Rapidity regulators for the CGC: F_L at NLO

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NLO CGC calculations : with standard cut-off

Many calculations have been performed at NLO in the CGC: evolution equations, DIS or pA observables, since NLO BK from Balitsky, Chirilli, 2007.

Most frequently used regularization technique (in particular in LFPT):

1. Perform transverse integration in dim. reg.
2. Expand in ϵ
3. And then perform integrations over k^+ momenta regulated by a cut off k^+_{min}

Issues with this regularization procedure:

- Does not distinguish clearly soft divergences from rapidity/low x divergences
- Difficult to compare results with other pQCD communities, like TMD, jets, etc...
- Biases us to consider BK/JIMWLK as evolutions along k^+ (related to projectile), instead of k^- (related to target), which is physically more natural for DIS. (Smoother transition to DGLAP in the collinear regime)

Rapidity regulators from pQCD/TMD

Many new regulators for rapidity divergences have been proposed by the TMD and SCET communities in the last 15 years

Some of them should be suitable as well in the context of low x physics/CGC, for example:
Chiu, Jain, Neill, Rothstein, 2011-2012
Becher, Neubert, 2011
Ebert, Moult, Stewart, Tackmann, Vita, Zhu, 2019

Such rapidity regulators have been used for CGC observables, but in the language of SCET, in Liu, Kang, Liu, 2020; Liu, Xie, Kang, Liu, 2022

A similar rapidity regulator has been proposed for CGC in LFPT in Liu, Ma, Chao, 2019, at the level of each energy denominator
$\rightarrow$ By experience, does not seem to work in full generality

Using rapidity regulators in NLO CGC calculations

3 versions of rapidity regularisation:

Introduce a factor in the loop integrand (with gluon momentum k)

- regulator in k^+: $\left(\frac{k^+}{\nu^+}\right)^\eta$
- regulator in k^-: $\left(\frac{\nu^-}{k^-}\right)^\eta \sim \left(\frac{2k^+\nu^-}{\mathbf{k}^2}\right)^\eta$
- true rapidity regulator: $\left(\frac{k^+}{k^-}\frac{\nu^-}{\nu^+}\right)^{\frac{\eta}{2}} \sim \left(\frac{2(k^+)^2\nu^-}{\mathbf{k}^2\nu^+}\right)^{\frac{\eta}{2}}$

In the 3 cases, transforms divergent dk^+/k^+ integrals over k^+ into $dk^+(k^+)^{-1+\eta}$.

Analogy with dim.reg. : $\eta \leftrightarrow \epsilon$ and $\nu^\pm \leftrightarrow \mu$

Order of limits: take $\eta \to 0$ at finite ϵ, and later expand in ϵ.

$\Rightarrow$ η regulates only rapidity/low x div., whereas ϵ regulates also soft div.

Aim: revisit the calculation of NLO DIS (F_L, massless quarks) (G.B., 2016-2017) with the $+$ and $-$ versions of the regulator validate their implementation in CGC in LFPT.

Remark: results with *true rapidity regulator* can be obtained from the average of the $+$ and $-$ versions.

Using rapidity regulators in NLO CGC calculations

From a diagram with dim. reg. and a rapidity regulator: typical expression of the form

$$I(\epsilon,\eta) = \int_0^1 d\xi\, \xi^{-1+\eta}\, f(\xi,\epsilon,\eta)$$

with ξ the k^+ momentum fraction of the gluon in the loop.

- First case: $f(0,\epsilon,\eta)=0 \Rightarrow$ rapidity regularization unnecessary: *rapidity safe*

$$I(\epsilon,\eta) = I(\epsilon,0) + O(\eta) = \int_0^1 d\xi\, \xi^{-1}\, f(\xi,\epsilon,0) + O(\eta)$$

- Second case: finite $f(0,\epsilon,\eta)\neq 0 \Rightarrow$ rapidity regularization necessary

$$\begin{aligned} I(\epsilon,\eta) &= \int_0^1 d\xi\, \xi^{-1+\eta}\, f(0,\epsilon,\eta) + \int_0^1 d\xi\, \xi^{-1+\eta}\, \left[f(\xi,\epsilon,\eta) - f(0,\epsilon,\eta)\right] \\ &= \frac{1}{\eta}\, f(0,\epsilon,\eta) + \int_0^1 \frac{d\xi}{(\xi)_+}\, f(\xi,\epsilon,0) + O(\eta) \end{aligned}$$

$\Rightarrow$ two contributions: η pole, and $+$ prescription

Quark off-shell self-energy diagram

One loop corrections to the $\gamma_L^* \to q\bar{q}$ Light-Front wave function found to factorize as:

$$\Psi^{NLO}_{\gamma_L^*\to q\bar{q}} = \left(1 + \frac{\alpha_s C_F}{2\pi}\,\mathcal{V}^L\right)\,\Psi^{LO}_{\gamma_L^*\to q\bar{q}}$$

Contribution of quark self-energy diagram, with dim. reg. only:

$$\mathcal{V}^L_{q\text{ S. E.}} = \int_0^1 \frac{d\xi}{\xi}\Big[-2+O(\xi)\Big]\,4\pi\,\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}\mathbf{K}}{(2\pi)^{2-2\epsilon}}\frac{1}{\Big[\mathbf{K}^2+\frac{\xi(1-\xi)}{(1-z)}(\mathbf{P}^2+\overline{Q}^2)\Big]}$$

$$=\Gamma(\epsilon)\left[\frac{\mathbf{P}^2+\overline{Q}^2}{4\pi\mu^2(1-z)}\right]^{-\epsilon}\int_0^1 d\xi\;\xi^{-1-\epsilon}\,(1-\xi)^{-\epsilon}\Big[-2+O(\xi)\Big]$$

Scale $\propto \xi$ in the denominator of $\mathbf{K}$ integral $\Rightarrow$ $\xi^{-\epsilon}$ factor regulating the $\xi=0$ IR div.

Dim. reg. enough in that case: no rapidity divergence!

Full result, with UV times IR double ϵ pole (with $S_\epsilon \equiv [4\pi\,e^{-\gamma_E}]^\epsilon$):

$$\mathcal{V}^L_{q\text{ S. E.}} = 2\,\frac{S_\epsilon}{\epsilon^2}\left[\frac{\mathbf{P}^2+\overline{Q}^2}{\mu^2(1-z)}\right]^{-\epsilon}+\frac{3}{2}\,\frac{S_\epsilon}{\epsilon}\left[\frac{\mathbf{P}^2+\overline{Q}^2}{\mu^2(1-z)}\right]^{-\epsilon}-\frac{\pi^2}{6}+\frac{\delta_s}{2}+3+O(\epsilon)$$

Vertex correction

3 LFPT diagrams with vertex correction topology: 2 different ordering of vertices and 1 instantaneous gluon exchange

Individual diagrams have power divergences at $\xi = 0$ on top of log divergences
But power divergences (and some log) cancel between vertex correction LFPT diagrams

Leftover in (half of) the vertex correction:

- Terms with no potential div at $\xi = 0 \Rightarrow$ dim. reg. enough (single ϵ UV pole)

$$\mathcal{V}^L_{\text{v. corr.}}\Big|_{\text{rap. safe}} = \frac{(z-2)}{2}\,\frac{S_\epsilon}{\epsilon}\left[\frac{\overline{Q}^2}{\mu^2}\right]^{-\epsilon} - \frac{3}{2}\log(1-z) - \frac{\delta_s\, z}{2} + \frac{z}{2} - 2 + O(\epsilon)$$

Vertex correction

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- Terms with no potential div at $\xi = 0 \Rightarrow$ dim. reg. enough (single ϵ UV pole)
- Terms of the same type as quark self-energy $\Rightarrow$ dim. reg. enough (double ϵ pole)

$$\mathcal{V}^L_{\text{v. corr.}}\Big|_{\mathcal{A}_0(\Delta_1)/\xi} = \int_0^1 \frac{d\xi}{\xi}\, 4\pi\, \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{K}}{(2\pi)^{2-2\epsilon}} \frac{1}{\left[\mathbf{K}^2 + \frac{\xi(1-\xi)}{(1-z)}(\mathbf{P}^2 + \overline{Q}^2)\right]}$$

$$= -\frac{S_\epsilon}{\epsilon^2}\left[\frac{\mathbf{P}^2 + \overline{Q}^2}{\mu^2(1-z)}\right]^{-\epsilon} + \frac{\pi^2}{12} + O(\epsilon)$$

Vertex correction

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Leftover in (half of) the vertex correction:

- Terms with no potential div at $\xi = 0 \Rightarrow$ dim. reg. enough (single ϵ UV pole)
- Terms of the same type as quark self-energy $\Rightarrow$ dim. reg. enough (double ϵ pole)
- Terms with potential div at $\xi = 0$ but finite $\mathbf{K}$ integral:

$$\mathcal{V}^L_{\text{v. corr.}}\Big|_{\mathcal{B}_0/\xi} = \int_0^1 \frac{d\xi}{\xi}\,(1-\xi)\left[\left(1+\frac{z\xi}{(1-z)}\right)\mathbf{P}^2 + (1-\xi)\,\overline{Q}^2\right]\mathcal{B}_0$$

$$\mathcal{B}_0 \equiv 4\pi\,(\mu^2)^\epsilon \int \frac{d^{2-2\epsilon}\mathbf{K}}{(2\pi)^{2-2\epsilon}}\,\frac{1}{[\mathbf{K}^2+\Delta_1]\,[(\mathbf{K}+\mathbf{L})^2+\Delta_2]}$$

Dim. reg. insufficient in such term: Rapidity regulator needed!

Remark need to calculate the finite integral $\mathcal{B}_0$ with full ϵ dependence because of the ordering of limits.

Rapidity singular contribution with $\eta+$ regulator

Introducing the factor $(\xi z q^+/\nu^+)^\eta$, performing the $\mathbf{K}$ integral thanks to Feynman parametrization, and changing variables:

$$\mathcal{V}^L_{\text{v. corr.}}\Big|^{\eta+}_{\mathcal{B}_0/\xi} = \left[\frac{zq^+}{\nu^+}\right]^\eta \Gamma(1+\epsilon)\,\left[4\pi\,\mu^2\right]^\epsilon \int_0^1 dy\, y^{-1-\epsilon+\eta} \int_0^1 d\zeta\, \zeta^{\eta-1}\,\left[1+\frac{z\zeta}{(1-z)}\right]^{-1-\epsilon}$$
$$\times\,\left[(1-y)\,\mathbf{P}^2+(1-y\zeta)\,\overline{Q}^2\right]^{-1-\epsilon}\left[\left((1-y)\,\mathbf{P}^2+(1-y\zeta)\,\overline{Q}^2\right)+y\mathbf{P}^2\left(1+\frac{z\zeta}{(1-z)}\right)\right]$$

Dim. reg. can regulate the $y=0$ div, but rapidity regulator needed for the $\zeta=0$ div.

Separating the η pole piece and the $+$ prescription piece:

$$\mathcal{V}^L_{\text{v. corr.}}\Big|^{\eta+}_{\mathcal{B}_0/\xi;\ \eta\text{ pole}} = \frac{1}{\eta}\left[\frac{zq^+}{\nu^+}\right]^\eta \Gamma(1+\epsilon)\,\left[4\pi\,\mu^2\right]^\epsilon \left[\mathbf{P}^2+\overline{Q}^2\right]\int_0^1 dy\, y^{-1-\epsilon+\eta}\,\left[(1-y)\,\mathbf{P}^2+\overline{Q}^2\right]^{-1-\epsilon}$$
$$= \left[\frac{1}{\eta}+\log\left(\frac{zq^+}{\nu^+}\right)\right]\left[-\frac{S_\epsilon}{\epsilon}\left[\frac{\overline{Q}^2}{\mu^2}\right]^{-\epsilon}+2\,\log\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\overline{Q}^2}\right)\right]$$
$$-\frac{S_\epsilon}{\epsilon^2}\left[\frac{\mathbf{P}^2+\overline{Q}^2}{\mu^2}\right]^{-\epsilon}-\text{Li}_2\left(\frac{\mathbf{P}^2}{\mathbf{P}^2+\overline{Q}^2}\right)-\frac{\pi^2}{12}+O(\epsilon)+O(\eta)$$

Note: double pole in ϵ is a consequence of expanding in η first, at finite ϵ.

Rapidity singular contribution with $\eta+$ regulator

Introducing the factor $(\xi z q^+/\nu^+)^\eta$, performing the $\mathbf{K}$ integral thanks to Feynman parametrization, and changing variables:

$$\mathcal{V}^L_{\text{v. corr.}}\Big|^{\eta+}_{\mathcal{B}_0/\xi} = \left[\frac{zq^+}{\nu^+}\right]^\eta \Gamma(1+\epsilon)\,\left[4\pi\,\mu^2\right]^\epsilon \int_0^1 dy\, y^{-1-\epsilon+\eta} \int_0^1 d\zeta\, \zeta^{\eta-1}\, \left[1+\frac{z\zeta}{(1-z)}\right]^{-1-\epsilon}$$
$$\times \left[(1-y)\,\mathbf{P}^2 + (1-y\zeta)\,\overline{Q}^2\right]^{-1-\epsilon} \left[\left((1-y)\,\mathbf{P}^2 + (1-y\zeta)\,\overline{Q}^2\right) + y\mathbf{P}^2\left(1+\frac{z\zeta}{(1-z)}\right)\right]$$

Dim. reg. can regulate the $y=0$ div, but rapidity regulator needed for the $\zeta=0$ div.

Separating the η pole piece and the $+$ prescription piece:

$$\mathcal{V}^L_{\text{v. corr.}}\Big|^{\eta+}_{\mathcal{B}_0/\xi;\,+\,\text{prescr.}} = \Gamma(1+\epsilon)\,\left[4\pi\,\mu^2\right]^\epsilon \int_0^1 \frac{d\zeta}{(\zeta)_+} \int_0^1 dy\, y^{-1-\epsilon} \left[1+\frac{z\zeta}{(1-z)}\right]^{-1-\epsilon}$$
$$\times \left[\left((1-y)\,\mathbf{P}^2 + (1-y\zeta)\,\overline{Q}^2\right)\right]^{-1-\epsilon} \left\{\left[(1-y)\,\mathbf{P}^2 + (1-y\zeta)\,\overline{Q}^2\right] + y\mathbf{P}^2\left(1+\frac{z\zeta}{(1-z)}\right)\right\} + O(\eta)$$
$$= -\log(1-z)\frac{S_\epsilon}{\epsilon}\left[\frac{\mathbf{P}^2+\overline{Q}^2}{\mu^2}\right]^{-\epsilon} - \frac{1}{2}\left[\log(1-z)\right]^2 - \mathrm{Li}_2\left(-\frac{z}{(1-z)}\right) + \mathrm{Li}_2\left(\frac{\mathbf{P}^2}{\mathbf{P}^2+\overline{Q}^2}\right) + O(\epsilon) + O(\eta)$$

Rapidity singular contribution with $\eta-$ regulator

Introducing instead the factor $(2\xi z q^+ \nu^- / \mathbf{K}^2)^\eta$, and following similar steps:

- The η pole piece is now obtained as

$$\mathcal{V}^L_{\text{v. corr.}}\Big|^{\eta-}_{\mathcal{B}_0/\xi;\ \eta\ \text{pole}} = \left[\frac{1}{\eta} + \log\left(\frac{2zq^+\nu^-}{\mathbf{P}^2 + \overline{Q}^2}\right)\right] \left[-\frac{S_\epsilon}{\epsilon}\left[\frac{\overline{Q}^2}{\mu^2}\right]^{-\epsilon} + 2\,\log\left(\frac{\mathbf{P}^2 + \overline{Q}^2}{\overline{Q}^2}\right)\right]$$
$$- \frac{\pi^2}{3} + O(\epsilon) + O(\eta)$$

- Same $+$ prescription piece is obtained as with the rapidity regulator in k^+

On-loop $\gamma_L^* \to q\bar{q}$ LFWF in momentum space

Collecting all one-loop corrections to the $\gamma_L^* \to q\bar{q}$ LFWF:

- Result with rapidity regulator in k^+:

$$\mathcal{V}^L\Big|^{\eta+} = \left[\frac{2}{\eta} + 2\log\left(\frac{q^+}{\nu^+}\right) + \log\left(z(1-z)\right) - \frac{3}{2}\right]\left[-\frac{S_\epsilon}{\epsilon}\left[\frac{\overline{Q}^2}{\mu^2}\right]^{-\epsilon} + 2\,\log\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\overline{Q}^2}\right)\right]$$
$$+\frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{\pi^2}{6} + \frac{(5+\delta_s)}{2} + O(\epsilon) + O(\eta)$$

$\to$ Very similar as earlier results with cut-off in k^+ from G.B., 2016.

- Result with rapidity regulator in k^-:

$$\mathcal{V}^L\Big|^{\eta-} = \left[\frac{2}{\eta} + 2\log\left(\frac{2q^+\nu^-}{\overline{Q}^2}\right) + \log\left(z(1-z)\right) - \frac{3}{2}\right]\left[-\frac{S_\epsilon}{\epsilon}\left[\frac{\overline{Q}^2}{\mu^2}\right]^{-\epsilon} + 2\log\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\overline{Q}^2}\right)\right]$$
$$+2\frac{S_\epsilon}{\epsilon^2}\left[\frac{\overline{Q}^2}{\mu^2}\right]^{-\epsilon} + 2\,\mathrm{Li}_2\left(\frac{\mathbf{P}^2}{\mathbf{P}^2+\overline{Q}^2}\right) - 3\left[\log\left(\frac{\mathbf{P}^2+\overline{Q}^2}{\overline{Q}^2}\right)\right]^2$$
$$+\frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{2\pi^2}{3} + \frac{(5+\delta_s)}{2} + O(\epsilon) + O(\eta)$$

$\to$ New: double pole in ϵ, and non-trivial dependence on relative momentum $\mathbf{P}$ of the dipole.

On-loop $\gamma_L^* \to q\bar{q}$ LFWF in mixed space

Taking Fourier transform from $\mathbf{P}$ to dipole size $\mathbf{x}_{01}$:

One loop corrections to the $\gamma_L^* \to q\bar{q}$ LFWF still factorizes in mixed space:

$$\tilde{\Psi}^{NLO}_{\gamma_L^* \to q\bar{q}} = \left(1 + \frac{\alpha_s C_F}{2\pi}\, \tilde{\mathcal{V}}^L\right) \tilde{\Psi}^{LO}_{\gamma_L^* \to q\bar{q}}$$

- With rapidity regulator in k^+ (with $c_0 \equiv 2e^{-\gamma_E}$):

$$\tilde{\mathcal{V}}^L\Big|^{\eta+} = -\left[\frac{2}{\eta} + 2\log\left(\frac{q^+}{\nu^+}\right) + \log\left(z(1-z)\right) - \frac{3}{2}\right] \frac{S_\epsilon}{\epsilon} \left[\frac{\mathbf{x}_{01}^2 \mu^2}{c_0^2}\right]^\epsilon + \frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{\pi^2}{6} + \frac{(5+\delta_s)}{2} + O(\epsilon) + O(\eta)$$

- With rapidity regulator in k^-:

$$\tilde{\mathcal{V}}^L\Big|^{\eta-} = -\left[\frac{2}{\eta} + 2\log\left(\frac{2q^+\nu^- \mathbf{x}_{01}^2}{c_0^2}\right) + \log\left(z(1-z)\right) - \frac{3}{2}\right] \frac{S_\epsilon}{\epsilon} \left[\frac{\mathbf{x}_{01}^2 \mu^2}{c_0^2}\right]^\epsilon + 2\frac{S_\epsilon}{\epsilon^2}\left[\frac{\mathbf{x}_{01}^2 \mu^2}{c_0^2}\right]^\epsilon + \frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{\pi^2}{3} + \frac{(5+\delta_s)}{2} + O(\epsilon) + O(\eta)$$

Differences: double pole term in ϵ, and scale for rapidity/low x log.

On-loop $\gamma_L^* \to q\bar{q}$ LFWF in mixed space

$q\bar{q}$ contribution to F_L structure function at NLO:

$$F_L|^{q\bar{q}} = 16Q^4\, N_c \sum_f e_f^2 \int_0^1 dz\, z^2(1-z)^2 \int \frac{d^{2-2\epsilon}\mathbf{x}_0}{(2\pi)^2} \int \frac{d^{2-2\epsilon}\mathbf{x}_1}{(2\pi)^2} \mathrm{Re}\left[1-\mathcal{S}_{01}\right]$$
$$\times \left(\frac{4\pi^2\mu^2\mathbf{x}_{01}^2}{\overline{Q}^2}\right)^{\epsilon} \left[\mathrm{K}_\epsilon\left(\overline{Q}|\mathbf{x}_{01}|\right)\right]^2 \left(1+\frac{\alpha_s C_F}{\pi}\tilde{\mathcal{V}}^L\right)$$

- With rapidity regulator in k^+ (with $c_0 \equiv 2e^{-\gamma_E}$):

$$\tilde{\mathcal{V}}^L\Big|^{\eta+} = -\left[\frac{2}{\eta} + 2\log\left(\frac{q^+}{\nu^+}\right) + \log\left(z(1-z)\right) - \frac{3}{2}\right] \frac{S_\epsilon}{\epsilon} \left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon}$$
$$+ \frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{\pi^2}{6} + \frac{(5+\delta_s)}{2} + O(\epsilon) + O(\eta)$$

- With rapidity regulator in k^-:

$$\tilde{\mathcal{V}}^L\Big|^{\eta-} = -\left[\frac{2}{\eta} + 2\log\left(\frac{2q^+\nu^-\mathbf{x}_{01}^2}{c_0^2}\right) + \log\left(z(1-z)\right) - \frac{3}{2}\right] \frac{S_\epsilon}{\epsilon} \left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon}$$
$$+2\frac{S_\epsilon}{\epsilon^2}\left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon} + \frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{\pi^2}{3} + \frac{(5+\delta_s)}{2} + O(\epsilon) + O(\eta)$$

Differences: double pole term in ϵ, and scale for rapidity/low x log.

$q\bar{q}g$ contribution to F_L: Rapidity safe terms

Other contributions to F_L at NLO at low x_{Bj}: with $q\bar{q}g$ Fock state scattering on the target

Can be split into regular terms and potentially log divergent terms at $\xi = 0$

Regular terms at $\xi = 0$ don't need rapidity regularization $\Rightarrow$ same results as G.B., 2017

Reminder: UV divergent terms for gluon close to the quark ($\mathbf{x}_2 \to \mathbf{x}_0$) or to the antiquark ($\mathbf{x}_2 \to \mathbf{x}_1$) have a $q\bar{q}$ dipole form, thanks to color coherence
$\to$ should cancel with UV divergences from the genuine $q\bar{q}$ Fock state contribution

- Extract UV divergent dipole-like contribution (to be combined with the $q\bar{q}$ contribution)

$$\tilde{\mathcal{V}}^{L}_{q\bar{q}g;\ \xi \text{ reg.};\ \text{UV}} = -\frac{3}{2}\frac{S_\epsilon}{\epsilon}\left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon} - \frac{\delta_s}{2} + O(\epsilon)$$

- Same UV-subtracted leftover from the terms regular terms at $\xi = 0$ as in G.B., 2017

$q\bar{q}g$ contribution to F_L: Rapidity safe terms

- Same UV-subtracted leftover from the terms regular terms at $\xi = 0$ as in G.B., 2017:

$$F_L\Big|^{q\bar{q}g\ \text{reg.}} = 16Q^2\, N_c \left(\frac{\alpha_s C_F}{\pi}\right) \sum_f e_f^2 \int_0^1 dz\, z(1-z) \int \frac{d^2\mathbf{x}_0}{(2\pi)^2} \int \frac{d^2\mathbf{x}_1}{(2\pi)^2} \int \frac{d^2\mathbf{x}_2}{2\pi} \int_0^1 d\xi$$
$$\times \left\{ (-2+\xi) \left[\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2} \cdot \left(\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2} - \frac{\mathbf{x}_{21}}{\mathbf{x}_{21}^2}\right)\right] \left\{ \Big[\mathrm{K}_0\left(QX_{012}\right)\Big]^2 \mathrm{Re}\left[1-\mathcal{S}_{012}^{(3)}\right] - \left(\mathbf{x}_2 \to \mathbf{x}_0\right) \right\} \right.$$
$$\left. + \,\xi\, \frac{\mathbf{x}_{20}\cdot\mathbf{x}_{21}}{\mathbf{x}_{20}^2\, \mathbf{x}_{21}^2} \Big[\mathrm{K}_0\left(QX_{012}\right)\Big]^2 \mathrm{Re}\left[1-\mathcal{S}_{012}^{(3)}\right] \right\} + O(\epsilon) + (q \leftrightarrow \bar{q})$$

$$X_{012}^2 \equiv \frac{1}{(q^+)^2}\left[k_0^+ k_1^+ \mathbf{x}_{01}^2 + k_2^+ k_0^+ \mathbf{x}_{20}^2 + k_2^+ k_1^+ \mathbf{x}_{21}^2\right] = z(1-z)(1-\xi)\mathbf{x}_{01}^2 + z^2\xi(1-\xi)\mathbf{x}_{20}^2 + z(1-z)\xi\mathbf{x}_{21}^2$$
$$\mathcal{S}_{012}^{(3)} \equiv \frac{1}{N_c C_F}\,\mathrm{Tr}\left(t^b U_F(\mathbf{x}_0) t^a U_F(\mathbf{x}_1)^\dagger\right)\, U_A(\mathbf{x}_2)_{ba}$$

Term $(q \leftrightarrow \bar{q})$: similar integrand, up to the exchanges $\mathbf{x}_0 \leftrightarrow \mathbf{x}_1$ and $z \leftrightarrow 1-z$.
$\Rightarrow$ same contribution to F_L, after the integrations.

$q\bar{q}g$ contribution to F_L: Rapidity sensitive terms

Rapidity divergent piece of the $q\bar{q}g$ contribution:

$$F_L|_{1/\xi}^{q\bar{q}g} = \frac{16Q^4}{2\pi} N_c \sum_f e_f^2 \int_0^1 dz\, z^2(1-z)^2 \int d^{2-2\epsilon}\mathbf{x}_0 \int d^{2-2\epsilon}\mathbf{x}_1 \frac{\alpha_s C_F}{\pi} \int d^{2-2\epsilon}\mathbf{x}_2$$
$$\times \mathrm{Re}\left[1-\mathcal{S}_{012}^{(3)}\right] \int_0^1 d\xi\, \frac{2}{\xi} \left\{|\mathcal{I}^j(a)|^2 - \mathrm{Re}\left(\mathcal{I}^j(a)^*\mathcal{I}^j(b)\right)\right\} + (q \leftrightarrow \bar{q})$$

with Fourier integral (and similar for $\mathcal{I}^j(b)$)

$$\mathcal{I}^j(a) \equiv \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}\mathbf{P}}{(2\pi)^{2-2\epsilon}} \frac{e^{i\mathbf{P}\cdot(\mathbf{x}_{01}+\xi\mathbf{x}_{20})}}{(\mathbf{P}^2+\overline{Q}^2)} \int \frac{d^{2-2\epsilon}\mathbf{K}}{(2\pi)^{2-2\epsilon}} \frac{\mathbf{K}^j\, e^{i\mathbf{K}\cdot\mathbf{x}_{20}}}{\left[\mathbf{K}^2 + \frac{\xi(1-\xi)}{(1-z)}(\mathbf{P}^2+\overline{Q}^2)\right]}$$

Remark on implementation of k^- rapidity reg. : different $\mathbf{K}$ gluon momentum before and after the target

$\Rightarrow$ Insert the factor $(2\xi z q^+\nu^-/\mathbf{K}^2)^{\frac{\eta}{2}}$ in each integral $\mathcal{I}^j(a)$ or $\mathcal{I}^j(b)$.

Observation: taking $\xi = 0$ in $\mathcal{I}^j(a)$ is equivalent to focusing on its UV regime $\mathbf{x}_2 \to \mathbf{x}_0$ (and $\mathbf{K} \to +\infty$).

$q\bar{q}g$ contribution to F_L: $+$ prescription piece

Both rapidity regulators in k^+ and k^- lead to the same $+$ prescription contribution:

$$F_L|^{q\bar{q}g}_{+\text{ prescr.}} = \frac{16Q^4}{2\pi} N_c \sum_f e_f^2 \int_0^1 dz\, z^2(1-z)^2 \int d^{2-2\epsilon}\mathbf{x}_0 \int d^{2-2\epsilon}\mathbf{x}_1 \frac{\alpha_s C_F}{\pi} \int d^{2-2\epsilon}\mathbf{x}_2$$
$$\times \operatorname{Re}\left[1-\mathcal{S}^{(3)}_{012}\right] \int_0^1 d\xi\, \frac{2}{(\xi)_+} \left\{|\mathcal{I}^j(a)|^2 - \operatorname{Re}\left(\mathcal{I}^j(a)^*\mathcal{I}^j(b)\right)\right\} + (q \leftrightarrow \bar{q})$$

But subtracting the $\xi = 0$ value of the bracket simultaneously subtracts its UV behavior $\Rightarrow$ Fully finite contribution, can take $\epsilon = 0$:

$$F_L|^{q\bar{q}g}_{+\text{ prescr.}} = 16Q^4 N_c \sum_f e_f^2 \int_0^1 dz\, z^2(1-z)^2 \int \frac{d^2\mathbf{x}_0}{(2\pi)^2} \int \frac{d^2\mathbf{x}_1}{(2\pi)^2} \frac{\alpha_s C_F}{\pi} \int \frac{d^2\mathbf{x}_2}{2\pi} \left[\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2}\cdot\left(\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2} - \frac{\mathbf{x}_{21}}{\mathbf{x}_{21}^2}\right)\right]$$
$$\times \operatorname{Re}\left[1-\mathcal{S}^{(3)}_{012}\right]\, 2\int_0^1 \frac{d\xi}{\xi} \left\{\left[\mathrm{K}_0\left(\overline{Q}\sqrt{(1-\xi)\mathbf{x}_{01}^2 + \xi\mathbf{x}_{21}^2 + \frac{z\xi(1-\xi)}{(1-z)}\mathbf{x}_{20}^2}\right)\right]^2 - \left[\mathrm{K}_0\left(\overline{Q}|\mathbf{x}_{01}|\right)\right]^2\right\} + (q \leftrightarrow \bar{q})$$

However, in the regime of large daughter dipoles $\mathbf{x}_{20}^2 \sim \mathbf{x}_{21}^2 \gg \mathbf{x}_{01}^2$, the ξ integration gives a large $\log(\mathbf{x}_{20}^2/\mathbf{x}_{01}^2)$.

$q\bar{q}g$ contribution to F_L: UV term from the η pole

From the rapidity sensitive $q\bar{q}g$ term, apart from the + prescription piece, one gets the η pole piece:
Contains UV divergences that can be isolated into a dipole-like combination by writing

$$\left(1-\mathcal{S}_{012}^{(3)}\right) = (1-\mathcal{S}_{01}) + \left(\mathcal{S}_{01}-\mathcal{S}_{012}^{(3)}\right)$$

- With rapidity regulator in k^+:

$$\tilde{\mathcal{V}}^{L}_{q\bar{q}g;\ \eta\ \text{pole.};\ \text{UV}}\Big|^{\eta+} = \left[\frac{2}{\eta} + 2\log\left(\frac{q^+}{\nu^+}\right) + \log\left(z(1-z)\right)\right] \frac{S_\epsilon}{\epsilon}\left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon} + O(\epsilon) + O(\eta)$$

- With rapidity regulator in k^-:

$$\tilde{\mathcal{V}}^{L}_{q\bar{q}g;\ \eta\ \text{pole.};\ \text{UV}}\Big|^{\eta-} = \left[\frac{2}{\eta} + 2\log\left(\frac{2q^+\nu^-\mathbf{x}_{01}^2}{c_0^2}\right) + \log\left(z(1-z)\right)\right] \frac{S_\epsilon}{\epsilon}\left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon}$$
$$-2\frac{S_\epsilon}{\epsilon^2}\left[\frac{\mathbf{x}_{01}^2\mu^2}{c_0^2}\right]^{\epsilon} + \frac{\pi^2}{6} + O(\epsilon) + O(\eta)$$

In both cases: total dipole-like contribution to NLO F_L ($q\bar{q}$ terms + dipole-like UV terms from $q\bar{q}g$):

$$\tilde{\mathcal{V}}^{L}_{\text{total}} = \frac{1}{2}\left[\log\left(\frac{z}{1-z}\right)\right]^2 - \frac{\pi^2}{6} + \frac{5}{2} + O(\epsilon) + O(\eta)$$

Same result, finite, as with cut-off in k^+, G.B., 2017.

UV subtracted η pole piece with $\eta+$ regulator

Expanding in η and then taking $\epsilon = 0$ in the leftover contribution, in the case of rapidity regulator in k^+:

$$F_L\big|^{q\bar{q}g\ ;\ \eta+}_{\eta\text{ pole, UV sub.}} = 16Q^4\, N_c \sum_f e_f^2 \int_0^1 dz\, z^2(1-z)^2 \int \frac{d^2\mathbf{x}_0}{(2\pi)^2} \int \frac{d^2\mathbf{x}_1}{(2\pi)^2} \Big[\mathrm{K}_0\left(\overline{Q}|\mathbf{x}_{01}|\right)\Big]^2$$
$$\times \frac{2\alpha_s C_F}{\pi} \int \frac{d^2\mathbf{x}_2}{2\pi} \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2\mathbf{x}_{21}^2} \mathrm{Re}\left[\mathcal{S}_{01} - \mathcal{S}_{012}^{(3)}\right] \left[\frac{1}{\eta} + \log\left(\frac{q^+\sqrt{z(1-z)}}{\nu^+}\right)\right] + O(\epsilon) + O(\eta)$$

Need to define a *rapidity subtracted* (or renormalized) dipole operator to absorb the $1/\eta$ into the LO, as

$$\mathcal{S}_{01}\big|_{\text{rap. sub.}} \equiv \mathcal{S}_{01}\big|_{\text{unsub.}} + \frac{1}{\eta} \frac{2\alpha_s C_F}{\pi} \left\{ \int \frac{d^2\mathbf{x}_2}{2\pi} \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2\mathbf{x}_{21}^2} \mathrm{Re}\left[\mathcal{S}_{01} - \mathcal{S}_{012}^{(3)}\right] + O(\epsilon) \right\}$$

The rapidity subtracted dipole operator should then depend on ν^+, according the standard BK equation.

Natural scale choice: $\nu^+ = q^+\sqrt{z(1-z)}$, to resum low x leading logs.

However: large collinear logs mentioned earlier for large daughter dipoles $\mathbf{x}_{20}^2 \sim \mathbf{x}_{21}^2 \gg \mathbf{x}_{01}^2$ still there.

UV subtracted η pole piece with $\eta-$ regulator

Expanding in η and then taking $\epsilon = 0$ in the leftover contribution, in the case of rapidity regulator in k^-:

$$F_L|^{q\bar{q}g\ ;\ \eta-}_{\eta\text{ pole, UV sub.}} = 16Q^4\, N_c \sum_f e_f^2 \int_0^1 dz\, z^2(1-z)^2 \int \frac{d^2\mathbf{x}_0}{(2\pi)^2} \int \frac{d^2\mathbf{x}_1}{(2\pi)^2} \Big[\mathrm{K}_0\left(\overline{Q}|\mathbf{x}_{01}|\right)\Big]^2$$
$$\times \frac{2\alpha_s C_F}{\pi} \int \frac{d^2\mathbf{x}_2}{2\pi} \mathrm{Re}\left[\mathcal{S}_{01} - \mathcal{S}^{(3)}_{012}\right] \left\{ \left[\frac{1}{\eta} + \log\left(\frac{2zq^+\nu^-\,\mathbf{x}_{20}^2}{c_0^2}\right)\right]\left[\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2}\cdot\left(\frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2} - \frac{\mathbf{x}_{21}}{\mathbf{x}_{21}^2}\right)\right]\right.$$
$$\left. + \left[\frac{1}{\eta} + \log\left(\frac{2(1-z)q^+\nu^-\,\mathbf{x}_{21}^2}{c_0^2}\right)\right]\left[\frac{\mathbf{x}_{21}}{\mathbf{x}_{21}^2}\cdot\left(\frac{\mathbf{x}_{21}}{\mathbf{x}_{21}^2} - \frac{\mathbf{x}_{20}}{\mathbf{x}_{20}^2}\right)\right]\right\} + O(\epsilon) + O(\eta)$$

After similar *rapidity subtraction* of dipole operator, it should depend on ν^-, according the standard BK equation.
Results reminiscent of Liu, Xie, Kang, Liu, 2022 for NLO single jet in pA.

Natural scale choice: $\nu^- = c_0^2/\left(2q^+\sqrt{z(1-z)}\mathbf{x}_{01}^2\right)$, to resum low x leading logs.

Leftover after this choice: terms in $\log(\mathbf{x}_{20}^2/\mathbf{x}_{01}^2)$ and in $\log(\mathbf{x}_{21}^2/\mathbf{x}_{01}^2)$:

- Cancel the large collinear logs mentioned earlier for large daughter dipoles $\mathbf{x}_{20}^2 \sim \mathbf{x}_{21}^2 \gg \mathbf{x}_{01}^2$
- Become new large logs in the small daughter dipole regimes $\mathbf{x}_{20}^2 \ll \mathbf{x}_{21}^2 \sim \mathbf{x}_{01}^2$ or $\mathbf{x}_{21}^2 \ll \mathbf{x}_{20}^2 \sim \mathbf{x}_{01}^2$

Summary and comments

- Rapidity regulators used to rederive:
  - $\gamma_L^* \to q\bar{q}$ LFWF at one loop
  - DIS structure function F_L at NLO
- Transverse photon case: calculations ongoing

In this calculation:

- LL BK equation recovered, with either scale ν^+ or ν^- as evolution variable, depending on the type of rapidity regulator used
- Expected scheme-dependent pattern of large collinear logs recovered

Using these rapidity regulators: new insights on collinear logs in BK/JIMWLK and their resummation?

Related work involving rapidity regulator: rederiving CSS

Altinoluk, G.B., Jalilian-Marian, arXiv:2505.20467 [hep-ph].

Rederivation of CSS from quark TMD operator definition, using

- Background field method, following Balistsky, Braun, 1989.
- Pure rapidity regulator, from Ebert, Moult, Stewart, Tackmann, Vita, Zhu, 2019
- *Target* light-cone gauge, so that the light-like parts of the gauge link are trivial, and only the transverse link at infinity can contribute

Observation: CSS evolution is recovered in that setup when using Mandelstam-Leibbrandt prescription for the gluon propagator in light-cone gauge, but not with other common prescriptions (retarded, advanced, principal value)

Seems to further confirm the Mandelstam-Leibbrandt prescription as the correct one in light-cone gauge, as derived by Bassetto, Dalbosco, Lazzizzera, Soldati, 2025.