Rapidity factorization and rapidity evolution in QCD

Ian Balitsky

From LHC to EIC
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Outline

1. Part 1: Warm-up talk: one more nice feature of BK and a possible path to NNLO BK in QCD
   - Conformal LO BK at $d = 4$
   - BK equation at Wilson-Fisher point $d = 4 - \varepsilon_*$
2. Part 2: Rapidity-only TMD factorization and power corrections
   - TMD factorization for particle production in hadron collisions.
   - Leading-N_c power corrections for DY hadronic tensor.
   - Results for Z-boson production: angular asymmetries.
   - Estimates for unpolarized SIDIS at EIC.
3. Part 3: Rapidity factorization and rapidity evolution of TMDs:
   - Rapidity-only cutoff *vs* UV+rapidity regularization
   - Rapidity evolution of TMDs in the Sudakov region.
   - Argument of coupling constant by BLM.
   - Rapidity-only factorization at one loop.
4. Conclusions

Warm-up: conformal BK equation at Wilson-Fisher point

Motivation: pQCD as a "nearly conformal" theory

QCD is a far cry from being a conformal theory.

But pQCD is not that far cry...

1. From $\mathcal{N} = 4$ SYM
   - Leading-order evolutions are conformally invariant ($SL(2R)$ for DGLAP evolution and $SL(2C)$ for the BFKL one)
   - The "most transcendental" part of NNLO anomalous dimensions of twist-two operators is the same as in $\mathcal{N} = 4$ SYM
   - Same is true for NLO BFKL pomeron intercept
2. From Wilson-Fisher point at $d = 4 - \varepsilon_*$
   - Anomalous dimensions of operators in $\overline{MS}$ are the same
   - Evolution kernels for light-ray operators are almost $SL(2, R)$ invariant $\Rightarrow$ 3 loops for PDFs $\rightarrow$ 3 loops for GPDs (Braun *el al*)

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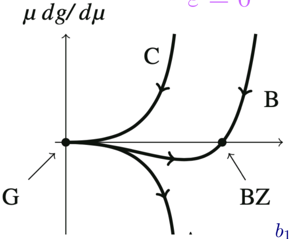
Goal: with NNLO BFKL in sights, what can we learn about NLO BFKL in QCD from $d = 4 - 2\varepsilon_*$ analysis?
(see “High-energy evolution in planar QCD to three loops: the non-conformal contribution” in today’s arXiv submission)

Wilson-Fisher point(s)

$$\frac{1}{g}\beta(g) \;=\; -\epsilon - b_1\frac{g^2}{16\pi^2} - b_2\Big(\frac{g^2}{16\pi^2}\Big)^2 + ..., \qquad b_1 = \big(\frac{11}{3}N_c - \frac{2}{3}n_f\big), \quad b_2 = \big(\frac{34}{3}N_c^2 - \frac{10}{3}N_cn_f - c_fn_f\big)$$

$\varepsilon = 0$



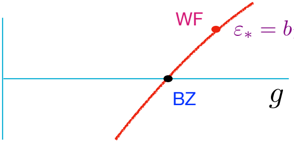
$$A: \qquad n_f < \frac{34N_c^3}{13N_c^2-3} \qquad - \quad \text{asym. freedom}$$

$$C: \qquad n_f > \frac{11N_c}{2} \qquad - \quad \text{like QED}$$

$$B: \qquad \frac{11N_c}{2} > n_f > \frac{34N_c^3}{13N_c^2-3} \quad - \quad \text{Conformal window}$$

$$b_1\frac{g_{\rm BZ}^2}{16\pi^2} + b_2\Big(\frac{g_{\rm BZ}^2}{16\pi^2}\Big)^2 + .. = 0 \qquad \text{Banks} - \text{Zaks point}$$

$\varepsilon = 2 - \frac{d}{2}$



$$\varepsilon_* = b_1\Big(\frac{g_{\rm BZ}^2}{16\pi^2} - \frac{g^2}{16\pi^2}\Big) - b_2\Big(\frac{g_{\rm BZ}^2}{16\pi^2} - \frac{g^2}{16\pi^2}\Big)^2 + ...$$

Wilson-Fisher point(s)

Conformal invariance of the LO BK equation

Formally, a light-like Wilson line

$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \; = \; \text{Pexp}\left\{ig\int_{-\infty}^{\infty} dx^+ \, A_+(x^+, x_\perp)\right\}$$

is invariant under inversion (with respect to the point with $x^- = 0$).

Indeed,
$(x^+, x_\perp)^2 = -x_\perp^2 \Rightarrow$ after the inversion $x_\perp \to x_\perp / x_\perp^2$ and $x^+ \to x^+ / x_\perp^2$

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$$[\infty p_1 + x_\perp, -\infty p_1 + x_\perp] \;\to\; \text{Pexp}\left\{ig\int_{-\infty}^{\infty} d\frac{x^+}{x_\perp^2}\, A_+\Big(\frac{x^+}{x_\perp^2}, \frac{x_\perp}{x_\perp^2}\Big)\right\} \;=\; [\infty p_1 + \frac{x_\perp}{x_\perp^2}, -\infty p_1 + \frac{x_\perp}{x_\perp^2}]$$

$\Rightarrow$The dipole kernel is invariant under the inversion $V(x_\perp) = U(x_\perp / x_\perp^2)$

$$\frac{d}{d\eta}\text{Tr}\{V_x V_y^\dagger\} = \frac{\alpha_s}{2\pi^2}\int \frac{d^2 z}{\not{z}^4}\, \frac{(x-y)^2\ \not{z}^4}{(x-z)^2(z-y)^2}[\text{Tr}\{V_x V_z^\dagger\}\text{Tr}\{V_z V_y^\dagger\} - N_c \text{Tr}\{V_x V_y^\dagger\}]$$

BFKL/BK equation at $d \neq 4$

Leading-order calculation yields

$$\frac{d}{d\eta}\mathcal{U}(x,y) = \frac{\alpha_s N_c \mu^{-2\epsilon}}{2\pi^{2-2\epsilon}}\Gamma^2(1-\epsilon)\int d^{2-2\epsilon}z_\perp\Big[\frac{X_i}{(X^2)^{1-\epsilon}} - \frac{Y_i}{(Y^2)^{1-\epsilon}}\Big]\Big[\frac{X_i}{(X^2)^{1-\epsilon}} - \frac{Y_i}{(Y^2)^{1-\epsilon}}\Big]$$
$$\times \big[\mathcal{U}(x,z)+\mathcal{U}(z,y)-\mathcal{U}(x,y)-\mathcal{U}(x,z)\mathcal{U}(z,y)\big] \qquad X\equiv x_\perp - z_\perp,\ Y\equiv y_\perp - z_\perp$$

Common wisdom:
The solutions of BFKL equation at $d_\perp = 2-2\epsilon$ are not known.
$\Rightarrow$ BFKL/BK equation at $d \neq 4$ is useless.

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$$\times \big[\mathcal{U}(x,z) + \mathcal{U}(z,y) - \mathcal{U}(x,y) - \mathcal{U}(x,z)\mathcal{U}(z,y)\big] \qquad X \equiv x_\perp - z_\perp,\ Y \equiv y_\perp - z_\perp$$

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Solutions at $d_\perp = 2 - 2\varepsilon_*$ with $\varepsilon_* \simeq -\frac{\alpha_s}{4\pi}b_1 - \frac{\alpha_s^2}{16\pi^2}b_1$ such that $\beta(\alpha_s) = 0$ are again powers due to conformal invariance!

NLO BK equation at Wilson-Fisher point $d = 4 - \varepsilon_*$

$$\frac{d}{d\eta}\mathcal{U}(x,y)$$

$$\stackrel{\rm NLO}{=} \frac{\alpha_s N_c \mu^{-2\epsilon}}{2\pi^{2-2\varepsilon_*}}\Gamma^2(1-\varepsilon_*)\int d^{2-2\varepsilon_*}z_\perp\Big[\frac{X_i}{(X^2)^{1-\varepsilon_*}} - \frac{Y_i}{(Y^2)^{1-\varepsilon_*}}\Big]\Big[\frac{X_i}{(X^2)^{1-\varepsilon_*}} - \frac{Y_i}{(Y^2)^{1-\varepsilon_*}}\Big]$$

$$+ b_0\frac{\alpha_s}{4\pi}\Big(\frac{(x-y)^2}{X^2Y^2}\ln(x-y)^2\mu^2 - \frac{X^2-Y^2}{X^2Y^2}\ln\frac{X^2}{Y^2}\Big)$$

$$\times\big[\mathcal{U}(x,z)+\mathcal{U}(z,y)-\mathcal{U}(x,y)-\mathcal{U}(x,z)\mathcal{U}(z,y)\big]\Big] + \Big(\frac{\alpha_s}{4\pi^2}\Big)^2\{\text{conformal part}\}$$

$$= \frac{\alpha_s N_c}{2\pi^{2-\varepsilon_*}}\Gamma(1-\varepsilon_*)\int d^{2-2\varepsilon_*}z_\perp\Big(\frac{(x-y)_\perp^2}{X^2Y^2}\Big)^{1-\varepsilon_*}\big[\mathcal{U}(x,z)+\mathcal{U}(z,y)-\mathcal{U}(x,y)-\mathcal{U}(x,z)\mathcal{U}(z$$

$$+ \frac{\alpha_s^2}{16\pi^4}\{\text{conformal part}\}$$

For now, this "{conformal part}" is known only at $d = 4$.

NLO evolution of “composite dipoles” in QCD

I. B. and G. Chirilli

$$a\frac{d}{da}[\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c\mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}]^{\mathrm{comp}}_a$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{z_{14}^2z_{23}^2+z_{24}^2z_{13}^2-4z_{12}^2z_{34}^2}{2(z_{14}^2z_{23}^2-z_{24}^2z_{13}^2)}\ln\frac{z_{14}^2z_{23}^2}{z_{24}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_3}U_{z_4}U^\dagger_{z_2}U_{z_3}U^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{23}^2z_{14}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{23}^2z_{14}^2}\Big)\ln\frac{z_{13}^2z_{24}^2}{z_{23}^2z_{14}^2}\Big]$$

$$\times[\mathrm{tr}\{U_{z_1}U^\dagger_{z_3}\}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}\mathrm{tr}\{U_{z_4}U^\dagger_{z_2}\} - \mathrm{tr}\{U_{z_1}U^\dagger_{z_4}U_{z_3}U^\dagger_{z_2}U_{z_4}U^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b=\tfrac{11}{3}N_c-\tfrac{2}{3}n_f$$

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Pomeron intercept at $d = 4 - 2\varepsilon_*$

Linear BFKL evolution of "forward" dipole $\mathcal{U}(x,y) = \mathcal{U}(x-y)$

$$\frac{d}{d\eta}\mathcal{U}(x) \;=\; \frac{\alpha_s N_c}{2\pi^{2-\varepsilon_*}}\Gamma(1-\varepsilon_*)\int d^{2-2\varepsilon_*}z\Big(\frac{x^2}{(x-z)^2 z^2}\Big)^{1-\varepsilon_*}\big[\mathcal{U}(x-z)+\mathcal{U}(z)-\mathcal{U}(x)\big]$$

Eigenvalues of BFKL eqn

$$\frac{\Gamma(d/2)}{2\pi^{d/2}}\int d^d z\big[2(z^2)^{\frac{d}{4}+i\nu}-(x^2)^{\frac{d}{4}+i\nu}\big] \;=\; \psi\big(\frac{d}{2}\big)-\psi\big(\frac{d}{4}+i\nu\big)-\psi\big(\frac{d}{4}-i\nu\big)-\gamma_E$$

Pomeron contribution to the evolution of $\mathcal{U}^\eta(x)$

$$U^{\eta_1}(x) \;=\; \frac{\Gamma(d/2)}{\pi^{d/2}}\int d^d z\int\frac{d\nu}{2\pi}\;(x^2)^{\frac{d}{4}+i\nu}(z^2)^{-\frac{3d}{4}-i\nu}U^{\eta_2}(z)\;e^{\aleph(\nu)(\eta_1-\eta_2)}$$

where $\aleph(\alpha_s,\nu,\varepsilon_*)$ is the pomeron intercept

$$\aleph(\alpha_s,\nu,\varepsilon_*) \;=\; \frac{\alpha_s N_c}{\pi}\Big[\psi(1-\varepsilon_*)-\psi\big(\frac{1}{2}-\frac{\varepsilon_*}{2}+i\nu\big)-\psi\big(\frac{1}{2}-\frac{\varepsilon_*}{2}-i\nu\big)-\gamma_E\Big] \;+\; O(\alpha_s^2)$$

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Eigenvalues of BFKL eqn

$$\frac{\Gamma(d/2)}{2\pi^{d/2}}\int d^dz\big[2(z^2)^{\frac{d}{4}+i\nu}-(x^2)^{\frac{d}{4}+i\nu}\big] \;=\; \psi\big(\frac{d}{2}\big)-\psi\big(\frac{d}{4}+i\nu\big)-\psi\big(\frac{d}{4}-i\nu\big)-\gamma_E$$

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$$\aleph(\alpha_s,\nu,\varepsilon_*) \;=\; \frac{\alpha_s N_c}{\pi}\Big[\psi(1-\varepsilon_*)-\psi\big(\frac{1}{2}-\frac{\varepsilon_*}{2}+i\nu\big)-\psi\big(\frac{1}{2}-\frac{\varepsilon_*}{2}-i\nu\big)-\gamma_E\Big] \;+\; O(\alpha_s^2)$$

Pomeron intercept is different but the corresponding anomalous dimensions are the same (G.A. Chirilli and I.B., last year)

Part 2: Rapidity-only TMD factorization and power corrections

TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$

$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(ff \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

Examples: Drell-Yan process with Q being the mass of DY pair and Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

The quantities $\mathcal{D}_{f/A}(x_A, k_\perp)$, $\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)$, and $C(q, k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A, k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS to small x. (Recently: LO BFKL from SCET)

It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

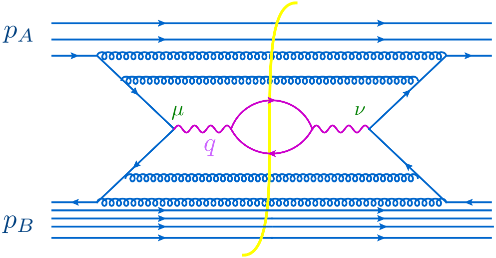
Example: full list of power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor.

They are not obtained (yet?) by CSS or SCET

Classical example: DY hadronic tensor

DY cross section is given by the product of leptonic tensor and hadronic tensor. The hadronic tensor $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}(p_A, p_B, q) = \frac{1}{(2\pi)^4}\int d^4x\, e^{-iqx}\langle p_A, p_B|J_\mu(x)J_\nu(0)|p_A, p_B\rangle$$



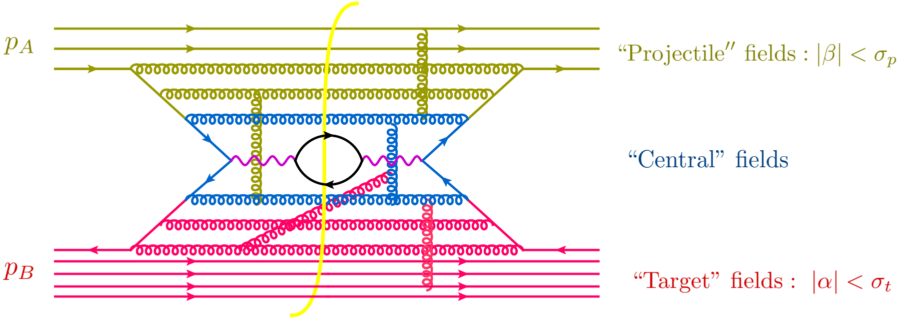
p_A, p_B = hadron momenta, q = the momentum of DY pair, and J_μ is the electromagnetic or Z-boson current.

There are four tensor structures W_T, W_L, W_Δ, $W_{\Delta\Delta}$

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p \; = \; \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A, \;\; p_2 \simeq p_B, \;\; p_1^2 = p_2^2 = 0$$



The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2}$ $\Rightarrow$ power corrections

TMD representation for W_i

The hadronic tensor in the Sudakov region $q^2 \equiv Q^2 \gg q_\perp^2$ can be studied by TMD factorization. For example, functions W_T and $W_{\Delta\Delta}$ can be represented as

$$W_i = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}^{(i)}_{f/A}(x_A, k_\perp)\mathcal{D}^{(i)}_{f/B}(x_B, q_\perp - k_\perp)C_i(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ Y} - \text{terms}$$

There is, however, a problem with this equation for the functions W_L and W_Δ.

W_T and $W_{\Delta\Delta}$ are determined by leading-twist quark TMDs, but W_Δ and W_L start from terms $\sim \frac{q_\perp}{Q}$ and $\sim \frac{q_\perp^2}{Q^2}$ determined by quark-quark-gluon TMDs.

The power corrections $\sim \frac{q_\perp}{Q}$ were found more than two decades ago but there was no calculation of power corrections $\sim \frac{q_\perp^2}{Q^2}$ until recently.

Goal: TMD factorization formula

TMD factorization formula structure :

$$\langle p_A', p_B'|J(x_1)J(x_2)|p_A, p_B\rangle = \sum_{\text{TMD operators}} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_i(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t)$$
$$\times \langle p_A'|\hat{\mathcal{O}}_i^{\sigma_p}(z_2^-, x_{2_\perp}; z_1^-, x_{1_\perp})|p_B\rangle\langle p_B'|\hat{\mathcal{O}}_i^{\sigma_t}(z_2^+, x_{2_\perp}; z_1^+, x_{1_\perp})|p_B'\rangle$$

$q_\perp^2 \ll Q^2 \Rightarrow$ no dynamics in the transverse space (to be demonstrated below)

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "projectile" TMD operators with $\beta < \sigma_p$ cutoff, e.g

$$\mathcal{O}(z_{1-}, z_{1_\perp}, z_{2-}, z_{2_\perp}) \equiv \bar\psi(z_{1-}, z_{1_\perp})[z_{1-}, -\infty]_{z_{1_\perp}}\Gamma[-\infty, z_{2+}]_{z_2}\psi(z_{2+}, z_{2_\perp})$$

$\hat{\mathcal{O}}_i^{\sigma_p}$ - "target" TMD operators with $\alpha < \sigma_t$ cutoff, e.g

$$\mathcal{O}(z_{1+}, z_{1_\perp}, z_{2+}, z_{2_\perp}) \equiv \bar\psi(z_{1+}, z_{1_\perp})[z_{1+}, -\infty]_{z_{1_\perp}}\Gamma[-\infty, z_{2+}]_{z_{2_\perp}}\psi(z_{2+}, z_{2_\perp}).$$

Standard notation for straight-line gauge link

$$[x, y] \equiv \mathrm{P}e^{ig\int_0^1 du\ (x-y)^\mu A_\mu(ux+(1-u)y)} - \text{gauge link}$$

Convenient notations

$$[x_+, y_+]_{z_\perp} \equiv [x_+, 0_-, z_\perp; y_+, 0_-, z_\perp], \quad [x_-, y_-]_{z_\perp} \equiv [x_-, 0_+, z_\perp; y_-, 0_+, z_\perp]$$

Means: "double operator expansion"

Intermediate step: double operator expansion

$$\hat{J}(x_1)\hat{J}(x_2) = \sum_{I,J} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t) \times \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-, x_{2_\perp}; z_1^-, x_{1_\perp}) \hat{\mathcal{O}}_J^{\sigma_t}(z_2^+, x_{2_\perp}; z_1^+, x_{1_\perp})$$

To find relevant operators and coefficients, it is convenient to consider "matrix" elements of the l.h.s. and r.h.s. in suitable background field

Suitable field $\mathbb{A}$: solution of classical YM equations with boundary condition that at the remote past the field is a sum of projectile and target fields

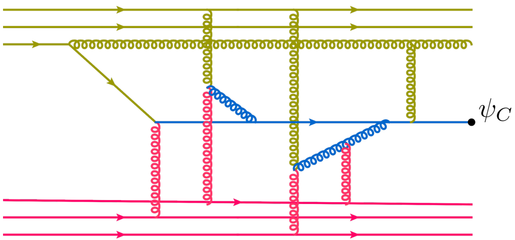
$$\langle \hat{J}(x_1)\hat{J}(x_2)\rangle_{\mathbb{A}} = \sum_{I,J} \int dz_1^- dz_2^- dw_1^+ dw_2^+ \mathfrak{C}_{IJ}(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t) \times \langle \hat{\mathcal{O}}_I^{\sigma_p}(z_2^-, x_{2_\perp}; z_1^-, x_{1_\perp}) \hat{\mathcal{O}}_J^{\sigma_t}(z_2^+, x_{2_\perp}; z_1^+, x_{1_\perp})\rangle_{\mathbb{A}}$$

Method of solution:

- Start with $\Psi_{\text{trial}} = \psi_A + \psi_B$ and $\mathbb{A}_{\text{trial}} = \bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0$, $A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(A + B)$

ψ_C in the tree approximation

It is convenient to choose projectile/target fields as
Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp),\ \psi_A(x^-, x_\perp)$
Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp),\ \psi_B(x^-, x_\perp)$



Classical background fields: Ψ , $\mathbb{A}_\mu$
ψ_C = sum of tree diagrams in external A, ψ_A and B, ψ_B fields with sources
$J_\psi = (\not{P} + m)(\psi_A + \psi_B)$, $\qquad J_\nu = D^\mu F^{\mu\nu}(A + B)$

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of such YM equations in general case is yet unsolved problem (goes under the name "glasma" $\Leftrightarrow$ scattering of two "color glass condensates").

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\Psi, \mathbb{A} \ = \ \text{series in } \frac{q_\perp}{Q}: \qquad \Psi = \psi^{(0)} + \psi^{(1)} + ..., \qquad \mathbb{A} = A^{(0)} + A^{(1)} + ...$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} \ = \ \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} \ = \ \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} \ + ...$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times\left(\frac{q_\perp}{Q}\text{ or }\frac{q_\perp^2}{Q^2}\right)\times\left(1+\frac{1}{N_c}+\frac{1}{N_c^2}\right)$.

NB: almost all $\bar{q}Gq$ TMDs not suppressed by $\frac{1}{N_c}$ are determined by the $\bar{q}q$ TMDs due to QCD equations of motion

Leading twist (for the projectile nucleon): $\varrho\equiv\sqrt{s/2}$

$$\frac{1}{8\pi^3 s}\int dx^- d^2x_\perp\, e^{-i\alpha\varrho x^- + i(k,x)_\perp}\,\langle N|\hat{\bar{\psi}}(x^-,x_\perp)\,\not{p}_2\hat{\psi}(0)|N\rangle\;=\;f_1(\alpha,k_\perp^2)$$

Power correction:

$$\frac{1}{8\pi^3 s}\int dx^- dx_\perp\, e^{-i\alpha\varrho x^- + i(k,x)_\perp}$$
$$\times\,\langle N|\hat{\bar{\psi}}(x^-,x_\perp)\hat{\not{A}}(x^-,x_\perp)\not{p}_2\gamma_i\hat{\psi}(0)|N\rangle$$
$$=\;k_i f_1(\alpha,k_\perp)-\alpha k_i\big[f_\perp(\alpha,k_\perp)+ig^\perp(\alpha,k_\perp)\big],$$

(Mulders & Tangerman, 1996)

Application: angular coefficients of Z-boson production

In CMS and ATLAS experiments $s = 8$ TeV, $Q = 80 - 100$ GeV and $Q_\perp$ varies from 0 to 120 GeV.

Our analysis is valid at $Q_\perp = 10 - 30$ GeV and $Y \simeq 0$ ($x_A \sim x_B \sim 0.1$) so that power corrections are small but sizable.

Angular distribution of DY leptons in the Collins-Soper frame ($c_\phi \equiv \cos\phi$, $s_\phi \equiv \sin\phi$ etc.)

$$\frac{d\sigma}{dQ^2 dy d\Omega_l} = \frac{3}{16\pi}\frac{d\sigma}{dQ^2 dy}\Big[(1+c_\theta^2) + \frac{A_0}{2}(1-3c_\theta^2) + A_1 s_{2\theta}c_\phi + \frac{A_2}{2}s_\theta^2 c_{2\phi} + A_3 s_\theta c_\phi + A_4 c_\theta + A_5 s_\theta^2 s_{2\phi} + A_6 s_{2\theta}s_\phi + A_7 s_\theta s_\phi\Big]$$

Back-of-the envelope estimation: take only f_1 contribution at large N_c, use "factorization hypothesis" for TMD $f_1(x,k_\perp) \simeq f(x)g(k_\perp)$ and calculate integrals over $k_\perp$ in the leading log approximation using $f_1(x,k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$

Comparison of A_0 with LHC results

Logarithmic estimate of A_0 (m_z - Z-boson mass, m - proton mass)

$$A_0 \;=\; \frac{Q_\perp^2}{m_z^2}\,\frac{1+2\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}}{1+\frac{Q_\perp^2}{m_z^2}\,\frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (*)$$

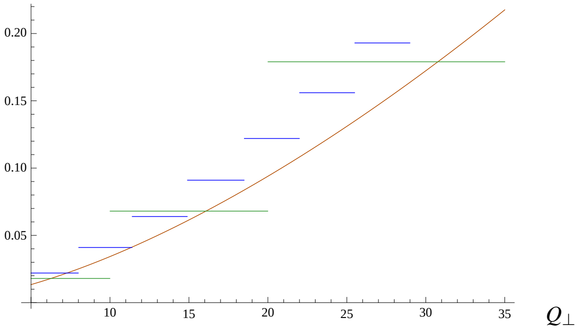


Figure: Comparison of prediction $(*)$ with lines depicting angular coefficient A_0 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Comparison of A_2 with LHC results

Logarithmic estimate of A_2

$$A_2 = \frac{Q_\perp^2}{m_z^2} \frac{1}{1 + \frac{Q_\perp^2}{m_z^2} \frac{\ln m_z^2/Q_\perp^2}{\ln Q_\perp^2/m^2}} \qquad (**)$$

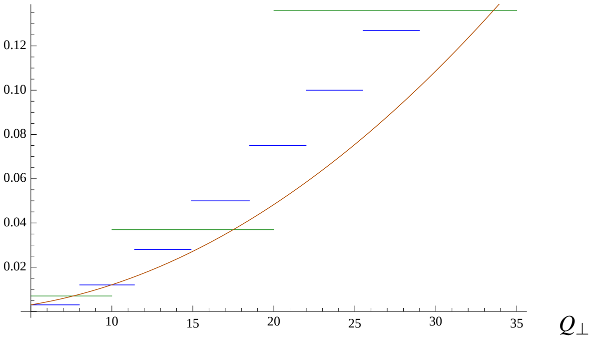


Figure: Comparison of prediction $(**)$ with lines depicting angular coefficient A_2 in bins of $Q_\perp$ and $Y < 1$ from CMS (arXiv:1504.03512) and ATLAS (arXiv1606.00689)

Result of estimation

Result of the estimation: A_0 and A_2 in agreement with data, rest of $A_i = 0$.
Experimentally, at $q_\perp \leq \frac{1}{4}m_z$ other A_i are an order of magnitude smaller that $A_0 \sim A_2$
$\Rightarrow$ it looks like f_1 is numerically the most important TMD for unpolarized DY cross sections.

Application: unpolarized SIDIS

Result for power corrections in SIDIS: $\mathrm{PC}^{\mathrm{SIDIS}}(q) = \mathrm{PC}^{\mathrm{DY}}(-q)$ with replacements $f_1(-\alpha_q, -q_\perp - k_\perp) \to \bar{D}_1\Big(z = \frac{1}{\alpha_q}, q_\perp + k_\perp\Big)$,
$h_1^\perp(-\alpha_q, -q_\perp - k_\perp) \to -\bar{H}_1^\perp\Big(z = \frac{1}{\alpha_q}, q_\perp + k_\perp\Big)$ *etc.*

Hopefully, our analysis is valid for EIC kinematics at $Q \geq 10$ GeV and $Q_\perp \sim 3$GeV so that power corrections are small but sizable.

The unpolarized cross section is parametrized by four functions $F_{UU,T}, F_{UU,L}, F_{UU}^{\cos\phi_h}, F_{UU,T}^{\cos 2\phi_h}$

Estimation: similarly to the DY case, take only f_1 and D_1 contribution at large N_c,

$$
\begin{aligned}
F_{UU,T} &= xz\int dk_\perp\Big(1 - \frac{2(q,k)_\perp}{Q^2}\Big)\Phi(q,k_\perp)\\
F_{UU,L} &= x\int dk_\perp \frac{4k_\perp^2}{Q^2}\Phi(q,k_\perp)\\
F_{UU}^{\cos\phi_h} &= x\int dk_\perp \frac{2(q,k)_\perp}{Qq_\perp}\Phi(q,k_\perp)\\
F_{UU}^{\cos 2\phi_h} &= -x\int dk_\perp \frac{2(q,k)_\perp}{Q^2}\Phi(q,k_\perp) \qquad \Rightarrow \quad F_{UU}^{\cos 2\phi_h} = -\frac{q_\perp}{Q}F_{UU}^{\cos\phi_h}
\end{aligned}
$$

$$
\Phi(q,k_\perp) \equiv D_1(z, q_\perp + k_\perp)f_1(x,k_\perp) + \bar{D}_1(z, q_\perp + k_\perp)\bar{f}_1(x,k_\perp)
$$

Back-of-the-envelope estimation

To understand the magnitude of power corrections, define

$$R_{UU,T} = \frac{F_{UU,T}}{F^{\text{l.t.}}_{UU,T}} - 1, \quad R_{UU,L} = \frac{F_{UU,L}}{F^{\text{l.t.}}_{UU,T}}, \quad R^{\cos\phi_h}_{UU} = \frac{F^{\cos\phi_h}_{UU}}{F^{\text{l.t.}}_{UU,T}}, \quad R^{\cos 2\phi_h}_{UU} = \frac{F^{\cos 2\phi_h}_{UU}}{F^{\text{l.t.}}_{UU,T}}$$

Back-of-the envelope estimation: similarly to the DY case, take only f_1 and D_1 contribution at large N_c, use "factorization hypothesis" for TMD PDFs and FFs $\phi(x,k_\perp) \simeq \phi(x)\psi(k_\perp)$ and calculate integrals over $k_\perp$ in the leading log approximation using $f_1(x,k_\perp^2) \simeq \frac{f(x)}{k_\perp^2}$ and $D_1(z,k_\perp^2) \simeq \frac{D(z)}{k_\perp^2}$

Result:

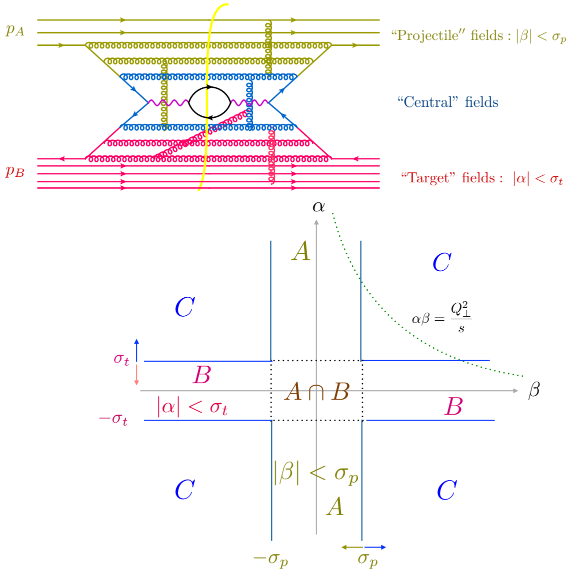
$$R_{UU,T} = R^{\cos 2\phi_h}_{UU} = \frac{q_\perp^2}{Q^2}, \quad R^{\cos\phi_h}_{UU} = -\frac{q_\perp}{Q}$$

$$R_{UU,L} = 2\frac{q_\perp^2}{Q^2} + \frac{\int dk_\perp 4(k,q+k)_\perp \Phi(q,k)}{Q^2 \int dk_\perp \Phi(q,k)} \simeq 2\frac{q_\perp^2}{Q^2}\frac{\ln\frac{Q^2}{m^2}}{\ln\frac{q_\perp^2}{m^2}}$$

This estimate does not depend on z and x.

Part 3: Rapidity factorization and rapidity evolution of TMDs

Rapidity-only cutoffs and matching of logs



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons
$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons cancel out

Rapidity-only cutoff

Typical diagram in the background field $\Psi(\beta_B, p_{B_\perp}) = \varrho\int dz^+ dz_\perp\ \Psi(z^+, z_\perp) e^{i\varrho\beta_B z^+ - i(p_B,z)_\perp}$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp)\rangle_\Psi = g^2 c_F \int d\!\!{}^-\beta_B d\!\!{}^- p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp})$$

$$\times \int_0^\infty d\!\!{}^-\alpha \int \frac{d\!\!{}^- p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,\Delta)}}{\alpha\beta_B s + (p-p_B)_\perp^2 + i\epsilon} \qquad \leftarrow \text{ divergent as } \alpha \to \infty$$

$$\langle [x^+, -\infty]_x \Gamma\psi(y^+, y_\perp, -\delta^-)\rangle_\Psi = g^2 c_F \int d\!\!{}^-\beta_B d\!\!{}^- p_{B_\perp} e^{-ip_B y}\Gamma\Psi(\beta_B, p_{B_\perp}) \qquad \sigma \equiv \frac{1}{\varrho\delta^-}$$

$$\times \int_0^\infty d\!\!{}^-\alpha \int \frac{d\!\!{}^- p_\perp}{p_\perp^2} \frac{\beta_B s e^{-i\frac{p_\perp^2}{\alpha s}\varrho\Delta^+ + i(p,x-y)_\perp}}{\alpha\beta_B s + (p-p_B)_\perp^2 + i\epsilon} e^{-i\frac{\alpha}{\sigma}} \qquad \leftarrow \text{ convergent as } \alpha \to \infty$$

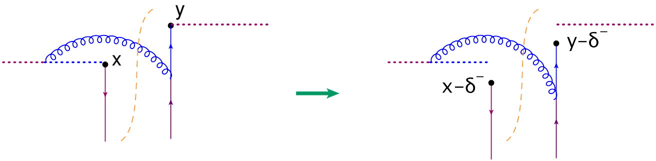


Figure: Point-splitting visualization of "smooth" rapidity-only cutoff.

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\not{d}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon} \int \not{d}\alpha \not{d}\beta \not{d}p_\perp \frac{1}{\beta - i\epsilon} \frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon} \frac{s(\beta - \beta_B)}{\alpha(\beta - \beta_B)s - p_\perp^2 + i\epsilon} \left(1 - e^{i(p,x)_\perp}\right)$$

$$= \mu^{-2\varepsilon} \int \frac{\not{d}p_\perp}{p_\perp^2} \left(1 - e^{i(p,x)_\perp}\right) \int_0^{\beta_B} \frac{\not{d}\beta}{\beta_B} \frac{\beta_B - \beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2} \frac{\Gamma(\varepsilon)}{(x_\perp^2 \mu^2)^\varepsilon} \int_0^{\beta_B} \frac{d\beta}{\beta_B} \frac{\beta_B - \beta}{\beta - i\epsilon}$$

Regularization with $A^-(z^+) \to A^-(z^+) e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2} \frac{\Gamma(\varepsilon)}{(x_\perp^2 \mu^2)^\varepsilon} \int_0^{\beta_B} \frac{d\beta}{\beta_B} \frac{\beta_B - \beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2} \left(-\frac{1}{\varepsilon} + \ln \mu^2 \frac{x_\perp^2}{4} + \gamma_E\right) \left(\ln \frac{\beta_B}{-i\delta} - 1\right)$$

Rapidity-only cutoff

$$-i \int \not{d}\alpha \not{d}\beta \not{d}p_\perp \frac{1}{\beta - i\epsilon} \frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon} \frac{s(\beta - \beta_B)}{\alpha(\beta - \beta_B)s - p_\perp^2 + i\epsilon} \left(1 - e^{i(p,x)_\perp}\right)$$

$$= \int \frac{\not{d}p_\perp}{p_\perp^2} \left(1 - e^{i(p,x)_\perp}\right) \int_0^\infty \not{d}\alpha \frac{\beta_B s}{\alpha\beta_B s + p_\perp^2} e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2} \ln^2\left(-i\beta_B \sigma s \frac{x_\perp^2}{4} e^{\gamma_E}\right)$$

Rapidity evolution of TMDs

Quark TMD operator

$$\mathcal{O}(z_{1+}, z_{1_\perp}, z_{2+}, z_{2_\perp}) \equiv \bar{\psi}(z_{1+}, z_{1_\perp})[z_{1+}, -\infty]_{z_1}\Gamma[-\infty, z_{2+}]_{z_2}\psi(z_{2+}, z_{2_\perp})$$

Sudakov regime: $Q^2 \gg Q_\perp^2 \Leftrightarrow z_{12+}z_{12-} \ll z_{12_\perp}^2$

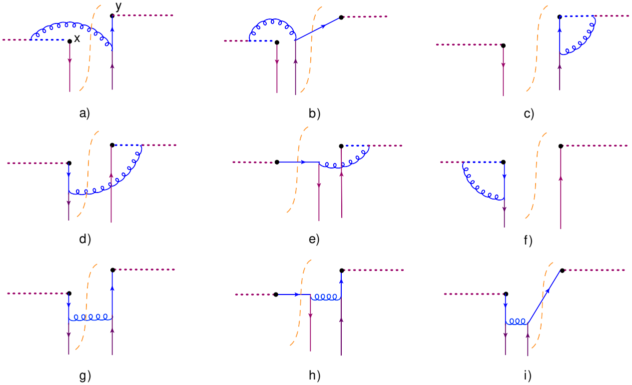


Figure: Diagrams for leading-order rapidity evolution of quark TMD in the Sudakov regime.

Conformally invariant evolution at the LO (G.A. Chirilli & I.B.,2019)

Evolution equation ($\lambda \equiv \sigma(x-y)_\perp^2 \frac{s}{4}$)

$$\lambda \frac{d}{d\lambda}\mathcal{O}(x_+, y_+; \lambda)$$
$$= \Big[\int_{x_+}^{\infty} dx'_+ \frac{1}{x'_+ - y_+} e^{i\frac{\lambda\sqrt{2/s}}{x'_+ - y_+}} \mathcal{O}(x_+, y_+; \lambda) - \int_{y_+}^{\infty} dy'_+ \frac{\mathcal{O}(x_+, y_+; \lambda) - \mathcal{O}(x_+ t, y'_+; \lambda)}{y'_+ - y_+}$$
$$+ \int_{y_+}^{\infty} dy'_+ \frac{1}{y'_+ - x_+} e^{i\frac{\lambda\sqrt{2/s}}{y'_+ - x_+}} \mathcal{O}(x_+, y_+; \lambda) - \int_{x_+}^{\infty} dx'_+ \frac{\mathcal{O}(x_+, y_+; \lambda) - \mathcal{O}(x'_+, y_+; \lambda)}{x'_+ - x_+}\Big]$$

If we use rapidity cutoff at $\sigma = \frac{8\varsigma}{|x-y|_\perp \sqrt{s}} \quad \Leftrightarrow \lambda = \varsigma |x-y|\sqrt{s}$,
the solution

$$\mathcal{O}(x_+, y_+; \sigma) = e^{-\frac{\tilde{\alpha}_s}{2}\left(\ln^2 \frac{2(x-y)_\perp^2 \varsigma^2}{x_+ y_+} - \ln^2 \frac{2(x-y)_\perp^2 \varsigma_0^2}{x_+ y_+}\right)} e^{4\tilde{\alpha}_s \psi(1) \ln \frac{\varsigma}{\varsigma_0}} \int dx'_+ dy'_+ \; \mathcal{O}(x'_+, y'_+; \sigma_0)$$
$$\times (x_+ y_+)^{-\tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}} \left[\frac{i\Gamma\left(1 - \tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}\right)}{(x_+ - x'_+ + i\epsilon)^{1-\tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1 - \tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}\right)}{(x_+ - x'_+ - i\epsilon)^{1-\tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}}}\right]$$
$$\times \left[\frac{i\Gamma\left(1 - \tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}\right)}{(y_+ - y'_+ + i\epsilon)^{1-\tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}}} - \frac{i\Gamma\left(1 - \tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}\right)}{(y_+ - y'_+ - i\epsilon)^{1-\tilde{\alpha}_s \ln \frac{\varsigma}{\varsigma_0}}}\right]$$

is obviously invariant under the inversion $x_+ \to \frac{x_+}{x_+^2}$, $y_+ \to \frac{y_+}{y_+^2}$.

Argument of coupling constant by BLM/renormalon method

A problem with leading-order rapidity evolution: what is the argument of coupling constant?

In CSS approach - no problem, argument is defined by renormgroup

With rapidity-only evolution (BFKL, BK and the like) - argument of α_s may be obtained from the NLO calculations. BLM approach: calculate the small part of the NLO result, namely quark loop contribution to gluon propagator, and promote $-\frac{2}{3}n_f$ to the full $b = \frac{11}{3}N_c - \frac{2}{3}n_f$.

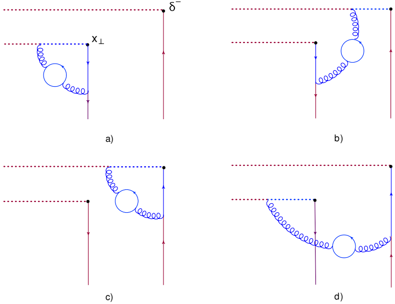
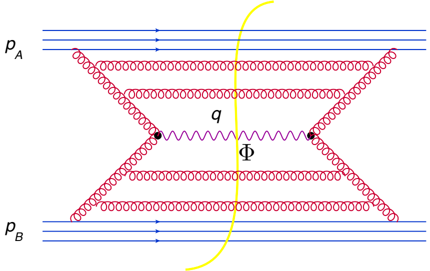


Figure: Quark loop correction to quark TMD evolution

Result: BLM optimal scale is logarithmically halfway between transverse momentum ($b_\perp^{-1/2}$) and energy ($\sigma\beta_B s$) of TMD both for quarks and gluons

Coefficient function for TMD factorization at one loop

Particle production by gluon fusion



$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

Goal: one-loop TMD factorization formula for hadronic tensor.

Result of calculations:

$$W(p_A, p_B; q) \;=\; \int db_\perp \; e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b)$$
$$\times \; \exp\Big\{ \frac{\alpha_s N_c}{2\pi} \Big[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2\Big(\ln \frac{x_A}{\sigma_t} + \gamma \Big) \Big(\ln \frac{x_B}{\sigma_p} + \gamma \Big) + \frac{\pi^2}{2} \Big] \Big\}$$
$$+ \text{ NLO terms} \sim O\big(\alpha_s^2\big) \; + \; \text{power corrections}$$

One-loop coefficient function

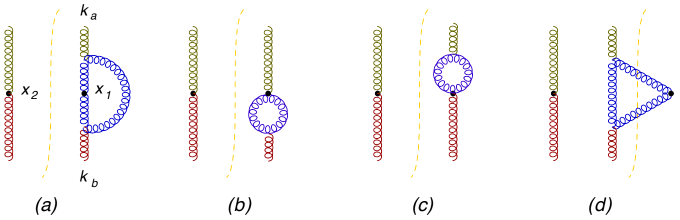
Calculation of coefficient function $\mathfrak{C}_1$ in the background field $\mathbb{A} = \bar{A} + \bar{B} + \bar{C}$

$$\int dz_2^- dz_{2_\perp} dz_1^- dz_{1_\perp} dw_1^+ dw_{1_\perp} dw_2^+ dw_{2_\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i_\perp}, w_i^+, w_{i_\perp}; \sigma_p, \sigma_t)$$
$$\times F^{-i,a}(z_2^+, z_{2_\perp}) F^{-j,a}(z_1^+, z_{1_\perp}) F^{+i,a}(z_2^-, z_{2_\perp}) F^{+j,a}(z_1^-, z_{1_\perp})$$
$$= \frac{N_c^2 - 1}{16} g^4 \langle \tilde{F}^a_{\mu\nu} \tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho} F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A} = \bar{A} + \bar{B}}$$
$$- \langle \hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2_\perp}; x_1^-, x_{1_\perp}) \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2_\perp}; x_1^+, x_{1_\perp}) \rangle_{\mathbb{A} = \bar{A} + \bar{B}}$$

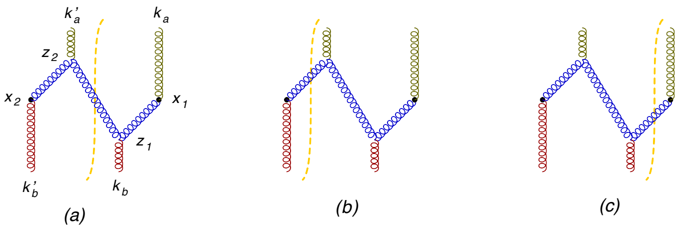
(for the purpose of calculating leading-twist coefficient function the "correction field" C can be neglected: $\mathbb{A} = \bar{A} + \bar{B}$)

Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2) F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



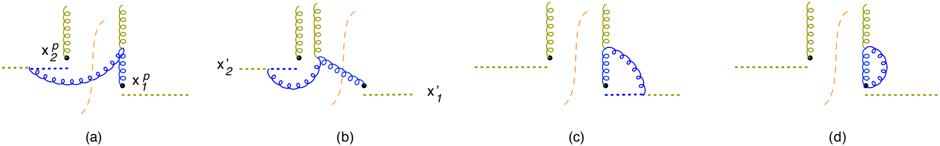
"Real" diagrams



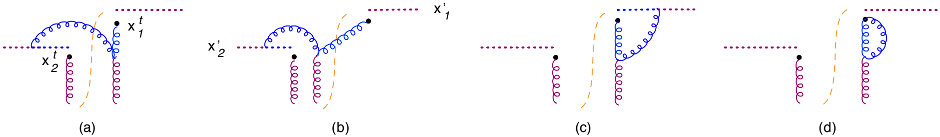
Diagrams for subtracted TMD matrix elements

“Projectile” TMD matrix elements.

The $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting: positions of $F's$ are separated from the beginnings of gauge links. (Violations of gauge invariance are power corrections).



“Target” TMD matrix elements. The $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



Result for the coefficient function

$$\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$= \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})|p_B\rangle$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p_B\rangle$$

$$\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_1,x_2;\sigma_p,\sigma_t)$$
$$=\ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}-\ln\frac{(-i\alpha'_a)e^{\gamma_E}}{\sigma_t}\ln\frac{(-i\beta'_b)e^{\gamma_E}}{\sigma_p}-\ln\frac{(-i\alpha_a)e^{\gamma_E}}{\sigma_t}\ln\frac{(-i\beta_b)e^{\gamma_E}}{\sigma_p}+\pi^2$$

The solution of TMD evolution equations compatible with this first-order result is

$$\mathfrak{C}(x_{1_\perp},x_{2_\perp};\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t) = e^{\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha'_a,\alpha_a,\beta'_b,\beta_b,x_{1_\perp},x_{2_\perp}) = \int đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b\, e^{\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_{12_\perp},\alpha'_a,\alpha_a,\beta'_b,\beta_b;\sigma_p,\sigma_t)}$$
$$\times\langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a,\alpha_a,x_{2_\perp},x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b,\beta_b,x_{2_\perp},x_{1_\perp})|p_B\rangle + \ldots$$

Forward case ($\equiv$ particle production by gluon fusion)

Recall $\alpha_q \equiv x_A$, $\beta_q \equiv x_B$.

$$
\begin{aligned}
W(p_A,p_B;q) &= \int db_\perp\ e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp), \\
W(p_A,p_B;\alpha_q,\beta_q,b_\perp) &= \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B) \\
&\times\ \exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p}+\gamma\Big)+\frac{\pi^2}{2}\Big]\Big\} \\
&+ \text{NLO terms} \sim O\big(\alpha_s^2\big)\ +\ \text{power corrections} \qquad (1)
\end{aligned}
$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$
\begin{aligned}
\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle &= -g^2\varrho^2\int_0^1 du\ u\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-, \\
\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle &= -g^2\varrho^2\int_0^1 du\ u\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-.
\end{aligned}
$$

Matching of coefficient function and TMDs

The r.h.s. of this evolution formula (1) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{x_A s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{x_B s}$. Thus, the result of double-log Sudakov evolution reads

$$
\begin{aligned}
W(p_A, p_B; x_A, x_B, b_\perp) \;&=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(x_A, b_\perp; p_A) \mathcal{G}^{ij;\tilde{\sigma}_t}(x_B, b_\perp; p_B) \\
&\times\; \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\Big[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\Big]\Big\} + \; O\big(\alpha_s^2\big) \text{ terms } + \text{ power corrections}
\end{aligned}
$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $x_B \tilde{\sigma}_p s \simeq b_\perp^{-2}$. After that:

- If $x_B \sim 1$ - DGLAP-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $x_B \ll 1$ - BFKL-type evolution from $\tilde{\sigma}_t = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$

Conclusions

1. The rapidity-only factorization is the most convenient tool for high-energy QCD.
   - Current status of the BFKL/BK evolution: NLO evolution and NLO impact factors. (For $\mathcal{N}=4$ SYM - NNLO)
   - Impact factors for various processes from shock-wave approach: LO and NLO
   - "Hybrid factorization" - phenomenological mix of DGLAP and BK evolutions
2. Rapidity-only TMD factorization works:
   - Full list of $\frac{1}{Q^2}$ power corrections for DY and SIDIS.
   - Back-of-the-envelope estimates of power corrections seems to agree with exp. data.
   - Rapidity-only evolution with BLM prescription for running coupling gives the same universal formula for Sudakov double logs at both small and moderate x for both quark and gluon TMDs.
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B

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Thank you for attention!