Understanding the collinearly-improved high-energy evolution

Alex Kovner*, Michael Lublinsky†, Maxim Nefedov†, Vladimir Skokov‡

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*University of Connecticut, USA
†Ben-Gurion University of the Negev, Israel
‡North Carolina State University, USA

High-energy (Regge) limit of QCD

Let us introduce the problem of High-Energy (Regge) limit of QCD on a particular example of a process – *production of Mueller-Navelet dijets* at hadron collider (X =anything):

$$p(P_1) + p(P_2) \to jet(k_{J1}) + X + jet(k_{J2}),$$

where $M_{JJ}^2 = (k_{J1} + k_{J2})^2 \gg |\mathbf{k}_{J1}|^2$ or $|\mathbf{k}_{J2}|^2 \gg \Lambda_{\rm QCD}^2$ **(bold–$\perp$-vectors!)**.

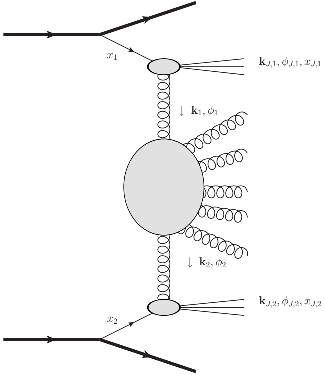
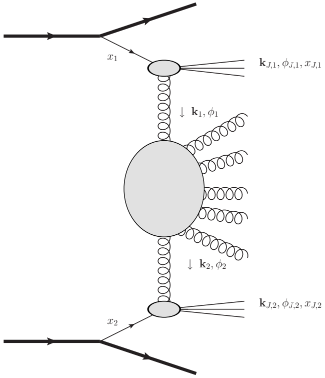


Figure from [hep-ph/1302.7012] .

- $d\sigma = \underbrace{\mathrm{PDF}_i \otimes \mathrm{PDF}_j}_{\text{"non-perturbative"}} \otimes \underbrace{d\hat{\sigma}_{ij}}_{\text{perturbative}}$,
- Fixed-order: $d\hat{\sigma}$ known up to NNLO in α_s,
- Regge limit: $Y = \ln \frac{M_{JJ}}{\sqrt{|\mathbf{k}_{J1}||\mathbf{k}_{J2}|}} \gg 1$, the $d\hat{\sigma}$ recieves corrections $\sim (\alpha_s Y)^n$, which should be resummed.
- The *Balitsky-Fadin-Kuraev-Lipatov (BFKL)* equation allows to rigirously resum them in the LLA ($\sum\limits_n (\alpha_s Y)^n$) and NLLA ($\sum\limits_n \alpha_s(\alpha_s Y)^n$).

BFKL equation



Gluons in t-channel are **Reggeised**.

- Up to NLLA, the partonic cross-section is:

$$d\hat{\sigma}_{ij} = \int d^2\mathbf{k}_1 d^2\mathbf{k}_2\, C_i(\mathbf{k}_1) G(Y, \mathbf{k}_1, \mathbf{k}_2) C_j(\mathbf{k}_2),$$

where $C_i(\mathbf{k})$ – *impact-factors*, G – *BFKL Green's functon*. From now on: $\int d^2\mathbf{k}_1 \to \int_{\mathbf{k}}$.

- The latter satisfies the BFKL equation:

$$\frac{\partial}{\partial Y} G(Y, \mathbf{k}_1, \mathbf{k}_2) = \int_{\mathbf{q}} K(\mathbf{k}_1, \mathbf{q}) G(Y, \mathbf{q}, \mathbf{k}_2),$$

The BFKL kernel is computed perturbatively:

$$K(\mathbf{k}, \mathbf{q}) = \frac{\alpha_s(\mu) N_c}{\pi} \left(K_0(\mathbf{k}, \mathbf{q}) + a_s(\mu) K_1(\mathbf{k}, \mathbf{q}) + \dots \right),$$

where $a_s = \alpha_s/(4\pi)$. The K_0 is known since [BFKL '76, '78] , the K_1 is also known in QCD [Fadin, Lipatov, '98] and $\mathcal{N} = 4$ SYM [Kotikov, Lipatov, 2000] . The K_2 is known in $\mathcal{N} = 4$ SYM [Caron-Huot, Herranen, 2016] and is being computed now in QCD [Gardi, Del Duca, Fadin, Papa, Caola, Falcioni, Byrne,Fucilla,...]

Characteristic functon

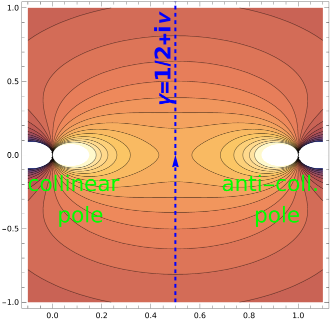
A useful mathematical tool is the *BFKL characteristic function*:

$$\int_{\mathbf{q}} K(\mathbf{k},\mathbf{q})(\mathbf{q}^2)^\gamma e^{in\phi_{\mathbf{q}}} = \frac{\alpha_s(\mathbf{k}^2)N_c}{\pi}\chi(n,\gamma)(\mathbf{k}^2)^\gamma e^{in\phi_{\mathbf{k}}},$$

where $\chi(n,\gamma) = \chi_0(n,\gamma) + a_s(\mathbf{k}^2)\chi_1(n,\gamma) + \ldots$

Then $G(Y,\mathbf{k}_1,\mathbf{k}_2) \propto \sum_n \int_\gamma \exp\left[Y\frac{\alpha_s N_c}{\pi}\chi(n,\gamma)\right]\left(\frac{\mathbf{k}_1^2}{\mathbf{k}_2^2}\right)^\gamma e^{in\phi}.$

Re $\chi_0(0,\gamma)$ in γ-plane:



$\chi(0,1/2+i\nu)$, dashed line – LO, dash-dotted line – NLO, solid line – a version of resummation for leading collinear poles:

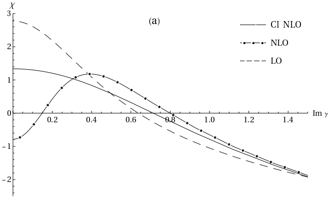


Figure from [Chernyshev,MN,Saleev, 2025]

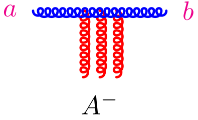
Corrections to χ are large and mostly come from poles at $\gamma = 0, 1$!

Let's "rise the sea level" *

Scattering amplitude of a gluon on a dense target (e.g. a nucleus A):

$$g_a(k) + A \to g_b(k') + A,$$

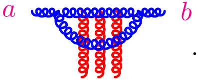
in the Regge limit ($k^+ \to \infty$) is described in the eikonal approximation, the S-matrix is:



$$= S_{ab}(\mathbf{x}) = \exp\left[ig \int_{-\infty}^{+\infty} dx^+ T^c A_c^-(x^+, 0^-, \mathbf{x})\right]_{ab}.$$

If the projectile gluon is boosted ($k^+ \to e^{\delta Y} k^+$) then the radiative correction includes:

$$\alpha_s \delta Y \times$$



.

Which leads to the JIMWLK-evolution for the projectile $\mathcal{O}$.

* A. Grothendieck, "Recoltes et Semailles"

JIMWLK equation

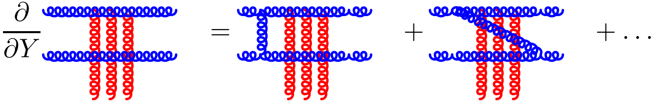
The operator of the projectile evolves with rapidity as:

$$\frac{\partial}{\partial Y}\mathcal{O} = \hat{H}_{JIMWLK}\mathcal{O},$$

where:

$$\hat{H}_{JIMWLK} = \frac{\alpha_s(\mu^2)}{2\pi^2}\int_{\mathbf{x},\mathbf{y},\mathbf{z}} K(\mathbf{x},\mathbf{y},\mathbf{z})\left[J_L^a(\mathbf{x})J_L^a(\mathbf{y}) + J_R^a(\mathbf{x})J_R^a(\mathbf{y})\right.$$
$$\left. - S^{ab}(\mathbf{z})\left(J_L^a(\mathbf{x})J_R^b(\mathbf{y}) + J_L^a(\mathbf{y})J_R^b(\mathbf{x})\right)\right],$$

with $K(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})/[(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2]$



If the target is **not** dense, the JIMWLK picture reduces to BFKL and Wilson lines S^{ab} can be expanded in the Reggeized gluon fields $\alpha^a(\mathbf{x})$ [Lipatov '95;...;Kovner, Lublinsky, 2005;...;Caron-Huot 2012;...] :

$$S^{ab}(\mathbf{x}) = \exp\left[igT^c\alpha_c(\mathbf{x})\right]^{ab},\ (ig)J^a_{(L,R)} = \left[F(\pm iT^c\alpha^c(\mathbf{x}))\right]^{ab}\frac{\delta}{\delta\alpha^b(\mathbf{x})},$$

with $F(x) = x/(1-e^{-x})$.

(anti-)Collinear resummation in JIMWLK (gluons)

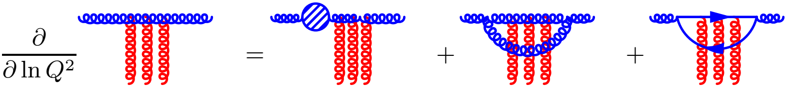
In [Kovner,Lublinsky,Skokov 2023] the method to resum large corrections to JIMWLK Hamiltonian, arising when $Q_T \gg Q_P$ was derived. One evolves the "bare" Wilson lines (S^{ab}) at the scale Q_T down to the scale Q_P ($\Rightarrow \mathbb{S}^{ab}_{Q_P}$):

$$\frac{\partial}{\partial \ln Q^2}\mathbb{S}_{ab}(\mathbf{x},Q) = -a\mathbb{S}_{ab}(\mathbf{x},Q)$$

$$-\frac{a}{\beta_0}\int_0^1 d\xi \int_0^{2\pi}\frac{d\phi}{2\pi}\Big[2N_c p_{gg}(\xi)\mathbb{D}_{ab}\left(\mathbf{x}+(1-\xi)Q^{-1}\mathbf{n}_\phi, \mathbf{x}-\xi Q^{-1}\mathbf{n}_\phi, Q\right)$$

$$+2T_F n_F p_{qg}(\xi)\mathbb{D}^{(F)}_{ab}\left(\mathbf{x}+(1-\xi)Q^{-1}\mathbf{n}_\phi, \mathbf{x}-\xi Q^{-1}\mathbf{n}_\phi, Q\right)\Big],$$

where $a = \alpha_s\beta_0/(4\pi)$, $\mathbb{D}_{ab}(\mathbf{x},\mathbf{y},Q) = \mathrm{tr}\left[T_a\mathbb{S}(\mathbf{x},Q)T_b\mathbb{S}^\dagger(\mathbf{y},Q)\right]/N_c$, $\mathbb{D}^{(F)}_{ab}(\mathbf{x},\mathbf{y},Q) = 2\,\mathrm{tr}\left[t_a\mathbb{V}(\mathbf{x},Q)t_b\mathbb{V}^\dagger(\mathbf{y},Q)\right]$.

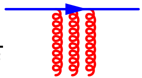
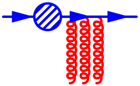
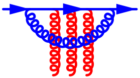


(anti-)Collinear resummation in JIMWLK (quarks)

In [Kovner,Lublinsky,Skokov 2023] the method to resum large corrections to JIMWLK Hamiltonian, arising when $Q_T \gg Q_P$ was derived. One evolves the "bare" Wilson lines (V^{ij}) at the scale Q_T down to the scale Q_P ($\Rightarrow \mathbb{V}^{ij}_{Q_P}$):

$$
\begin{aligned}
\frac{\partial}{\partial \ln Q^2}\mathbb{V}_{ij}(\mathbf{x},Q) = &-\frac{3C_F}{2}\frac{2a}{\beta_0}\mathbb{V}_{ij}(\mathbf{x},Q) \\
&-\frac{2a}{\beta_0}\int_0^1 d\xi \int_0^{2\pi}\frac{d\phi}{2\pi}p_{gq}(\xi)\mathbb{D}^{(FA)}_{ij}\left(\mathbf{x}+(1-\xi)Q^{-1}\mathbf{n}_\phi, \mathbf{x}-\xi Q^{-1}\mathbf{n}_\phi, Q\right),
\end{aligned}
$$

where $a = \alpha_s\beta_0/(4\pi)$, $\mathbb{D}^{(FA)}_{ij}(\mathbf{x},\mathbf{y},Q) = [t_a\mathbb{V}(\mathbf{y},Q)t_b]_{ij}\,\mathbb{S}_{ab}(\mathbf{x},Q)$.

$\frac{\partial}{\partial \ln Q^2}$  $=$  $+$ 

Resummed JIMWLK Hamiltonian

$$\hat{H}^{(\mathrm{res.})}_{JIMWLK} = \frac{\alpha_s(\mu^2)}{2\pi^2} \int_{\mathbf{x},\mathbf{y},\mathbf{z}} K(\mathbf{x},\mathbf{y},\mathbf{z}) \left[J^a_L(\mathbf{x}) J^a_L(\mathbf{y}) + J^a_R(\mathbf{x}) J^a_R(\mathbf{y}) \right.$$
$$\left. - \mathbb{S}^{ab}_Q(\mathbf{z}) \left(J^a_L(\mathbf{x}) J^b_R(\mathbf{y}) + J^a_L(\mathbf{y}) J^b_R(\mathbf{x}) \right) \right],$$

with the scale-choice [Kovner, Lublinsky, Skokov 2023] :

$$\boxed{Q^2 = \max((\mathbf{x}-\mathbf{z})^{-2}, (\mathbf{y}-\mathbf{z})^{-2}, Q_0^2)}$$

The task: linearize $\hat{H}^{(\mathrm{res.})}_{JIMWLK} \alpha_a(\mathbf{x}) \alpha_a(\mathbf{y})$, obtain the resummed BFKL equation, do the Fourier transform to momentum space and compute the cherecteristic function.

Fixed-coupling approximation

Linearising the resummed JIMWLK

Write an ansatz for $\mathbb{S}$ and $\mathbb{V}$ in terms of Reggeon fields ($\alpha(\mathbf{x})$):

$$\mathbb{S}_{ab}(\mathbf{z},Q) = \delta_{ab} + igT^{c_1}_{ab}\int_{\mathbf{z}_1} R^{(1)}_Q(\mathbf{z}-\mathbf{z}_1)\alpha_{c_1}(\mathbf{z}_1)$$

$$+\frac{(ig)^2}{2}(T^{c_1}T^{c_2})_{ab}\int_{\mathbf{z}_1\mathbf{z}_2} R^{(2,1)}_Q(\mathbf{z}-\mathbf{z}_1,\mathbf{z}-\mathbf{z}_2)\alpha_{c_1}(\mathbf{z}_1)\alpha_{c_2}(\mathbf{z}_2)+\ldots,$$

Other possible colour structures at $O(g^2)$ do not contribute when contracted with $\delta_{c_1c_2}$. Expand the resummed JIMWLK Hamiltonian up to $O(\alpha^2)$:

$$\hat{H}_{\text{lin.},Q} = \frac{\alpha_s}{2\pi^2}\int_{\mathbf{x},\mathbf{y},\mathbf{z}_{(1,2)}}\Bigg\{f^{[c_1ab_1}f^{c_2]ab_2}\Bigg[-K\alpha^{c_1}(\mathbf{x})\alpha^{c_2}(\mathbf{y}) - K^{(2)}_Q\alpha^{c_1}(\mathbf{z}_1)\alpha^{c_2}(\mathbf{z}_2)$$

$$+K^{(1)}_Q\Big(\alpha^{c_1}(\mathbf{z})\alpha^{c_2}(\mathbf{y})+\alpha^{c_1}(\mathbf{x})\alpha^{c_2}(\mathbf{z})\Big)\Bigg]\frac{\delta^2}{\delta\alpha^{b_1}(\mathbf{x})\delta\alpha^{b_2}(\mathbf{y})}$$

$$+C_A\delta^{(2)}(\mathbf{x}-\mathbf{y})[K\alpha^b(\mathbf{x}) - K^{(1)}_Q\alpha^b(\mathbf{z})]\frac{\delta}{\delta\alpha^b(\mathbf{x})}\Bigg\},$$

where $K^{(1)}_Q(\mathbf{x},\mathbf{y},\mathbf{z}) = \int_{\bar{\mathbf{z}}} K(\mathbf{x},\mathbf{y},\bar{\mathbf{z}})R^{(1)}_Q(\bar{\mathbf{z}}-\mathbf{z})$,

$K^{(2)}_Q(\mathbf{x},\mathbf{y},\mathbf{z}_1,\mathbf{z}_2) = \int_{\bar{\mathbf{z}}} K(\mathbf{x},\mathbf{y},\bar{\mathbf{z}})R^{(2,1)}_Q(\bar{\mathbf{z}}-\mathbf{z}_1,\bar{\mathbf{z}}-\mathbf{z}_2)$.

Resummed BFKL kernel

The scale-choice [Kovner, Lublinsky, Skokov, 2023] :

$\boxed{Q^2 = \max((\mathbf{x}-\mathbf{z})^{-2}, (\mathbf{y}-\mathbf{z})^{-2}, Q_0^2)}$. Act $\hat{H}_{\text{lin.},Q}\alpha_c(\mathbf{x})\alpha_c(\mathbf{y})$ and do the Fourier-transform:

$$
\begin{aligned}
K_D^{(\text{res.})}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{2\pi^2}\frac{\mathbf{k}^4}{\mathbf{q}^4}\Bigg\{ &-\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2_+} \\
&-2\frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{q})}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2}\int_{Q_0^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial\ln Q^2}J_0\left(\frac{|\mathbf{k}|}{Q}\right)J_0\left(\frac{|\mathbf{k}+\mathbf{q}|}{Q}\right)+\frac{1}{\mathbf{k}^2}\int_{Q_0^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R_Q^{(2,1)}(\mathbf{q})}{\partial\ln Q^2}J_0^2\left(\frac{|\mathbf{k}|}{Q}\right) \\
-\frac{1}{2}\int_{Q_0^2}^{\infty}\frac{dQ^2}{(Q^2)^{3/2}}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial\ln Q^2}\Bigg[&\frac{1}{|\mathbf{k}|}J_1\left(\frac{|\mathbf{k}|}{Q}\right)\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}+\mathbf{q}|}{q}\right)+\frac{1}{|\mathbf{k}+\mathbf{q}|}J_1\left(\frac{|\mathbf{k}+\mathbf{q}|}{Q}\right)\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}|}{q}\right)\Bigg] \\
&+\frac{1}{2|\mathbf{k}|}\int_{Q_0^2}^{\infty}\frac{dQ^2}{(Q^2)^{3/2}}\frac{\partial R_Q^{(2,1)}(\mathbf{q})}{\partial\ln Q^2}J_1\left(\frac{|\mathbf{k}|}{Q}\right)\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}|}{q}\right) \\
&-\pi\delta^{(2)}(\mathbf{k}+\mathbf{q})\Bigg[(1-R_{Q_0}^{(1)}(\mathbf{q}))(\ln 2-\gamma_E)-\int_{Q_0^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial\ln Q^2}\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}|}{q}\right)\Bigg]\Bigg\}.
\end{aligned}
$$

where $R_Q^{(1)}(\mathbf{q}) = \int_{\mathbf{z}} R_Q^{(1)}(\mathbf{z})e^{i\mathbf{q}\mathbf{z}}$, $R_Q^{(2,1)}(\mathbf{q}) = \int_{\mathbf{z}_{1,2}} R_Q^{(2,1)}(\mathbf{z}_1,\mathbf{z}_2)e^{i\mathbf{q}(\mathbf{z}_1-\mathbf{z}_2)}$.

Resummed BFKL kernel

The scale-choice [Kovner, Lublinsky, Skokov, 2023] :

$\boxed{Q^2 = \max((\mathbf{x}-\mathbf{z})^{-2}, (\mathbf{y}-\mathbf{z})^{-2}, Q_0^2)}$. Act $\hat{H}_{\mathrm{lin.},Q}\alpha_c(\mathbf{x})\alpha_c(\mathbf{y})$ and do the Fourier-transform:

$$
\begin{aligned}
K_D^{(\mathrm{res.})}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{\pi^2}\frac{\mathbf{k}^4}{\mathbf{q}^4}\Big\{ &-\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2_+} \\
&-2\frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{q})}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2}\int_{Q_0^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial\ln Q^2}J_0\left(\frac{|\mathbf{k}|}{Q}\right)J_0\left(\frac{|\mathbf{k}+\mathbf{q}|}{Q}\right) \boxed{+\frac{1}{\mathbf{k}^2}\int_{Q_0^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R_Q^{(2,1)}(\mathbf{q})}{\partial\ln Q^2}J_0^2\left(\frac{|\mathbf{k}|}{Q}\right)} \\
&-\frac{1}{2}\int_{Q_0^2}^{\infty}\frac{dQ^2}{(Q^2)^{3/2}}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial\ln Q^2}\Big[\frac{1}{|\mathbf{k}|}J_1\left(\frac{|\mathbf{k}|}{Q}\right)\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}+\mathbf{q}|}{q}\right)+\frac{1}{|\mathbf{k}+\mathbf{q}|}J_1\left(\frac{|\mathbf{k}+\mathbf{q}|}{Q}\right)\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}|}{q}\right)\Big] \\
&+\frac{1}{2|\mathbf{k}|}\int_{Q_0^2}^{\infty}\frac{dQ^2}{(Q^2)^{3/2}}\frac{\partial R_Q^{(2,1)}(\mathbf{q})}{\partial\ln Q^2}J_1\left(\frac{|\mathbf{k}|}{Q}\right)\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}|}{q}\right) \\
&-\pi\delta^{(2)}(\mathbf{k}+\mathbf{q})\Big[(1-R_{Q_0}^{(1)}(\mathbf{q}))(\ln 2-\gamma_E)-\int_{Q_0^2}^{\infty}\frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial\ln Q^2}\int_0^{Q^2}\frac{dq^2}{q^2}J_0\left(\frac{|\mathbf{k}|}{q}\right)\Big]\Big\}.
\end{aligned}
$$

where $R_Q^{(1)}(\mathbf{q}) = \int_{\mathbf{z}} R_Q^{(1)}(\mathbf{z})e^{i\mathbf{q}\mathbf{z}}$, $R_Q^{(2,1)}(\mathbf{q}) = \int_{\mathbf{z}_{1,2}} R_Q^{(2,1)}(\mathbf{z}_1,\mathbf{z}_2)e^{i\mathbf{q}(\mathbf{z}_1-\mathbf{z}_2)}$.

Resumed BFKL kernel, θ-approximation

Let's substitute $J_0(x) \to \theta(1-x)$, $J_1(x) \to (x/2)\theta(1-x)$.

$$
\frac{\mathbf{q}^4}{\mathbf{k}^4} K_D^{(\text{res.},\,\theta)}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{\pi^2} \Bigg\{ -\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})_+^2}
$$

$$
-2\frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{q})}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})^2}\left(1 - R^{(1)}_{\max(Q_0^2,\mathbf{k}^2,(\mathbf{k}+\mathbf{q})^2)}(\mathbf{q})\right) \boxed{+\frac{1}{\mathbf{k}^2}\left(1 - R^{(2,1)}_{\max(Q_0^2,\mathbf{k}^2)}(\mathbf{q})\right)}
$$

$$
-\frac{1}{4}\int\limits_{\max(Q_0^2,\mathbf{k}^2,(\mathbf{k}+\mathbf{q})^2)}^{\infty} \frac{dQ^2}{(Q^2)^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial \ln Q^2}\left[\ln\frac{Q^2}{(\mathbf{k}+\mathbf{q})^2} + \ln\frac{Q^2}{\mathbf{k}^2}\right]
$$

$$
+\frac{1}{4}\int\limits_{\max(Q_0^2,\mathbf{k}^2)}^{\infty} \frac{dQ^2}{(Q^2)^2}\frac{\partial R_Q^{(2,1)}(\mathbf{q})}{\partial \ln Q^2}\ln\frac{Q^2}{\mathbf{k}^2}
$$

$$
-\pi\delta^{(2)}(\mathbf{k}+\mathbf{q})\left[\left(1 - R^{(1)}_{Q_0}(\mathbf{q})\right)\left(\ln 2 - \gamma_E\right) - \int\limits_{\max(Q_0^2,\mathbf{k}^2)}^{\infty} \frac{dQ^2}{Q^2}\frac{\partial R_Q^{(1)}(\mathbf{q})}{\partial \ln Q^2}\ln\frac{Q^2}{\mathbf{k}^2}\right]\Bigg\}.
$$

Linearised evolution for $\mathbb{S}_Q$ and $\mathbb{V}_Q$: $R^{(1)}$ and $r^{(1)}$

The quark S-matrix is expanded in a similar way:

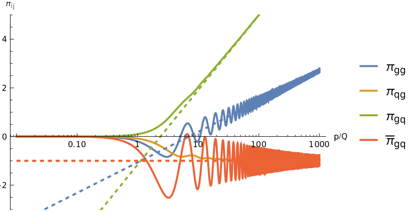
$$\mathbb{V}_{ij}(\mathbf{z}) = \delta_{ab} + igt^{c_1}_{ij} \int_{\mathbf{z}_1} r^{(1)}_Q(\mathbf{z}-\mathbf{z}_1)\alpha_{c_1}(\mathbf{z}_1)$$

$$+ \frac{(ig)^2}{2}(t^{c_1}t^{c_2})_{ij} \int_{\mathbf{z}_1\mathbf{z}_2} r^{(2,1)}_Q(\mathbf{z}-\mathbf{z}_1, \mathbf{z}-\mathbf{z}_2)\alpha_{c_1}(\mathbf{z}_1)\alpha_{c_2}(\mathbf{z}_2) + \dots,$$

Then the evolution for $\mathcal{R}^{(1)}_Q(\mathbf{p}) = \begin{pmatrix} R^{(1)}_Q(\mathbf{p}) \\ r^{(1)}_Q(\mathbf{p}) \end{pmatrix}$ is:

$$\frac{\partial}{\partial \ln Q^2}\mathcal{R}^{(1)}_Q(\mathbf{p}) = -\frac{a}{\beta_0}\Pi_1(|\mathbf{p}|/Q)\mathcal{R}^{(1)}_Q(\mathbf{p}),$$

$$\Pi_1(p) = \begin{pmatrix} -\frac{11N_c}{3}\pi_{gg}(p) & \frac{2n_F}{3}\pi_{qg}(p) \\ -\frac{3}{2}N_c\bar{\pi}_{gq}(p) & \frac{3}{2}\frac{1}{N_c}\pi_{gq}(p) \end{pmatrix},$$



with $\pi_{ij}(p) = \dfrac{\int_0^1 d\xi\, p_{ij}(\xi)\, J_0(p\xi)}{\int_0^1 d\xi\, p_{ij}(\xi)} - 1,$

$$\bar{\pi}_{ij}(p) = \frac{\int_0^1 d\xi\, p_{ij}(1-\xi)\, J_0(p\xi)}{\int_0^1 d\xi\, p_{ij}(\xi)} - 1.$$

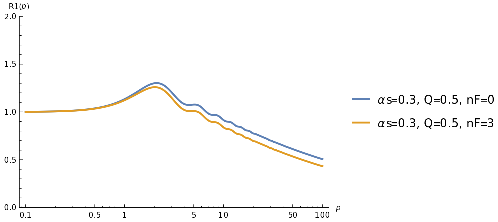
$$\begin{aligned}
\pi_{gg}(p) &\simeq \frac{6}{11}\left(\ln p - \frac{11}{6} - \ln 2 + \gamma_E\right), \\
\pi_{gq}(p) &\simeq \frac{4}{3}\left(\ln p - \frac{3}{4} - \ln 2 + \gamma_E\right), \\
\bar{\pi}_{gq}(p) &\simeq \pi_{qg}(p) \simeq -1.
\end{aligned}$$

Sudakov suppression for $R^{(1)}$

Without quarks, the LLA solution for $R^{(1)}$ looks as follows:

$$
\begin{aligned}
R_Q^{(1)}(\mathbf{p}) &= \theta(Q > |\mathbf{p}|) + \theta(Q_T > |\mathbf{p}| > Q) \exp\left[-\frac{\alpha_s}{2\pi}\frac{N_c}{4}\ln^2\left(\frac{\mathbf{p}^2}{Q^2}\right)\right] \\
&+ \theta(|\mathbf{p}| > Q_T) \exp\left[-\frac{\alpha_s}{2\pi} N_c \ln\frac{Q_T^2}{Q^2} \ln\frac{|\mathbf{p}|}{\sqrt{QQ_T}}\right].
\end{aligned}
$$

The numerical effect of quarks is small:



Linearised evolution for $\mathbb{S}_Q$ and $\mathbb{V}_Q$: $R^{(2,1)}$ and $r^{(2,1)}$

$$\frac{\partial}{\partial \ln Q^2}\mathcal{R}_Q^{(2,1)}(\mathbf{p}) = -\frac{a}{\beta_0}\left[\Pi_2\mathcal{R}_Q^{(2,1)}(\mathbf{p}) + J_0\left(\frac{|\mathbf{p}|}{Q}\right)\rho_Q^{(1)}(\mathbf{p})\right],$$

$$\rho_Q^{(1)}(\mathbf{p}) = \begin{pmatrix} \frac{11}{3}N_c[R_Q^{(1)}(\mathbf{p})]^2 + \frac{2n_F}{3N_c^2}[r_Q^{(1)}(\mathbf{p})]^2 \\ 3N_c R_Q^{(1)}(\mathbf{p}) r_Q^{(1)}(\mathbf{p}) \end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix} \beta_0 - 2N_c\frac{11}{3} & \frac{4C_F n_F}{3N_c} \\ -3N_c & 0 \end{pmatrix}.$$

We approximate $J_0(x) \to \theta(1-x)$ and then we need only that $R^{(1)}(|\mathbf{p}| < Q) = r^{(1)}(|\mathbf{p}| < Q) = 1$. This way we obtain the LLA solution for the $R_Q^{(2,1)}$ (for $Q_T \to \infty$):

$$R_Q^{(2,1)}(\mathbf{p}) = \frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)}\left(\frac{\max(Q^2,\mathbf{p}^2)}{Q^2}\right)^{\frac{a}{\beta_0}\lambda_-} + (\lambda_+ \leftrightarrow \lambda_-),$$

where $\lambda_\pm = \frac{\beta_0}{2} - \frac{11}{3}N_c \pm \sqrt{\left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right)^2 - 4C_F n_F}$.

LLA-resummed BKFL kernel

Only one of the terms contribute in the LLA:

$$K_D^{(\text{res., LLA})}(\mathbf{k},\mathbf{q}) = \frac{\alpha_s N_c}{2\pi^2}\frac{\mathbf{k}^4}{\mathbf{q}^4}\left\{-\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}+\mathbf{q})_+^2}+\frac{1}{\mathbf{k}^2}\big(1-R^{(2,1)}_{\max(Q_0^2,\mathbf{k}^2)}(\mathbf{q})\big)\right\},$$

with $Q_0^2 = \mathbf{k}^2$. The LLA solution for the $R_Q^{(2,1)}$:

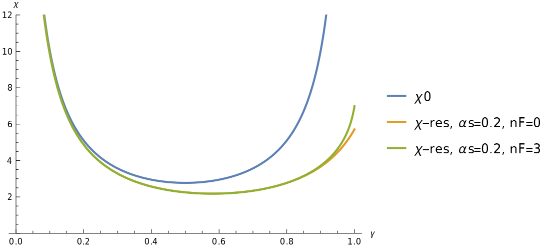
$$R_Q^{(2,1)}(\mathbf{p}) = \frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)}\left(\frac{\max(Q^2,\mathbf{p}^2)}{Q^2}\right)^{\frac{a}{\beta_0}\lambda_-} + (\lambda_+ \leftrightarrow \lambda_-),$$

where $\lambda_\pm = \frac{\beta_0}{2} - \frac{11}{3}N_c \pm \sqrt{\left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right)^2 - 4C_F n_F}$.

$$\int_{\mathbf{q}} K(\mathbf{k},\mathbf{q})(\mathbf{q}^2)^\gamma e^{in\phi_{\mathbf{q}}} = \frac{\alpha_s \cancel{(\mathbf{k}^2)} N_c}{\pi}\chi(n,\gamma)(\mathbf{k}^2)^\gamma e^{in\phi_{\mathbf{k}}},$$

LLA-resummed characteristic function

$$\chi^{(\text{res.})}(\gamma) = \chi_0(n,\gamma) + \delta_{n,0}\left[\frac{1}{\gamma-1} - \frac{N_c(\gamma-1) - \frac{4}{3}a_s C_F n_F}{N_c\left[(\gamma-1)^2 - a_s(\gamma-1)\left(\frac{22}{3}N_c - \beta_0\right) + 4a_s^2 C_F n_F\right]}\right].$$

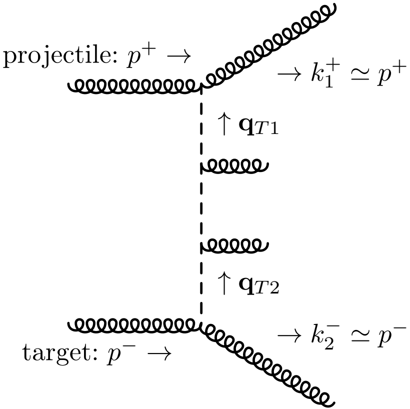


- The anti-collinear pole is removed. Nothing happens to the collinear pole
- The χ-value at the saddle point (*Pomeron intercept*) is decreased, which is good for phenomenology
- The effect of quarks is numerically small

Expanding in α_s we obtain:

$$\chi^{(\text{res.})}(\gamma,n) = \chi_0(\gamma) - \frac{\alpha_s N_c}{4\pi}\frac{11 + 2n_F/N_c^3}{3(1-\gamma)^2}\delta_{n,0} + O(\alpha_s^2),$$

Cheat-sheet on asymmetric schemes in BKFL



$$s = p^+ p^-,\ k_1^- = \frac{\mathbf{q}_{T1}^2}{p^+},\ k_2^+ = \frac{\mathbf{q}_{T2}^2}{p^-},$$

$$L_+ = \ln \frac{k_1^+}{k_2^+} = \ln \frac{s}{\mathbf{q}_{T2}^2},\ \text{(JIMWLK)}$$

$$L_- = \ln \frac{k_2^-}{k_1^-} = \ln \frac{s}{\mathbf{q}_{T1}^2},\ \text{(DIS)}$$

$$Y = \frac{1}{2} \ln \frac{k_1^+}{k_1^-} - \frac{1}{2} \ln \frac{k_2^+}{k_2^-} = \ln \frac{s}{\sqrt{\mathbf{q}_{T1}^2 \mathbf{q}_{T2}^2}}.\ \text{(BFKL)}$$

$$\chi_{\mp}(\gamma, n) = \chi_0(\gamma, n) \mp \frac{\alpha_s N_c}{4\pi} 2\chi_0'(\gamma, n)\chi_0(\gamma, n) + O(\alpha_s^2),$$

So in $(+)$ scheme the $1/(1-\gamma)^3$-pole cancels and in the $(-)$-scheme the $1/\gamma^3$-pole cancels.

Comparison with NLO BKFL

Pole structure of the NLO characteristic function from [Kotikov, Lipatov, 2000] **(symmetric scheme!)**:

$$\chi(n,\gamma) = \chi_0(n,\gamma) + \frac{\alpha_s(\mathbf{k}^2)}{4\pi}\delta(n,\gamma),$$

$$\delta(n,\gamma) = -\frac{2}{(\gamma+n/2)^3} - \frac{\delta_{n,0}}{\gamma^2}\left(\frac{11}{3}+\frac{2n_F}{3N_c^3}\right) + 2\frac{\psi(1)-\psi(n+1)}{\left(\gamma+\frac{n}{2}\right)^2}$$

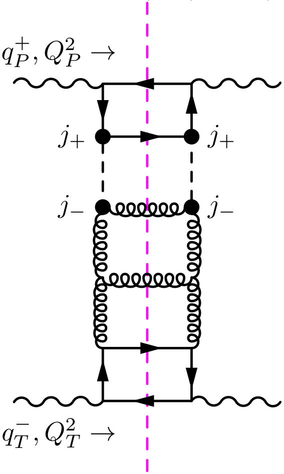
$$-\frac{2}{\left(1-\gamma+\frac{n}{2}\right)^3}\boxed{-\frac{\delta_{n,0}}{(1-\gamma)^2}\left(\frac{11}{3}+\frac{2n_F}{3N_c^3}\right)} + \frac{-\frac{11}{3}+\frac{2n_F}{3N_c}+2\big(\psi(1)-\psi(n+1)\big)}{\left(1-\gamma+\frac{n}{2}\right)^2}$$

$$+\ldots$$

The $1/(1-\gamma+n/2)^3$ pole cancels in the (+)-scheme. The $\delta_{n,0}$-term is reproduced by the resummation. The highlighted term is related with the running-coupling effects.

Cross-check: Bjorken limit of the target

We can cross-check the result by considering the high-energy limit of the $\gamma^*(Q_P) + \gamma^*(Q_T)$-scattering amplitude in the limit $Q_T \gg Q_P$.



- In the BFKL approach, the amplitude $\propto \int\limits_\omega (x_B^{(T)})^\omega \left(\frac{Q_P^2}{Q_T^2}\right)^{\gamma_\star^{(T)}(\omega)}$, where $x_B^{(T)} = Q_T^2/s$ and the anomalous dimansion $\gamma_\star$ is determined by the equation:

$$\frac{\alpha_s N_c}{\pi}\chi(0, 1-\gamma_\star^{(T)}(\omega)) = \omega.$$

- The LO solution of this equation:

$$\gamma_\star^{(T)}(\omega) = \frac{\alpha_s}{2\pi}\left[\frac{2N_c}{\omega} + \left(\frac{\beta_0}{2} - \frac{11}{3}N_c + \frac{2n_F}{3N_c}C_F\right) + O(\omega)\right] + \ldots,$$

agrees with the expectation from DGLAP.

$$\frac{\partial}{\partial \ln \mu^2}\begin{pmatrix} f_g(\omega,\mu^2) \\ f_q(\omega,\mu^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\Gamma\begin{pmatrix} f_g(\omega,\mu^2) \\ f_q(\omega,\mu^2)\end{pmatrix},\ \Gamma = \begin{pmatrix} \frac{2N_c}{\omega} + \left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right) & \frac{4n_F C_F}{\omega} - 3n_F C_F \\ \frac{1}{3} & 0 \end{pmatrix} + O(\omega),$$

Running-coupling effects

Isolating β_0-terms in the linearized equation

Let's start with the gluon and quark dipole terms of the NLO JIMWLK Hamiltonian:

$$\hat{H}^{(\text{dip.})}_{\text{NLO}} = \int_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}'} J^a_L(\mathbf{x})\big[K_{JSSJ}\,\mathrm{tr}[T^a S(\mathbf{z}) T^b S^\dagger(\mathbf{z}')]$$
$$+2n_F K_{q\bar{q}}\,\mathrm{tr}[V^\dagger(\mathbf{z}) t^a V(\mathbf{z}') t^b]\big] J^b_R(\mathbf{y}),$$

and **subtract** from it the real-emission term of the resummation:

$$\hat{H}^{(\text{dip.-res.})}_{\text{NLO}} = \hat{H}^{(\text{dip.})}_{\text{NLO}} - \frac{\alpha_s}{2\pi^2}\int_{\mathbf{x},\mathbf{y},\bar{\mathbf{z}}} K_{LO}(\mathbf{x},\mathbf{y},\bar{\mathbf{z}}) 2J^a(\mathbf{x})\big(\mathbb{S}^{ab}_Q(\bar{\mathbf{z}}) - S^{ab}(\bar{\mathbf{z}})\big) J^b(\mathbf{y}),$$

at the NLO in α_s.

Isolating β_0-terms in the linearized equation

After linearization we obtain:

$$\langle \hat{H}^{(\text{dip.-res.})}_{\text{NLO, exp.}} \alpha_c(\mathbf{x})\alpha_c(\mathbf{y})\rangle =$$

$$-N_c^2 \int\limits_{\mathbf{z},\mathbf{z}'} \left(K_{JSSJ} - \frac{n_F}{N_c^3} K_{q\bar{q}} + \frac{\alpha_s^2}{4\pi^4}\left[\frac{11}{3} + \frac{2n_F}{3N_c^3}\right] K_{\text{sub.}} \right) \langle \alpha_c(\mathbf{z})\alpha_c(\mathbf{z'})\rangle$$

$$+N_c \int\limits_{\mathbf{z}} \left[\int\limits_{\mathbf{z}'} \left(N_c K_{JSSJ} - n_F K_{q\bar{q}} + \frac{\alpha_s^2}{4\pi^4}\left[\frac{11N_c}{3} - \frac{2n_F}{3}\right] K_{\text{sub.}} \right) \right]$$

$$\times \big(\langle \alpha_c(\mathbf{x})\alpha_c(\mathbf{z})\rangle + \langle \alpha_c(\mathbf{y})\alpha_c(\mathbf{z})\rangle \big),$$

where (in $\xi = 1/2$-approximation):

$$K^{(\text{int.})}_{\text{sub.}}(\mathbf{x},\mathbf{y},\mathbf{z}) = \int\limits_{Z} \frac{\theta(Z^2 < Q^{-2})}{Z^2} \frac{(X - \frac{Z}{2}) \cdot (Y - \frac{Z}{2})}{(X - \frac{Z}{2})^2 (Y - \frac{Z}{2})^2}.$$

For $\boxed{Q^2 \gg \max(X^{-2}, Y^{-2})}$:

$$K^{(\text{int.})}_{\text{sub.}}(\mathbf{x},\mathbf{y},\mathbf{z}) = \pi \frac{X \cdot Y}{X^2 Y^2} \left[-\frac{1}{\epsilon} + \gamma_E + \ln \pi + \ln(\mu^2 Q_R^{-2}) + O(\epsilon) \right].$$

β_0-structure at "large-"Q^2

The virtual term of the NLO JIMWLK, combined with the virtual term of the resummation gives:

$$K_{JSJ}^{(\mathrm{NLO})} = \frac{\alpha_s^2}{4\pi^3} \frac{X \cdot Y}{X^2 Y^2} \frac{\beta_0}{2} \left[\ln(X^2\mu^2) + \ln(Y^2\mu^2) - \ln \frac{\mu^2}{Q_V^2} \right],$$

we obtain for the action on a "dipole" $\langle(\alpha(\mathbf{x}) - \alpha(\mathbf{y}))^2\rangle$:

$$\frac{\alpha_s^2 \beta_0}{8\pi^3} \frac{(X-Y)^2}{X^2 Y^2} \left[\ln[\mu^2 (X-Y)^2] - \frac{Y^2 - X^2}{(X-Y)^2} \ln \frac{Y^2}{X^2} + \ln \frac{Q_V^2}{Q_R^2} \right],$$

which agrees with the structure of β_0-terms in [Balitsky, Chirilli, 2007] . In the limit $X^2 \gg Y^2$ (or $X^2 \ll Y^2$) this structure can be simplified as:

$$\ln\left(\mu^2 \min(X^2, Y^2)\right),$$

which suggests the scale for running coupling: $\mu^2 = \max(X^{-2}, Y^{-2})$.

Conclusions and outlook

- The (anti-)collinear resummation for the JIMWLK Hamiltonian [Kovner, Lublinsky, Skokov, 2023] has implications for BFKL evolution.
- The effect for the characteristic function is asymmetric, only anti-collinear pole is affected. The agreement with known NLO BFKL results and with the Bjorken limit of the target, known from DGLAP, has been checked.
- The Pomeron intercept is decreased, the effect of quarks is numerically small
- The symmetrization of this resummation and inclusion of α_s-running effects is needed for phenomenology. Work in progress...

Thank you for your attention!

Backup: Phenomenology of Mueller-Navelet dijets

[Chernyshev, MN, Saleev, 2025]