Entanglement entropy, Krylov complexity and Deep inelastic scattering data

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Based on:

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Eur.Phys.J.C 82 (2022) 2, 111
M. Hentschinski, K. Kutak

Eur.Phys.J.C 82 (2022) 12, 1147
M. Hentschinski, K. Kutak, R. Straka

PRL’23
H. Hentschinski, D. Kharzeev. K. Kutak, Z. Tu

Rept.Prog.Phys. 87 (2024) 12, 120501
M. Hentschinski, D. Kharzeev, K. Kutak, Z. Tu

Phys.Rev.D 110 (2024) 8, 085011
P. Caputa, K. Kutak

Ongoing project with M. Praszałowicz

My motivation

Properties of entanglement entropy may provide some new insight on understanding of behavior of parton density functions

Links to other areas (thermodynamics, gravity, quantum information, conformal field theory)

Interesting in context of parton saturation and thermalization problem of Quark Gluon Plasma

Recent progress in the field comes from applying these ideas in the context of Deep Inelastic scattering

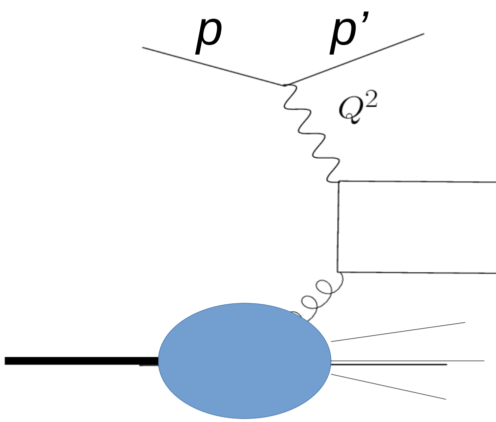
DIS proton structure function and dipole cross section

In the k_t factorization

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2}\alpha_s \sum_q e_q^2 \int d^2k\, \mathcal{F}(x,k^2)\left(S_L(k^2,Q^2,m_q^2) + S_T(k^2,Q^2,m_q^2)\right)$$

dipole gluon density

impact factors ~ hard coefficients



$$xg(x,Q) = \int_0^{Q^2} d\mathbf{k}^2 \mathcal{F}(x,\mathbf{k}^2)$$

Momentum density of gluons as "seen" at scale Q, cumulative distribution

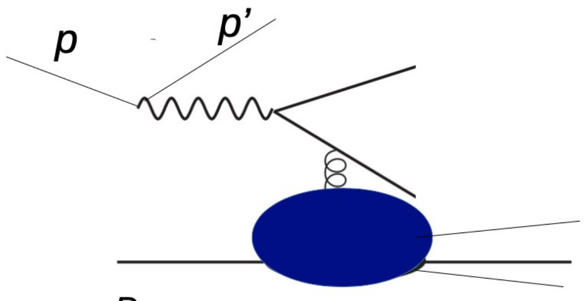
boosted proton

In the dipole formalism

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \int d^2b \int_0^1 dz \int d^2r \left(|\psi_L(z,r)|^2 + |\psi_T(z,r)|^2\right) N(x,r,b)$$

wave function

dipole amplitude



proton in rest frame

Entropy in stat. mech. – reminder

In statistical physics the entropy S of macrostate is given by the *log* of number *W* of distinct microstates that compose it

$$S = -\sum_{i=1}^{W} p(i) \ln p(i)$$

Gibbs entropy

For uniform distribution $p(i) = \frac{1}{W}$ the entropy is maximal

Boltzmann entropy

$$S = \ln W$$

If probability of state is 1 entropy is 0. Entropy in the information sense theory tells us about the amount of missing information.

Since partons are introduced as the microscopic constituents that compose the macroscopic state of the proton, it is natural to evaluate the corresponding entropy or entropy corresponding to parton density.

K. Kutak ‘11, Peschanski ’12

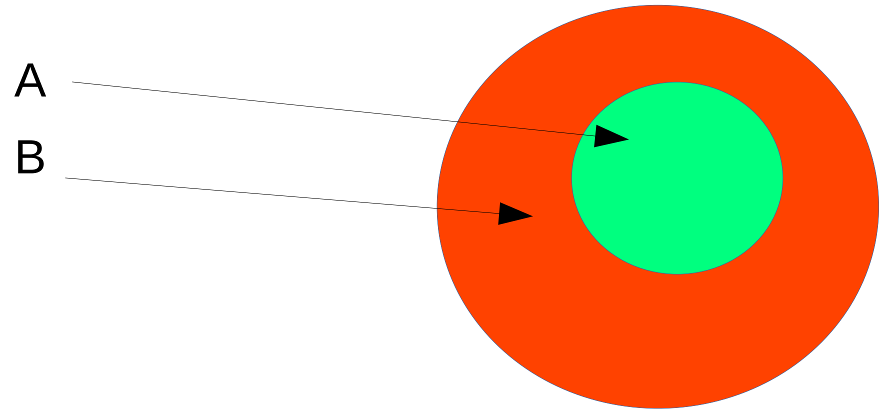
I. Zahed, A. Stoffers ’13

A. Kovner, M. Lublinsky ‘15,

D. Kharzeev, E. Levin ‘ 17, …..

Entanglement

$\mathcal{H}_A \otimes \mathcal{H}_B$



The composite system is described by

$|\Psi_{AB}\rangle$ in $A \cap B$

general definitions

entangled
if the product can not be expressed as separable product state

$$|\Psi_{AB}\rangle = \sum_{i,j} c_{ij} |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle$$

separable
if the product can be expressed as separable product state

$$|\Psi_{AB}\rangle = |\varphi^A\rangle \otimes |\varphi^B\rangle$$

$\mathcal{H}_B$ of dimension n_B.

$\mathcal{H}_A$ of dimension n_A

Schmidt decomposition

$$|\Psi_{AB}\rangle = \sum \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

orthonormal states belonging to A

orthonormal states belonging to B

related to matrix C

Entanglement and entropy

$$|\Psi_{AB}\rangle = \sum \alpha_n |\Psi_n^A\rangle |\Psi_n^B\rangle$$

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

density matrix of a pure sate

$$\rho_A = \mathrm{tr}_B \rho_{AB} = \sum_n \alpha_n^2 |\Psi_n^A\rangle\langle\Psi_n^A|$$

the reduced density matrix
of the mixed state after tracing
out some degrees of freedom

The von Neuman entropy is

$$S_A = -\rho_A \ln(\rho_A) = S_B$$

$$S_A = -\sum_n p_n \ln p_n \qquad \alpha_n^2 \equiv p_n$$

entropy results from the entanglement between states
in A and states in B, and can thus be interpreted as the
entanglement entropy.

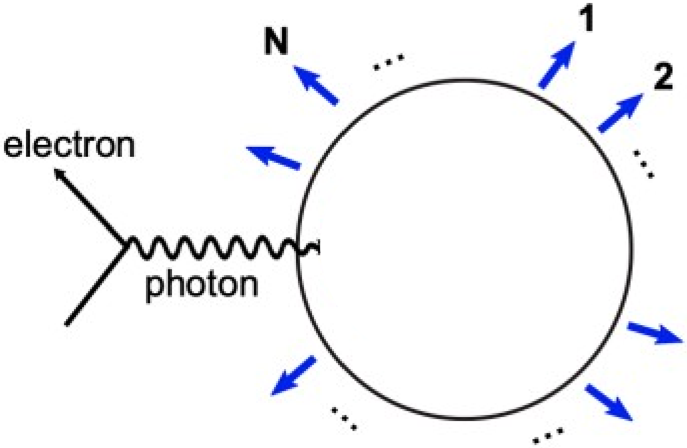
Analogy

black hole + coffee → Hawking radiation + back hole

electron + proton → electron +radiation of hadrons

J. A. Wheller thought experiment.

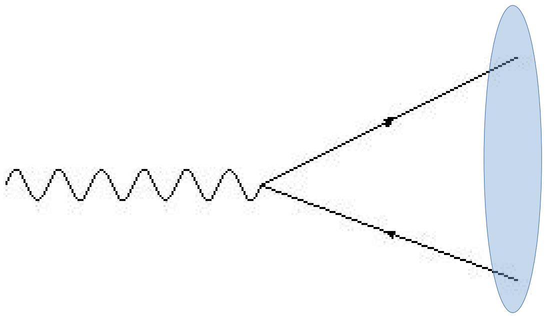
Dall-E

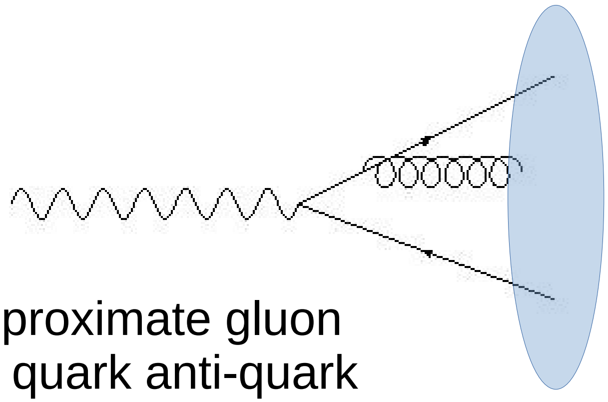


Bekenstein: BH has to have entropy because coffee has entropy and overall entropy would decrease if we toss coffee into black hole without increasing entropy

Comment:
better analogy would be proton – ion collision

Cascade of dipoles

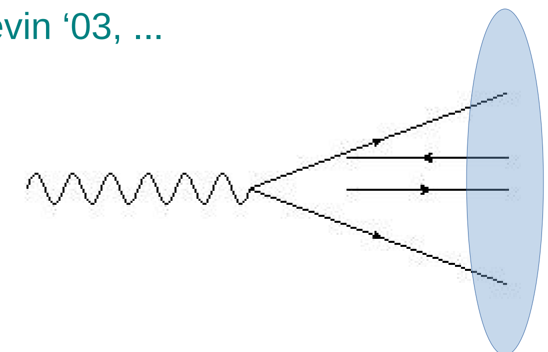


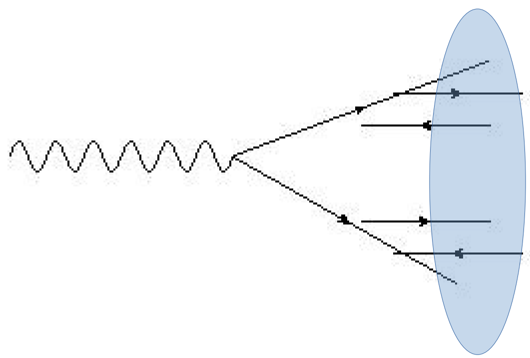


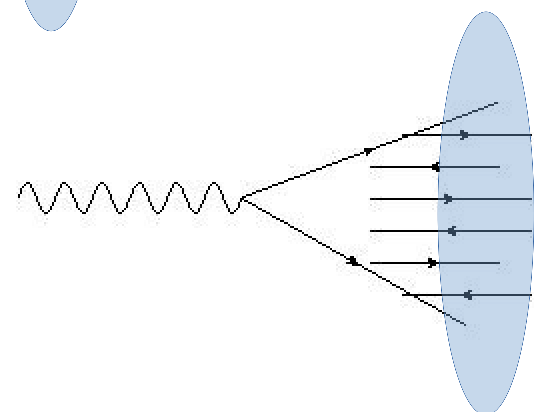
Microstates p_n – probabilities for given number of dipoles

approximate gluon by quark anti-quark

Mueller 95,
Lublinsky, Levin ‘03, ...





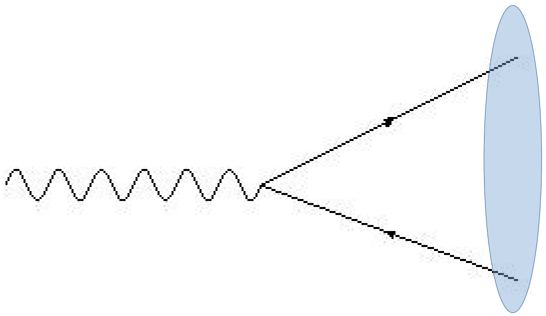


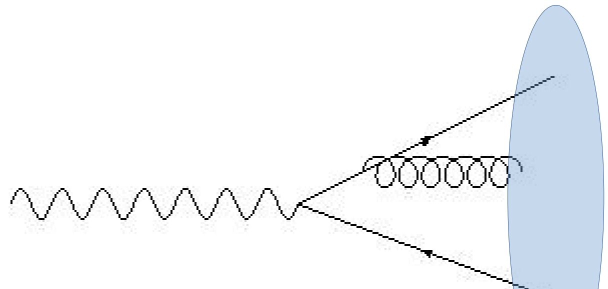
BFKL dipole cascade

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

- One considers state of $\bar{q}$ q and successive emission of gluons.
- Tracing over coordinates of quarks and transverse coordinates of gluons and colors will give reduced will density matrix of soft gluons from which one can the calculate entropy. Explicit construction in Liu Nowak, Zahed ‘22

Cascade of dipoles

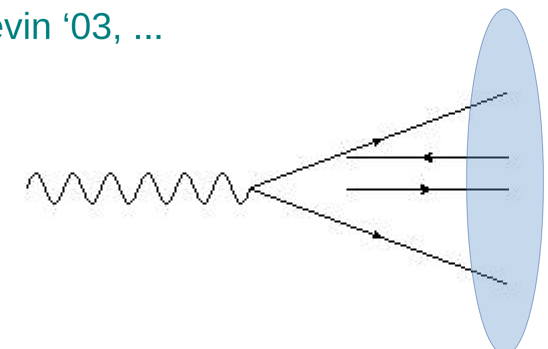


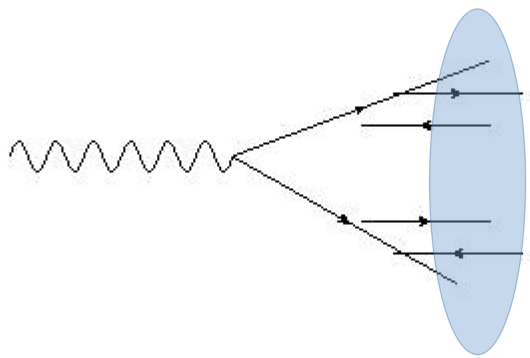


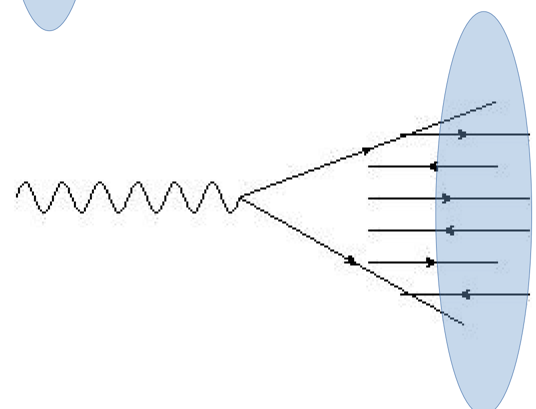
approximate gluon
by quark anti-quark

Microstates p_n – probabilities for given number of dipoles

Mueller 95,
Lublinsky, Levin ‘03, ...







BFKL dipole cascade

$$\partial_y p_n(y,\{r\}) = \sum_m K \otimes p_m(y,\{r\})$$

probability to find *n* dipoles at rapidity *y*

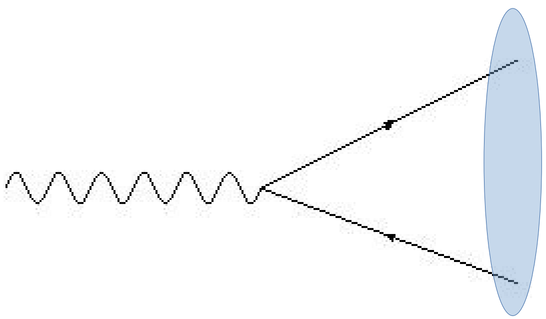
transverse sizes of dipoles

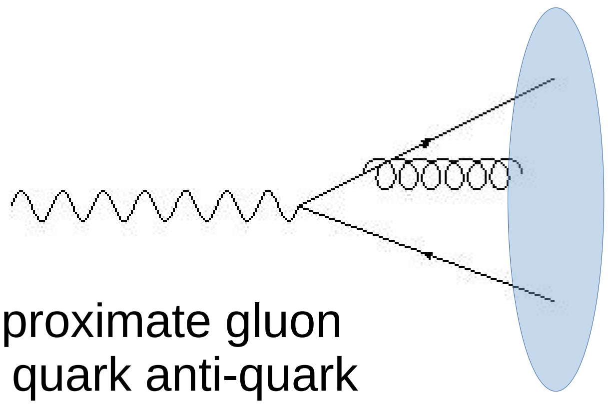
set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles

the growth due to the splitting of (n − 1) dipoles into n dipoles

Cascade of dipoles – 1 D

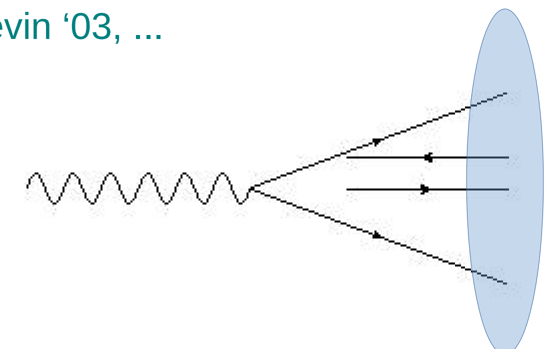


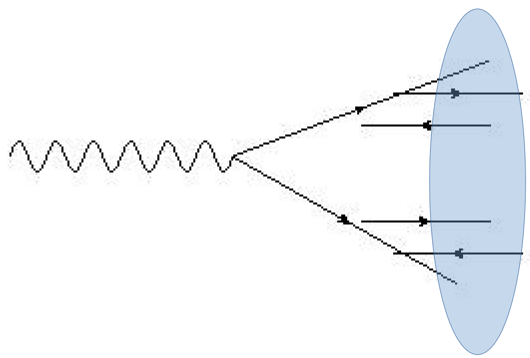


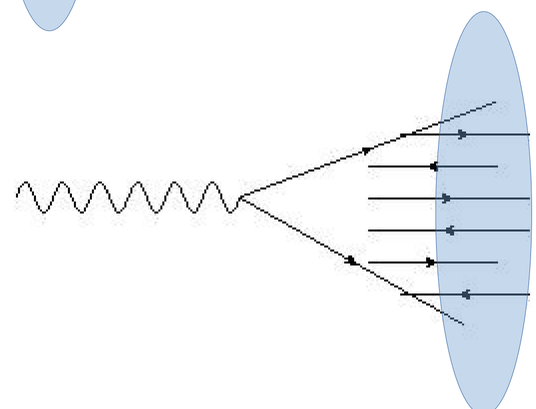
approximate gluon
by quark anti-quark

Microstates p_n – probabilities for given number of dipoles

Mueller 95,
Lublinsky, Levin ‘03, ...







BFKL dipole cascade

set of partons is described by set of dipoles with fixed sizes ,Y is rapidity and is related to energy

$$|\Psi_n\rangle = \sum_{x_0 \gg x_1 \gg x_2 \dots \gg x_n \gg x_{\min}} \sqrt{a^n n!}\, \frac{e^{-\frac{ay + a\sum_{i=1}^{n} y_i}{2}}}{\sqrt{(\Lambda^-)^n x_1 .. x_n}} |x_1\rangle |x_2\rangle .....|x_n\rangle$$

Liu, Nowak, Zahed ‘22

$$\hat{\rho}_1 = \sum_{n=0}^{\infty} |\Psi_n\rangle\langle\Psi_n|$$

State of n dipoles with
longitudinal momenta

$$y_i = \ln \frac{x_i}{x_{\min}}$$

Cascade of dipoles – fixed dipole size

Initial conditions

$$\frac{dp_n(y)}{dy} = -\lambda\, n\, p_n(y) + (n-1)\, \lambda\, p_{n-1}(y)$$

$p_1(0) = 1$ at initial rapidity there is only 1 dipole

$p_{n>1}(0) = 0$

Lublinsky, Levin ‘03

rate at which number of dipoles grow. The phenomenological value is λ = 0.3.
It is an observable.

depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles.

the growth due to the splitting of (n − 1) dipoles into n dipoles.

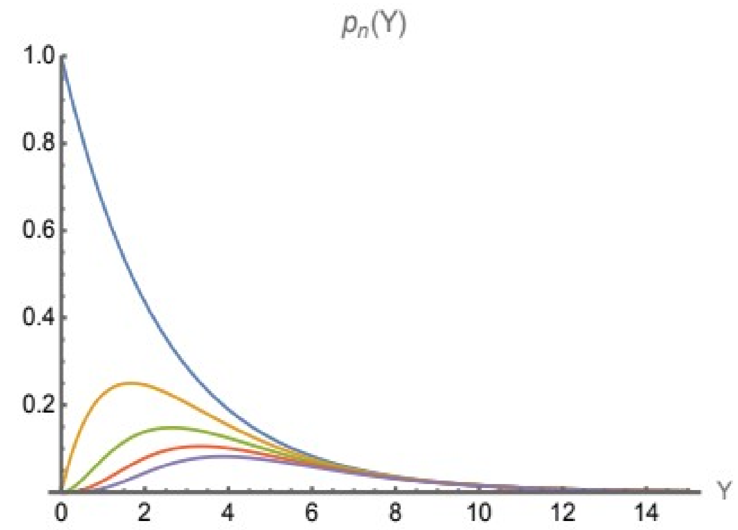
See for density matrix and 3+1 dimensional case in DLL and KNO function
Liu, Nowak, Zahed ’22 PRD
Liu, Nowak, Zahed ’22 PRD

Exact analytical solution is:

$$p_n(y) = e^{-\lambda y}(1 - e^{-\lambda y})^{n-1}$$

$$y = \ln\left(\frac{1}{x}\right)$$

Cascade of dipoles – fixed dipole size



$$\frac{dp_n(y)}{dy} = -\lambda\, n\, p_n(y) + (n-1)\,\lambda\, p_{n-1}(y)$$

rate at which number of dipoles grow. The phenomenological value is λ = 0.3.
It is an observable.

The eq. with dipole sises and BFKL kernel has the same structure

depletion of the probability to find n dipoles due to the splitting into (n + 1) dipoles.

the growth due to the splitting of (n − 1) dipoles into n dipoles.

See for density matrix and 3+1 dimensional case in DLL and KNO function
Liu, Nowak, Zahed ’22 PRD
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Exact analytical solution is:

$$p_n(y) = e^{-\lambda y}(1 - e^{-\lambda y})^{n-1} \qquad y = \ln\left(\frac{1}{x}\right)$$

KL entropy formula - interpretation

$$p_n(y) = e^{-\lambda y}(1 - e^{-\lambda y})^{n-1}$$

At low x partonic microstates have equal probabilities.

In this equipartitioned state the entropy is maximal – the partonic state at small *x* is maximally entangled.

In terms of information theory as Shanon entropy:

- equipartitioning in the maximally entangled state means that all “signals” with different number of partons are equally likely

- it is impossible to predict how many partons will be detected in a give event.

- structure function at small x should become universal for all hadrons.

.

Cascade of dipoles – entropy

$$\langle n \rangle_y = \sum_n n p_n(y) \equiv \bar{n}(x)$$

$$S_{inc.}(\bar{n}) = -\sum_n p_n(\bar{n}) \ln p_n(\bar{n}) = \ln \bar{n} - (\bar{n} - 1) \ln \left(1 - \frac{1}{\bar{n}}\right)$$

In the low x limit
dominant contribution

$$S_{inc.}(\bar{n}) = \ln xg(x)$$

$$y = \ln \left(\frac{1}{x}\right)$$

Assuming $xg(x) = \left(\frac{1}{x}\right)^{\lambda}$

$$S_{inc.}(y) \approx \lambda y$$

In agreemant with

K. Kutak 1103.3654v1

In DLL approximation i.e when subsequent dipoles are strongly ordered in size and rapidity one gets:

$$S(x, Q) = \ln(xg(x, Q))$$

Krylov subspace, complexity – motivation

"The complexity of the task is defined as the minimum number of gates used to construct the circuit that accomplishes it" L. Susskind

State complexity

Visvanath, Muller '63
Altman, Avdoshkin, Cao, Parker, Scaffidi '19
Balasubramanian, Caputa, Magan, Wu '22,...

Simple reference quantum state spreads and becomes complex in Hilbert space

The key point is to expand the state or the operator in the minimal basis that supports its unitary evolution.

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}|\Psi_n\rangle \qquad |\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

Projection of a high-dimensional problem onto a lower-dimensional Krylov subspace.

Used by Krylov to understand how to efficiently diagonalize matrices.

One can introduce cost function and show that the minimalisation is achieved in Krylov subspace.

Krylov subspace, complexity, entropy

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\Psi_n\rangle$$

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

n consequtive application of Hamiltonian

Gram-Schmidt orthogonalization procedure.
Construct vector K_2 by subtracting the previous two vectors, vector K_3 by subtracting the previous 3 vectors, and so forth

vs.

Lanczos algorithm
n + 1 is determined by n and n – 1.
Low memory requirements Visvanath, Muller ‘63

In the Krylov basis

$$H_{nm} := \langle K_n|\hat{H}|K_m\rangle = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ 0 & 0 & b_3 & a_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n \phi_n(t)|K_n\rangle$$

Is at most linear in n

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

$$p_n(t) = |\phi_n(t)|^2$$

probability amplitudes for each vector

Balasubramanian, Caputa, Magan, Wu ‘22

$$\mathcal{C}_K(t) = \langle n\rangle = \sum_n n\, p_n(t)$$

$$S_K(t) = -\sum_n p_n(t)\log p_n(t)$$

Krylov subspace – construction

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}|\Psi_n\rangle$$

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

n consequtive application of Hamiltonian

$$|K_0\rangle = |\psi(0)\rangle = |\psi_0\rangle$$

Gram-Schmidt orthogonalization procedure. Construct with K_2 by subtracting the previous two vectors, K_3 by subtracting the previous 3 vectors, and so forth

$$|z_1\rangle = \hat{H}|K_0\rangle - a_0|K_0\rangle \qquad |K_1\rangle = \frac{|z_1\rangle}{\langle z_1|z_1\rangle}$$

$$|z_{n+1}\rangle = (\hat{H} - a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

Strength of the Lanczos algorithm. n + 1 is determined by n and n – 1. Low memory requirements Visvanath, Muller ‘63

$$|K_n\rangle = b_n^{-1}|z_n\rangle \qquad b_n = \langle z_n|z_n\rangle^{\frac{1}{2}} \qquad a_n = \langle K_n|\hat{H}|K_n\rangle$$

Project a high-dimensional problem onto a lower-dimensional Krylov subspace

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$\langle K_n|K_m\rangle = \delta_{nm}$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n \phi_n(t)|K_n\rangle$$

$$H_{nm} := \langle K_n|\hat{H}|K_m\rangle = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ 0 & 0 & b_3 & a_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

probability amplitudes for each vector

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

Krylov basis and dipole evolution

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

The equation above can be solved for $a_n = 0 \quad b_n = \alpha n$

P. Caputa, K. Kutak, 2404.07657

Displacement operator

SL(2R)

$$|\psi(t)\rangle = e^{-i\alpha(L_1+L_{-1})t}\,|0\rangle \otimes |0\rangle$$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{\tanh^n(\alpha t)}{\cosh^{2h}(\alpha t)} \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\,|n\rangle \otimes |n\rangle$$

One can calculate density matrix and reduced density matrix to get

$$\rho(t) = \sum_{n=0}^{\infty} p_n(t)\,|n\rangle\,\langle n|$$

vacuum of part that we are interested in

vacuum of compliment

$$\mathcal{C}_K(t) = \sum_{n=0}^{\infty} n|\phi_n(t)|^2$$

After expressing it in terms rapidity and probabilities rapidity variable one gets

$$\mathcal{C}_K(Y) = e^{\alpha Y} - 1 \qquad Y = \ln(1/x)$$

$$p_n(Y) = \frac{\Gamma(2h+n)}{n!\Gamma(2h)}(e^{-\alpha Y})^{2h}\left(1 - e^{-\alpha Y}\right)^n$$

$$\mathcal{C}_K = xg(x)$$

$$\partial_Y p_n(Y) = \alpha n\, p_{n-1}(Y) - \alpha(n+1)\, p_n(Y)$$

QI measures and dipole equations

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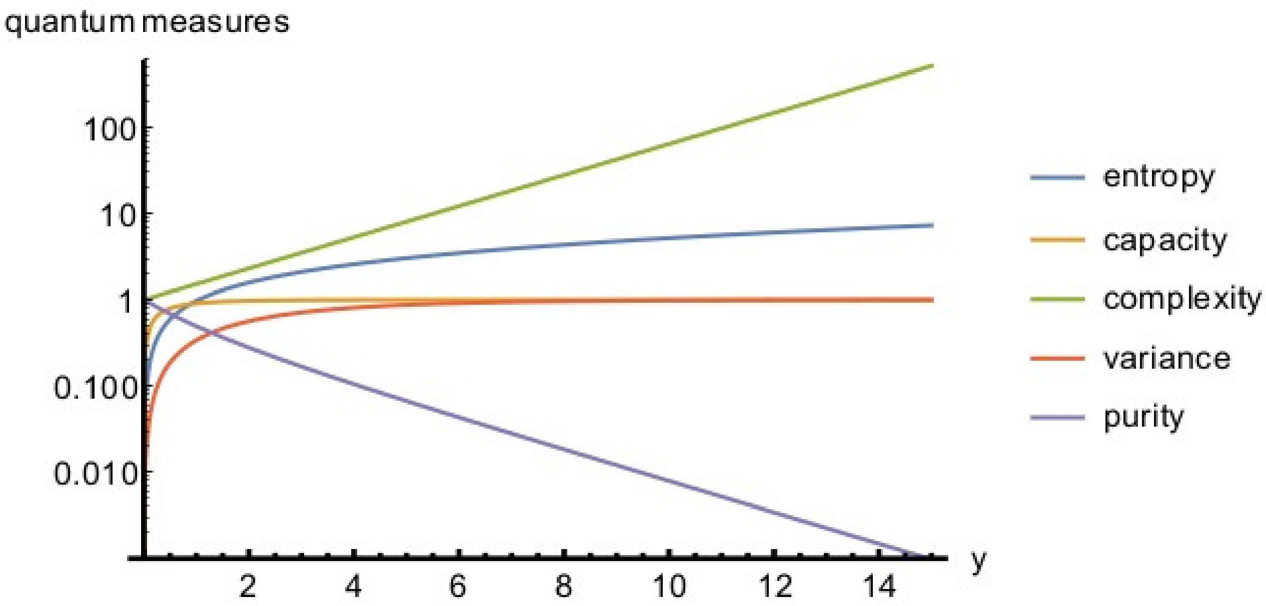


$$S_K(t) = -\sum_n p_n(t) \log p_n(t)$$

$$C_E = \lim_{m\to 1} m^2 \partial_m^2 [(1-m)\, S_K^{(m)}].$$

$$\mathcal{C}_K(t) = \langle n \rangle = \sum_n n\, p_n(t)$$

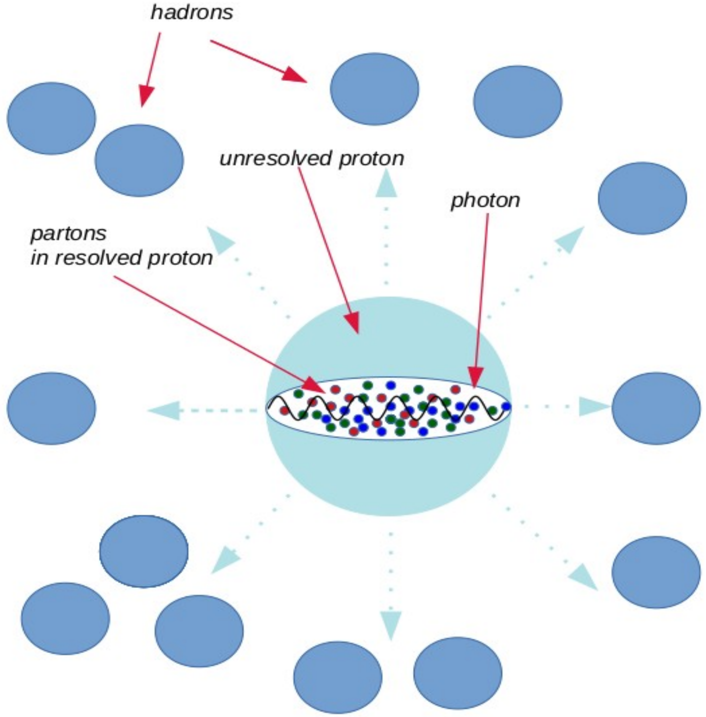
$$\delta_K^2 = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle^2}$$

$$\gamma_K = \sum_n p_n^2(t).$$

$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1) p_{n-1}(Y)$$

Entanglement entropy – calculation and measurement

For DIS at high energies, this entanglement entropy can be calculated using



$$S(x, Q^2) = \ln \left\langle n \left(\ln \frac{1}{x}, Q \right) \right\rangle$$

Calculated using parton/dipole density

Conjecture that these entropies are the same

$$S_{hadron} = \sum P(N) \ln P(N)$$

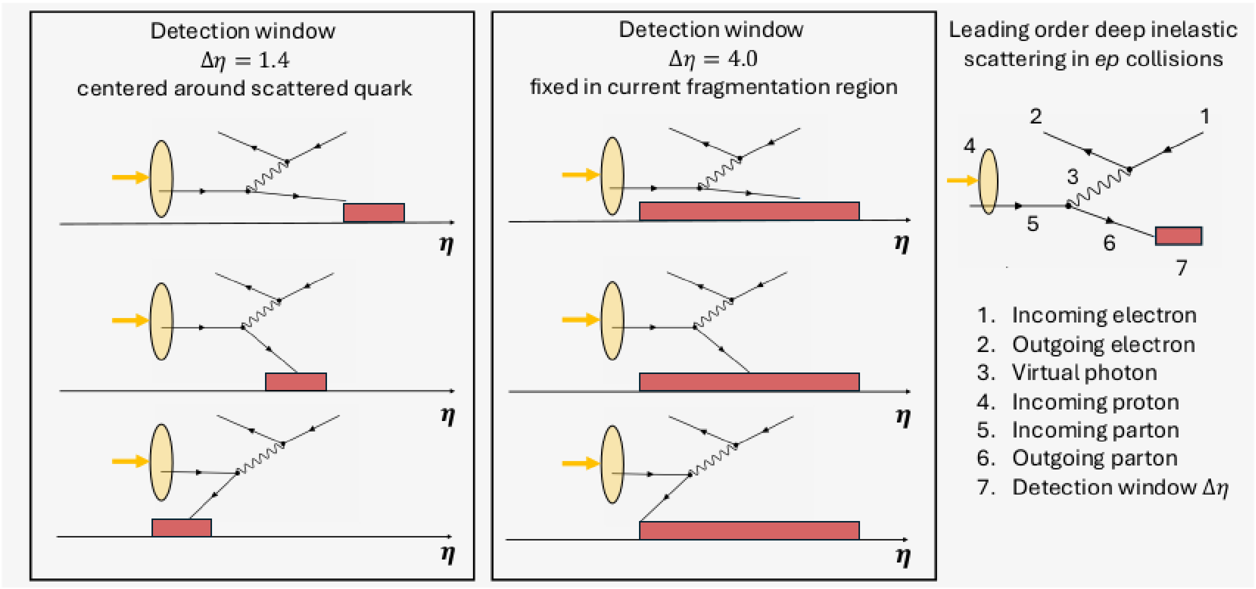
Measured by counting hadrons

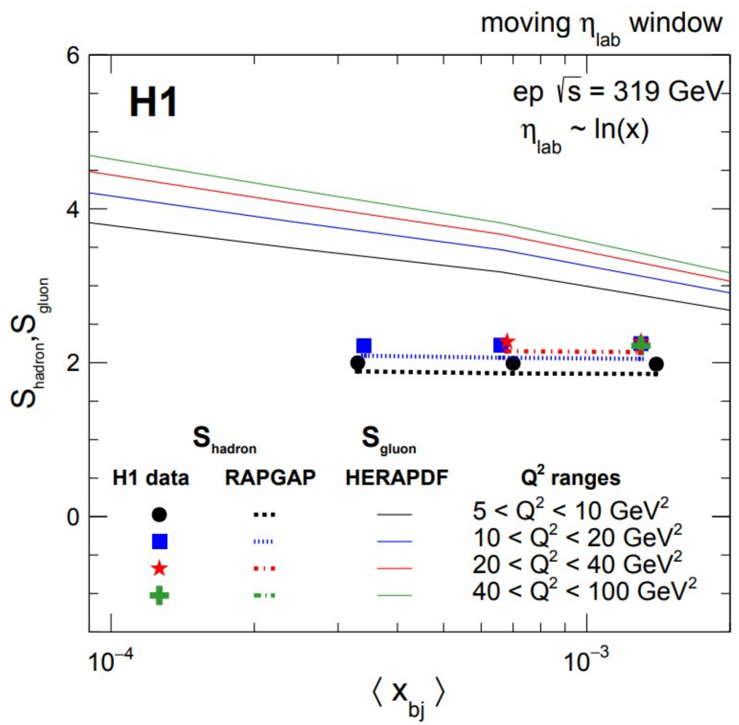
number of measured hadrons

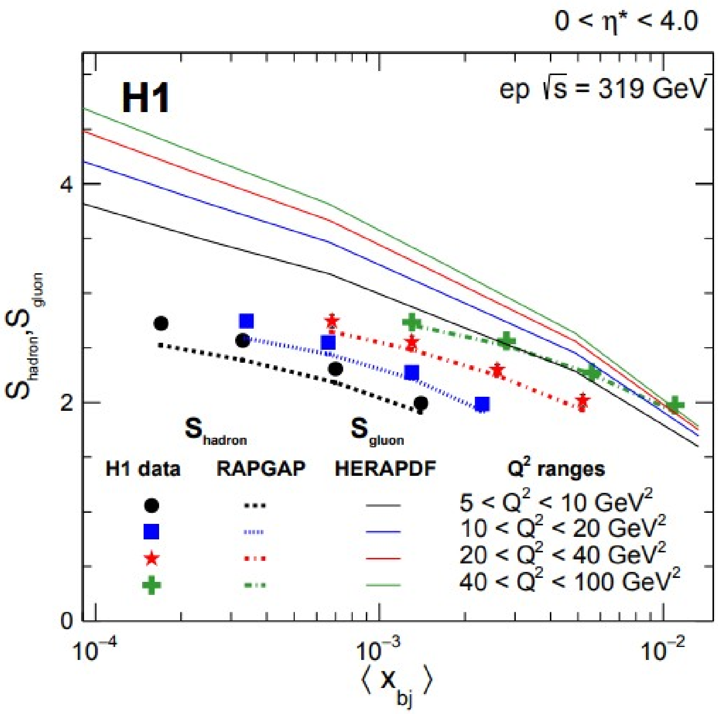
The charged particle multiplicity distribution measured in either the current fragmentation region or the target fragmentation region.

Fraction of events with charged hadron

Entropy measurements

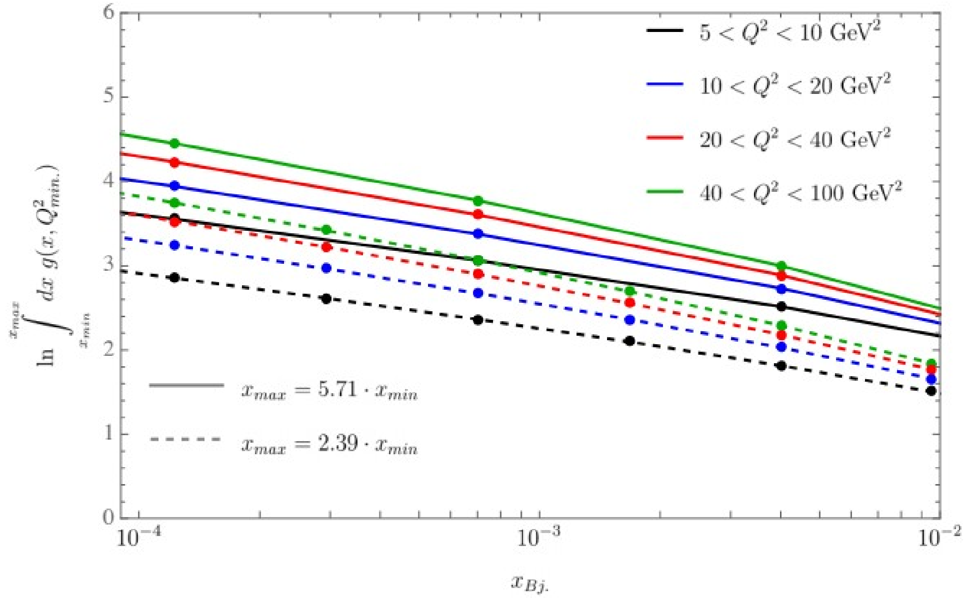






Problem with H1 escription of data...

Bining and KL formula



plot showing dependence of the result on the sie of bins if binig is naive

$$n_g(Q^2) = \int_0^1 dx g(x,Q^2)$$

Formal definition of number of gluons

$$n_g(\bar{x}) = \int_{x_{\min}}^{x_{\max}} dx g(x,Q^2)$$

$\bar{x} \in [x_{\min}, x_{\max}]$

$$\bar{x} = \frac{\int_{x_{\min}}^{x_{\max}} dx\, x g(x,Q^2)}{\int_{x_{\min}}^{x_{\max}} dx g(x,Q^2)}$$

average *x*

Data binning takes place in rapidity

$$\bar{n}_g(\bar{x}) = \frac{1}{y_{max} - y_{min}} \int_{y_{min}}^{y_{max}} dy \frac{dn_g}{dy} = \frac{n_g(y_{max}) - n_g(y_{min})}{y_{max} - y_{min}}$$

$$y_{max,min} = \ln 1/x_{min,max}$$

for small bins

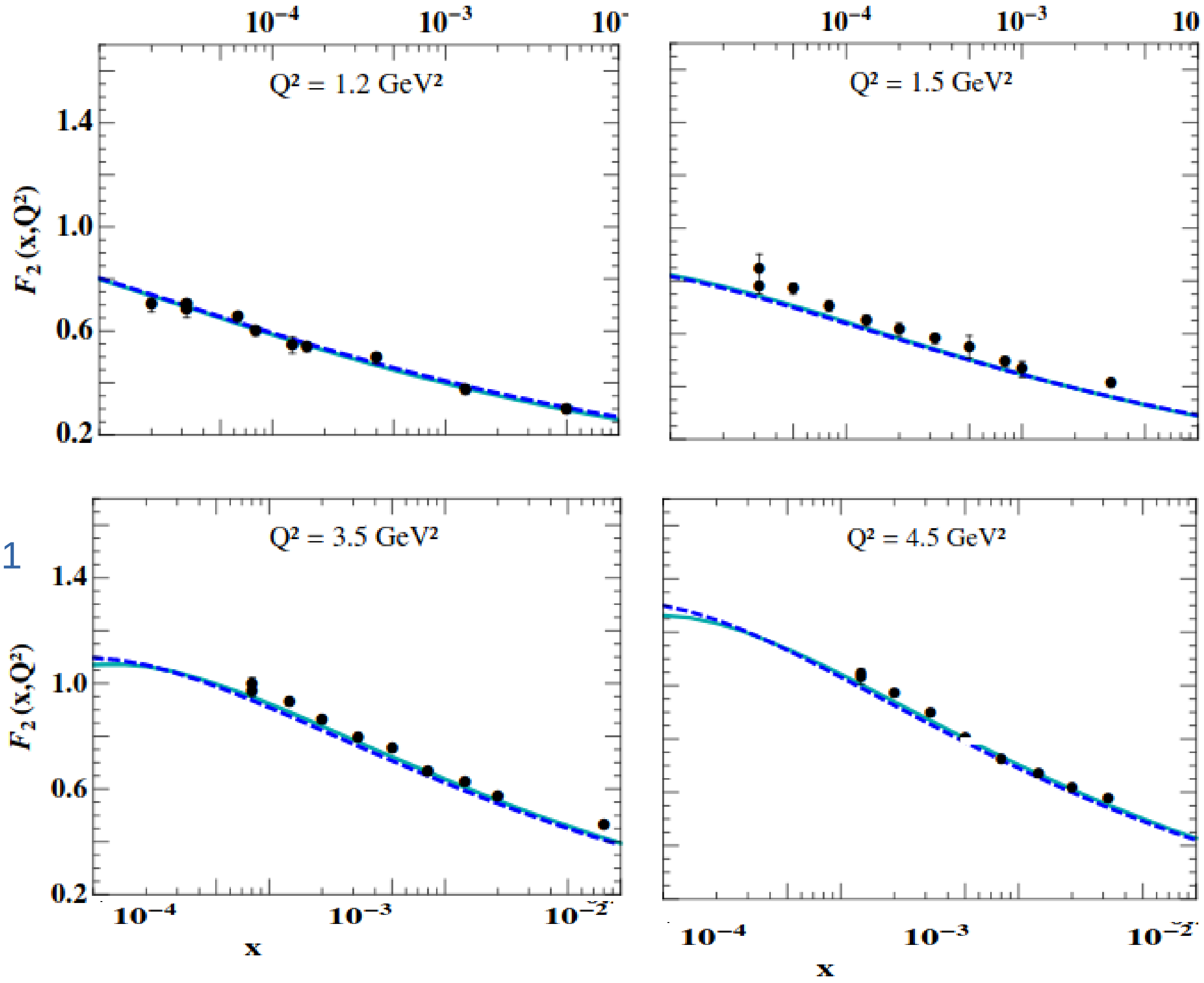
$$\bar{n}_g(x,Q^2) = \frac{dn_g}{d\ln(1/x)} = xg(x,Q^2)$$

$$\langle \bar{n}(x,Q^2) \rangle_{Q^2} = \frac{1}{Q^2_{\max} - Q^2_{\min}} \int_{Q^2_{\min}}^{Q^2_{\max}} dQ^2 \left[xg(x,Q^2) + x\Sigma(x,Q^2) \right]$$

$$\langle S(x,Q^2) \rangle_{Q^2} = \ln \langle \bar{n}(x,Q^2) \rangle_{Q^2}$$

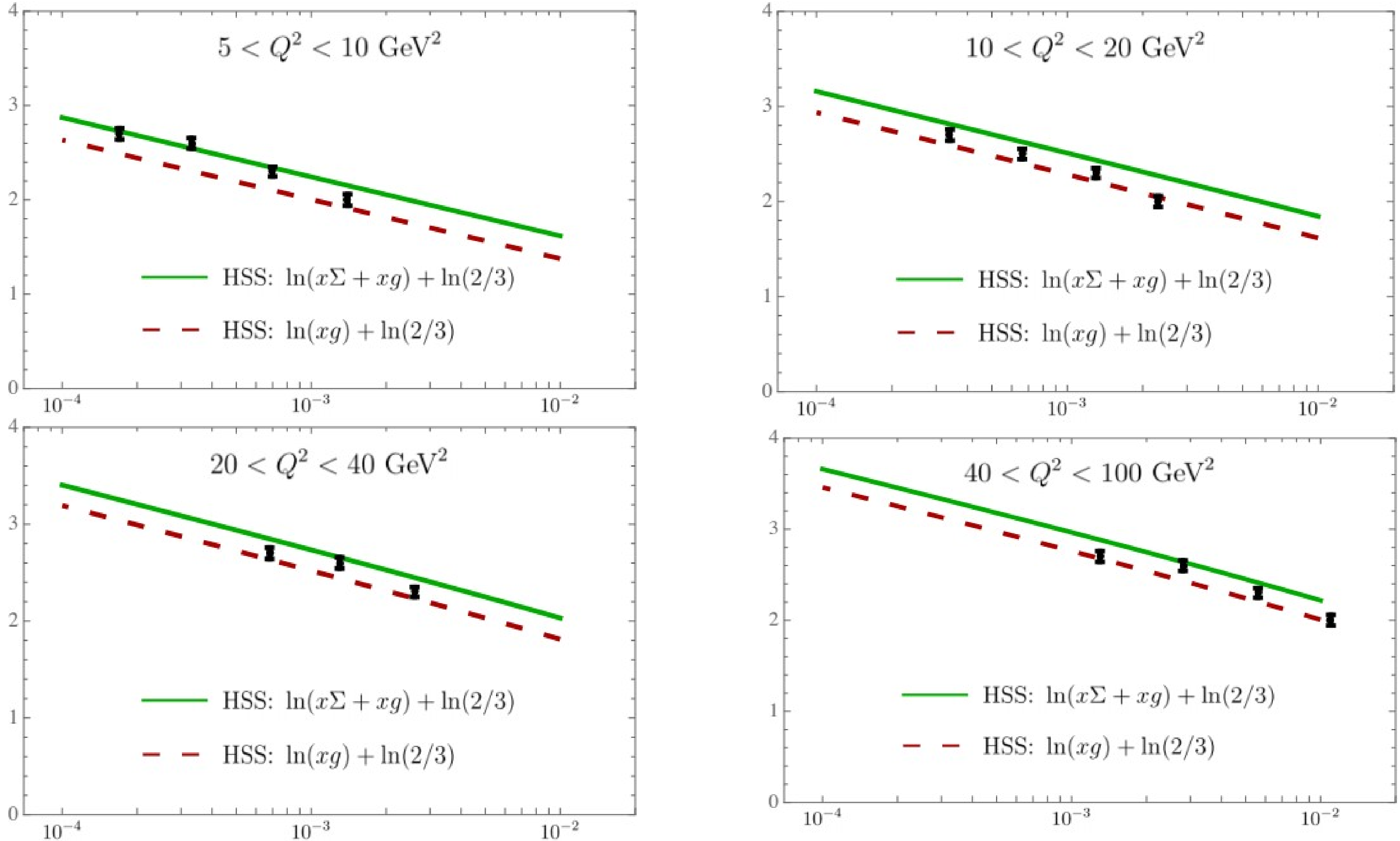
Proton structure function from HSS fit

Hentschinski, Sabio-Vera, Salas.
Phys.Rev.D 87 (2013) 7, 076005
Phys.Rev.Lett. 110 (2013) 4, 041601



Eur.Phys.J.C 82 (2022) 2, 111 M. Hentschinski, K. Kutak

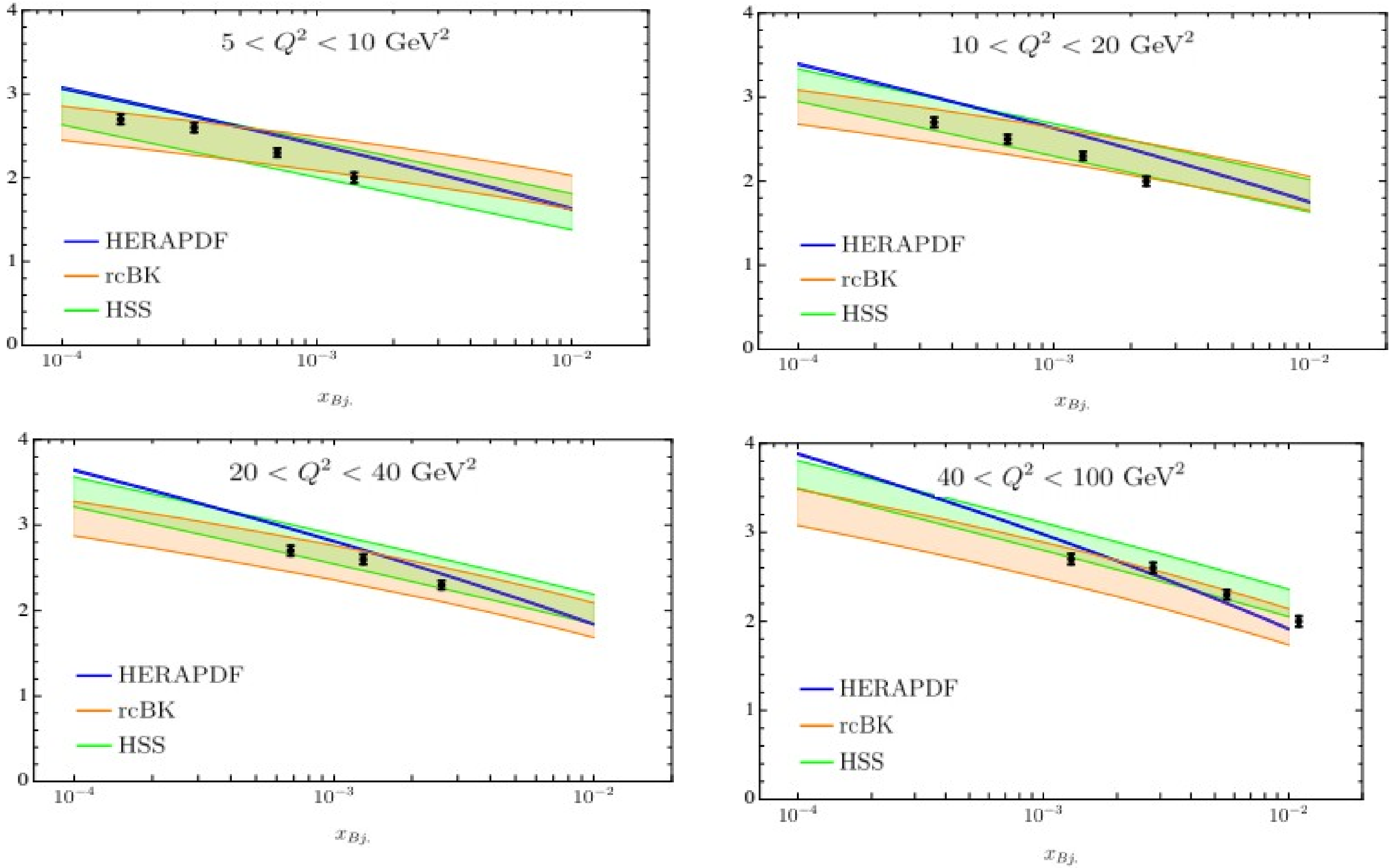
Results – fixed rapidity window



Hint that the general idea works. Gluon dominates over quarks.
One has to also take into account that only charged hadrons were measured.

Large scales - description

Martin Hentschinski, K. Kutak, Robert Straka ‘23



Generalization of KL to moving rapidity window

Hentschinski, Kharzeev, Kutak, Tu ‘24

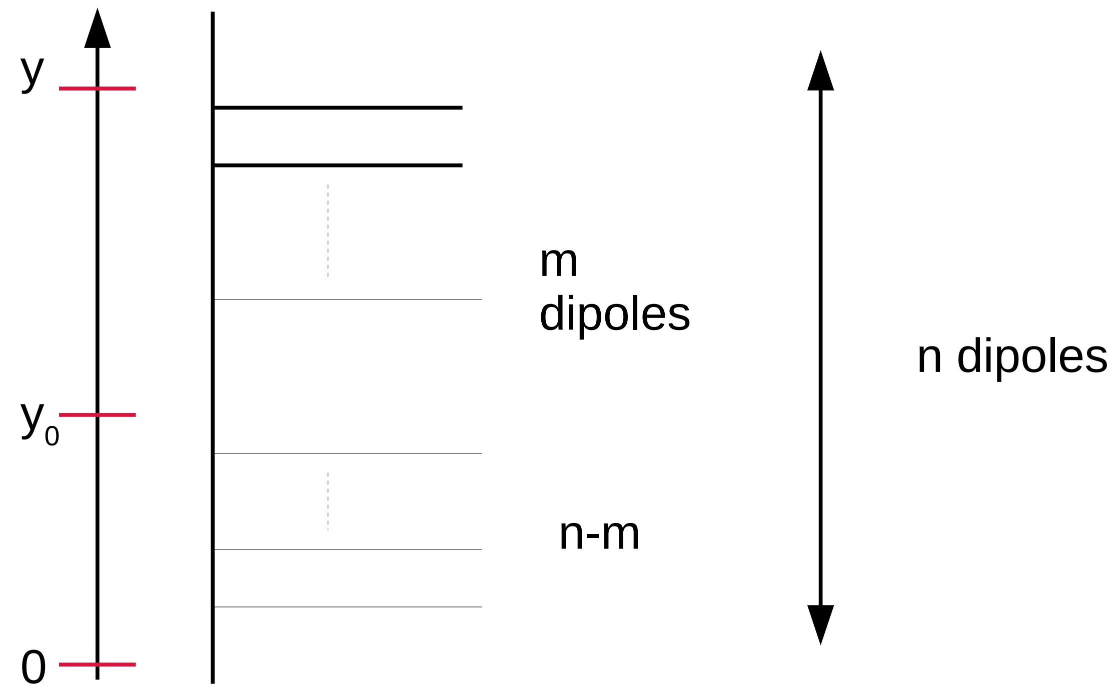
Generalize the KL formula to address measurement in rapidity window

$$p_n(y) = \sum_{m=0}^{n} p_{n-m}(y_0)\tilde{p}_m(y, y_0)$$

$$\tilde{p}_0(y, y_0) = e^{-\lambda(y-y_0)}$$

$$\tilde{p}_{n\geq 1}(y, y_0) = p_n(y) \cdot (1 - \tilde{p}_0(y, y_0))$$

$$\sum_{m=0} \tilde{p}_m(y, y_0) = 1$$



probability to have m dipoles in $[y_0,y]$

n-m dipoles in the range $[0,y_0]$

probability to have n dipoles in $[0,y]$

Generalization of KL to moving rapidity window

Hentschinski, Kharzeev, Kutak, Tu '24

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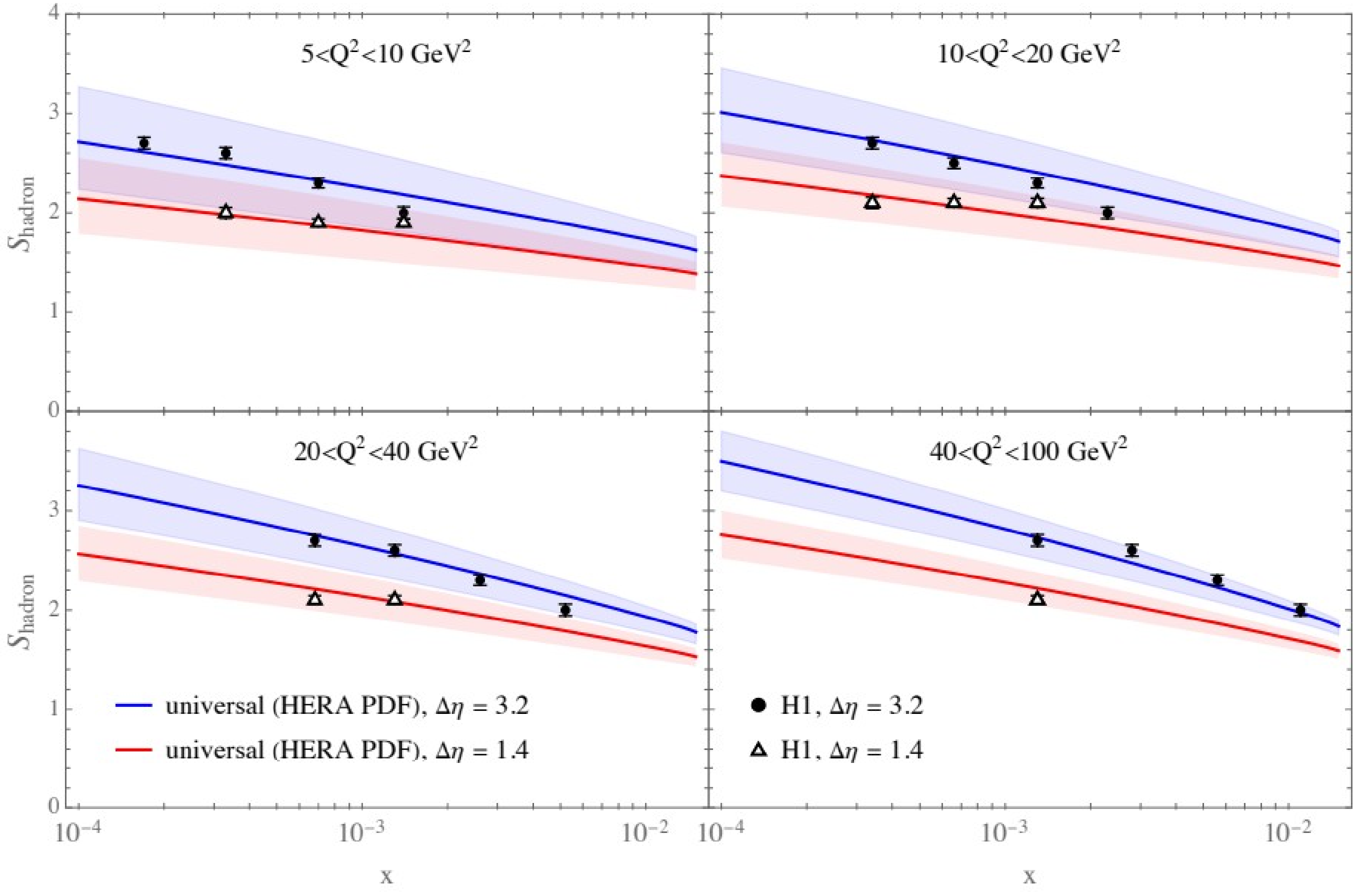
$$\langle n \rangle_{y;y_0} = \sum_n n\tilde{p}_n(y, y_0)$$

$$S_{loc}(\bar{n}, \tilde{p}_0) = -\sum_{n=0} \tilde{p}_n(\bar{n}, \tilde{p}_0) \ln \tilde{p}_n(\bar{n}, \tilde{p}_0)$$
$$= -\tilde{p}_0 \ln \tilde{p}_0 - (1 - \tilde{p}_0) \ln(1 - \tilde{p}_0) + (1 - \tilde{p}_0) S_{inc.}(\bar{n})$$

For large rapidity window this formula reduces to $S^{\text{univ.}}_{\text{inc.}}(\bar{n}) = \ln(\bar{n})$

Fixed and moving rapidity window description

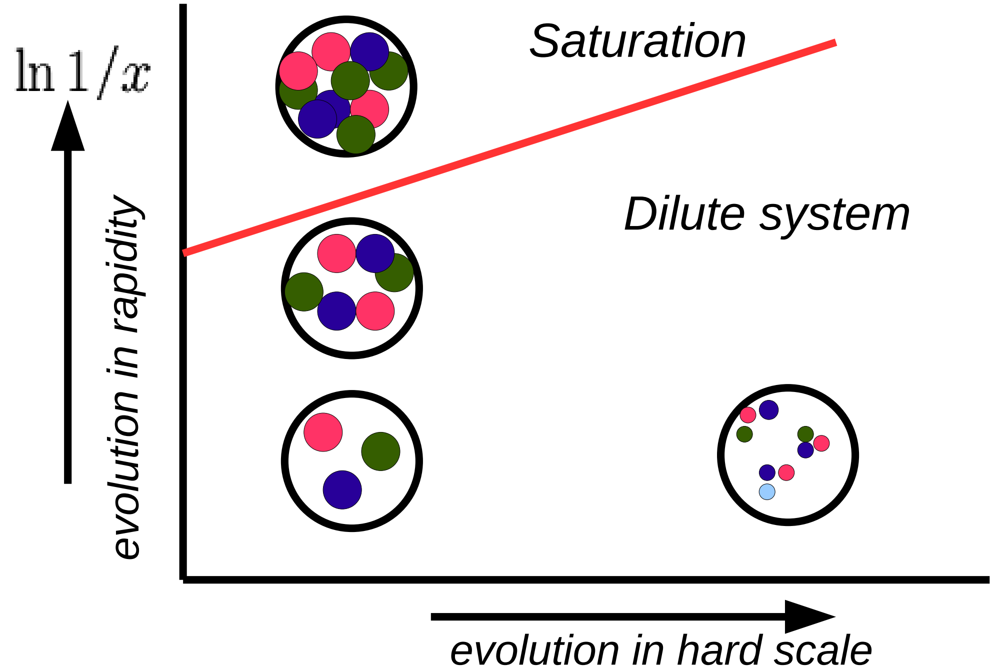


blue – fixed rapidity window

red - moving rapidity window

Gluons at high energies

Saturation – state where number of gluons stops growing due to high occupation number. Way to fulfill unitarity requirements in high energy limit of QCD.



L.V. Gribov, E.M. Levin, M.G. Ryskin
Phys.Rept. 100 (1983) 1-150

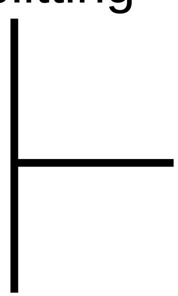
Larry D. McLerran, Raju Venugopalan
Phys.Rev. D49 (1994) 3352-3355

Phenomenological model:
Golec-Biernat, Wusthoff '99

On microscopic level it means that
gluon apart splitting recombine

splitting

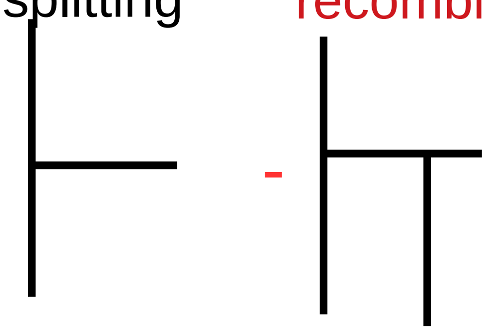
Linear evolution
Equation
BFKL



Nonlinear evolution equations
BK, JIMWLK
Balitcky-Kovchegov,

Jailian-Marian,Iancu
McLerran,Weigert,Leonidov,Kovner

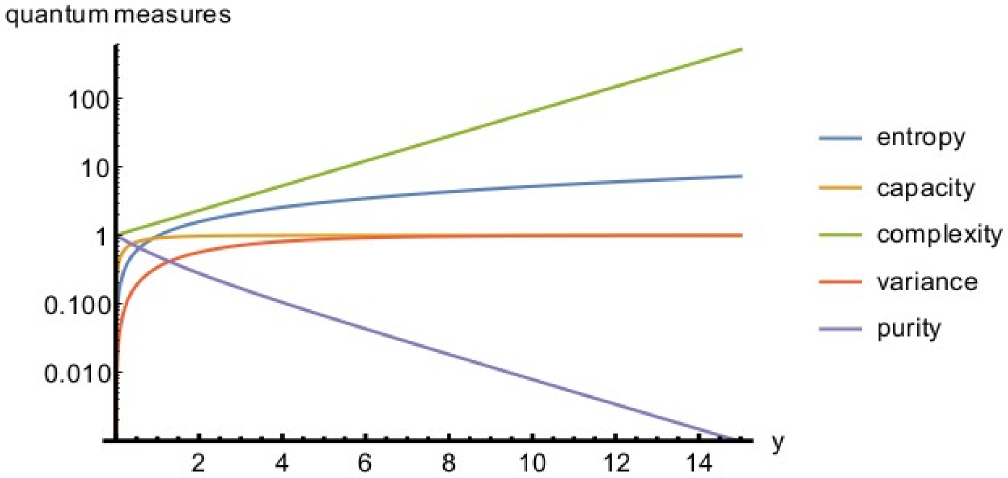
splitting recombination

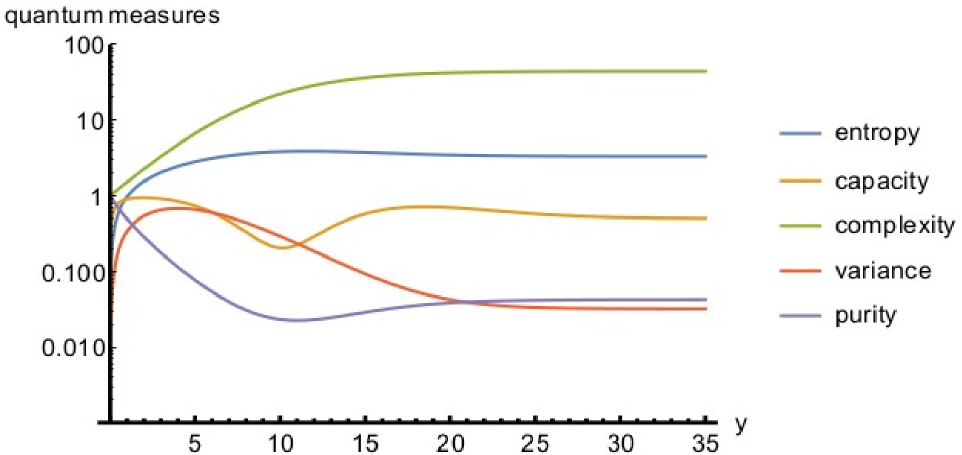


Bartels,
Wusthoff ‘93

QI measures and dipole equations

P. Caputa, K. Kutak, 2404.07657





$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y)$$

$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y) + \beta n(n+1)p_{n+1}(Y) - \beta n(n-1)p_n(Y)$$

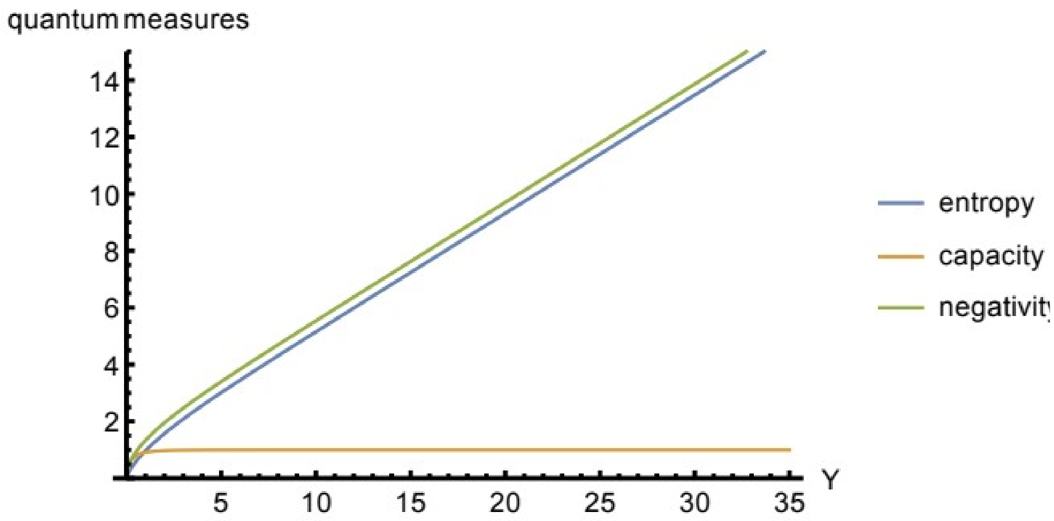
E. Iancu, D.N. Triantafyllopoulos '05

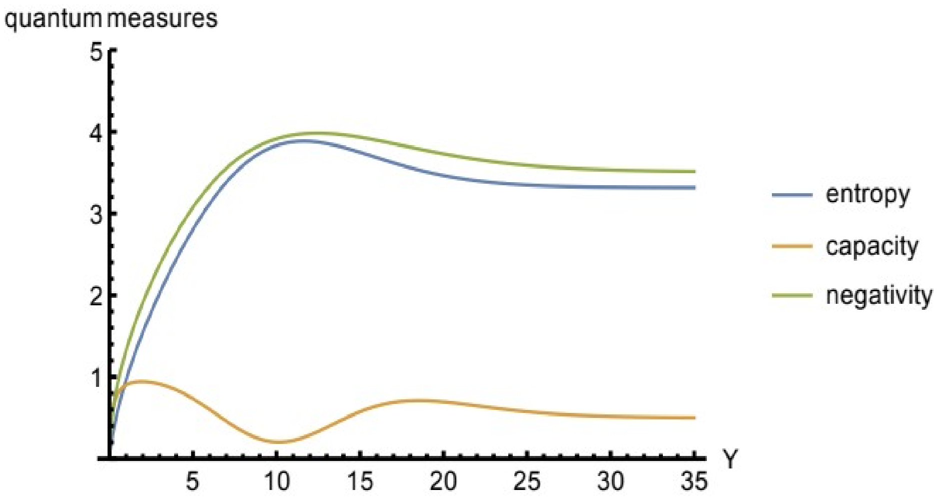
Bondarenko, Motyka, Mueller, Shoshi, Xiao '07

Hagiwara,Hatta,Xiao,Yuan '18

QI measures and dipole equations

P. Caputa, K. Kutak, 2404.07657





$$\partial_Y p_n(Y) = -\lambda n p_n(Y) + \lambda(n-1)p_{n-1}(Y)$$

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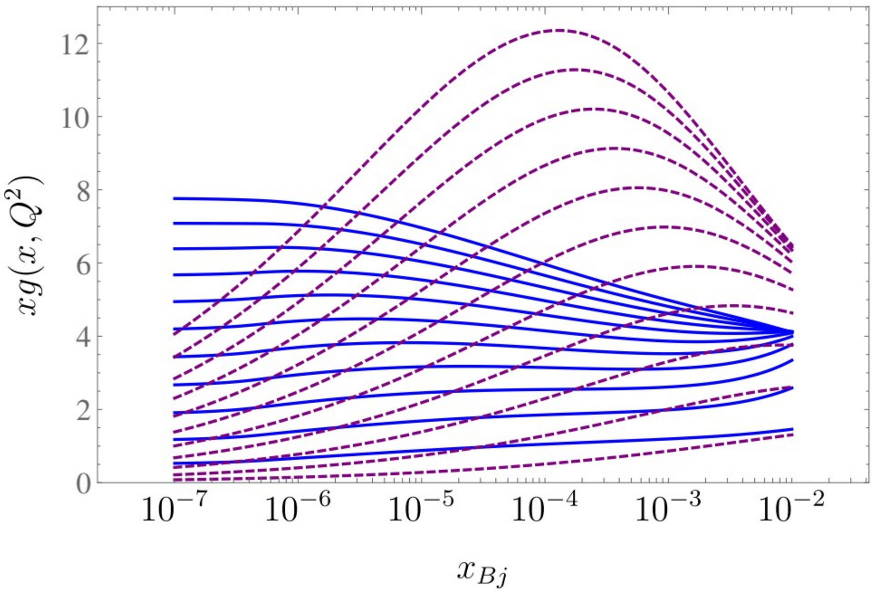
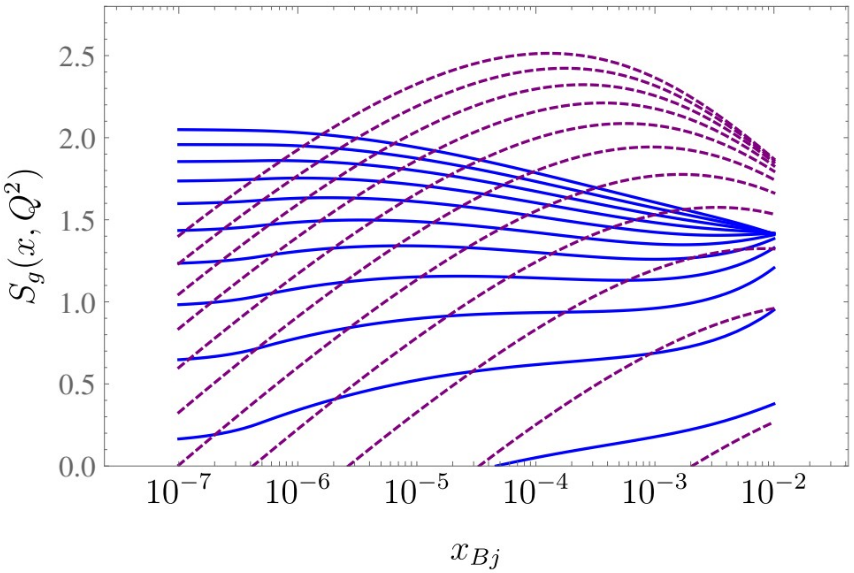
E. Iancu, D.N. Triantafyllopoulos ’05

Bondarenko, Motyka, Mueller, Shoshi, Xiao ‘07

Hagiwara,Hatta,Xiao,Yuan ‘18

Integrated gluon and entropy

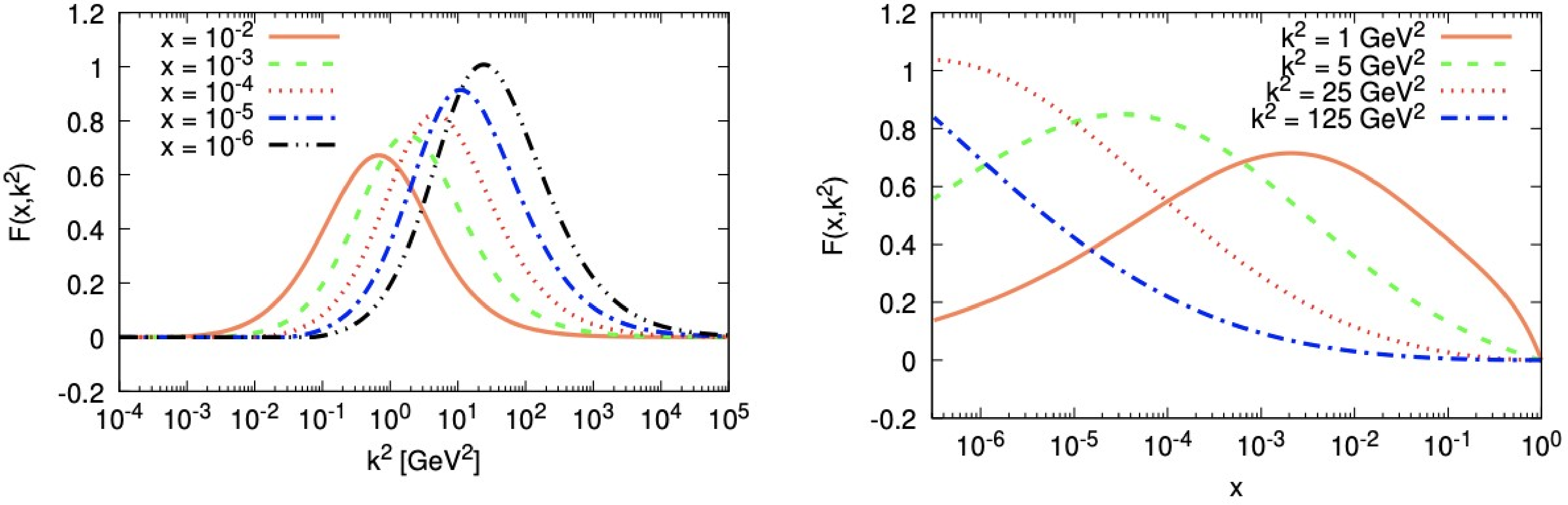




$$S(x) = \ln(xg(x))$$

Photon can not resolve anymore therefore the EE vanishes.
But it might be that the formalism breaks down for low scales.
There might be another source of entropy that keep the total entropy not vanishing →
generalized second law Bekenstein

Gluon density from BK - Saturon



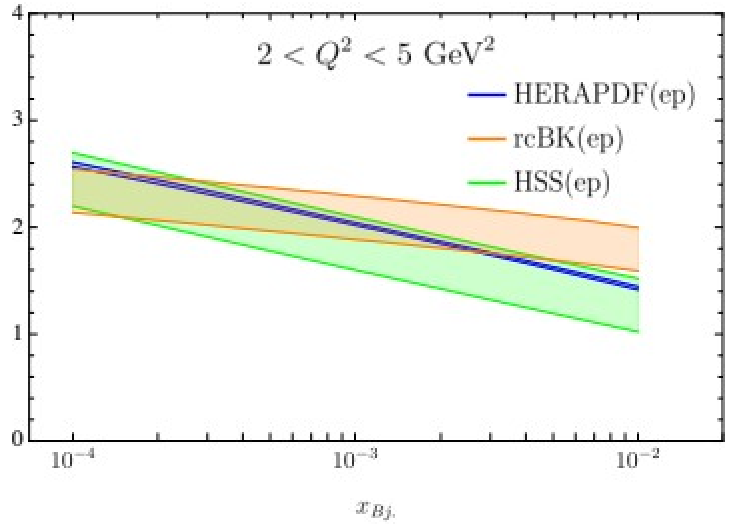
"Saturons are macroscopic objects with maximal microstate entropy" G. Dvali in context of black holes. Area proportional to entropy.

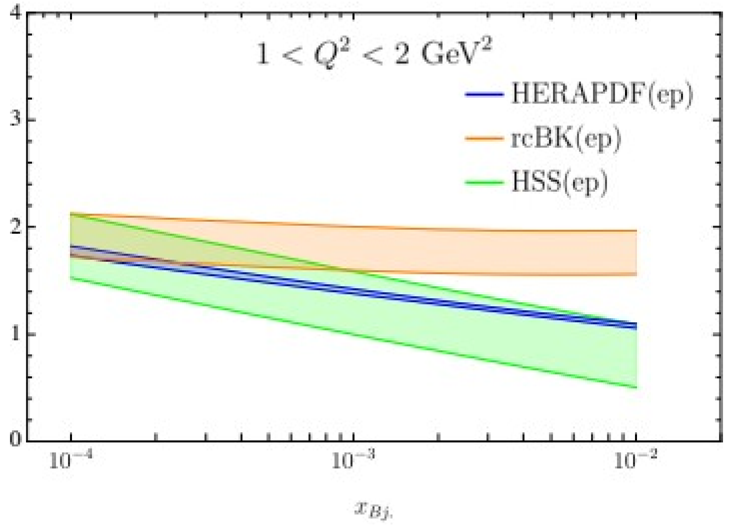
$$S = \frac{6C_F A_\perp}{\pi\alpha_s} Q_s^2(x) + S_0$$

Dipole gluon density has a soliton like shape but it is not a soliton. This term I heard from Leszek Motyka in 2005 and he was referring to private discussion with Maciej Nowak.

Kutak '11 PLB

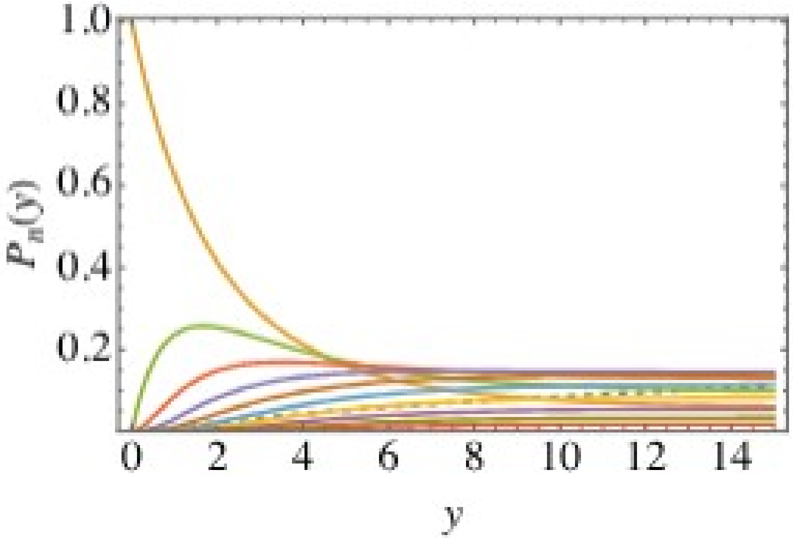
Small scales - prediction

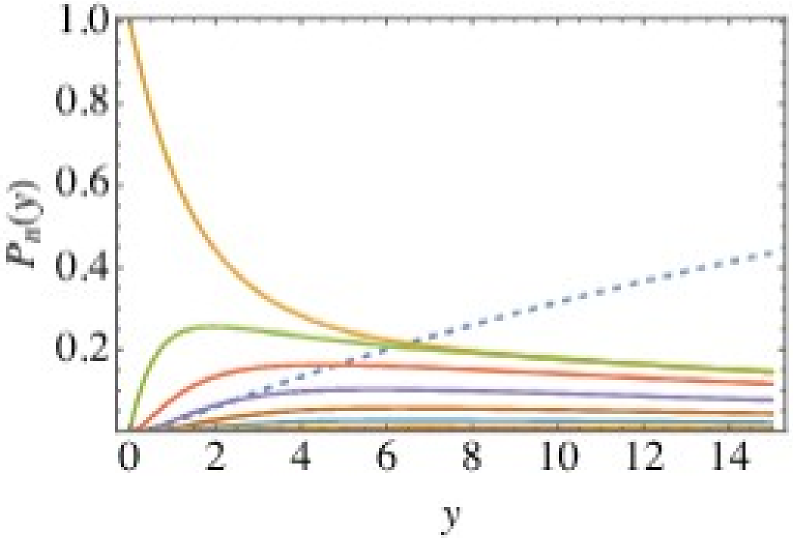




Entropy saturates as a consequence of gluon saturation

Production, recombination and transitions to vacuum – new cascade



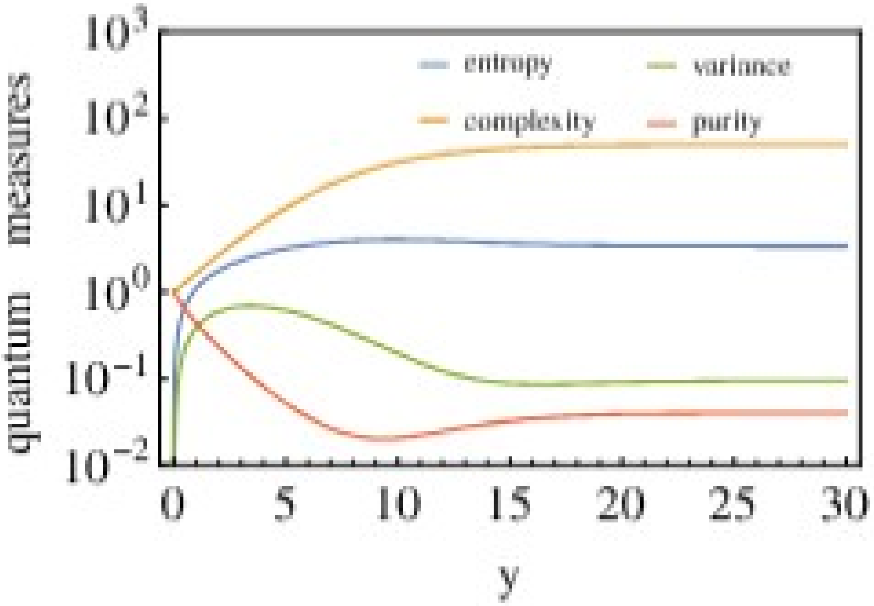


$$\begin{aligned}\partial_y p_n(y) &= -\alpha n p_n(y) + \alpha(n-1)p_{n-1}(y) \\ &+\beta n(n+1)p_{n+1}(y) - \beta n(n-1)p_n(y) \\ &+\gamma(n+1)(n+2)p_{n+2}(y) - \gamma n(n-1)p_n(y)\end{aligned}$$

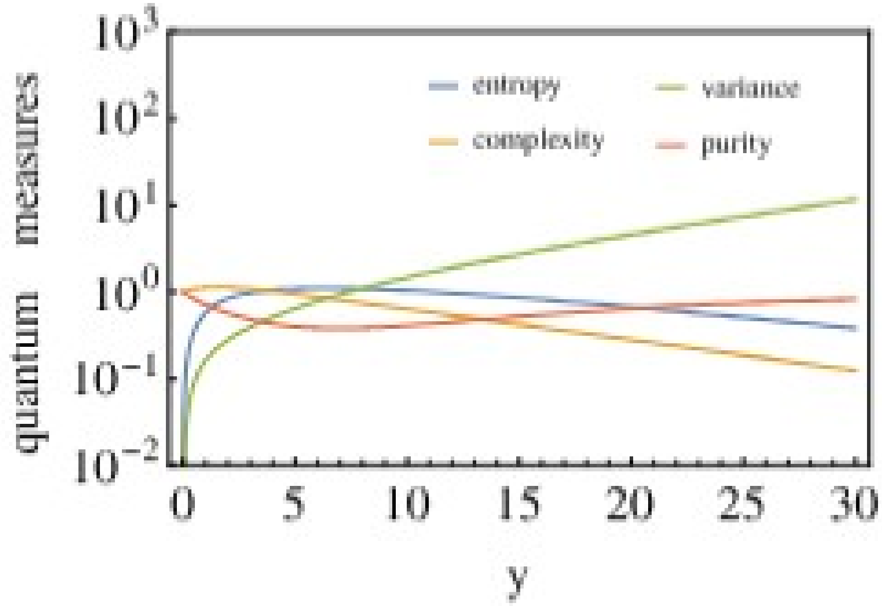
Kutak, Praszałowicz in preparation

Transition to vacuum or unmeasured emissions

Production, recombination and transitions to vacuum – QI measures



β - cascade

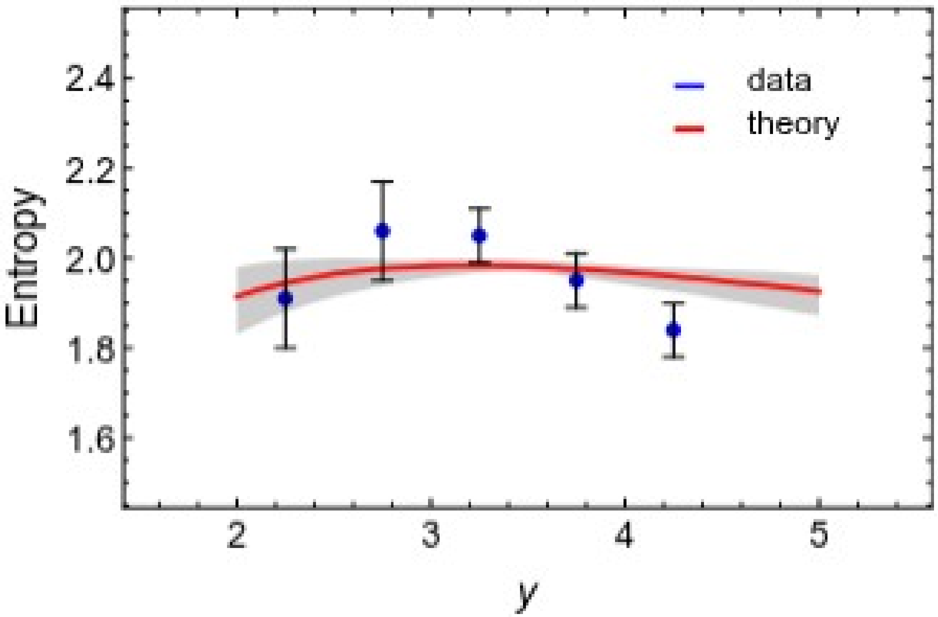


γ - cascade

Kutak, Praszałowicz in preparation

LHCb and entropy in forward direction

y	$\bar{n}_{ch}(y)$	$S(y)$
2.25	2.01 ± 0.12	1.91 ± 0.11
2.75	2.42 ± 0.10	2.06 ± 0.11
3.25	2.41 ± 0.10	2.05 ± 0.06
3.75	2.12 ± 0.09	1.95 ± 0.06
4.25	1.85 ± 0.07	1.84 ± 0.06



LHCb ‘14

Kutak, Praszałowicz in preparation

Conclusions and outlook and comments

- We show further evidences for the proposal for low x maximal entanglement entropy of proton constituents.
- The KL formalism after generalization to narrow rapidity windows describes data
- We related the 1 D evolution equation to equation for probabilities that follow from Lanczos/Krylov construction
- We generalized the dipole eqn. to account for transitions to vacuum and descrined the LHCb data
- We applied various quantum measures to the multiplicity densities of the dipole model. Possibly new handle to look for saturation
- interesting in the context of Electron Ion Collider at BNL and LHC
- Work in progress to get entropy from complete dipole model i.e. accounting for transverse d.o.f

$$[L_0, \phi(z)] = \left(z\frac{d}{dz} + h \right) \phi(z)$$ scaling

$$[L_{-1}, \phi(z)] = \frac{d}{dz}\phi(z)$$ translations

$$[L_1, \phi(z)] = \left(z^2\frac{d}{dz} + 2hz \right) \phi(z)$$ special conformal

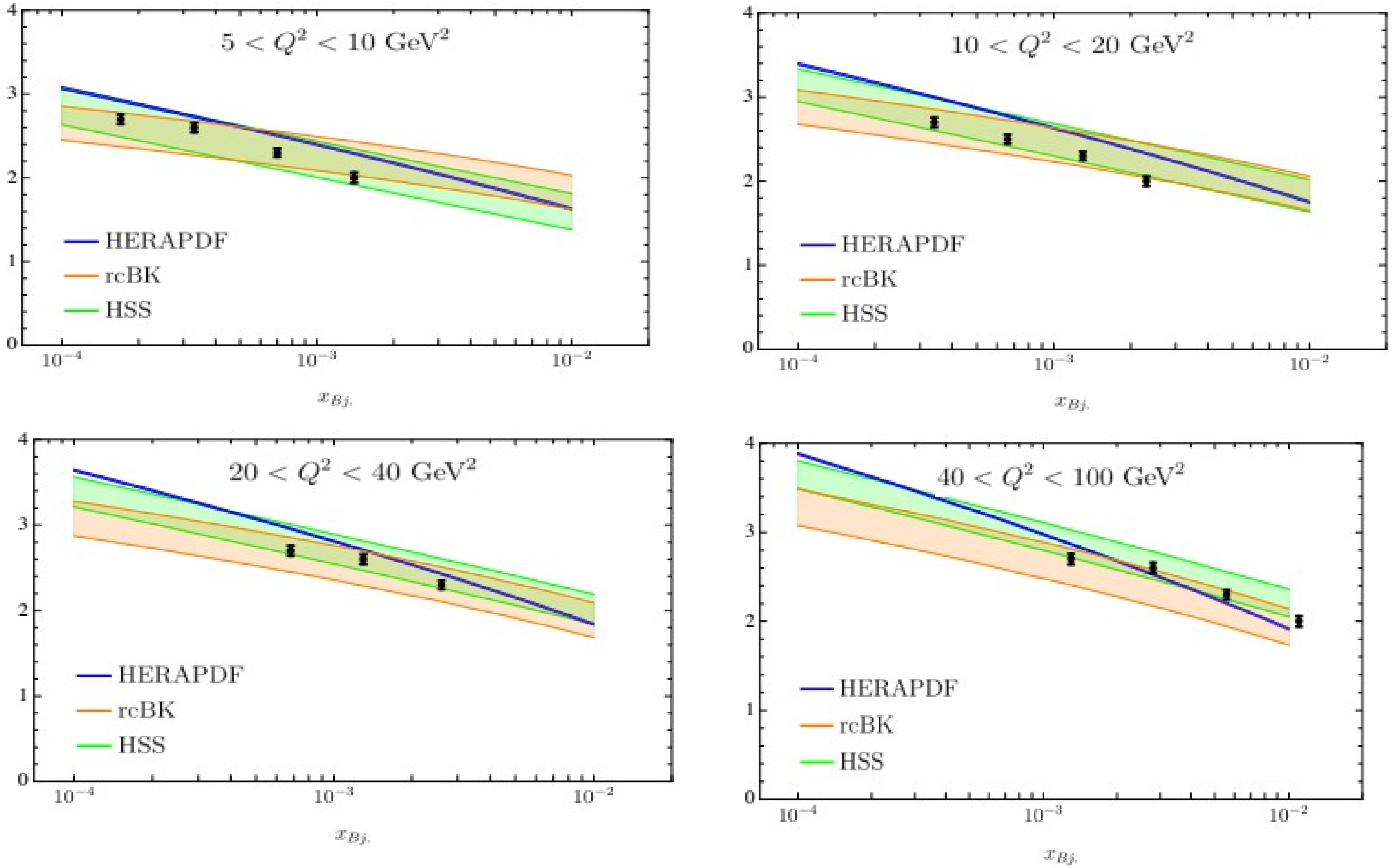
$$L_0 = \frac{1}{2}(a^\dagger a + b^\dagger b + 1), \qquad L_{-1} = a^\dagger b^\dagger, \qquad L_1 = ab$$

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_1, L_{-1}] = 2L_0$$

$$[a, a^\dagger] = [b, b^\dagger] = 1$$

Large scales - description

Martin Hentschinski, K. Kutak, Robert Straka '23



Krylov subspace, complexity – motivation

Simple reference quantum state spreads and becomes complex in Hilbert space

Visvanath, Muller ‘63
Altman, Avdoshkin, Cao, Parker, Scaffidi ‘19
Balasubramanian, Caputa, Magan, Wu ‘22,...

$$i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \qquad |\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle$$

Krylov basis

$$|\Psi(t)\rangle = e^{-iHt} |\Psi_0\rangle = \sum_n \phi_n(t) |K_n\rangle \qquad p_n(t) = |\phi_n(t)|^2 \qquad \sum_n |\phi_n(t)|^2 = 1$$

Comes from studies of efficient diagonalization of matrices and computation of characteristic polynomial coefficients

The complexity has simple form in Krylov basis.
It can be to used to quantify chaotic behavior of quantum systems.

$$\mathcal{C}_K(t) = \langle n \rangle = \sum_n n\, p_n(t)$$

Krylov subspace – construction

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\Psi_n\rangle$$

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

n consequtive application of Hamiltonian

$$|K_0\rangle = |\psi(0)\rangle = |\psi_0\rangle$$

Gram-Schmidt orthogonalization procedure. Construct with K_2 by subtracting the previous two vectors, K_3 by subtracting the previous 3 vectors, and so forth

$$|z_1\rangle = \hat{H}|K_0\rangle - a_0|K_0\rangle \qquad |K_1\rangle = \frac{|z_1\rangle}{\langle z_1|z_1\rangle}$$

$$|z_{n+1}\rangle = (\hat{H} - a_n)|K_n\rangle - b_n|K_{n-1}\rangle$$

Strength of the Lanczos algorithm. n + 1 is determined by n and n – 1. Low memory requirements Visvanath, Muller '63

$$|K_n\rangle = b_n^{-1}|z_n\rangle \qquad b_n = \langle z_n|z_n\rangle^{\frac{1}{2}} \qquad a_n = \langle K_n|\hat{H}|K_n\rangle$$

Project a high-dimensional problem onto a lower-dimensional Krylov subspace

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle$$

$$\langle K_n|K_m\rangle = \delta_{nm}$$

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_n \phi_n(t)|K_n\rangle$$

$$H_{nm} := \langle K_n|\hat{H}|K_m\rangle = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots \\ b_1 & a_2 & b_2 & 0 & \cdots \\ 0 & b_2 & a_3 & b_3 & \cdots \\ 0 & 0 & b_3 & a_4 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

probability amplitudes for each vector

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

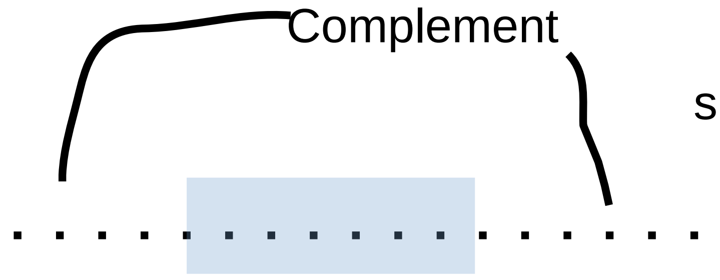
Comments

CFT result for EE

central charge

$$S = \frac{c}{3} \ln \frac{L}{\epsilon}$$

UV cutoff



spnis in 1 D

Region A of length L

Entanglement entropy obtained from CFT
calculations as well as from gravity using
Ryu-Takayanagi formula

See also
Callan, Wilczek '94
Calabrese, Cardy '04

and lectures by
Headrick

Relation to Kharzeev-Levin formula

$$L = (mx)^{-1} \qquad \epsilon \equiv 1/m$$

Length of tube probed in DIS

Proton's Compton wave length

$$S = \ln \left(\frac{1}{x} \right)^{1/3}$$

$$S(x) = \ln(xg(x))$$

Studied also in the context of 2 D QCD

Liu, Nowak, Zahed, '22

Casini, Huerta, Hosco '05

Gluon production and entropy – another assumptions

Kutak '11

Bialas; Janik; Fialkowski, Wit; Iancu, Blaizot, Peschanski,...

$$\phi = \frac{\alpha_s C_F}{\pi}\frac{1}{k^2}$$

$$M_G(x) = Q_s(x)$$ energy dependent gluon's mass

$$M(x) = N_G(x) M_G(x)$$ mass of system of gluons

$$N_G(x) \equiv \frac{dN}{dy} = \frac{1}{S_\perp}\frac{d\sigma}{dy}$$ number of gluons

$$dE = TdS$$

$$dM = TdS$$

Many-body interactions

Medium generated mass of gluon.
Framework of Hard Thermal Loops.

$$d\left[N_G(x)\, M_G(x)\right] = \frac{Q_s(x)}{2\pi} dS$$

Entropy due to less dense hadron

$$S = \frac{6 C_F A_\perp}{\pi \alpha_s} Q_s^2(x) + S_0$$

Similarly in QED. Cut on photon`s kt
Is equivalent to introducing mass.

$$S = 3\pi\left[N_G(x) + N_{G0}\right]$$

In presented approach mass is not fixed it is x dependent