Heavy quarkonium produciton impact-factors at NLO

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Outline

1. NRQCD factorisation
2. Lipatov's High-Energy EFT
3. Virtual corrections to S-wave impact-factors
4. Real corrections

Quarkonium pairs at large Y

Heavy quarkonium pair production at hadron colliders is an interesting process:

$$p(P_1) + p(P_2) \to J/\psi(p_1) + X + J/\psi(p_2),$$

where $M^2_{\psi\psi} = (p_1 + p_2)^2 \gg |\mathbf{p}_{1T}|^2$ or $|\mathbf{p}_{2T}|^2 \gg \Lambda^2_{\mathrm{QCD}}$.

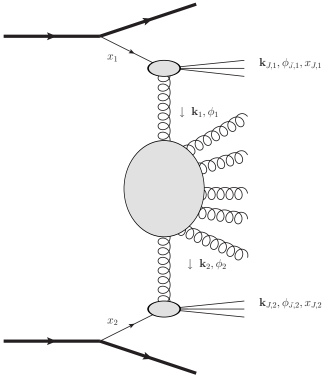


Figure from [hep-ph/1302.7012] .

- $d\sigma = \underbrace{\mathrm{PDF}_i \otimes \mathrm{PDF}_j}_{\text{“non-perturbative”}} \otimes \underbrace{d\hat{\sigma}_{ij}}_{\text{perturbative}}$,
- Fixed-order: $d\hat{\sigma}$ known up to NLO in α_s for double-colour-singlet channel,
- Regge limit: $Y = \ln \frac{M_{\psi\psi}}{\sqrt{|\mathbf{p}_{1T}||\mathbf{p}_{2T}|}} \gg 1$, the $d\hat{\sigma}$ recieves corrections $\sim (\alpha_s Y)^n$, which should be resummed.
- The *Balitsky-Fadin-Kuraev-Lipatov (BFKL)* equation allows to rigirously resum them in the LLA ($\sum_n (\alpha_s Y)^n$) and NLLA ($\sum_n \alpha_s (\alpha_s Y)^n$).
- Large DPS contamination exists at $Y \gg 1$, but even at largest Y experiments see correlations which can not be explained by the DPS models

The hybrid approximation

The hadronic cross section formula ("hybrid approximation"):

$$d\sigma = \int_0^1 dx_1 dx_2 f_a(x_1, \mu_F) f_b(x_2, \mu_F) d\hat{\sigma}.$$

The partonic cross section is re-factorised as:

$$\frac{d\hat{\sigma}}{dy_{1,2} d|\mathbf{p}_{1,2T}| d\phi_{1,2}} = \int d^2\mathbf{k}_T\, d^2\mathbf{k}'_T V_{\mathcal{Q},a}(\mathbf{k}_T, \mathbf{p}_{1T}, x)\, G(\mathbf{k}_T, \mathbf{k}'_T, Y)\, V_{\mathcal{Q},b}(\mathbf{k}'_T, \mathbf{p}_{2T}, x')\,,$$

where G is the (NLL) BFKL Green's function, which is universal. We want to compute quarkonium production impact-factors $V_{\mathcal{Q},a}$ up to NLO.

The hybrid approximation with collinear initial-state PDFs might be not a great idea. It is probably better to put the TMD initial-state PDFs instead, to absorb some of the NLO corrections to impact-factors into them [Mueller, Szymanowski, Wallon, Xiao, Yuan 2025; ... Altinoluk, Armesto, Kovner, Lublinsky 2023; ... Chernyshev, MN, Saleev 2025] .

To compute the IF, we need to first produce the $Q\bar{Q}$-pair perturbatively and then hadronise it $Q\bar{Q} \to \mathcal{Q} + X$. To describe the hadronisation stage we will use the NRQCD factorisation formalism [Bodwin, Braaten, Lepage '95] .

Non-Relativistic QCD factorisation

Charmonia and bottomonia

In the quark model, heavy quarkonia (charmonia and bottomonia) are described as bound states of heavy quark and antiquark ($c\bar{c}$ and $b\bar{b}$ respectively).

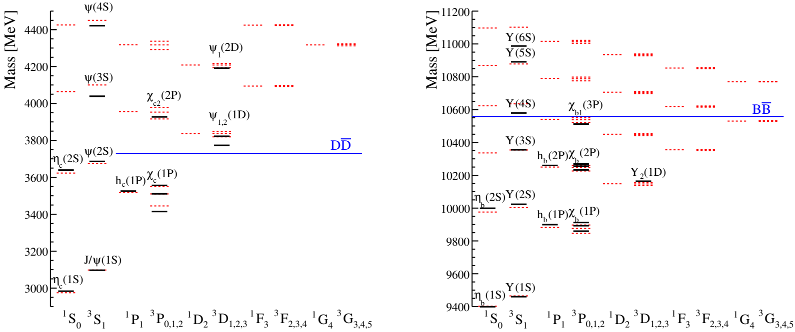


Figure from [hep-ph/1708.04012] , red lines – Godfrey-Isgur relativized potential model

They are "Hydrogen atoms of QCD" (recall the spectroscopic notation $^{2S+1}L_J$!), because the simple non-relativistic picture of two heavy quarks bound by a confining potential works well for the states below open-heavy-flavor ($D\bar{D}$ or $B\bar{B}$) threshold.

Non-relativistic QCD

The velocity-expansion for quarkonium eigenstate in the rest frame:

$$\begin{aligned} |J/\psi\rangle &= O(1)\left|c\bar{c}\left[{}^3S_1^{(1)}\right]\right\rangle + O(v)\left|c\bar{c}\left[{}^3P_J^{(8)}\right]+g\right\rangle \\ &+ O(v^2)\left|c\bar{c}\left[{}^1S_0^{(8)}\right]+g\right\rangle + O(v^2)\left|c\bar{c}\left[{}^3S_1^{(8)}\right]+gg\right\rangle + \ldots, \end{aligned}$$

NRQCD = *non-relativistic EFT for heavy quarks*, light quarks and gluons are still relativistic! Dynamics conserves number of heavy quarks. In such EFT, $Q\bar{Q}$-pair is produced in a point, by local operator (× Light-like WL factors)

$$\mathcal{A}_{\mathrm{NRQCD}} = \langle J/\psi + X|\, \chi^\dagger(0)\kappa_n\psi(0)\,|0\rangle\,,$$

Different operators "couple" to different $Q\bar{Q}$ states, e.g.:

$$\chi^\dagger(0)\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^1S_0^{(1)}\right]\right\rangle + \ldots,\ \chi^\dagger(0)\sigma_i\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^3S_1^{(1)}\right]\right\rangle + \ldots,$$

$$\chi^\dagger(0)\sigma_i T^a\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^3S_1^{(8)}\right]\right\rangle + \ldots,\ \chi^\dagger(0)D_i\psi(0) \leftrightarrow \left|c\bar{c}\left[{}^1P_1^{(8)}\right]\right\rangle + \ldots$$

squared NRQCD amplitude (=LDME):

$$\sum_X |\mathcal{A}_{\mathrm{NRQCD}}|^2 = \langle 0|\underbrace{\psi^\dagger\kappa_n^\dagger\chi a^\dagger_{J/\psi}a_{J/\psi}\chi^\dagger\kappa_n\psi}_{\mathcal{O}_n^{J/\psi}}|0\rangle = \left\langle \mathcal{O}_n^{J/\psi}\right\rangle,$$

where $n = {}^{2S+1}L_J^{(1,8)}$.

Non-relativistic QCD

Velocity-scaling of LDMEs follow from velocity-scaling of corresponding Fock states and of operators $\chi^\dagger \kappa_n \psi$:

$\mathcal{H}\backslash n$	${}^1S_0^{(1)}$	${}^3S_1^{(1)}$	${}^1S_0^{(8)}$	${}^3S_1^{(8)}$	${}^1P_1^{(1)}$	${}^3P_0^{(1)}$	${}^3P_1^{(1)}$	${}^3P_2^{(1)}$	${}^1P_1^{(8)}$	${}^3P_0^{(8)}$	${}^3P_1^{(8)}$	${}^3P_2^{(8)}$
η_c	1		v^4	v^4					v^4			
J/ψ		1	v^4	v^4						v^4	v^4	v^4
h_c			v^2		v^2							
χ_{c0}				v^2		v^2						
χ_{c1}				v^2			v^2					
χ_{c2}				v^2				v^2				

Note that:

- Colour-singlet LDMEs are LO in v for S-wave states $\Rightarrow$ *Colour-Singlet Model*
- For P-wave states the CS and CO LDMEs are of the same order $\Rightarrow$ *mixing*
- Connection between LDMEs for η_c and J/ψ through *Heavy-Quark Spin Symmetry*

Matching between QCD amplitude and NRQCD expansion:

$$v \ll 1 : \mathcal{A}_{\text{QCD}}(e^+e^- \to Q\bar{Q}(v)+X) = \sum_n f_n \left\langle Q\bar{Q}(v) + X \right| \chi^\dagger(0)\kappa_n\psi(0) \left|0\right\rangle + O(v^\#),$$

replace $\left|Q\bar{Q}(v) + X\right\rangle \to \left|\mathcal{H} + X\right\rangle \Rightarrow$ NRQCD factorization formula [BBL '95] :

$$\sigma(e^+e^- \to \mathcal{H} + X) = \sum_n \sigma(e^+e^- \to Q\bar{Q}[n] + X) \left\langle \mathcal{O}_n^{\mathcal{H}} \right\rangle,$$

where $\sigma(e^+e^- \to Q\bar{Q}[n] + X) \propto |f_n|^2$.

Task for today

Our task for today is tocompute the NLO impact-factors for the processes:

$$g + R \to Q\bar{Q}\left[{}^1S_0^{[1]}\right],$$
$$g + R \to Q\bar{Q}\left[{}^1S_0^{[8]}, {}^3S_1^{[8]}\right],$$

+ quark channels arising at NLO. R is the Reggeised gluon.

The ${}^1S_0^{[1]}$-one is the LO in v^2 contribution to the production of η_c (η_b), the ${}^1S_0^{[8]}$ and ${}^3S_1^{[8]}$ operators contribute to J/ψ ($\Upsilon(1S)$) production at $O(v^4)$. The P-wave CS and CO contributions should be computed also, but this is a more complicated problem.

Lipatov's High-Energy EFT

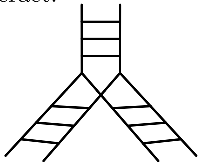
Reggeon Field Theory

The idea of RFT had been proposed by Gribov [Gribov, '68] . We introduce Reggeon fields which depend on *rapidity* y ($\sim \ln s$) and *transverse coordinates* $\mathbf{x}_T$: $R_\pm(y, \mathbf{x}_T)$. Then the "Reggeized" t-channel exchange follows from the Lagrangian:

$$L_{\rm RFT}^{(\rm kin.)} = R_+(y, \mathbf{x}_T) \left(\frac{\partial}{\partial y} - \omega_s(\mathbf{x}_T^2) \right) \partial_T^2 R_-(y, \mathbf{x}_T)$$

$$\Rightarrow \langle R_-(y_1, \mathbf{q}_T^2) R_+(y_2, \mathbf{q}_T^2) \rangle = \frac{i}{\mathbf{q}_T^2} \theta(y_1 - y_2) \exp[(y_1 - y_2)\omega_s(\mathbf{q}_T^2)].$$

Reggeons also can interact:



In phenomenological RFT the local interactions of Pomerons, Odderons etc is assumed, e.g.:

$$L_{\rm RFT}^{(\rm int.)} = g[R_+(y, \mathbf{x}_T) R_-(y, \mathbf{x}_T) R_-(y, \mathbf{x}_T) + (R_+ \leftrightarrow R_-)] + \dots,$$

which is probably a crude approximation.

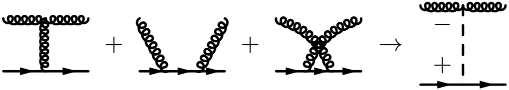
Our goal is to construct RFT from QCD and use it to do perturbative resummations for various observables.

Reggeized gluon

Studying tree-level amplitudes one quickly finds that **t-channel gluon exchanges dominate at the leading power at $s \to \infty$.** Often (e.g. for $qq' \to qq'$), just the replacement of the t-channel gluon propagator (Gribov's trick) extracts the LP contribution:

$$g^{\mu\nu} \to \frac{1}{2}(n_-^\mu n_+^\nu + n_+^\mu n_-^\nu).$$

But in a generic gauge, all 3 diagrams contribute to $qg \to qg$ amplitude:



The $R_\pm gg$ vertex reads:

$$\begin{aligned} \Gamma^{abc}_{\mu_1\mu_2-} \quad &= \quad f^{abc}\Big[(k_1^+ - k_2^+)g_{\mu_1\mu_2} \\ &+ n^+_{\mu_1}(2k_1+k_2)_{\mu_2} + n^+_{\mu_2}(2k_2+k_1)_{\mu_1} - \frac{q^2}{k_1^+}n^+_{\mu_1}n^+_{\mu_2}\Big]. \end{aligned}$$

- The vertex satisfies Slavnov-Taylor identity: $\varepsilon^{\mu_1}(k_1)k_2^{\mu_2}\Gamma^{abc}_{\mu_1\mu_2-} = 0 = k_1^{\mu_1}\varepsilon^{*\mu_2}(k_2)\Gamma^{abc}_{\mu_1\mu_2-}$
- It contains nonlocal "*induced*" term
- Terms in the second row are zero in the gauge $A_+ = 0$

Action of the Rg interaction

$$S_{\text{int.}} \supset \int dy \int d^4x\ \delta(x_+) 2\,\text{tr}\left\{R_-(y,\mathbf{x}_T) j_+[A_\mu^{[y,y+\eta]}](x)\right\}$$

where $1 \ll \eta \ll Y$. The j_+ in the gauge $\overline{A}_+ = 0$ is given by the

$$\Gamma^{abc}_{\mu_1\mu_2 -} = f^{abc}(k_1^+ - k_2^+) g_{\mu_1\mu_2} \leftrightarrow j_+[A_\mu] = i g_s \left[\overline{A}_\mu, \partial_+ \overline{A}^\mu\right]$$
$$= -\left[\overline{D}_\mu, \overline{G}_{\mu+}\right] - \partial_\mu \partial_+ \overline{A}^\mu,$$

where the first term can be dropped (at tree level) due to equations of motion $\left[\overline{D}_\mu, \overline{G}_{\mu+}\right] = 0$.

Relation with LCPT ($\Pi^- = -\partial_+ A_- = -\partial_+^{-1}(\mathcal{D}_i \partial_+ A_i)$):

$$\hat{H}_{\text{LCPT}} = \int dx^- d^2\mathbf{x}_T \left(\text{tr}\left[\Pi^-(\mathbf{x}_T, x^-)\Pi^-(\mathbf{x}_T, x^-)\right] + \frac{1}{2}\,\text{tr}\left[G^{ij}(\mathbf{x}_T, x^-) G^{ij}(\mathbf{x}_T, x^-)\right]\right)$$
$$\supset -\int d^3x\ \text{tr}\left[A_-\left(\partial_+\partial_i A_i + ig[A_i, \partial_+ A_i]\right)\right].$$

If one treats $A_-(x^-, \mathbf{x}_T) = \alpha(x^-, \mathbf{x}_T)$ as an external classical field, then one have to choose one of the interaction terms: $\partial_+\partial_i A_i$ or $ig[A_i, \partial_+ A_i]$ to avoid double-counting.

One can identify: $R_-^a(\mathbf{x}_T) \leftrightarrow \alpha^a(\mathbf{x}_T)$ and

$$\int dx_- j_+[\overline{A}](x_-, x_+ = 0, \mathbf{x}_T) \leftrightarrow \rho^a(\mathbf{x}_T) = \int_{k^+} a_b^\dagger(k^+, \mathbf{x}_T) T^a_{bc} a_c(k^+, \mathbf{x}_T).$$

Action of the Rg interaction

$$S_{\text{int.}} \supset \int dy \int d^4x\ \delta(x_+) 2\,\mathrm{tr}\left\{R_-(y, \mathbf{x}_T) j_+[A_\mu^{[y,y+\eta]}](x)\right\}$$

where $1 \ll \eta \ll Y$. The j_+ in the gauge $\overline{A}_+ = 0$ is given by the

$$\Gamma^{abc}_{\mu_1\mu_2-} = f^{abc}(k_1^+ - k_2^+) g_{\mu_1\mu_2} \leftrightarrow j_+[A_\mu] = ig_s\left[\overline{A}_\mu, \partial_+\overline{A}^\mu\right]$$
$$= -\left[\overline{D}_\mu, \overline{G}_{\mu+}\right] - \partial_\mu\partial_+\overline{A}^\mu,$$

where the first term can be dropped (at tree level) due to equations of motion $\left[\overline{D}_\mu, \overline{G}_{\mu+}\right] = 0$.
The field in the gauge $\overline{A}_+ = 0$ can be obtained from the field in arbitrary gauge A_μ by the following gauge transformation:

$$\overline{A}_\mu = \frac{-i}{g_s} W^\dagger[A_+](x) D_\mu W[A_+](x),$$

where $D_\mu = \partial_\mu + ig_s A_\mu$ and

$$W[A_\pm](x) = P\exp\left[\frac{-ig_s}{2}\int\limits_{-\infty}^{x_\mp} dx'_\mp A_\pm(x_\pm, x'_\mp, \mathbf{x}_T)\right]$$
$$= \left(\hat{1} + ig_s\partial_\pm^{-1}A_\pm\right)^{-1}\hat{1} = \hat{1} - ig_s(\partial_\pm^{-1}A_\pm) - (ig_s)^2(\partial_\pm^{-1}A_\pm\partial_\pm^{-1}A_\pm) + \ldots,$$

Action of the Rg interaction

The interaction term can be further simplified:

$$
\begin{aligned}
j_+[A_\mu](x) \quad \to \quad & -\partial_\mu \partial_+ \overline{A}^\mu = \frac{i}{g_s} \partial_\mu \partial_+ W^\dagger[A_+](x) D_\mu W[A_+](x) \\
= \quad & \frac{i}{g_s} \partial_\mu \partial_+ \left(\hat{1} + i g_s (\partial_+^{-1} A_\pm) + \ldots \right) \left[\partial^\mu + i g_s A^\mu \right] W[A_+](x) \\
= \quad & \frac{i}{g_s} \partial^2 \partial_+ W[A_+](x) - \partial_+ \partial_\mu \left((\partial_+^{-1} A_\pm) + \ldots \right) D^\mu W[A_+](x),
\end{aligned}
$$

the last term gives the vanishing contribution due to the conservation of the k_+-momentum component and $\partial^2 \to \partial_T^2$. Finally the interaction term takes the form

$$
S_{\text{int.}} \supset \frac{i}{g_s} \int dy \int d^4x \; \delta(x_+) 2\, \mathrm{tr} \left\{ R_-(y, \mathbf{x}_T) \partial_T^2 \partial_+ W[A_+^{[y,y+\eta]}] \right\},
$$

clearly it is *non-Hermitian*, which does not cause problems at tree level. Beyond tree level, the simplest Hermitian form, compatible with *negative signature* of R-exchange is [Lipatov '97; Bondarenko, Zubkov, '18] :

$$
S_{\text{int.}} \supset \frac{i}{g_s} \int dy \int d^4x \; \delta(x_+) \, \mathrm{tr} \left\{ R_-(y, \mathbf{x}_T) \partial_T^2 \partial_+ \left(W[A_+^{[y,y+\eta]}] - W^\dagger[A_+^{[y,y+\eta]}] \right) \right\}.
$$

Feynman rules

Rg**-transition vertex** ("nonsense polarisation"):

$$L_{Rg} \supset \frac{i}{g_s}\mathrm{tr}\left[R_- \partial_\rho^2 \partial_+ (-2ig_s)\partial_+^{-1} A_+\right] \to \Delta^{ab}_{-\mu}(q) = (-iq^2) n^+_\mu \delta_{ab},$$

Rgg **induced vertex:**

$$\frac{i}{g_s}\mathrm{tr}\left[R_- \partial_\rho^2 \partial_+ (-g_s^2)\left(T^{b_1}T^{b_2} - T^{b_2}T^{b_1}\right)\partial_+^{-1} A_+^{b_1} \partial_+^{-1} A_+^{b_2}\right] = -ig_s \frac{if^{ab_1b_2}}{2} R_-^a \partial_\rho^2 A_+^{b_1} \partial_+^{-1} A_+^{b_2}$$

$$\to \Delta^{ab_1b_2}_{-\mu_1\mu_2}(q, k_1) = g_s (n^+_{\mu_1} n^+_{\mu_2}) \frac{q^2}{2}\left(\frac{f^{ab_1b_2}}{k_2^+ + i\varepsilon} + \frac{f^{ab_2b_1}}{k_1^+ + i\varepsilon}\right) = g_s q^2 (n^+_{\mu_1} n^+_{\mu_2}) \frac{f^{ab_1b_2}}{[k_1^+]},$$

$Rggg$ **and** $Rgggg$ **induced vertices:**

$$\Delta^{ab_1b_2b_3}_{-\mu_1\mu_2\mu_3} = -ig_s^2 q^2 (n^+_{\mu_1} n^+_{\mu_2} n^+_{\mu_3}) \sum_{(i_1,i_2,i_3)\in S_3} \frac{\mathrm{tr}\left[T^a\left(T^{b_{i_1}}T^{b_{i_2}}T^{b_{i_3}} + T^{b_{i_3}}T^{b_{i_2}}T^{b_{i_1}}\right)\right]}{(k^+_{i_3} + i\varepsilon)(k^+_{i_3} + k^+_{i_2} + i\varepsilon)},$$

$$\Delta^{ab_1b_2b_3b_4}_{-\mu_1\mu_2\mu_3\mu_4} = -ig_s^3 q^2 (n^+_{\mu_1} n^+_{\mu_2} n^+_{\mu_3} n^+_{\mu_4})$$

$$\times \sum_{(i_1,i_2,i_3,i_4)\in S_4} \frac{\mathrm{tr}\left[T^a\left(T^{b_{i_1}}T^{b_{i_2}}T^{b_{i_3}}T^{b_{i_4}} - T^{b_{i_4}}T^{b_{i_3}}T^{b_{i_2}}T^{b_{i_1}}\right)\right]}{(k^+_{i_4} + i\varepsilon)(k^+_{i_4} + k^+_{i_3} + i\varepsilon)(k^+_{i_4} + k^+_{i_3} + k^+_{i_2} + i\varepsilon)},$$

and so on... The Hermitian Rg interaction satisfies properties of **signature** and of **sign-**$i\varepsilon$ **independence**, see backup.

Relation with $\ln W$-definition

In [Caron-Huot, '12] an alternative definition of the Reggeized gluon operator had been proposed:

$$S_{Rg} \supset \int d^2\mathbf{x}_T \frac{f^{abc}}{C_A g_s} R_-^a(\mathbf{x}_T) \left\{ \ln \left[W^{\text{adj.}}_{(-\infty_-,+\infty_+,\mathbf{x}_T)}[A_+] \right] \right\}_{bc},$$

where the infinite lightlike adjoint Wilson line is:

$$W^{\text{adj.}}_{(-\infty_-,+\infty_+,\mathbf{x}_T)}[A_+] = 1 +$$

$$\sum_{n=1}^{\infty} (-g_s)^n f^{ba_1c_1} f^{c_1a_2c_2} \ldots f^{c_{n-1}a_nc} \int_{-\infty}^{+\infty} dx_- \partial_+ (\partial_+^{-1} A_+^{a_1} \ldots \partial_+^{-1} A_+^{a_n}).$$

For tree-level $Rg \ldots g$ vertices (i.e. without $i\varepsilon$) all three definitions agree (checked up to $n = 4$, MH has the all-order proof)

Two definitions differ if one takes into account $i\varepsilon$ prescriptions. For $Rggg$ vertex the difference between all three approaches is proportional to:

$$\delta(k_1^+)\delta(k_2^+) \sum_{(i_1,i_2,i_3)\in S_3} \text{tr}\left[T^a T_{i_1} T_{i_2} T_{i_3}\right],$$

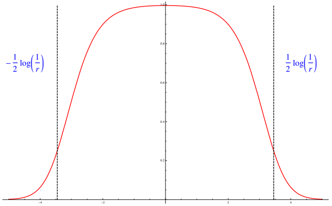
which **does not contribute to 2-loop Regge trajectory** but starts to contribute at 3 loops.

Regularisation by tilted Wilson lines

The *Eikonal propagators* $\partial_{\pm}^{-1} \to -i/(k^{\pm})$ lead to **rapidity divergences**, which are regularized by tilting the Wilson lines from the light-cone [Hentschinski, Sabio Vera, Chachamis *et. al.*, '12-'13; M.N. '19]:

$$n_{\pm}^{\mu} \to \tilde{n}_{\pm}^{\mu} = n_{\pm}^{\mu} + r n_{\mp}^{\mu},\ r \ll 1:\ \tilde{k}^{\pm} = \tilde{n}^{\pm} k.$$

To keep the action Gauge-invariant at finite r one has to substitute $\delta(x_{\pm}) \to \delta(x_{\pm} - r x_{\mp})$ [MN, 2019]
For real emissions this is equivalent to a smooth cutoff in rapidity ($\eta = \ln r$):



The square of regularized Lipatov's $(R_+ R_- g)$ vertex:

$$\Gamma_{+\mu-}\Gamma_{+\nu-}P^{\mu\nu} = \frac{16\mathbf{q}_{T1}^2\mathbf{q}_{T2}^2}{\mathbf{k}_T^2} f(y),$$

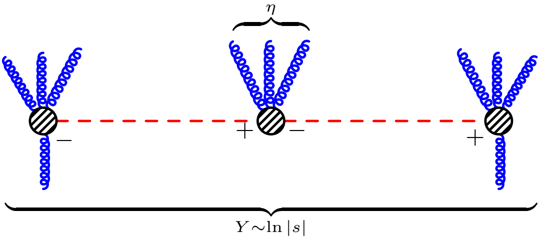
$$\longleftarrow f(y) = \frac{1}{(re^{-y} + e^{y})(re^{y} + e^{-y})},$$

$$\int\limits_{-\infty}^{+\infty} dy\ f(y) = -\ln r + O(r)$$

The pre-RFT action

$$S = \int dy \int d^2\mathbf{x}_T 2\,\mathrm{tr}\left\{R_+(y,\mathbf{x}_T)\partial_T^2 \frac{\partial}{\partial y} R_-(y,\mathbf{x}_T)\right\}$$
$$+\frac{i}{g_s}\int dy \int d^4x\ \ \mathrm{tr}\left\{\delta(x_+)R_-(y-\frac{\eta}{2},\mathbf{x}_T)\partial_T^2\partial_+\left(W[A_+^{[y,y+\eta]}] - W^\dagger[A_+^{[y,y+\eta]}]\right)\right.$$
$$\left. +(+\leftrightarrow -, +\frac{\eta}{2}\to -\frac{\eta}{2})\right\} + \int dy\left(S_{\mathrm{QCD}}\left[A_\mu^{[y,y+\eta]}\right] + S_{\mathrm{RFT}}^{(\mathrm{int.})}\left[R_+(y,\mathbf{x}_T), R_-(y,\mathbf{x}_T)\right]\right),$$

Integrating-out usual gluons (A_μ) and quarks we will obtain the RFT in QCD.



Bare Reggeon propagator:

$$\langle R_-^a R_+^b\rangle = \frac{i\delta^{ab}}{2\mathbf{q}_T^2}\theta(y_1-y_2)$$

Regulator:

$$1 \ll \eta \ll Y.$$

The dependence on the regulator η should cancel between integrations in y and the dependence of vertices on $\eta \Rightarrow$ *Rapidity renormalization group.*

Building the RFT

We construct the RFT interactions:

$$S_{\rm RFT}^{\rm (int.)} = \int\limits_{\mathbf{x}_{1,2,3T}} R_+(y,\mathbf{x}_{1T}) Z_{-+}(\mathbf{x}_{1T},\mathbf{x}_{2T}) \left(\delta_{\mathbf{x}_{2T},\mathbf{x}_{3T}} \frac{\partial}{\partial y} - \omega_g(\mathbf{x}_{2T},\mathbf{x}_{3T})\right) R_+(y,\mathbf{x}_{3T})$$

$$+ \int d^2\mathbf{x}_T d^2\mathbf{x}'_{T1} d^2\mathbf{x}'_{T2}\ R_-(y,\mathbf{x}_T) K_{-++}(\mathbf{x}_T,\mathbf{x}'_{T1},\mathbf{x}'_{T2}) R_+(y,\mathbf{x}'_{T1}) R_+(y,\mathbf{x}'_{T2})$$

$$+ \int d^2\mathbf{x}_{T1} d^2\mathbf{x}_{T2} d^2\mathbf{x}'_T\ R_-(y,\mathbf{x}_{T1}) R_-(y,\mathbf{x}_{T2}) K_{--+}(\mathbf{x}_{T1},\mathbf{x}_{T2},\mathbf{x}'_T) R_+(y,\mathbf{x}'_T)$$

$$+ \int d^2\mathbf{x}_{T1} d^2\mathbf{x}_{T2} d^2\mathbf{x}'_{T1} d^2\mathbf{x}'_{T2}\ R_-(y,\mathbf{x}_{T1}) R_-(y,\mathbf{x}_{T2}) K_{--++} R_+(y,\mathbf{x}'_{T1}) R_+(y,\mathbf{x}'_{T2})$$

$$+ \ldots$$

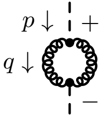
in such a way that the η-dependence cancels.

2-point function

The quadratic part of the RFT action leads to the "Reggeized" propagator:

$$\langle R_-(y_1, \mathbf{q}_T) R_+(y_2, \mathbf{q}_T)\rangle = \frac{iZ_{+-}(\mathbf{q}_T)}{2\mathbf{q}_T^2}\theta(y_1 - y_2) e^{\omega_g(\mathbf{q}_T^2)(y_1-y_2)},$$

while the Reggeon self-energy contains a rapidity-divergent contribution:



$$= g_s^2 C_A \delta_{ab} \int \frac{d^d q}{(2\pi)^D} \frac{\left(\mathbf{p}_T^2 (n_+ n_-)\right)^2}{q^2 (p-q)^2 [q^+][q^-]} \theta\left(\eta - \frac{1}{2}\mathrm{Re}\ln\frac{q_+}{q_-}\right)$$

$$= \eta\omega_g^{(1)}(\mathbf{p}_T^2) \text{ (or } \omega_g^{(1)}(\mathbf{p}_T^2)\ln r \text{ in TWL regularization).}$$

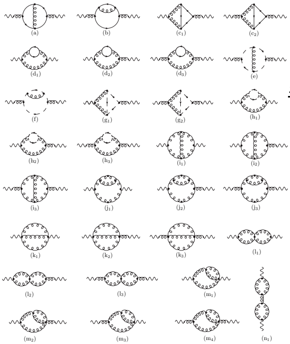
where $\frac{1}{[q_\pm]} = \frac{1}{2}\left(\frac{1}{q_\pm + i\varepsilon} + \frac{1}{q_\pm - i\varepsilon}\right)$ and *one-loop Regge trajectory of a gluon* is:

$$\omega_g^{(1)}(\mathbf{p}_T^2) = C_A g_s^2 \int \frac{\mathbf{p}_T^2 d^{D-2}\mathbf{q}_T}{\mathbf{q}_T^2 (\mathbf{p}_T - \mathbf{q}_T)^2}.$$

The cancellation of η-dependence requires $\omega_g(\mathbf{p}_T^2) = \omega_g^{(1)}(\mathbf{p}_T^2) + O(\alpha_s^2)$.
The Z_{+-} cancels the non-RD part of the self-energy.
For the 3-point and 4-point functions see the backup.

2-loop Regge trajectory from the EFT

The EFT formalism had been tested at 2 loops in [Chachamis, *et al.*, 2013] .



$$\frac{g^4 N_c^2}{(4\pi)^4}\left(\left\{ \frac{2}{\epsilon^2} + \frac{4(1-\Xi)}{\epsilon} + 4(1-\Xi)^2 - \frac{\pi^2}{3} \right\} \ln^2 r + \left\{ \frac{7}{\epsilon^2} - \frac{14\Xi}{\epsilon} - \frac{1-\pi^2}{3\epsilon} - 2\frac{\Xi(\pi^2-1)}{3} + 14(1+\Xi^2) + \frac{2}{9} - \frac{\pi^2}{2} - 2\zeta(3) - i\pi\left[\frac{2}{\epsilon^2} + 4\frac{1-\Xi}{\epsilon} + \frac{1}{3}(12(1-\Xi)^2 - \pi^2)\right] \right\} \ln r \right),$$

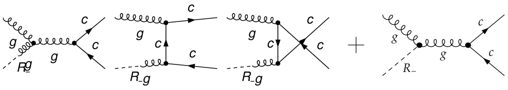
$$\text{where } \Xi = 1 - \gamma_E - \ln\frac{\mathbf{q}_T^2}{4\pi\mu^2}.$$

The coefficient in front of $\ln^2 r$ coincides with $[\omega_g^{(1)}(\mathbf{q}_T^2)]^2/2$ (exponentiation!). After subtracting it, the coefficient in front of $\ln r$ reproduces the QCD result for 2-loop Regge trajectory [Fadin, Fiore, Kotsky '96] .

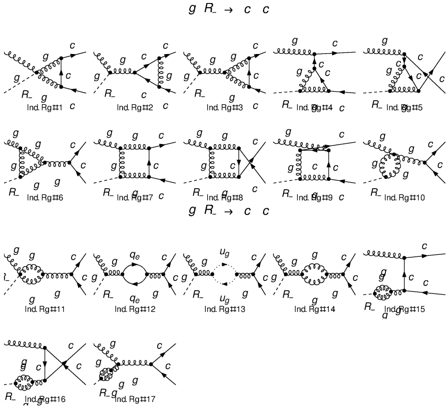
Loop corrections to S-wave impact-factors

$Rg \to c\bar{c}\left[{}^1S_0^{[1]}\right]$ and $c\bar{c}\left[{}^3S_1^{[8]}\right]$ @ 1 loop

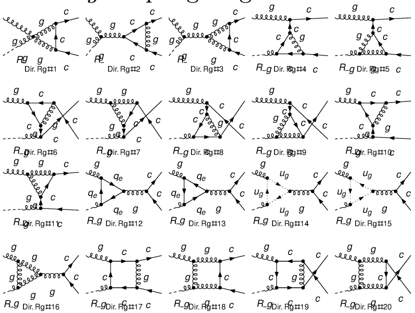
Interference with LO:



Induced Rgg coupling diagrams:



Some Rg-coupling diagrams:



and so on...

- Diagrams had been generated using custom `FeynArts` model-file, projector on the $c\bar{c}\left[{}^1S_0^{[1]}\right]$-state is inserted
- heavy-quark momenta $= p_{\mathcal{Q}}/2 \Rightarrow$ need to resolve linear dependence of quadratic denominators in some diagrams before IBP
- IBP reduction to master integrals has been performed using `FIRE`
- Master integrals with linear and massless quadratic denominators are expanded in $r \ll 1$ using Mellin-Barnes representation. The differential equations technique is used when the integral depends on more than one scale of virtuality.
- In presence of the linear denominator the massive propagator can be converted to the massless one:

$$\frac{1}{((\tilde{n}_+l)+k_+)(l^2-m^2)} = \frac{1}{((\tilde{n}_+l)+k_+)(l+\kappa\tilde{n}_+)^2} + \frac{2\kappa\cancel{\left[(\tilde{n}_+l)+\frac{m^2+\tilde{n}_+^2\kappa^2}{2\kappa}\right]}}{\cancel{((\tilde{n}_+l)+k_+)}(l+\kappa\tilde{n}_+)^2(l^2-m^2)}$$

$\Rightarrow$ all the masses can be moved to integrals with **only quadratic propagators**.

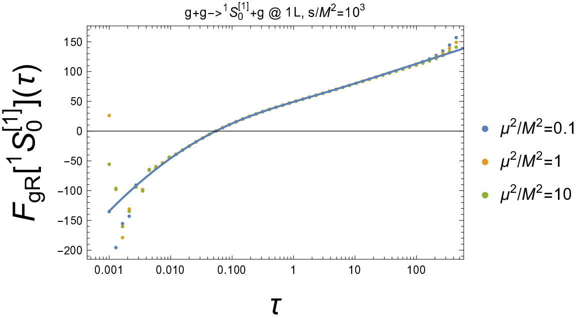
See [hep-ph/2408.06234] for details.

Result: $Rg \to c\bar{c}\left[{}^1S_0^{[1]}\right]$ @ 1 loop

Result [MN, 2024] for $2\Re\left[\frac{V_{\text{1L x LO}}(\mathbf{q}_T) - (\text{On-shell mass CT})}{(\alpha_s/(2\pi))V_{\text{LO}}(\mathbf{q}_T)}\right]$:

$$\left(\frac{\mu^2}{\mathbf{q}_T^2}\right)^{\epsilon}\left[-\frac{C_A}{\epsilon^2} + \frac{1}{\epsilon}\left(C_A \ln\frac{p_+^2}{r\mathbf{q}_T^2} + \beta_0 + 3C_F - C_A\right)\right] - \frac{10}{9}n_F + F_{{}^1S_0^{[1]}}(\mathbf{q}_T^2/M^2)$$

Cross-check against the Regge limit of one-loop amplitude ($\tau = \mathbf{q}_T^2/M^2$):



Points – the function $F_{{}^1S_0^{[1]}}(\tau)$ extracted form numerical results for interference between **exact** one-loop and tree-level QCD amplitudes of $g + g \to c\bar{c}[{}^1S_0^{[1]}] + g$ at $s = 10^3 M^2$. Solid line – analytic result from the EFT.

$$F_{{}^1S_0^{[1]}}(\mathbf{q}_T^2/M^2) = C_F C[gR \to {}^1S_0^{[1]}, C_F] + C_A C[gR \to {}^1S_0^{[1]}, C_A].$$

$$\begin{aligned}
C[gR \to {}^1S_0^{[1]}, C_F] = \frac{1}{6(\tau+1)^2}\Bigg\{ & -12\tau(\tau+1)\mathrm{Li}_2(-2\tau-1) + \frac{6L_2}{\tau}(-2L_1\tau + L_1 + 6\tau(\tau+1)) \\
& + \frac{1}{(2\tau+1)^2}\Bigg[(\tau+1)12\ln(2)(\tau+1)\left(6\tau^2+8\tau+3\right) - 8\tau^3\left(9\ln(\tau+1)+2\pi^2+15\right) \\
& -4\tau^2\left(30\ln(\tau+1)+\pi^2+63\right) + 8\tau\left(-6\ln(\tau+1)+\pi^2-21\right) \\
& +18(\tau+1)(2\tau+1)^2\ln(\tau) + 3\pi^2 - 36\Bigg]\Bigg\},
\end{aligned}$$

$$\begin{aligned}
C[gR \to {}^1S_0^{[1]}, C_A] = & \frac{2(\tau(\tau(\tau(7\tau+8)+2)-4)-1)}{(\tau-1)(\tau+1)^3}\mathrm{Li}_2(-\tau) - \frac{\tau(\tau(4\tau+5)+3)}{(\tau+1)^3}\mathrm{Li}_2(-2\tau-1) \\
& -\frac{L_2^2}{2\tau(\tau+1)^2} + \frac{1}{18(\tau-1)(\tau+1)^3}\Bigg\{-2\left(\tau^2-1\right)(18\ln(2)(\tau-1)\tau - 67(\tau+2)\tau - 67) \\
& +18[\ln(\tau)\left(-2\tau^4 + \left(\tau\left(-\tau^3+\tau+3\right)+2\right)\tau\ln(\tau) + 2\tau^2 + \ln(\tau)\right) \\
& -(\tau-1)^2(\tau+1)^3\ln^2(\tau+1) + 2(\tau-1)(\tau+1)^2\left(\tau + (\tau+1)^2\ln(\tau)\right)\ln(\tau+1)] \\
& +\pi^2(3\tau(\tau(\tau(15\tau+14)-3)-12)-6)\Bigg\},
\end{aligned}$$

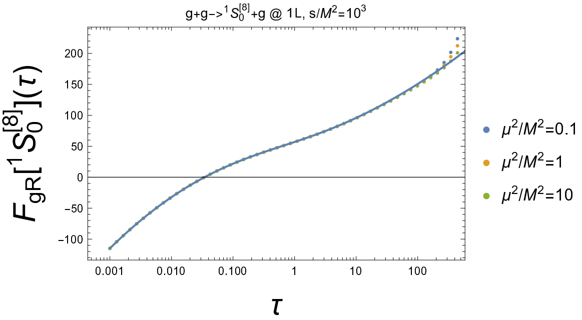
where $L_1 = L_1^{(+)} - L_1^{(-)} - L_2/2$ with $L_1^{(\pm)} = \sqrt{\tau(1+\tau)}\ln\left(\sqrt{1+\tau} \pm \sqrt{\tau}\right)$ and $L_2 = \sqrt{\tau(1+\tau)}\ln\left(1 + 2\tau + 2\sqrt{\tau(1+\tau)}\right)$.

Result: $Rg \to c\bar{c}\left[{}^1S_0^{[8]}\right]$ @ 1 loop

Result [MN, 2024] for $2\Re\left[\frac{V_{\mathbf{1L\ x\ LO}}(\mathbf{q}_T) - \mathbf{(On\text{-}shell\ mass\ CT)}}{(\alpha_s/(2\pi))V_{\rm LO}(\mathbf{q}_T)}\right]$:

$$\left(\frac{\mu^2}{\mathbf{q}_T^2}\right)^\epsilon \left[-\frac{C_A}{\epsilon^2} + \frac{1}{\epsilon}\left(C_A \ln\frac{p_+^2}{r\mathbf{q}_T^2} + C_A \ln\left(1 + \frac{\mathbf{q}_T^2}{M^2}\right) + \beta_0 + 3C_F - 2C_A\right)\right] - \frac{10}{9}n_F + F_{{}^1S_0^{[8]}}(\mathbf{q}_T^2/M^2)$$

Cross-check against the Regge limit of one-loop amplitude ($\tau = \mathbf{q}_T^2/M^2$):

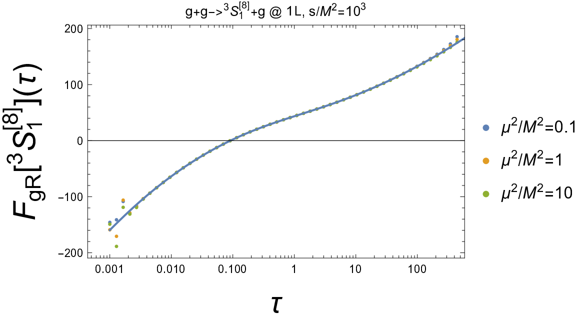


Result: $Rg \to c\bar{c}\left[{}^3S_1^{[8]}\right]$ @ 1 loop

Result [MN, 2024] for $2\Re\left[\frac{V_{\mathbf{1L\ x\ LO}}(\mathbf{q}_T) - \mathbf{(On\text{-}shell\ mass\ CT)}}{(\alpha_s/(2\pi))V_{\mathrm{LO}}(\mathbf{q}_T)}\right]$:

$$\left(\frac{\mu^2}{\mathbf{q}_T^2}\right)^{\epsilon}\left[-\frac{C_A}{\epsilon^2} + \frac{1}{\epsilon}\left(C_A \ln\frac{p_+^2}{r\mathbf{q}_T^2} + C_A \ln\left(1 + \frac{\mathbf{q}_T^2}{M^2}\right) + \beta_0 + 3C_F - 2C_A\right)\right]$$
$$-\frac{10}{9}n_F + F_{{}^3S_1^{[8]}}(\mathbf{q}_T^2/M^2)$$

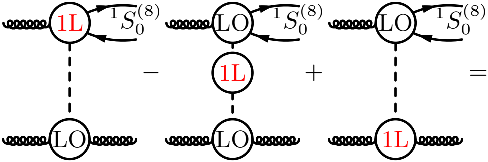
Cross-check against the Regge limit of one-loop amplitude ($\tau = \mathbf{q}_T^2/M^2$):



Relation to the QCD 1-loop amplitude

In the following combination of EFT results, the $\ln r$-divergence cancels:

$\mathcal{M}_{\text{EFT}} =$



$= \mathcal{M}^{(\mathbf{8}_A,-)}_{\text{QCD}} + O(s/t).$

In BFKL approach the one-Reggeon-exchange part of $\mathcal{M}^{(\mathbf{8}_A,-)}_{\text{QCD}}$ is expressed as:

$$\mathcal{M}^{(\mathbf{8}_A,-)}_{\text{QCD}} = \frac{s}{t}\Gamma_{gR\mathcal{Q}}\left[\left(\frac{s}{s_0}\right)^{\omega_g(-t)} + \left(\frac{-s}{s_0}\right)^{\omega_g(-t)}\right]\Gamma_{gRg},$$

the Γ_{gRg} and $\omega_g(-t)$ are known up to NLO in α_s, which allows one to extract $\Gamma_{gR\mathcal{Q}}$ up to NLO. The explicit formula for the one-loop correction to the IF is:

$$\Gamma_{gR\mathcal{Q}} = \text{Re}\, C[\ldots + R \to \ldots] - \frac{1}{2}\Pi^{(1)}_{\text{non-RD}}(-t) + \frac{\bar{\alpha}_s C_A}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left[\ln r - 2\ln\frac{p_+}{\sqrt{s_0}}\right],$$

where $\Pi^{(1)}_{\text{non-RD}}(-t) = \frac{\bar{\alpha}_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left[\frac{1}{\epsilon}(\beta_0 - 2C_A) - \frac{8}{3}C_A + \frac{5}{3}\beta_0 + O(\epsilon)\right]$.

Real-emission corrections to S-wave quarkonium impact-factors

Real-emission corrections

We will extract the IR and collinear divergences from the real-emission part, using a subtractive approach:

$$V_i^{(\text{NLO, finite})}(\mathbf{q}_T,\mathbf{p}_T,z)=\frac{\alpha_s(\mu_R)}{2\pi}\frac{z|\mathbf{p}_T|}{\pi}h_g^{(\mathcal{Q},\text{LO})}(\mathbf{p}_T^2)$$
$$\times\left[\frac{\tilde{H}_{Ri}^{(\mathcal{Q})}(\mathbf{q}_T,\mathbf{p}_T,z)}{z(1-z)\mathbf{q}_T^2}-\mathcal{J}_{Ri}^{(\text{sub.}),\mathcal{Q}}(\mathbf{q}_T,\mathbf{p}_T,z,r=0)\right],$$

where $\mathcal{Q}={}^1S_0^{[1]}$, ${}^1S_0^{[8]}$ or ${}^3S_1^{[8]}$. The **subtraction term** $\mathcal{J}_{Ri}^{(\text{sub.}),\mathcal{Q}}$ should remove the non-integrable singularities from the "exact matrix element" $\tilde{H}_{Ri}^{(\mathcal{Q})}(\mathbf{q}_T,\mathbf{p}_T,z)$. The latter is given by all Feynman diagrams for the process:

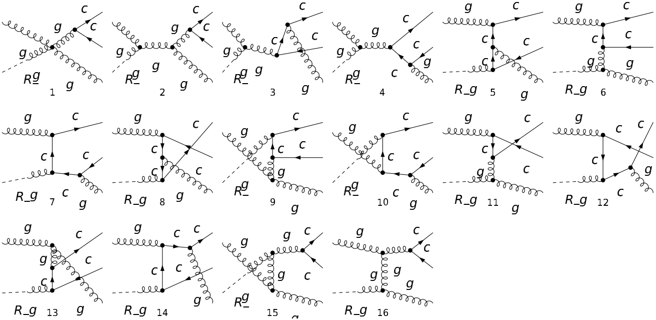
$$g+R_-\to Q\bar{Q}[\mathcal{Q}]+g,$$

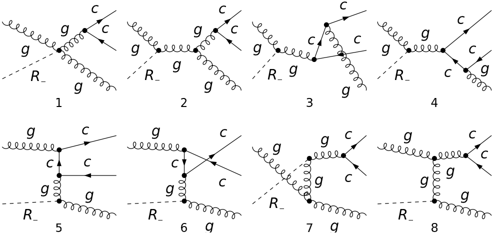
as, e.g. for ${}^1S_0^{[1]}$:

$$\tilde{H}_{Rg}=\frac{C_A\left(M^2+\mathbf{p}_T^2\right)^2}{s^2t^2u^2\left(M^2-s\right)^2\left(M^2-t+t_1\right)^2\left(M^2-u\right)^2}\sum_{n=-1}^{3}(1-z)^n w_n(s,t,u),$$

where $t_1=\mathbf{q}_T^2$ and s, t and u are expressed through $\mathbf{p}_T$, $\mathbf{q}_T$, M and z.

Diagrams for $\tilde{H}_{Rg}$





Subtraction terms

Subtraction term for colour-singlet $Q\bar{Q}$-state:

$$\mathcal{J}_{Rg}^{(\text{sub.}),\mathbf{1}} = \frac{2C_A}{\mathbf{k}_T^2}\left[\frac{1-z}{(1-z)^2 + r\frac{\mathbf{k}_T^2}{p_+^2}} + z(1-z)\right.$$

$$\left. + \frac{\mathbf{k}_T^2\mathbf{p}_T^2 - (\mathbf{k}_T\mathbf{p}_T)^2}{\mathbf{k}_T^2\mathbf{p}_T^2}\left(\frac{2}{z} + \frac{2\epsilon(1-z)}{z} + O(\epsilon^2)\right) - \frac{3\mathbf{k}_T^2\mathbf{p}_T^2 - 2(\mathbf{k}_T\mathbf{p}_T)^2}{\mathbf{k}_T^2\mathbf{p}_T^2}\right],$$

where $\mathbf{k}_T = \mathbf{q}_T - \mathbf{p}_T$, and for the colour-octet final state:

$$\mathcal{J}_{Rg}^{(\text{sub.}),\mathbf{8}} = \mathcal{J}_{Rg}^{(\text{sub.}),\mathbf{1}} + \frac{2C_A}{z\mathbf{k}_T^2}\frac{(\mathbf{k}_T\mathbf{p}_T)[(\mathbf{k}_T-(1-z)\mathbf{p}_T)^2 + (1-z)^2M^2] - (1-z)M^2\mathbf{k}_T^2}{[(\mathbf{k}_T-(1-z)\mathbf{p}_T)^2 + (1-z)^2M^2]^2}.$$

They reproduce the non-integrable singularities of the exact ME in the limits:

- Initial-state collinear limit $\mathbf{k}_T \to 0$,
- Regge limit: $z \to 1$, $\mathbf{k}_T$-finite
- Soft limit: $1-z \sim \lambda$, $\mathbf{k}_T \sim \lambda\mathbf{p}_T$, $\lambda \to 0$.

Final-state collinear singularity is regulated by the mass of a heavy quark $m_Q = M/2$. The new **final-state soft divergence** appears in the octet case, due to emission of a soft gluon from $(Q\bar{Q})_{\mathbf{8}}$.

Isolating the divergences

Expand the r-dependence in a distributional sense:

$$\frac{1-z}{(1-z)^2 + r\frac{\mathbf{k}_T^2}{p_+^2}} = \frac{1}{(1-z)_+} - \delta(1-z)\frac{1}{2}\ln\frac{r\mathbf{k}_T^2}{p_+^2} + O(r).$$

And do another add-subtract trick:

$$V_i^{(\text{NLO, sub.})}(\mathbf{q}_T, \mathbf{p}_T, z) = \frac{\alpha_s(\mu_R)}{2\pi}\underbrace{\frac{z|\mathbf{p}_T|}{\pi}h_g^{(\mathcal{Q},\text{LO})}(\mathbf{p}_T^2)}_{\tilde{V}_{\text{LO}}}\mathcal{J}_{Ri}^{(\text{sub.}),\mathcal{Q}}(\mathbf{q}_T, \mathbf{p}_T, z, r)$$

$$= \frac{\alpha_s(\mu_R)}{2\pi}\tilde{V}_{\text{LO}}\int d^{2-2\epsilon}\mathbf{k}_T\mathcal{J}_{Ri}^{(\text{sub.}),\mathcal{Q}}(\mathbf{q}_T, \mathbf{q}_T-\mathbf{k}_T, z, r)\delta^{(2-2\epsilon)}(\mathbf{q}_T-\mathbf{p}_T-\mathbf{k}_T)$$

$$\left.\begin{aligned} &= \tfrac{\alpha_s(\mu_R)}{2\pi}\tilde{V}_{\text{LO}}\int d^{2-2\epsilon}\mathbf{k}_T\mathcal{J}_{Ri}^{(\text{sub.}),\mathcal{Q}}(\mathbf{q}_T, \mathbf{q}_T-\mathbf{k}_T, z, r) \\ &\times\left[\delta^{(2-2\epsilon)}(\mathbf{q}_T-\mathbf{p}_T-\mathbf{k}_T) - \tfrac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2}\delta^{(2-2\epsilon)}(\mathbf{q}_T-\mathbf{p}_T)\right]\end{aligned}\right\} V_i^{(\text{NLO, sub. 1})}$$

$$+\frac{\alpha_s(\mu_R)}{2\pi}\tilde{V}_{\text{LO}}\delta^{(2-2\epsilon)}(\mathbf{p}_T-\mathbf{q}_T)$$

$$\times\int d^{2-2\epsilon}\mathbf{k}_T\mathcal{J}_{Ri}^{(\text{sub.}),\mathcal{Q}}(\mathbf{q}_T, \mathbf{q}_T-\mathbf{k}_T, z, r)\frac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2} \to V_i^{(\text{NLO, sub. 2})}$$

The square-bracket makes $\mathbf{k}_T$-integration safe:

$$\int d^2\mathbf{q}_T G(\mathbf{q}_T)\left[\delta(\mathbf{q}_T-\mathbf{p}_T-\mathbf{k}_T) - \frac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2}\delta(\mathbf{q}_T-\mathbf{p}_T)\right] = G(\mathbf{p}_T+\mathbf{k}_T) - \frac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2}G(\mathbf{p}_T).$$

Isolating the divergences

The IR/collinear-finite piece for the colour-singlet $Q\bar{Q}$

$$V_g^{(\text{NLO sub. 1}),\mathbf{1}}(\mathbf{q}_T,\mathbf{p}_T,z) = \frac{\alpha_s(\mu_R)C_A}{\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)$$

$$\times \int \frac{d^2\mathbf{k}_T}{\pi\mathbf{k}_T^2}\left[\delta^{(2)}(\mathbf{q}_T-\mathbf{k}_T-\mathbf{p}_T) - \frac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2}\delta^{(2)}(\mathbf{q}_T-\mathbf{p}_T)\right]\left[\frac{1}{(1-z)_+} + z(1-z)\right.$$

$$\left. +2\frac{\mathbf{k}_T^2\mathbf{p}_T^2-(\mathbf{k}_T\mathbf{p}_T)^2}{z\mathbf{k}_T^2\mathbf{p}_T^2} - \frac{3\mathbf{k}_T^2\mathbf{p}_T^2-2(\mathbf{k}_T\mathbf{p}_T)^2}{\mathbf{k}_T^2\mathbf{p}_T^2} - \delta(1-z)\frac{1}{2}\ln\frac{r\mathbf{k}_T^2}{p_+^2}\right],$$

and for colour-octet $Q\bar{Q}$:

$$V_g^{(\text{NLO sub. 1}),\mathbf{8}} = V_g^{(\text{NLO sub. 1}),\mathbf{1}} + \frac{\alpha_s(\mu_R)C_A}{\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)$$

$$\times \int \frac{d^2\mathbf{k}_T}{\pi\mathbf{k}_T^2}\left[\delta^{(2)}(\mathbf{q}_T-\mathbf{k}_T-\mathbf{p}_T) - \frac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2}\delta^{(2)}(\mathbf{q}_T-\mathbf{p}_T)\right]$$

$$\times\left[\frac{2C_A}{z\mathbf{k}_T^2}\frac{(\mathbf{k}_T\mathbf{p}_T)[(\mathbf{k}_T-(1-z)\mathbf{p}_T)^2+(1-z)^2M^2]-(1-z)M^2\mathbf{k}_T^2}{[(\mathbf{k}_T-(1-z)\mathbf{p}_T)^2+(1-z)^2M^2]^2}\right].$$

The IR/collinear divergent part

For the colour-singlet $Q\bar{Q}$:

$$V_g^{(\text{NLO sub. 2}),\mathbf{1}}(\mathbf{q}_T, z, \mathbf{p}_T) = \frac{\bar{\alpha}_s(\mu)}{2\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)\delta^{(2-2\epsilon)}(\mathbf{q}_T - \mathbf{p}_T)$$

$$\times \left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \left\{ -\frac{1}{\epsilon} P_{gg}(z) + \delta(1-z)\left[\frac{C_A}{\epsilon^2} + \frac{\beta_0}{2}\frac{1}{\epsilon} + \frac{C_A}{\epsilon}\ln\frac{r\mathbf{p}_T^2}{q_+^2} - \frac{\pi^2}{6}C_A\right] + O(\epsilon^2)\right\},$$

and colour-octet $Q\bar{Q}$:

$$V_g^{(\text{NLO sub. 2}),\mathbf{8}} = V_g^{(\text{NLO sub. 2}),\mathbf{1}} + V_g^{(\text{NLO sub. 2}),\mathbf{8}\text{-fin.}}$$

$$+z|\mathbf{p}_T| h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)\delta^{(2-2\epsilon)}(\mathbf{q}_T - \mathbf{p}_T)\frac{\bar{\alpha}_s(\mu_R)}{2\pi}\left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon}$$

$$\times\left\{\delta(1-z)\left[\frac{C_A}{\epsilon}\left(1 - \ln\frac{M^2+\mathbf{p}_T^2}{M^2}\right) + C_A\left(\text{Li}_2\left(-\frac{\mathbf{p}_T^2}{M^2}\right) - \ln\frac{M^2+\mathbf{p}_T^2}{\mathbf{p}_T^2}\ln\frac{M^2+\mathbf{p}_T^2}{M^2}\right)\right]\right.$$

$$\left. -\frac{2C_A}{z(1-z)_+}\left[1 - \ln\frac{M^2+\mathbf{p}_T^2}{M^2}\right] + O(\epsilon)\right\}.$$

$$V_g^{(\text{NLO sub. 2}),\mathbf{8}\text{-fin.}} = -\frac{\alpha_s(\mu_R)}{2\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)\delta^{(2)}(\mathbf{q}_T - \mathbf{p}_T)$$

$$\times\left[\frac{2C_A}{z(1-z)}\right]\int\frac{d^2\mathbf{k}_T}{\pi}\frac{(\boldsymbol{k}_T\cdot\boldsymbol{p}_T)[(\boldsymbol{k}_T-\boldsymbol{p}_T)^2+M^2] - M^2\boldsymbol{k}_T^2}{[\mathbf{k}_T^2+\mathbf{p}_T^2/(1-z)^2][(\boldsymbol{k}_T-\boldsymbol{p}_T)^2+M^2]^2}$$

The $V_g^{(\text{NLO sub. 2}),\mathbf{8}\text{-fin.}}$-integral is **finite and smooth at** $z \to 1$.

Transition to BFKL scheme

When $y_g \ll y_{Q\bar{Q}}$ $(z \to 1)$, the real-emission part of the IF simplifies as:

$$V_g^{(\text{NLO, Regge})}(\mathbf{q}_T, z, \mathbf{p}_T) = \frac{\alpha_s(\mu_R)}{\pi} \frac{z|\mathbf{p}_T|}{(2\pi)^{1-2\epsilon}} h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)\mathcal{J}_{RR}(k),$$

where the squared Lipatov's vertex is:

$$\mathcal{J}_{RR}(k) = \frac{1}{1-z}\frac{2C_A}{k^+k^-}.$$

In the TWL approach the pole $1/k^+$ is regularised as $1/(k^+ + rk^-)$, while in the BFKL approach, the rapidity of a gluon is restricted as $y_g > y_0$ with $y_0 = \ln \frac{p^+}{\sqrt{s_0}}$. The scheme-transition term is just a difference of two regularisation prescriptions:

$$\mathcal{J}_{RR}^{(\text{BFKL})}(k) - \mathcal{J}_{RR}^{(\text{TWL})}(k) = \frac{-2C_A(1-z)}{\mathbf{k}_T^2\left[(1-z)^2 + r\frac{\mathbf{k}_T^2}{p_+^2}\right]}\theta\left(1 - z < \frac{|\mathbf{k}_T|}{\sqrt{s_0}}\right).$$

We should do the same transformations with this expression as in the previous slide.

Transition to BFKL scheme, UV-renormalization

Putting the result of the previous slide together with the scheme-transformation terms for the virtual correction, we get:

$$V_g^{(\text{BFKL sch.})} = \frac{\bar{\alpha}_s(\mu)}{2\pi}|\mathbf{p}_T| h_g^{(\mathcal{Q},\text{ LO})}(\mathbf{p}_T^2)\delta(1-z)$$
$$\times\left\{ \left(\frac{\mu^2}{\mathbf{p}_T^2}\right)^{\epsilon} \left[-\frac{1}{2\epsilon}\left(\beta_0 - 2C_A\right) + \frac{4}{3}C_A - \frac{5}{6}\beta_0 \right] \delta^{(2-2\epsilon)}(\mathbf{q}_T - \mathbf{p}_T) \right.$$
$$\left. +2C_A \int \frac{d^2\mathbf{k}_T}{\pi \mathbf{k}_T^2}\left[\delta^{(2)}(\mathbf{q}_T - \mathbf{k}_T - \mathbf{p}_T) - \frac{\mathbf{p}_T^2}{\mathbf{k}_T^2 + \mathbf{p}_T^2}\delta^{(2)}(\mathbf{q}_T - \mathbf{p}_T)\right]\left[\frac{1}{2}\ln\frac{r\mathbf{k}_T^2}{p_+^2} + \ln\frac{\sqrt{s_0}}{|\mathbf{k}_T|}\right]\right\}.$$

The UV counter-terms to include:

- The $\overline{MS}$ renormalisation of α_s ($h_g^{(\mathcal{Q},\text{ LO})} \sim \alpha_s^2$):

$$2\left(-\frac{\beta_0}{2\epsilon}\right),$$

- Wave function renormalisation for 2 external heavy quarks in the on-shell scheme:

$$2\left[-\frac{3C_F}{2\epsilon} - C_F\left(2 + \frac{3}{2}\ln\frac{\mu^2}{m_Q^2}\right)\right].$$

Result, ${}^1S_0^{[1]}$-state

The full NLO correction to IF consists of the analytic and numerical pieces:

$$V_g^{(\mathrm{NLO},\ {}^1S_0^{[1]},\ \mathrm{analyt.})} = V_g^{(\mathrm{NLO},\ {}^1S_0^{[1]},\ \mathrm{finite})} + V_g^{(\mathrm{NLO},\ {}^1S_0^{[1]},\ \mathrm{analyt.})}.$$

Where the explicit analytic piece is:

$$\begin{aligned}
V_g^{(\mathrm{NLO},\ {}^1S_0^{[1]},\ \mathrm{analyt.})}(\mathbf{q}_T,\mathbf{p}_T,z) &= \frac{\alpha_s(\mu_R)C_A}{\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\mathrm{LO})}(\mathbf{p}_T^2) \\
&\times \int \frac{d^2\mathbf{k}_T}{\pi \mathbf{k}_T^2}\left[\delta^{(2)}(\mathbf{q}_T-\mathbf{k}_T-\mathbf{p}_T) - \frac{\mathbf{p}_T^2}{\mathbf{p}_T^2+\mathbf{k}_T^2}\delta^{(2)}(\mathbf{q}_T-\mathbf{p}_T)\right] \\
\times \Bigg[\frac{1}{(1-z)_+} + z(1-z) &+ 2\frac{\mathbf{k}_T^2\mathbf{p}_T^2-(\mathbf{k}_T\mathbf{p}_T)^2}{z\mathbf{k}_T^2\mathbf{p}_T^2} - \frac{3\mathbf{k}_T^2\mathbf{p}_T^2-2(\mathbf{k}_T\mathbf{p}_T)^2}{\mathbf{k}_T^2\mathbf{p}_T^2} + \delta(1-z)\ln\left(\frac{\sqrt{s_0}}{|\mathbf{k}_T|}\right)\Bigg] \\
&+ \frac{\alpha_s(\mu_R)}{2\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\mathrm{LO})}(\mathbf{p}_T^2)\delta^{(2)}(\mathbf{q}_T-\mathbf{p}_T)\left\{-\ln\frac{\mu_F^2}{\mathbf{p}_T^2}P_{gg}(z)\right. \\
\left. +\delta(1-z)\left[-\frac{\pi^2}{6}C_A + \frac{4}{3}C_A - \frac{5}{6}\beta_0 - 2C_F\left(2+\frac{3}{2}\ln\frac{\mathbf{p}_T^2}{m_c^2}\right) \right.\right. &\left.\left. + \beta_0\ln\frac{\mu_R^2}{\mathbf{p}_T^2} + F_{{}^1S_0^{[1]}}(\mathbf{p}_T^2/M^2)\right]\right\}.
\end{aligned}$$

Result, colour-octet $Q\bar{Q}$

For the CO states (${}^1S_0^{[8]}$ and ${}^3S_1^{[8]}$) one has to replace: $V_g^{(\text{NLO},\ {}^1S_0^{[1]},\ \text{finite})}$ and $F_{{}^1S_0^{[1]}}$ by the functions for the corresponding states and add new terms:

$$\Delta V_g^{(\text{NLO},\ \mathbf{8})}(\mathbf{q}_T, \mathbf{p}_T, z) = \frac{\alpha_s(\mu_R)}{\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\ \text{LO})}(\mathbf{p}_T^2)$$
$$\times \int \frac{d^2\mathbf{k}_T}{\pi \mathbf{k}_T^2} \left[\delta^{(2)}(\mathbf{q}_T - \mathbf{k}_T - \mathbf{p}_T) - \frac{\mathbf{p}_T^2}{\mathbf{p}_T^2 + \mathbf{k}_T^2} \delta^{(2)}(\mathbf{q}_T - \mathbf{p}_T) \right]$$
$$\times \left[\frac{2C_A}{z\mathbf{k}_T^2} \frac{(\mathbf{k}_T \mathbf{p}_T)[(\mathbf{k}_T - (1-z)\mathbf{p}_T)^2 + (1-z)^2 M^2] - (1-z) M^2 \mathbf{k}_T^2}{[(\mathbf{k}_T - (1-z)\mathbf{p}_T)^2 + (1-z)^2 M^2]^2} \right]$$
$$+ V_g^{(\text{NLO sub. 2}),\mathbf{8}\text{-fin.}} + z|\mathbf{p}_T| h_g^{(\mathcal{Q},\ \text{LO})}(\mathbf{p}_T^2) \delta^{(2)}(\mathbf{q}_T - \mathbf{p}_T) \frac{\alpha_s(\mu_R)}{2\pi}$$
$$\times \left\{ C_A \delta(1-z) \left[\text{Li}_2\left(-\frac{\mathbf{p}_T^2}{M^2} \right) - \ln \frac{M^2 + \mathbf{p}_T^2}{\mathbf{p}_T^2} \ln \frac{M^2 + \mathbf{p}_T^2}{M^2} \right] - \frac{2C_A}{z(1-z)_+} \left[1 - \ln \frac{M^2 + \mathbf{p}_T^2}{M^2} \right] \right\}.$$

$$V_g^{(\text{NLO sub. 2}),\mathbf{8}\text{-fin.}} = -\frac{\alpha_s(\mu_R)}{2\pi} z|\mathbf{p}_T| h_g^{(\mathcal{Q},\ \text{LO})}(\mathbf{p}_T^2) \delta^{(2)}(\mathbf{q}_T - \mathbf{p}_T)$$
$$\times \left[\frac{2C_A}{z(1-z)} \right] \int \frac{d^2\mathbf{k}_T}{\pi} \frac{(\boldsymbol{k}_T \cdot \boldsymbol{p}_T)[(\boldsymbol{k}_T - \boldsymbol{p}_T)^2 + M^2] - M^2 \boldsymbol{k}_T^2}{[\mathbf{k}_T^2 + \mathbf{p}_T^2/(1-z)^2][(\boldsymbol{k}_T - \boldsymbol{p}_T)^2 + M^2]^2}.$$

Conclusions and outlook

- The complete NLO impact factors in **(symmetric) BFKL scheme** for the $g + R \to Q\bar{Q}[{}^1S_0^{[1]}, {}^1S_0^{[8]}, {}^3S_1^{[8]}]$ processes are computed, including one-loop and real-emission corrections. The results in **asymmetric BFKL schemes** are also known.
- The computation for other NRQCD-factorisation intermediate states: $Q\bar{Q}[{}^3P_J^{[1,8]}]$ are in progress. The $Q\bar{Q}[{}^3S_1^{[1]}]$ at NLO is more challenging.
- The result in the "shockwave" scheme, corresponding to the cut in "projectile" light-cone component (k^+) is easy to obtain. However this is $1R$-exchange only.
- The same computaion technology can be applied to the central production vertices $RR \to Q\bar{Q}[n]$.

Thank you for your attention!

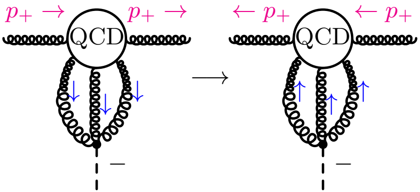
Signature of induced vertices

In the LLA the Reggeized gluon has negative signature w.r.t. $s \to -s$:

$$\mathcal{M}_{1R}(gg \to gg) \propto f^{a_1 c a_2} f_{a_3 c a_4} \frac{s}{t} \left[\left(\frac{s}{-t} \right)^{\omega_g^{(1)}(-t)} + \left(\frac{-s}{-t} \right)^{\omega_g^{(1)}(-t)} \right] \delta_{\lambda_1 \lambda_2} \delta_{\lambda_3 \lambda_4},$$

we want to keep this property to all orders in the EFT.

Signature $p_+ \to -p_+$:



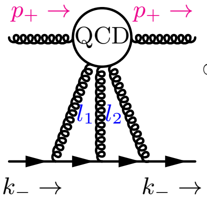
Simple graph-theoretic arguments show that the signature of $Rg \dots g$ vertex with n-gluons ($O(g_s^{n-1})$) is $(-1)^{n-1}$.
This property should be respected by $i\varepsilon$-prescriptions for Eikonal poles.

Signature of QCD part is $(-1)^{(\#\text{ 3g vert.})}$

The vertices from Hermitian version of the EFT satisfy the signature property.

The $\mathrm{sgn}(\varepsilon)$ independence

The induced vertices come from QCD diagrams like:



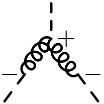
$$\propto \sum_{\text{perms.}} \frac{T^{a_1} \dots T^{a_n}}{(l_1^+ + i\varepsilon/k_-)(l_1^+ + l_2^+ + i\varepsilon/k_-) \dots},$$

factorisation requires **independence** on the sign of $k_- \leftrightarrow$ sign of ε.
This property is automatically satisfied by the EFT vertices:

$$S_{Rg} \supset \frac{i}{g_s} \int d^2\mathbf{x}_T \int dx_- \mathrm{tr}\left\{ R_-(\mathbf{x}_T) \partial_T^2 \frac{\partial}{\partial x_-} \left[W_{(-\infty_-, x_-)}[A_+] - W^\dagger_{(-\infty_-, x_-)}[A_+] \right] \right\}$$
$$= \frac{i}{g_s} \int d^2\mathbf{x}_T \mathrm{tr}\left\{ R_-(\mathbf{x}_T) \partial_T^2 \left[W_{(-\infty_-, +\infty_-)}[A_+] - W_{(+\infty_-, -\infty_-)}[A_+] \right] \right\}.$$

Additionally in [Hentschinski, '11] the *maximal anti-symmetry* of the colour factor in the induced vertices had been imposed. The physical motivation for this choice is less clear for me.

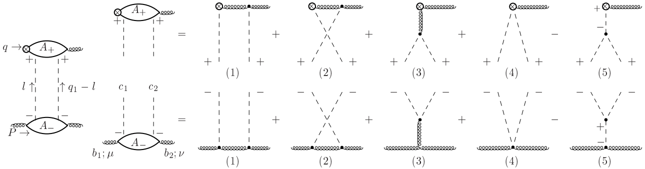
3-point function



$$= 4g_s f^{abc}\mathbf{p}_T^2 \int \frac{dk^-}{[k^-]}.$$

Two interpretations:

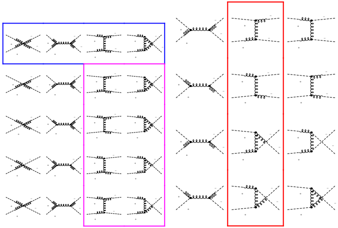
- Put integral $\int \frac{dk^-}{[k^-]} = 0 \Rightarrow$ all **even-odd** transitions are forbidden (*Gribov's signature conservation rule*).
- Put this vertex into RFT with the opposite sign, as the subtraction term for ill-defined light-cone momentum integrals [Hentchinski PhD thesis; M.N. 2019] .



$$f^{abc}\int_{-\infty}^{+\infty} dl_- \left\{ \frac{1}{l_- + \mathbf{l}_T^2/P_+ - i\varepsilon} + \frac{1}{l_- + (2\mathbf{l}_T\mathbf{q}_{T1} - \mathbf{l}_T^2)/P_+ + i\varepsilon} - \frac{1}{l_- - i\varepsilon} - \frac{1}{l_- + i\varepsilon} \right\} = 0.$$

4-point function, BFKL equation

Connected diagrams:



Lead to the rapidity-divergent contribution [Bartels, Lipatov, Vacca, 2012] :

$$\langle R_+^{a_1}(\mathbf{p}_{T1}) R_+^{a_2}(\mathbf{p}_{T2}) R_-^{b_1}(\mathbf{k}_{T1}) R_-^{b_2}(\mathbf{k}_{T2})\rangle$$

$$= -i\alpha_s \eta [f^{a_1 c b_1} f^{c a_2 b_2} K_0 + (b_1 \leftrightarrow b_2, \mathbf{k}_{T1} \leftrightarrow \mathbf{k}_{T2})]$$

$$K_0 = \frac{\mathbf{k}_{T2}^2 \mathbf{p}_{T1}^2 + \mathbf{k}_{T1}^2 \mathbf{p}_{T2}^2}{\mathbf{k}_T^2} - \mathbf{q}_T^2,$$

where $\mathbf{k}_T = \mathbf{k}_{T1} - \mathbf{p}_{T1}$,
$\mathbf{q}_T = \mathbf{k}_{T2} - \mathbf{k}_{T1}$.

Together with the disconnected part form Regge trajectory, we get *the BFKL equation* for $2R$ Green's function, e.g. for **1** pair [BFKL, '76] :

$$\frac{\partial}{\partial Y} G_Y(\mathbf{p}_{T1}, \mathbf{p}_{T2}) = \frac{\alpha_s C_A}{\pi} \int d^{2-2\epsilon} \mathbf{k}_{T1,2} \Bigg[K_0(\mathbf{p}_{T1}, \mathbf{p}_{T2}, \mathbf{k}_{T1}, \mathbf{k}_{T2}) G_Y(\mathbf{k}_{T1}, \mathbf{k}_{T2})$$

$$+ (\omega_g^{(1)}(\mathbf{p}_{T1}) + \omega_g^{(1)}(\mathbf{p}_{T2})) G_Y(\mathbf{p}_{T1}, \mathbf{p}_{T2}) \Bigg],$$

For the **1** RR-pair the IR divergence at $\mathbf{k}_T \to 0$ cancels within the kernel.