \log \rho_A(q)]$$

$$S_1 = \underbrace{-\sum_i p_{q_i} \log p_{q_i}}_{\text{number entropy}} + \underbrace{\sum_i p_{q_i} S_1(q_i)}_{\text{configuration entropy}}$$

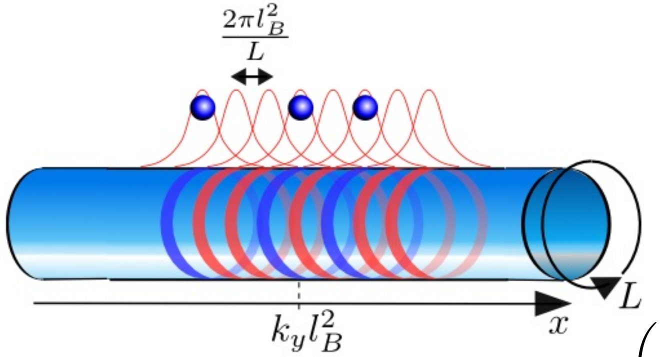
Equipartition of entanglement: $S_n(q)$ *does not depend on* q *in the thermodynamic limit.*

J. Xavier, F. Alcaraz, and G. Sierra, PRB (2018)

Quantum Hall Effect



$$\left(\omega_c = \frac{eB}{m}\right)$$



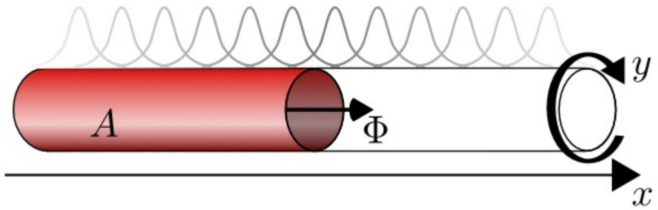
$$\left(l_B^2 = \frac{\hbar c}{eB}\right)$$

(Spinless) charged particles in a uniform magnetic field $\vec{B}$ fill Landau levels

The filling factor $\nu = \frac{hn}{eB} = \frac{N}{N_\Phi}$.

In Landau gauge on the cylinder, these orbitals are ring-like and centered around $k_m l_B^2$, with k_m quantized.

Integer Quantum Hall Effect (IQHE): **filled** lowest Landau level (LLL)



Φ flux along the cylinder axis is another knob to turn.

The region A covers the cylinder for $x < 0$.

The wavefunction in the LLL is

$$\phi_{k_m}(x,y) = \frac{1}{\sqrt{L\sqrt{\pi}}} e^{ik_m y} e^{-(x-k_m)^2/2}, \quad k_m \in \frac{2\pi}{L}(\mathbb{Z}+\Phi)$$

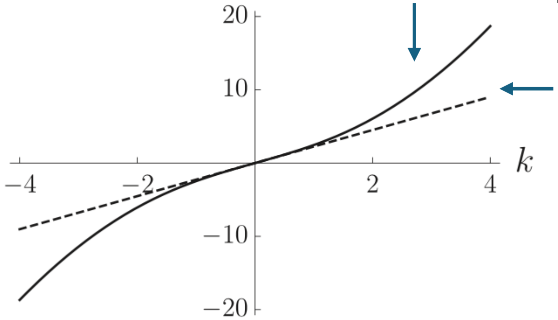
The $\nu = 1$ ground state is then the Slater determinant of the LLL: $|\Omega\rangle = \bigwedge_{k_m \in \frac{2\pi}{L}(\mathbb{Z}+\Phi)} |\phi_{k_m}\rangle$

We define $\hat{Q}_A \equiv \hat{N}_A - \langle : \hat{N}_A : \rangle$, where $\langle : \hat{N}_A : \rangle = \frac{1}{2} - \Phi$ up to Gaussian corrections in L.

(from here on out, taking $l_B = 1$ w.o.l.o.g.)

IQHE: Entanglement spectrum

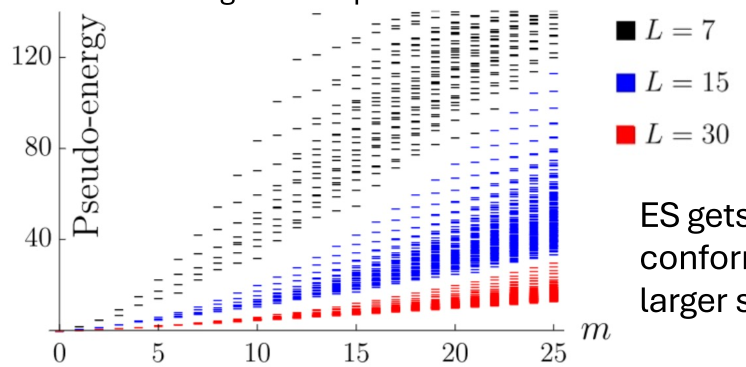
$$\epsilon(k) \equiv \log\left[\frac{\mathrm{erfc}(-k)}{\mathrm{erfc}(k)}\right]$$



Linear approximation good for small k:

$$\epsilon(k) = \frac{4}{\sqrt{\pi}}k + O(k^3)$$

Entanglement spectra



ES gets closer to conformal with larger system size

B. Oblak, N. Regnault, and B. Estienne, PRB (2021)

The IQHE entanglement Hamiltonian (EH) will be given by

$$H_A = \sum_{m\in\mathbb{Z}} \epsilon(k_m)\ :c_m^\dagger c_m: \qquad \text{(Peschel)}$$

The edge CFT is a chiral Dirac fermion:

$$H_{\mathrm{CFT}} = v\sum_{m\in\mathbb{Z}} k_m :c_m^\dagger c_m:$$

We have a Li-Haldane correspondence!

$H_A = \beta_{\mathrm{effective}} H_{\mathrm{CFT}} +$ more irrelevant terms

Plus important irrelevant terms, that shape the spectrum at finite size. More details later

J. Dubail, N. Read, E.H. Rezayi, PRB (2012)

IQHE: FCS and SREE analytical results

Approximate charged moment for small α:
(parameters can be computed analytically)

$$\widehat{Z}_n(\alpha) \sim e^{-L(a_n + b_n \alpha^2 + c_n \alpha^4) + O(L\alpha^6)}$$

$$Z_n(q) \equiv \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-i\alpha q} \widehat{Z}_n(\alpha) = \mathrm{Tr}\left(\Pi_q \rho_A^n\right)$$

FCS: $p_q = Z_1(q)$

SREE: $S_n(q) = \frac{1}{1-n} \log \frac{Z_n(q)}{Z_1(q)^n}$

With saddle point approximation,

$$p_q \propto \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \quad \sigma^2 = \frac{L}{(2\pi)^{3/2}}$$

(valid for $q = O(\sqrt{L})$)

B. Oblak, N. Regnault, and B. Estienne, PRB (2021)

We obtain

$$S_n(q) \sim S_n - \frac{1}{2}\log L + A_n - B_n \frac{q^2}{L} + C_n \frac{q^4}{L^3}$$

Equipartition for large L, small q !

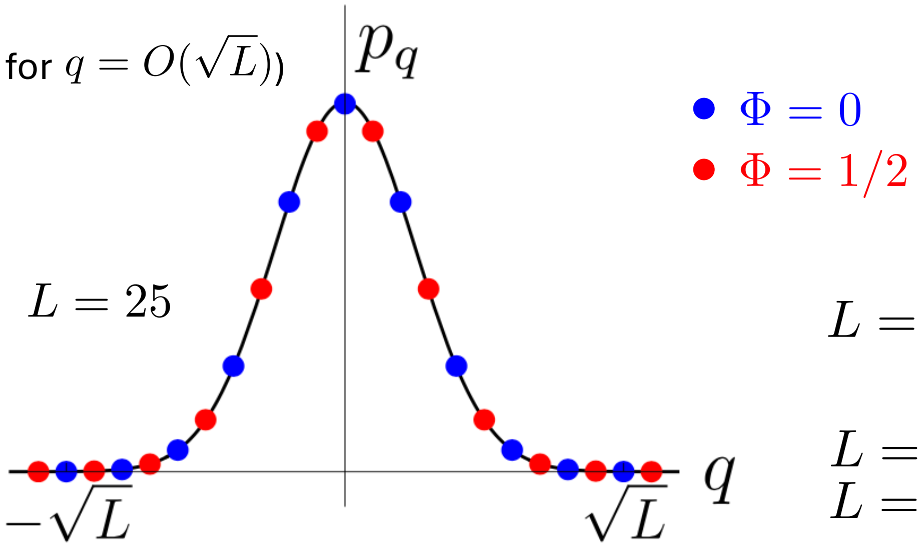
(Note that $S_n = \alpha_n L - \gamma$ satisfies an area law, with $\gamma = 0$ for IQHE.)

A.Kitaev, J.Preskill (2006)

IQHE: FCS and SREE numerical results

$$p_q \propto \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

(valid for $q = O(\sqrt{L})$)



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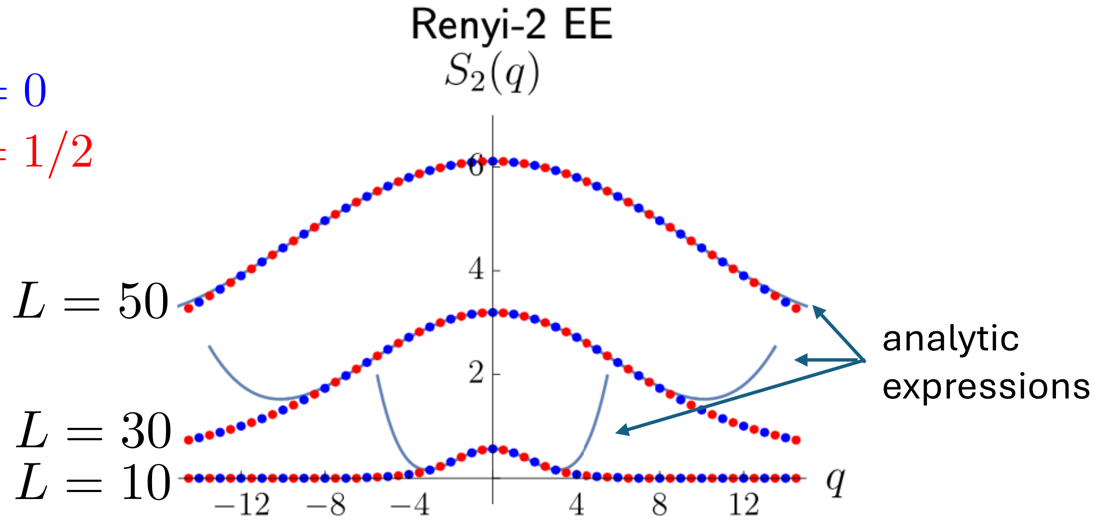


B. Oblak, N. Regnault, and B. Estienne, PRB (2021)

IQHE: Synthetic entanglement spectrum

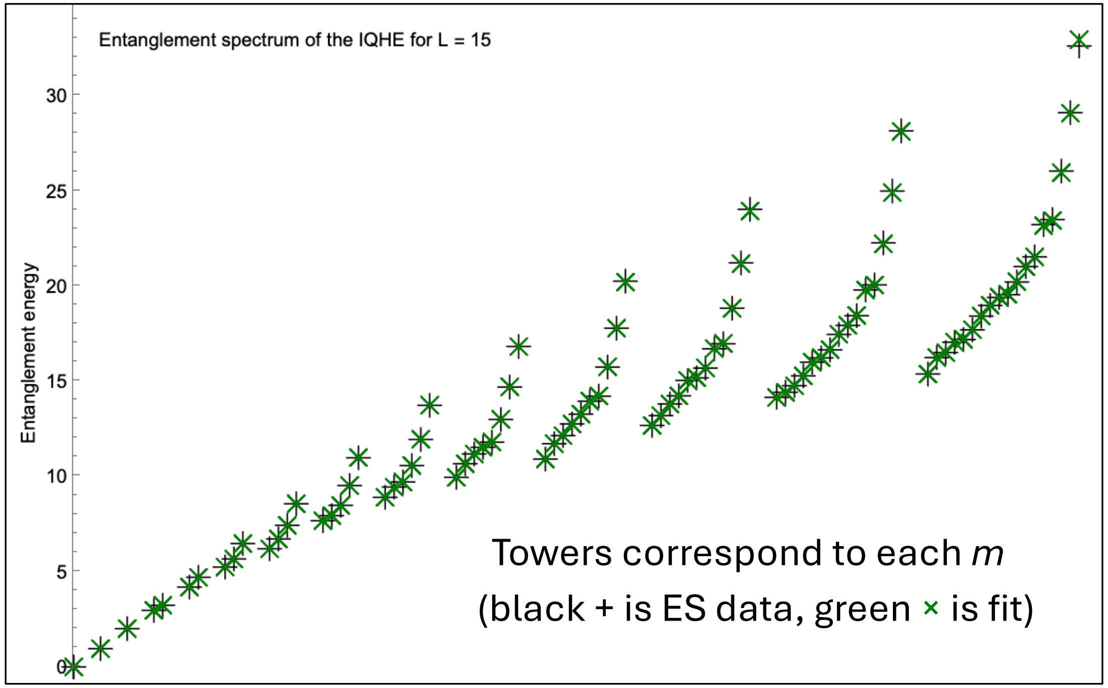
Expand the EH $H_A = \sum_{m \in \mathbb{Z}} \epsilon(k_m) \; : c_m^\dagger c_m :$

as a series: $H_A = \sum_{j \geq 0} g_j \sum_{m \in \mathbb{Z}} k_m^{2j+1} \; : c_m^\dagger c_m :$

The term for $j = 0$ is $\beta_{\text{effective}} H_{\text{CFT}}$.

Truncating this series can approximate the ES.

E.g.: we fit the ES with the first 4 terms by choosing the g_j to minimize weighted squares of differences between spectrum levels.



The IQHE can serve as benchmark for ES fitting approaches.

Fractional quantum Hall effect (FQHE)

The fractional quantum Hall effect occurs for partially filled Landau levels at certain filling fractions v in the presence of interactions.

For FQH states captured by a CFT, the entanglement Hamiltonian is:

$$H_A = \underbrace{\frac{2\pi v}{L}\left(L_0 - \frac{c}{24}\right)}_{\beta_{\text{effective}} H_{\text{CFT}}} + \underbrace{\sum_j g_j \left(\frac{\pi}{L}\right)^{\Delta_j - 1} V_j}_{\text{more irrelevant terms}}$$

(per corrected Li-Haldane)

(central charge $c = 1$ for Laughlin)

As an example, for Laughlin states (filling fraction $\nu = 1/p$),

$$L_0 = \frac{1}{2} J_0^2 + \sum_{n>0} J_{-n} J_n$$

J_n is a mode of the bosonic U(1) current, $J_0 = \sqrt{p}\hat{Q}_A$

topological sectors $\delta = 0, \ldots, p-1$ with $q = \frac{\delta}{p} + \mathbb{Z}$

MPS on the cylinder for the FQHE

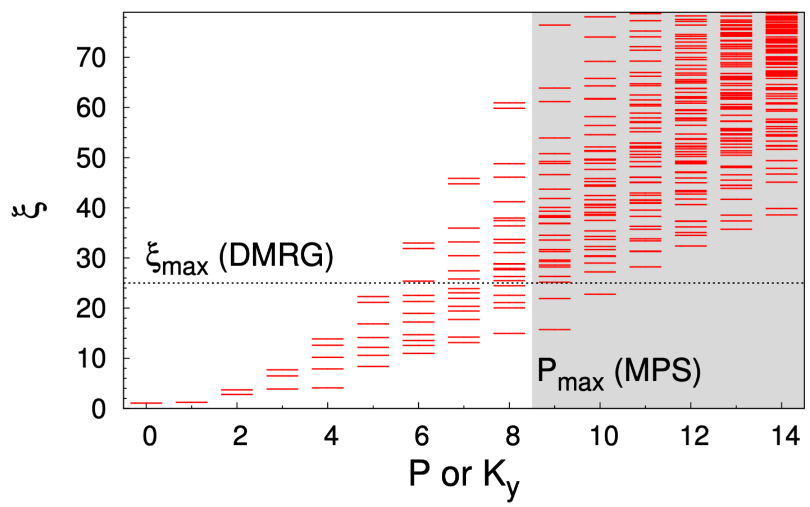
FQHE states are interacting, so this demands a more sophisticated numerical approach: MPS

$$|\Psi\rangle = \sum_{\{m_i\}} \langle \alpha_L | A^{[m_1]} \dots A^{[m_{N_{\mathrm{orb}}}]} | \alpha_R \rangle | m_1, \dots, m_{N_{\mathrm{orb}}} \rangle$$

Physical indices $[m_i]$ correspond to orbital occupation.
Auxiliary indices range from 1 to χ.
The auxiliary space is the **Hilbert space of the edge CFT**.
The truncation is at conformal weight $P_{\max}$.

In general, $\chi \sim \exp S_A$

Zaletel, Mong (2012)



Bosonic Laughlin state ($p = 2$) MPS results

$$\widehat{Z}_n(\alpha) \sim p^{\frac{n-1}{2}} e^{-L(a_n + b_n \alpha^2 + c_n \alpha^4) + O(L\alpha^6)}$$

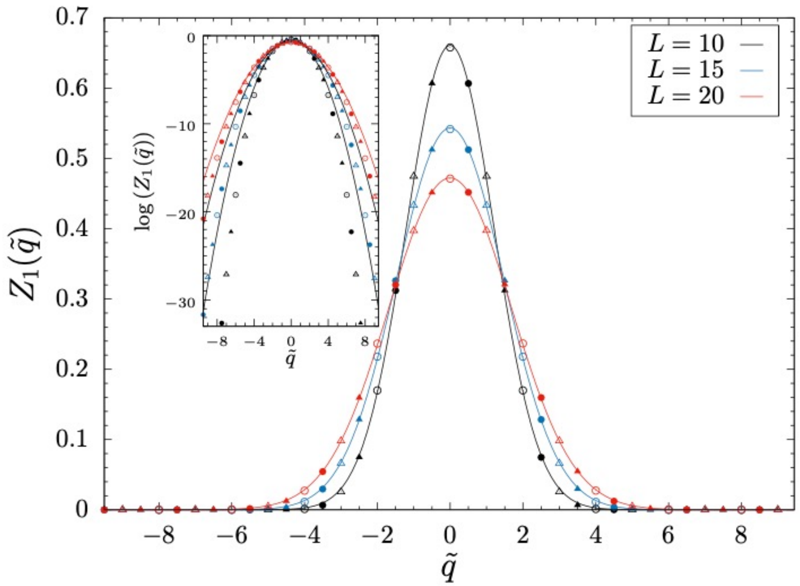
analogous to IQHE, though now params. not known analytically

$$p_q \propto \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \quad \sigma^2 = \frac{L}{2\pi p v}$$

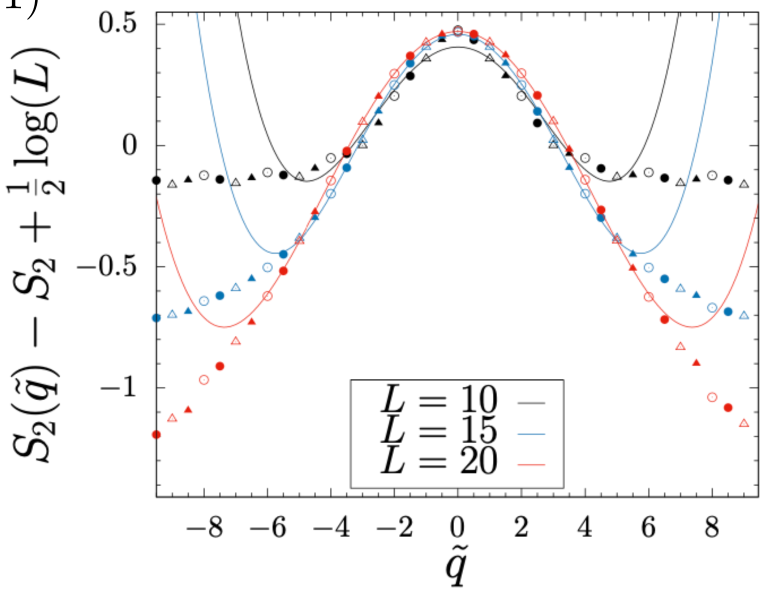
assuming Li-Haldane, can extract from fits that $v = 0.144(1)$

$$S_n(q) - S_n + \frac{1}{2}\log L \sim A_n - B_n \frac{q^2}{L} + C_n \frac{q^4}{L^3}$$

FCS



Renyi-2 EE



B. Oblak, N. Regnault, and B. Estienne, PRB (2021)

$(\tilde{q} = q/2)$

Bosonic Moore-Read (MR) state at ν = 1: CFT

In addition to the bosonic U(1) current J, we also have a real fermion ψ.

3 topological sectors: two Abelian (vacuum and ψ) and one non-Abelian (σ).

$$H_{\mathrm{CFT}} = \frac{2\pi v}{L}\left(L_0 - \frac{c}{24}\right) = \frac{\pi v_\varphi}{L}(JJ)_0 - \frac{\pi v_\psi}{L}(\psi\partial\psi)_0$$, where $c = 3/2$.

(v_φ and v_ψ are the boson and fermion velocities, respectively)

We can write down U(1)-charged moments for the non-Abelian sector and for fixed (even and odd) fermionic parity in the Abelian sectors:

$$\widehat{Z}_{n,a}(\alpha) \sim p_a^{\frac{n-1}{2}} e^{-L(a_{n,a}+b_{n,a}\alpha^2+c_{n,a}\alpha^4)+O(L\alpha^6)}$$, analogous to Laughlin and IQHE

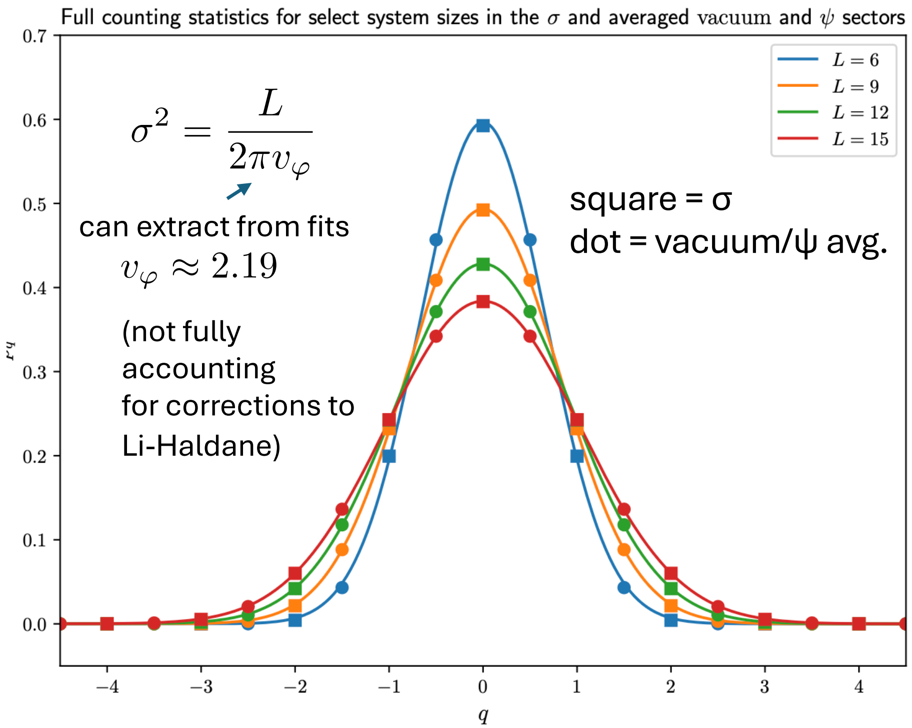
↑ prefactor depends on Abelian or non-Abelian sector

G. Moore, N. Read, Nucl. Phys. B (1991)

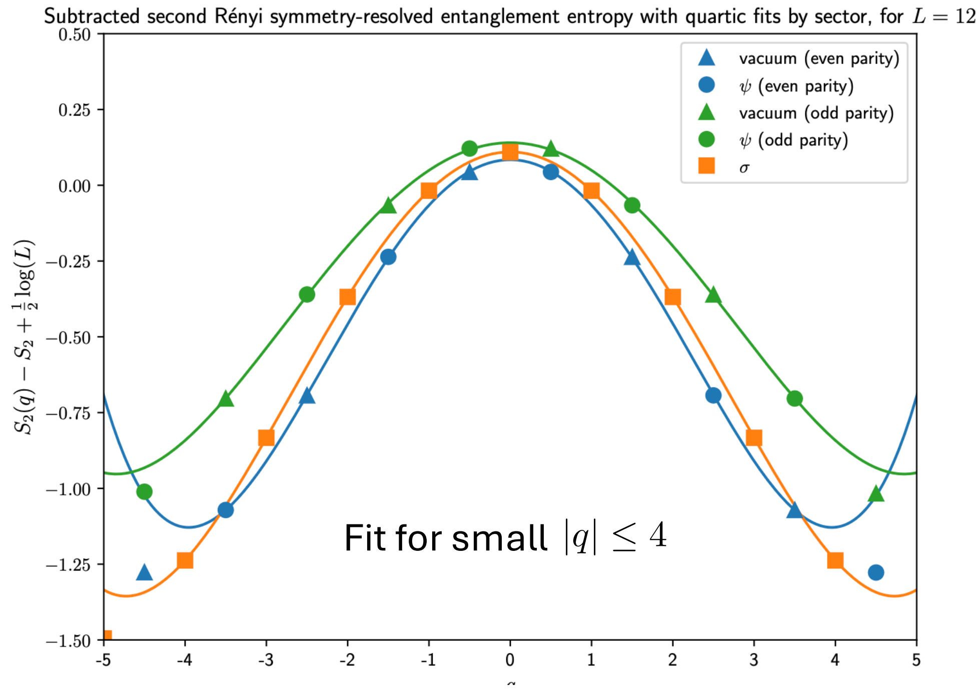
Bosonic MR state: FCS and SREE from MPS

$$\frac{p_{q,\text{vacuum}} + p_{q,\psi}}{2} = \frac{p_{q,\text{even}} + p_{q,\text{odd}}}{2} \sim p_{q,\sigma} \sim \frac{e^{-\frac{q^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

$$S_{n,a}(q) - S_{n,a} + \frac{1}{2}\log L \sim A_{n,a} - B_{n,a}\frac{q^2}{L} + C_{n,a}\frac{q^4}{L^3}$$



MJA, V. Crépel, N. Regnault, B. Estienne (in prep.)



We expect equipartition in the thermodynamic limit!

Bosonic MR state: synthetic ES

$$H_A = \sum_{i=0}^{\infty} g_i \int_0^L \phi_i(y) dy$$

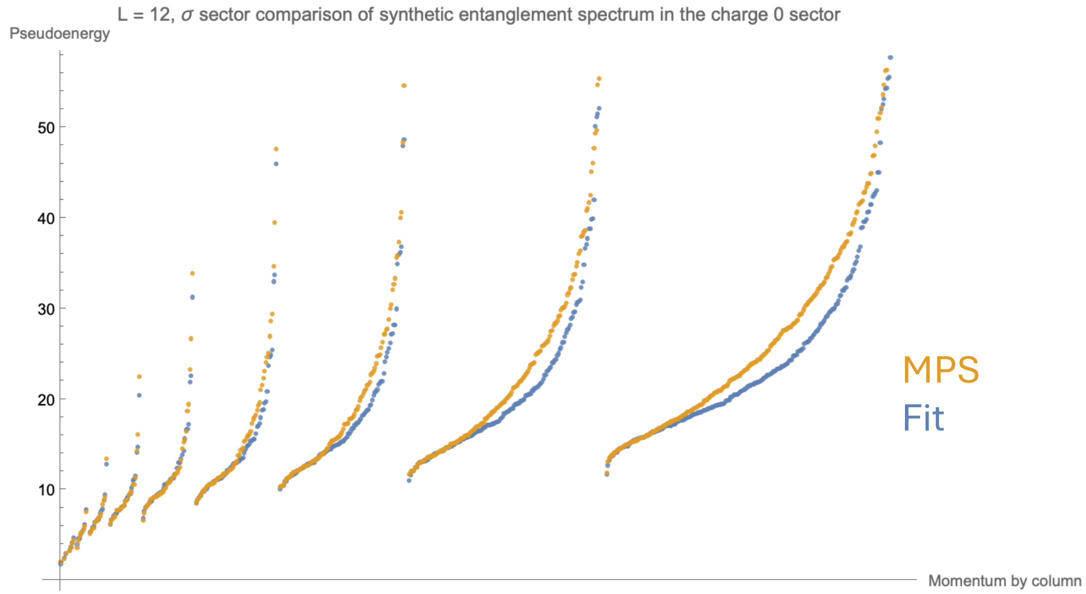
We approximate with operators of $\Delta_i \leq 4$:

i	Δ_i	$\phi_i(y)$
0	2	$(JJ)(y)$
1	2	$-(\psi\partial\psi)(y)$
2	4	$(\partial J \partial J)(y)$
3	4	$-(\partial\psi\partial^2\psi)(y)$
4	4	$((JJ)(JJ))(y)$
5	4	$-((JJ)(\psi\partial\psi))(y)$

H_{CFT} (rows 0–1)

g_i can be fit by minimizing the weighted sum of squares of differences with the ES from MPS.

MJA, V. Crépel, N. Regnault, B. Estienne (in prep.)



Can calculate topological EE, FCS, SREE, ...

Fit coefficients

g_0	g_1	g_2	g_3	g_4	g_5
1.95318	1.14884	2.36093	1.11988	0.313558	1.50018

This gives v_φ and v_ψ accounting for Li-Haldane corrections!

(cf. $v_\varphi \approx 2.19$ from FCS)

Conclusions

- The Li-Haldane correspondence and entanglement equipartition are powerful principles, and their corrections help us understand the structure of entanglement
- The quantum Hall states provide an excellent platform for exploration of symmetry-resolved entanglement
- IQHE, Laughlin, and bosonic Moore-Read all satisfy entanglement equipartition!
- Outlook:
  - Can any QH state violate entanglement equipartition?