Born-Oppenheimer RG for high energy evolution

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Motivation

The eventual goal is to unify the eikonal High energy evolution (BFKL) with the "partonic" high virtuality evolution (DGLAP).

We often think about the BFKL or JIMWLK evolution in terms of their different time scales. As we increase energy, time dilates, faster modes live longer and contribute to scattering. **This is how the original JIMWLK papers pictured the physics of the evolution.**

But in fact JIMWLK does not work quite like that: one also includes in the evolution eikonal contribution of very low transverse momentum modes, as long as their longitudinal momentum is high enough. These modes may have low frequency irrespective of the boost. They do not "freeze" as a result of the boost (evolution), but are frozen all the way through.

The practical effect of including these extraneous modes in the evolution is the appearance of (some of the) large transverse logs in NLO JIMWLK: too many low transverse momentum modes are produced.

What if we implement mathematically this physical picture, i.e. evolve directly in frequency?

Frequency evolution has been discussed in the context of BFKL Salam (1998), Sabio Vera (2005).

In NLO BFKL switching to frequency (k^-) evolution eliminates higher poles in the characteristic function at $\gamma = 0$, and makes the kernel better behaved.

Beyond BFKL frequency or "Ioffe time" evolution was discussed in Altinoluk et.al. (2014), Ducloe et. al. (2019), but these discussions are incomplete.

Balitsky and Tarasov (I think) are using a very similar approach, but I am still far from understanding their papers.

Frequency evolution.

What should happen if we evolve in frequency?

In the eikonal limit frequency ordering should impose life time ordering.

But that's not the whole story. DGLAP evolution is in fact also an evolution in frequency. Increasing Q^2 increases the frequency of the fluctuations in the target resolved by the hard scattering:

DGLAP collinear splittings
$(k^+, k_\perp \sim 0) \to (p^+ \sim k^+, p_\perp \gg k_\perp) + (k^+ - p^+ \sim k^+, -p_\perp)$ increase the frequency of the relevant modes $\frac{p_\perp^2}{2p^+} \gg \frac{k_\perp^2}{2k^+}$.

And that's exactly what we want!

Frequency increase $\leftrightarrow$ Born-Oppenheimer physics

The physics principle for frequency evolution is exactly the same as for the famous Born-Oppenheimer approximation. As the external frequency with which we probe the system is increased (be it the total energy E or the transverse resolution scale Q^2), faster modes participate in the process.

Thus to understand the evolution, we need to solve for "fast" modes (higher frequency) on the background of the slower modes (lower frequency).

Of course as we go higher and higher in the resolution (energy, Q^2 ...) we need to include faster and faster modes. We call this procedure "the Born-Oppenheimer RG". The BO RG should unify the BFKL-type and the DGLAP-type evolution in a single framework.

The setup I

To be clear about terminology: we will be deriving the evolution of the wave function of a hadron, which we refer to as "projectile".

The projectile is a right mover and has a large longitudinal momentum P^+.

The target hadron is present only in the sense of providing the "Ioffe time" resolution τ_{Ioffe}, i.e. the typical time scale over which the projectile gluons have to live in order to contribute to the scattering amplitude.

This defines the frequency "cutoff" up to which we account for gluon modes in the projectile wave function $p^- \equiv \frac{\mathbf{p}^2}{2p^+} < E \equiv 1/\tau_{Ioffe}$.

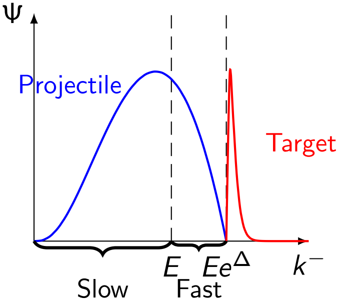
In this discussion we neglect quarks for simplicity, even though to take the full DGLAP physics into account we will have to restore them already at LO.

The Setup II

As always, we assume factorization between the projectile and target degrees of freedom, so that the wave function before scattering

$$|\Psi_{in}\rangle = |\Psi_P\rangle \otimes |\Psi_T\rangle$$

The projectile wave function contains modes with frequencies below those of the target:



The setup III

Here we assume dilute limit, i.e. projectile fields are small.

The BO paradigm: need to find the LCWF of the fast modes in the slow background.

1. Identify the interaction Hamiltonian that couples slow and fast modes H^I_{SF}.

2. Find the QCD time evolution operator that evolves the vacuum of the fast modes due to this coupling.

$$U(0,\tau) = \mathcal{T}\exp\{i\int_0^\tau dx^+ H^I_{SF}(x^+)\}$$

3. According to Low's theorem, the operator that diagonalizes the fast mode sector (with appropriate regulator) is

$$\Omega = \lim_{\tau\to\infty} U(0,\tau)$$

The LCWF of the projectile is then

$$|\Psi_P\rangle = \Omega\,|0\rangle_F \otimes |\psi_0\rangle_S$$

We follow this procedure through perturbatively to leading order for fast modes on a small interval $E < p^- < E + \Delta E$, and then iterate as E changes .

The interaction Hamiltonian I

This is straightforward - just staring at the QCD Hamiltonian and separating out the highest frequency mode $p^- > k^-, (p-k)^-$:

$$\mathcal{H}_I(p) = -ig \int_{max(k^-,(k-p)^-)<p^-} A_i^a(k^+, \mathbf{k}) f^{abc} \times$$
$$\times \Big\{ \Big[\delta_{ki}\delta_{jl} \left(\frac{2k^+}{p^+} - 1 \right) + \epsilon_{ki}\epsilon_{jl} \Big] \boldsymbol{P}_j^{bd} A_l^{\dagger d}(p^+, \boldsymbol{p}) A_k^{\dagger c}(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p}) +$$
$$+ \Big[\delta_{ki}\delta_{jl} \left(\frac{2k^+}{k^+ - p^+} - 1 \right) + \epsilon_{ki}\epsilon_{jl} \Big] (\boldsymbol{K} - \boldsymbol{P})_j^{bd} A_l^{\dagger d}(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p}) A_k^{\dagger c}(p^+, \boldsymbol{p}) \Big\} + h$$

Here

$$P_i^{ab} \equiv \boldsymbol{p}_i \delta^{ab} + igf^{abc} \int_{k^+ \ll p^+; k^- \ll p^-} \left[\alpha_i^{\dagger c}(k^+, \mathbf{k}) + \alpha_i^c(k^+, -\mathbf{k}) \right]$$

α^i - are *very slow and soft* fields with all components of momentum small - "soft fields" in the SCET language. This interaction is outside of either BFKL or DGLAP framework. In a dense projectile the soft field background may be large and will have to be taken into account. In the dilute limit at LO α's are unimportant and we neglect them.

The interaction Hamiltonian II

The no-soft - fields Hamiltonian is

$$\mathcal{H}_I(p) = -ig \int_{max(k^-,(k-p)^-)<p^-} A_i^a(k^+,\boldsymbol{k}) f^{abc} \times$$

$$\times \quad \Big\{ \Big[\delta_{ki}\delta_{jl} \Big(\frac{2k^+}{p^+} - 1 \Big) + \epsilon_{ki}\epsilon_{jl} \Big] \boldsymbol{p}_j A_l^{\dagger b}(p^+,\boldsymbol{p}) A_k^{\dagger c}(k^+ - p^+, \boldsymbol{k}-\boldsymbol{p})$$

$$- \quad \Big[\delta_{ki}\delta_{jl} \Big(\frac{2k^+}{k^+ - p^+} - 1 \Big) + \epsilon_{ki}\epsilon_{jl} \Big] \boldsymbol{p}_j A_l^{\dagger b}(k^+ - p^+, \boldsymbol{k}-\boldsymbol{p}) A_k^{\dagger c}(p^+,\boldsymbol{p}) \Big\} + h.c.$$

To find the LO wave function we need the energy denominator. For $k^- < p^-, (k-p)^-$ (holds in BFKL and DGLAP limits) it is

$$D^{-1} \equiv k^- - p^- - (k-p)^- = \frac{\boldsymbol{k}^2}{2k^+} - \frac{\boldsymbol{p}^2}{2p^+} - \frac{(\boldsymbol{k}-\boldsymbol{p})^2}{2(k^+-p^+)} \approx -\frac{\boldsymbol{p}^2 k^+}{2p^+(k^+-p^+)}$$

The LCWF for a fast mode p

We then find the perturbative LCWF

$$\Omega_p = 1 + iG(p^+, \boldsymbol{p}) \approx e^{iG(p^+, \boldsymbol{p})};$$

where

$$G(p^+, \boldsymbol{p}) = A_i^\dagger(p^+, \boldsymbol{p}) C_i(p^+, \boldsymbol{p}) + A_i(p^+, \boldsymbol{p}) C_i^\dagger(p^+, \boldsymbol{p})$$

$$C_i^a(p^+, \boldsymbol{p}) = g \int_{k^- < p^-; \quad (k-p)^- < p^-} F^i_{lk}(k, p) A_l^\dagger(k^+ - p^+, \boldsymbol{k} - \boldsymbol{p}) T^a A_k(k^+, \boldsymbol{k})$$

$$F^i_{lk}(k, p) = \frac{4p^+(k^+ - p^+)}{k^+} \left\{ \delta_{kl}\delta_{ji} \frac{k^+}{p^+} + \delta_{ki}\delta_{jl} \frac{k^+}{k^+ - p^+} - \delta_{kj}\delta_{il} \right\} \frac{\boldsymbol{p}_j}{\boldsymbol{p}^2}$$

No matter the details: C_i is the "classical field" produced by the slow background (Ω is a coherent operator) (in the eikonal limit reduces to the usual $\frac{\boldsymbol{p}_i}{\boldsymbol{p}^2} \rho^a(\boldsymbol{p})$).

The evolved LCWF

Now that we have diagonalized the Hamiltonian for a single fast mode, we write the wave function evolved over a finite range of frequency

$$|\Psi_P\rangle_E = \mathcal{P}\exp\left\{ i\int_{E_0}^{E} dp^- \mathcal{G}(p^-)\right\} |\Psi_P\rangle_{E_0}$$

where

$$\mathcal{G}(p^-) \equiv \int_p \delta(p^- - \frac{\boldsymbol{p}^2}{2p^+}) G(p^+, \boldsymbol{p}^2);$$

$$G(p^+, \boldsymbol{p}) = A_i^\dagger(p^+, \boldsymbol{p}) C_i(p^+, \boldsymbol{p}) + A_i(p^+, \boldsymbol{p}) C_i^\dagger(p^+, \boldsymbol{p})$$

Given the wave function we can discuss evolution!

What do we evolve?

So what do we want to evolve?

In BFKL we are interested in gluonic observables that depend on the softest gluons that are allowed in the wave function.

For example $\langle \rho(\boldsymbol{x})\rho(\boldsymbol{y})\rangle$ is most important to the eikonal total cross section, and is dominated by the softest gluons, $\propto G(x_{min)}$.

But these are not the only interesting observables. E.g. gluon TMD at a higher value of x - we heard about it many times last week. In JIMWLK - H.Duan, A.K. and M. Lublinsky 2407.15960 (PRD, 2025) considered evolution when the LCWF is evolved to values lower than x. This is the CSS regime.

As it turns out it is easier to evolve the "CSS type" operators (that depend on values of $x \gg x_{min}$) than the "BFKL type" operators (that live at $x \sim x_{min}$).

On to the evolution

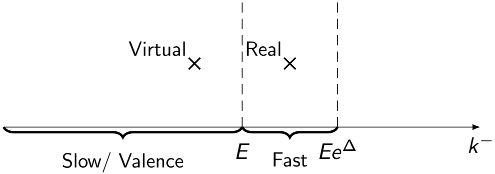
For any operator $\hat{O}$ in the projectile Hilbert space

$$\langle\hat{O}\rangle_E = \langle\Psi_P|\hat{O}|\Psi_P\rangle_E$$

The evolution equation follows from

$$\frac{d}{\eta}\langle\hat{O}\rangle = \lim_{\Delta\to 0}\frac{1}{\Delta}\left[\langle\hat{O}\rangle_{Ee^{\Delta}} - \langle\hat{O}\rangle_E\right]; \qquad \eta \equiv \ln E/E_0$$

Quite generally for an arbitrary $\hat{O}$ there are two types of contributions to the evolution: virtual (or Lindblad) - due to gluonic degrees of freedom in $\hat{O}$ that live below E, and real - due to those that live between E and Ee^{Δ}.



For TMD $T(p^+, \boldsymbol{p})$: $\frac{\boldsymbol{p}^2}{2p^+} < E$ only the Lindblad term contributes (CSS) - easier.

For $\langle\rho(\boldsymbol{x})\rho(\boldsymbol{y})\rangle$ - both contributions are present (BFKL) - more difficult.

Evolution of gluon TMD

Consider

$$\hat{T}(k) \equiv a_i^{\dagger a}(k) a_i^a(k); \quad \mathcal{T}(k) \equiv \langle \Psi_P | \hat{T}(k) | \Psi_P \rangle_E; \quad k^- < E$$

Note - the LCWF is evolved well past the frequency of the gluon, and we are interested in its evolution to even higher frequencies.

Direct calculation is easy - the operator does not depend on the gluonic degrees of freedom in the evolution interval $E < p^- < Ee^\Delta$:

$$\delta\mathcal{T}(k) = \delta\mathcal{T}^L(k) + \delta\mathcal{T}^{NL}(k)$$

with

$$\begin{aligned}
\delta\mathcal{T}^L(k) &= \frac{g^2 N_c}{2} \int_{E<p^-<Ee^\Delta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^+} \\
&\Big[\frac{1}{4k^+(k^+ + p^+)} F^l_{st}(k+p,p) F^l_{st}(k+p,p) \mathcal{T}(k+p) \qquad \text{gain} \\
&- \frac{1}{4k^+(k^+ - p^+)} F^l_{st}(k,p) F^l_{st}(k,p) \mathcal{T}(k)\Big] \qquad \text{loss}
\end{aligned}$$

The physics of the linear term is very simple. The loss term is due to decay of gluons with momentum k into soft gluons in the phase space open by the evolution. The gain term is due to decay of gluons with $k+p$ into k and additional soft gluon. Note that for $p < k$ there is practically complete cancellation between the loss and gain terms. For $p > k$ we are in the DGLAP regime, and the loss (virtual) term dominates.

Evolution of gluon TMD II

The nonlinear term is

$$\delta\mathcal{T}^{NL}(k) = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3l}{(2\pi)^3}\frac{1}{2p^+}\times$$

$$\Big[F^i_{lk}(l,p)F^i_{nm}(k+p,p)\times$$

$$\langle : A^\dagger_l(l^+-p^+,\boldsymbol{l}-\boldsymbol{p})T^aA_k(l^+,\boldsymbol{l})A^\dagger_m(k^++p^+,\boldsymbol{k}+\boldsymbol{p})T^aA_n(k^+,\boldsymbol{k}) :\rangle$$

$$-F^i_{lk}(l,p)F^i_{nm}(k,p)\times$$

$$\langle : A^\dagger_l(l^+-p^+,\boldsymbol{l}-\boldsymbol{p})T^aA_k(l^+,\boldsymbol{l})A^\dagger_m(k^+,\boldsymbol{k})T^aA_n(k^+-p^+,\boldsymbol{k}-\boldsymbol{p}) :\rangle\Big] +$$

This looks pretty long but makes perfect sense - see below.

TMD and CSS

The linear term first:

$$F^i_{ln}(k,p)F^i_{ln}(k,p) = 32(k^+)^2\zeta(1-\zeta)\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\frac{1}{\boldsymbol{p}^2}; \quad \zeta \equiv \frac{p^+}{k^+}$$

The "loss term""

$$\delta\mathcal{T}^L_{loss}(k) = -\frac{g^2N_c}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2p^+}\frac{1}{4k^+(k^+-p^+)}F^l_{st}(k,p)F^l_{st}(k,p)\mathcal{T}(\boldsymbol{k},x) =$$
$$-\frac{\Delta}{2\pi^2}\frac{g^2N_c}{2}\int_0^{1/2}d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\mathcal{T}(\boldsymbol{k},x)$$

The same simplification for the "gain term" : it regulates the loss below $\zeta \approx k^-/E$

So that

$$\frac{\partial}{\partial\eta}\mathcal{T}(\boldsymbol{k},x) = -\frac{g^2N_c}{4\pi^2}\int_{\frac{k^-}{E}}^{1/2}d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\mathcal{T}(\boldsymbol{k},x)+\mathrm{NL}$$

Looks a lot like CSS equation, but only a single equation, and the evolution parameter is $\ln k^-$.

BO and CSS I

The physics of the TMD evolution is very similar

One can define TMD in many ways via a cascade that allows splittings into a part of phase space limited by a "transverse" and a "longitudinal" resolution scales: "parton branching TMD" (Hautmann et, al. 2017-2024)

For example let us limit the allowed phase space by a transverse momentum $\boldsymbol{p}^2 < \mu^2$ and longitudinal momentum fraction $p^+/k^+ < \xi/k^+$ (call it DGLAP cascade). At the initial point of the evolution a gluon with momentum $\boldsymbol{k}, k^+$ is not allowed to split into any constituents. At this point clearly the transverse resolution scale is $\mu_0^2 = \boldsymbol{k}^2$, and the longitudinal resolution scale is $\xi_0/k^+ = 1$. We can then allow DGLAP parton branchings into gluons with transverse momenta up to μ^2 and longitudinal resolution down to ξ/k^+.
An initial gluon with momentum $\boldsymbol{k}, k^+$ thus splits into all possible pairs in the allowed phase space. The evolution of such TMD is given by a pair of equations

$$\frac{\partial T(k^+, \boldsymbol{k}^2; \mu^2; \xi)}{\partial \ln \mu^2} = -\frac{\alpha_s}{2\pi} N_c \int_{\xi/k^+}^{1-\xi/k^+} d\zeta \left[\frac{1-\zeta}{\zeta} + \frac{\zeta}{1-\zeta} + \zeta(1-\zeta)\right] T\left(k^+, \boldsymbol{k}^2; \mu^2; \xi\right)$$

$$\frac{\partial T(k^+, \boldsymbol{k}^2; \mu^2; \xi)}{\partial \ln \frac{1}{\xi}} = -\frac{\alpha_s}{2\pi} 2N_c \int_{\boldsymbol{k}^2}^{\mu^2} \frac{dp^2}{p^2} T\left(k^+, \boldsymbol{k}^2; \mu^2; \xi\right)$$

Note that the DGLAP kinematics is not imposed here artificially, rather the "loss" and "gain" terms compensate each other for emission of gluons with $\boldsymbol{p} < \boldsymbol{k}$.

BO and CSS II

The phase space occupied by this DGLAP cascade :

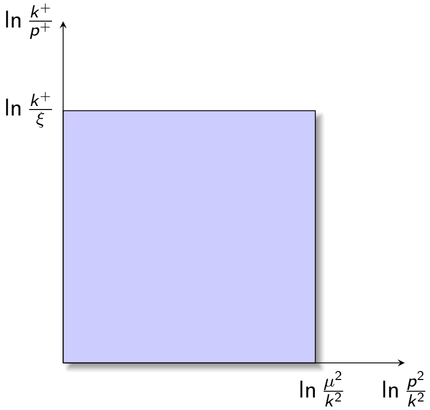


Figure: The phase space of the DGLAP cascade with resolution scales μ^2 and ξ. Only gluons with momenta $(p^+, \boldsymbol{p}^2)$ inside the blue rectangle are allowed in the cascade/wave function.

The BO evolution has a single evolution parameter, since we limit the cascade only by the frequency of allowed gluons. This gave us

$$\frac{\partial}{\partial\eta}\mathcal{T}(\boldsymbol{k},x) \;=\; -\frac{g^2N_c}{4\pi^2}\int_{\frac{k^-}{E}}^{1/2} d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right]\mathcal{T}(\boldsymbol{k},x)+\mathrm{NL}$$

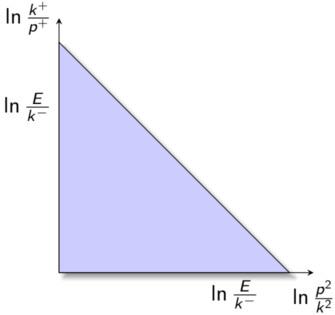


Figure: The phase space of the BO cascade. Only gluons with $p^- < E$, $p^+ < k^+$ and $\boldsymbol{p}^2 > \boldsymbol{k}^2$ are present in the wave function.

BO and CSS IV

Nevertheless the two cascades are exactly equivalent (as far as the evolution of TMD is concerned) if we identify

$$\mathcal{T}_{BO}(\boldsymbol{k}, k^+; E) = T_{DGLAP}(k^+, \boldsymbol{k}; \mu^2(E); \xi(E))$$

$$\ln \frac{\mu^2(E)}{\boldsymbol{k}^2} = \ln \frac{E}{k^-}; \qquad \ln \frac{k^+}{\xi(E)} - \frac{11}{12} = \frac{1}{2}\left[\ln \frac{E}{k^-} - \frac{11}{12}\right]$$

The pair of the two evolution equations for the DGLAP cascade then collapse onto the BO evolution.

BO and CSS V

The CSS equation describes a cascade that is limited by the transverse resolution and the angle of emission of gluons with the two cutoffs μ^2 and $\zeta \equiv \theta^2\mu^2$

$$\frac{\partial T(k^+,\boldsymbol{k}^2;\mu^2;\zeta)}{\partial \ln \mu^2} = -\frac{\alpha_s}{2\pi}2N_c\int_{\mu/\sqrt{\zeta}}^{1/2} d\xi\Big[\frac{1-\xi}{\xi}+\xi(1-\xi)\Big]\,T\left(k^+,\boldsymbol{k}^2;\mu^2;\zeta\right)$$

$$\frac{\partial T(k^+,\boldsymbol{k}^2;\mu^2;\zeta)}{\partial \ln \sqrt{\zeta}} = -\frac{\alpha_s}{2\pi}2N_c\int_{\boldsymbol{k}^2}^{\mu^2}\frac{dp^2}{p^2}\,T\left(k^+,\boldsymbol{k}^2;\mu^2;\zeta\right)$$

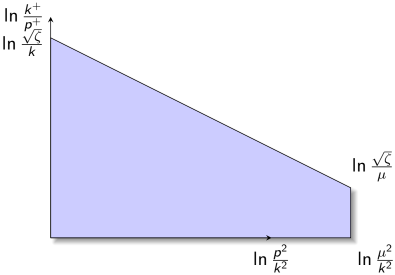


Figure: The phase space of the CSS cascade. Only gluons with $p^2 < \mu^2$ and $|\boldsymbol{p}|/p^+ < \sqrt{\zeta}/\mu$ are present in the cascade/wave function.

BO and CSSVI

Again, we get the same evolution of TMD in the BO and CSS cascades if we identify

$$\mathcal{T}_{BO}(\boldsymbol{k}, k^+; E) = T_{CSS}(k^+, \boldsymbol{k}; \mu^2(E); \zeta(E))$$

$$\ln \frac{\mu^2(E)}{\boldsymbol{k}^2} = \ln \frac{E}{k^-}; \qquad \ln \frac{\zeta(E)}{\mu^2(E)} - \frac{11}{12} = \frac{1}{2}\left[\ln \frac{E}{k^-} - \frac{11}{12}\right]$$

The Nonlinear terms - the stimulated emission I

Now a couple of words about the nonlinear terms.

Recall that we got nonlinear contributions to the evolution

$$\delta\mathcal{T}^{NL}(k) = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3l}{(2\pi)^3}\frac{1}{2p^+}\times$$

$$\Big[F^i_{lk}(l,p)F^i_{nm}(k+p,p)\langle : A^\dagger_l(l^+-p^+,\boldsymbol{l}-\boldsymbol{p})T^aA_k(l^+,\boldsymbol{l})A^\dagger_m(k^++p^+,\boldsymbol{k}+\boldsymbol{p})T^aA_n(k^+$$

$$-F^i_{lk}(l,p)F^i_{nm}(k,p)\langle : A^\dagger_l(l^+-p^+,\boldsymbol{l}-\boldsymbol{p})T^aA_k(l^+,\boldsymbol{l})A^\dagger_m(k^+,\boldsymbol{k})T^aA_n(k^+-p^+,\boldsymbol{k}-\boldsymbol{p})$$

To interpret these let us use the "dilute approximation".

We assume that spectators don't matter, and that the two particles that scatter here must be in the same final state as in the initial state in order to contribute to the forward scattering amplitude. Then the average factorizes and we get a product of two TMD's on the RHS.

The Stimulated emission II

The things then simplify, and when the dust settles

$$\frac{\partial}{\partial\eta}[x\mathcal{T}(\boldsymbol{k},x)] = -\frac{g^2N_c}{4\pi^2}\Bigg[\int_{\boldsymbol{k}^2/Q^2}^{1/2} d\zeta\left[\frac{\zeta}{1-\zeta}+\frac{1-\zeta}{\zeta}+\zeta(1-\zeta)\right] + \frac{1}{4\pi(N_c^2-1)}\frac{1}{Q^2S_\perp}\int_{\boldsymbol{p}^2=\boldsymbol{k}^2}^{\boldsymbol{p}^2=Q^2}\frac{d^2\boldsymbol{p}}{(2\pi)^2}\,[x\mathcal{T}(\boldsymbol{p},x)]\Bigg][x\mathcal{T}(\boldsymbol{k},x)]$$

Here we have introduced a natural transverse resolution $Q^2 = 2Ek^+$. This is the highest transverse momentum allowed in the BO wave function for particles with longitudinal momentum k^+.

The nonlinear term is just a stimulated emission: the probability of splitting $(k) \to (p) + (k-p)$ is enhanced if there is already a particle with momentum p in the wave function!

It is a nonlinear (higher twist) effect that depletes the number of particles at k. It has nothing to do with low x physics - indeed our "dilute approximation" in the low x regime is hardly justified.

PDF and DGLAP

Where there is TMD, there is PDF.

For PDF we need to include the real corrections due to the gluons in the "window" between E and Ee^{Δ}. This does not present difficulties. We find that at moderate x the DGLAP equation is reproduced where $Q^2 = 2Ek^+$.

$$\frac{\partial}{\partial \ln Q^2}\left[xG(x,Q^2)\right]^L = \frac{\alpha_s}{2\pi}\int_x^1 d\zeta P_{gg}(\zeta)\left[\frac{x}{\zeta}G\left(\frac{x}{\zeta},\frac{Q^2}{\zeta}\right)\right] \quad (1)$$

At low x this deviates from the standard BFKL, as the resolution scale on the RHS becomes different from Q^2.

In addition the stimulated emission corrections also contribute, and are dominated by their virtual terms.

$$\frac{\partial}{\partial \ln Q^2}[xG(Q^2,x)]^{NL} = -\frac{\alpha_s N_c}{4\pi}\frac{1}{(2\pi)^3}\frac{1}{N_c^2-1}\frac{1}{Q^2 S_\perp}[xG(Q^2,x)]^2$$

A higher twist (obviously), but not the GLR-MQ term! No $\ln x$, but leading in α_s. A completely different physical effect, but also leads to slowing down of the evolution.

Future ...

Our ultimate goal is of course to unify DGLAP with BFKL. We need to consider the multiple soft scatterings, a.k.a. Wilson lines. That is still some way off.

The next step is to look at soft scattering off a dilute target, meaning consider the evolution of $\langle \rho^a(\boldsymbol{x}) \rho^a(\boldsymbol{y}) \rangle$. We thought we had it, but we don't - still working on it. It is not straightforward, but we are slowly progressing, and we will do it.

Then there is the question of soft fields, which we have set aside for now. Physically these describe rescattering of emitted gluons on the fields of the projectile when the projectile is dense. Are they important? Probably.