High-Energy QCD from Eikonal to Sub-Eikonal Corrections

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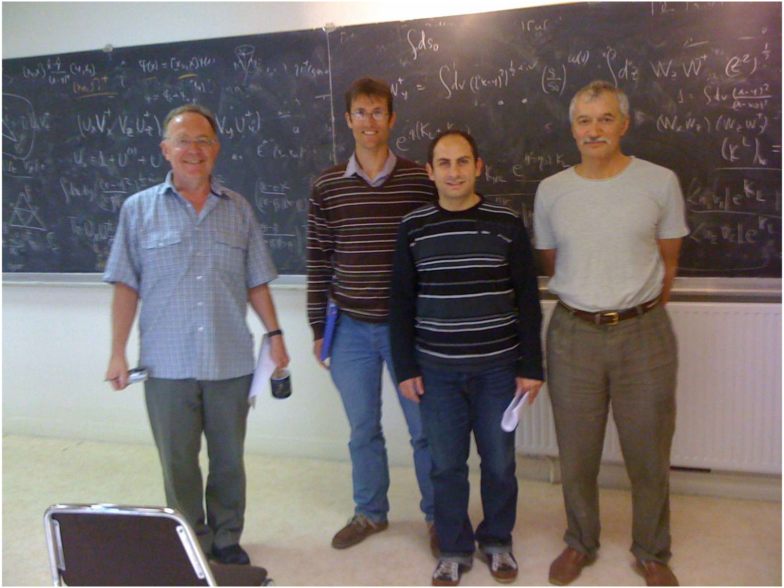
- Ian's style to answer questions
- Do not explain him in words why you think your result is correct

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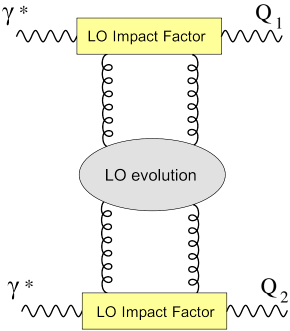
Ian, Happy Birthday!

Story of a Diagram

LO Impact Factor: Ian's PhD thesis (1978)

LO $\gamma^*\gamma^*$ scattering



LO Impact Factor: Ian's PhD thesis (1978)

LO $\gamma^*\gamma^*$ scattering

$$\sigma_{\gamma\gamma} = \frac{4\pi^2\alpha^2\alpha_s^2}{9Q_1Q_2}\left\{F_1(\kappa,\xi) + [(e_1\cdot e_2)^2 - \frac{1}{2}]F_2(\kappa,\xi)\right\}$$

$$F_1(\kappa,\xi) = \frac{1}{2^8}\int d\nu \frac{\sinh^2\pi\nu}{\nu^2\cosh^4\pi\nu}\frac{(9+4\nu^2)^2}{(1+\nu^2)^2}e^{\xi\chi(0,\nu)+i\kappa\nu}$$

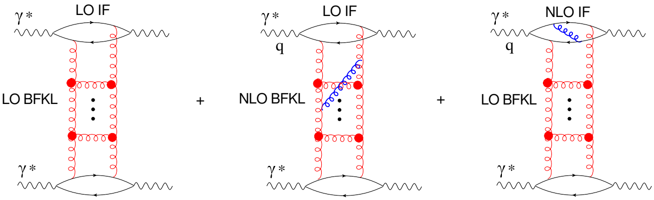
$$F_2(\kappa,\xi) = \frac{1}{2^8}\int d\nu \frac{\sinh^2\pi\nu}{\nu^2\cosh^4\pi\nu}\frac{(1+4\nu^2)^2}{(1+\nu^2)^1}e^{\xi\chi(2,\nu)+i\kappa\nu}$$

$\left(\sum e_i^2\right)^2 = \frac{4}{9}, \quad \xi \equiv \frac{N_c}{2}\ln\frac{s}{Q_1Q_1}$

$\chi(n,\nu) \equiv \Re\left[\psi(1) - \psi(i\nu + \frac{|n|+1}{2})\right]$, and $\kappa \equiv \ln\frac{Q_1^2}{Q_2^2}$.

LO BFKL formalism in full glory

Factorization at NLO



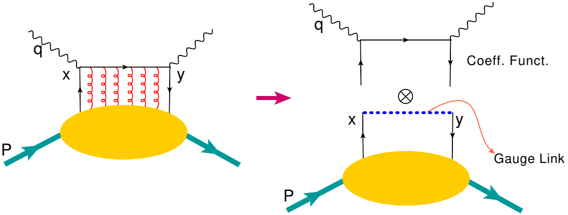
- NLO BFKL: $\alpha_s \left(\alpha_s \ln \frac{1}{x_B}\right)^n$ Fadin-Lipatov (1997)
- NLO Impact factor contains contributions prop. to $\alpha_s \ln \frac{1}{x_B}$ which can be included in the LLA.
- NLO IF from *standard* QCD: very difficult (see Bartels and collaborators)

Road map towards the NLO: it started from far away

- DGLAP original derivation: Leading log resummation
- Modern view: DGLAP corresponds to a renomalization group equation
  - Ian may tell you offline the story of Politzer visiting PNPI in Gatchina

- BFKL original derivation: Leading log resummation
- Ian motivation was to get BFKL from an opratorial point of view
  - $\Rightarrow$ Operator expansion for high-energy scattering (more than 2000 citations)

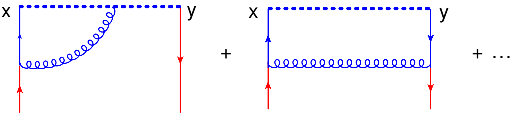
- Idea: semi-classical approach: quantum particles propagates in the background of classical fields
  - Similar to Wilsonian renomalization group: but not quite the same



$$T\{j_\mu(x)j_\nu(y)\} = C_\xi(x,y)\ \bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y) + O((x-y)^2)$$

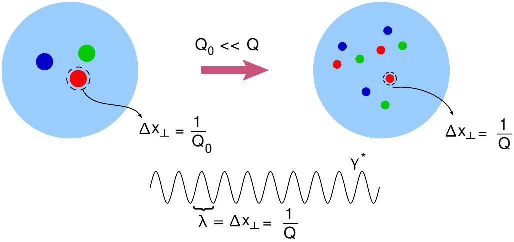
- $C_\xi(x,y)$ is the coefficient function calculable in pQCD
- $\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)$ is the non local operator

DGLAP evolution equation



Renorm-group equation for light-ray operators: DGLAP evolution equation

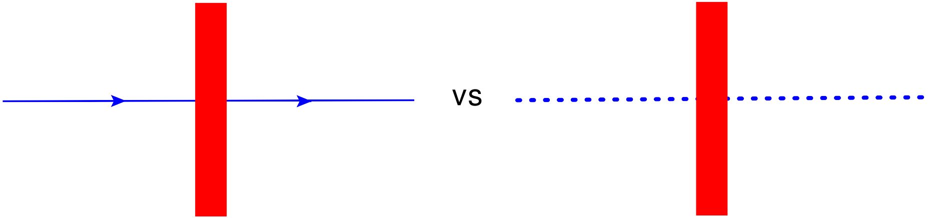
$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \;=\; K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \dots$$



- What Ian had in mind was to use a similar idea of the non-local OPE and apply it to the small-x regime
- Background field method: now field are not separated in transverse momenta but in longitudinal Sudakov component

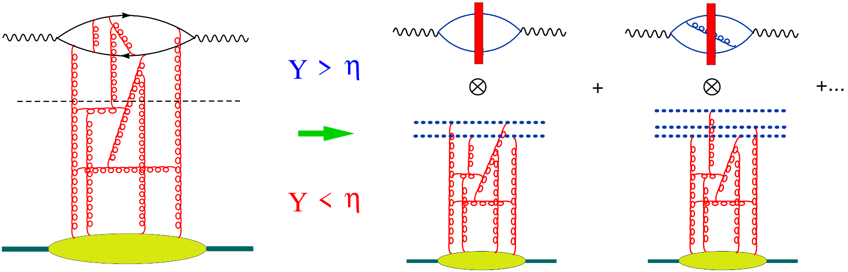
- Motivation: Get BFKL from an operator point of view in a gauge invariant way

- Result: Wilson-line/shock wave formalism



- Solid line: needed for scattering of a quark in the background of a Lorentz contracted gluon field.
- Dotted line: Wilson line (just the operator) in the background of a Lorentz contracted gluon field.

High-energy Operator Product Expansion

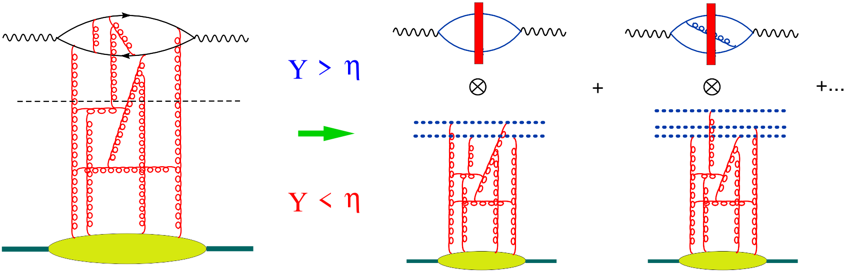


factorization scale: rapidity η

Rapidity Y > η - coefficient function ("impact factor")

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta \;=\; \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\Big]$$



The high-energy operator product expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ \mathcal{I}^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\int d^2z_1 d^2z_2 d^2z_3\ \mathcal{I}^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

DIS at Leading Log Approximation at high-energy

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\, \mathcal{I}^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

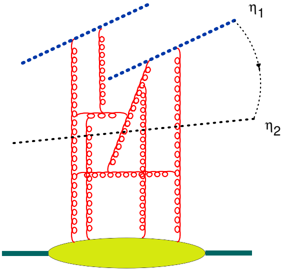
- Calculate LO Imapct factor: $\mathcal{I}^{\rm LO}_{\mu\nu}(z_1,z_2,x,y)$
- Calculate evolution of matrix element ${\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$: BK/JIMWLK equation
- Solve the evolution equation with initial condition: GBW/MV model
- Convolute the solution of the evolution equation with the impact factor

Evolution Equation

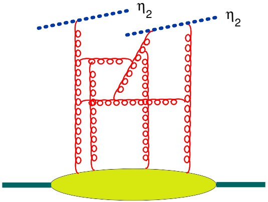
$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle\mathrm{tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

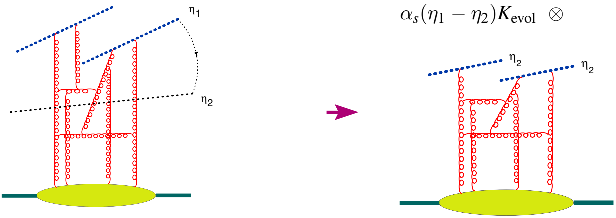
In the frame || to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.



$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \;\otimes$



Particles with different rapidity perceive each other as Wilson lines.



- Separate fields in quantum and classical according to low and large rapidity. Formally we may write:

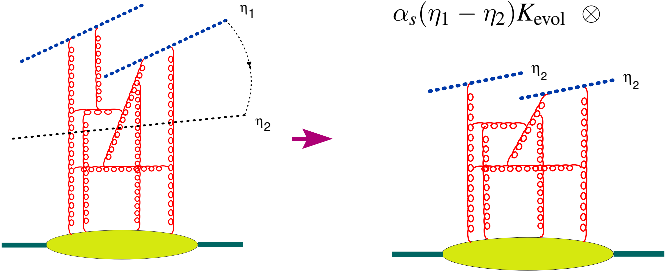
$$\langle B|\mathcal{O}^{\eta_1}|B\rangle \to \langle \mathcal{O}^{\eta_1}\rangle_A \to \langle \mathcal{O}'^{\eta_2} \otimes \mathcal{O}'^{\eta_1}\rangle_A$$

- Integrate over the quantum fields and get one-loop rapidity evolution of the operator $\mathcal{O}$

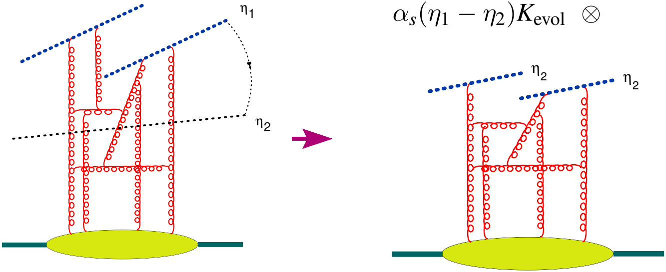
$$\langle \mathcal{O}^{\eta_1}\rangle_A = \alpha_s(\eta_1 - \eta_2) K_{\rm evol} \otimes \langle \mathcal{O}'^{\eta_2}\rangle_A$$

- Where in principle $\mathcal{O}$ and $\mathcal{O}'$ are different operators.

Non-linear evolution equation



- Linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta\, K_{\rm evol} \otimes \mathcal{O}^{\eta_2}$



- Linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\rm evol} \;\otimes\; \mathcal{O}^{\eta_2}$
- Non-linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\rm evol} \;\otimes\; \{\mathcal{O}^{\eta_2}\mathcal{O}^{\eta_2}\}$

Non-linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)-\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\Big\}$$

- LLA for DIS in pQCD $\Rightarrow$ BFKL
  - (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$): Ladder type of diagrams: proliferation of gluons.
- LLA for DIS in semi-classical-QCD $\Rightarrow$ BK/JIMWLK eqn
  - background field method: describes recombination process.

Non-linear evolution equation at NLO

$$\frac{d}{d\eta}Tr\{U_xU^\dagger_y\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s\frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_xU^\dagger_z\}Tr\{U_zU^\dagger_y\} - N_c Tr\{U_xU^\dagger_y\}] +$$
$$\alpha_s^2\int d^2z d^2z'\left(K_4(x,y,z,z')\{U_x, U^\dagger_{z'}, U_z, U^\dagger_y\} + K_6(x,y,z,z')\{U_x, U^\dagger_{z'}, U_{z'}, U_z, U^\dagger_z, U^\dagger_y\}\right)$$

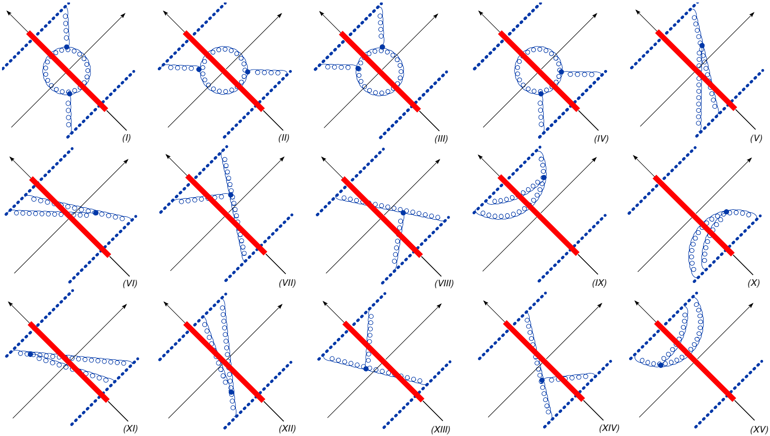
- Conformal composite operator $\Rightarrow$ Preserve $SL(2,C)$ invariance;
  - $\Rightarrow$ allows book-keeping procedure for higher order calculation;

NLO BK in QCD Balitsky and G.A.C. (2007)

NLO BK in $\mathcal{N}$=4 SYM Balitsky and G.A.C. (2009)

NLO Balitsky-JIMWLK Balitsky and G.A.C (2013); Kovner, Lublinsky, Mulian (2013)

Non-linear evolution equation at NLO: sample diagrams



$\sim 100s$ diagrams

- NLO BK in $\mathcal{N}$=4. It was a necessary step
  - Understand the terms in the running coupling not prop to b_0
  - $\Rightarrow$ Composite conformal operators

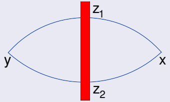
- Composite conformal operator where motivated by the NLO Impact factor
  - First done in $\mathcal{N}$=4
  - then in QCD

LO and NLO Impact Factor

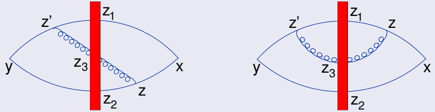
$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\; I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\; I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

LO Impact Factor diagram: $I^{\rm LO}$



NLO Impact Factor diagrams: $I^{\rm NLO}$



NLO Photon Impact Factor

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\,\left[I_1^{\mu\nu}(z_1,z_2,z_3) + I_2^{\mu\nu}(z_1,z_2,z_3)\right] \times[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

where $I_2^{\mu\nu}(z_1,z_2,z_3)$ is finite and conformal, while

$$I_1^{\mu\nu}(z_1,z_2,z_3) = \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3}$$

is rapidity divergent.

How to get the NLO Impact factor

$$\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A$$

$$+ \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\; I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}] + \dots$$

$$\Rightarrow$$

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm NLO} - \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO}$$

$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\; I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

$$\left[\langle {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}\rangle_A\right]^{\rm LO} = \frac{\alpha_s}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]\int_0^{e^\eta}\frac{d\alpha}{\alpha}$$

How to get the NLO Impact factor

$$\int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\; I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta)[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

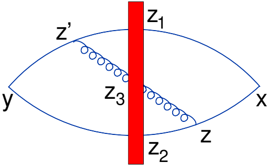
$$= \int \frac{d^2z_1 d^2z_2}{z_{12}^4} d^2z_3\; \Big\{ I_2^{\mu\nu}(z_1,z_2,z_3) + \frac{\alpha_s}{2\pi^2} I^{\rm LO}_{\mu\nu}\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big[\int_0^{+\infty}\frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta}\frac{d\alpha}{\alpha}\Big]\Big\}$$
$$\times[{\rm tr}\{U_{z_1}U^\dagger_{z_3}\}{\rm tr}\{U_{z_3}U^\dagger_{z_2}\} - N_c{\rm tr}\{U_{z_1}U^\dagger_{z_2}\}]$$

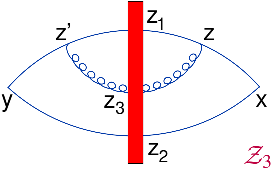
$$\Big[\int_0^{+\infty}\frac{d\alpha}{\alpha}\, e^{i\frac{s\alpha}{4}\mathcal{Z}_3} - \int_0^{e^\eta}\frac{d\alpha}{\alpha}\Big] \to -\ln\frac{\sigma s}{4}\mathcal{Z}_3 - \frac{i\pi}{2} + C$$

where $\sigma = e^\eta$ and C is the Euler constant

$$\mathcal{Z}_3 \equiv \frac{(x-z_3)^2_\perp}{x^+} - \frac{(y-z_3)^2_\perp}{y^+}$$

$\mathcal{Z}_3$ is not conformal invariant in the transverse 2-d coordinate space, but QCD at tree level has to be conformal invariant.





$$\mathcal{Z}_3 \equiv \frac{(x-z_3)^2_\perp}{x^+} - \frac{(y-z_3)^2_\perp}{y^+}$$

$$I^{\rm NLO}_{\mu\nu}(x,y;z_1,z_2,z_3;\eta) \;=\; -\,I^{\rm LO}_{\mu\nu} \times \frac{\alpha_s}{2\pi}\frac{z^2_{13}}{z^2_{12}z^2_{23}} \ln\frac{\sigma s}{4}\mathcal{Z}_3 \;+\; \text{conf.}$$

The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln\sigma$ is not invariant.

Conformal Composite Operator

Define a composite operator

$$[\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

(a - analog of μ^{-2} for usual OPE)
the impact factor becomes conformal at the NLO.

Conformal Composite Operator

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$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2z_{23}^2} + O(\alpha_s^2)$$

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the impact factor becomes conformal at the NLO.

$$[\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} \ = \ \mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\} + \ BK_{\mathrm{LO}}\ln\sqrt{\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2z_{23}^2}} \ + \ O(\alpha_s^2)$$

Conformal Composite Operator

Define a composite operator

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$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\ \frac{z_{12}^2}{z_{13}^2z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\}-N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2z_{23}^2}+O(\alpha_s^2)$$

(a - analog of μ^{-2} for usual OPE)
the impact factor becomes conformal at the NLO.

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + BK_{\rm LO}\ln\sqrt{\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2z_{23}^2}} + O(\alpha_s^2)$$

$$[\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]^{\rm conf} = \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + BK_{\rm LO}\int_0^{\sqrt{\frac{a\ z_{12}^2}{z_{13}^2z_{23}^2}}}\frac{d\alpha}{\alpha}+O(\alpha_s^2)$$

Conformal Composite Operator

$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_{a,\eta}$$
$$= \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + \frac{\alpha_s}{4\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2 z_{23}^2}$$

choose a rapidity-dependent constant $a \to ae^{-2\eta} \Rightarrow [\mathrm{Tr}\{\hat U^\sigma_{z_1}\hat U^{\dagger\sigma}_{z_2}\}]^{\rm conf}_a$ does not depend on $\eta = \ln\sigma$ and all the rapidity dependence is encoded into a-dependence:

$$[\mathrm{Tr}\{\hat U_{z_1}\hat U^\dagger_{z_2}\}]^{\rm conf}_{a}$$
$$= \mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\} + \frac{\alpha_s}{4\pi^2}\int d^2z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}]\ln\frac{4\ a\ z_{12}^2}{\sigma^2\ s\ z_{13}^2 z_{23}^2}$$

Conformal Composite Operator

Using the leading-order evolution equation

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$\Rightarrow \frac{d}{d\eta}[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^{\dagger}_{z_2}\}]_a^{\mathrm{conf}} = 0$ (with $O(\alpha_s^2)$ accuracy).

$$2a\frac{d}{da}[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^{\dagger}_{z_2}\}]_a^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^{\eta}_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}^{\eta}_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

Operator expansion in conformal dipoles

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm tr}[\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\rm conf}$$

$$+\int d^2z_1 d^2z_2 d^2z_3\ I^{\rm NLO}_{\mu\nu}(z_1,z_2,z_3,x,y)[\frac{1}{N_c}{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}{\rm tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - {\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]$$

$$I^{\rm NLO}_{\mu\nu} = \ - I^{\rm LO}_{\mu\nu}\ \frac{\alpha_s N_c}{4\pi}\int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\ln\frac{z_{12}^2 e^{2\eta} a s^2}{z_{13}^2 z_{23}^2}\mathcal{Z}_3^2 + {\rm conf.}$$

The new NLO impact factor is conformally invariant.

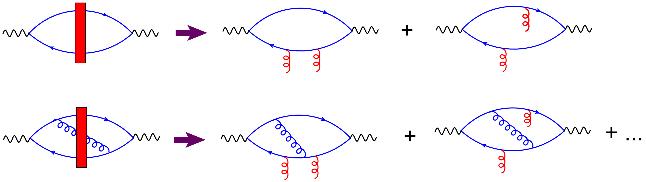
In conformal $\mathcal{N}=4$ SYM theory one can construct the composite conformal dipole operator order by order in perturbation theory.

$$2a\frac{d}{da}[\mathrm{tr}\{\hat{U}_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}$$
$$\times\,\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{12}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\big)\Big]$$
$$+\,\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{14}}^2z_{23}^2+{z_{24}}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{14}}^2z_{23}^2-{z_{24}}^2z_{13}^2)}\ln\frac{{z_{14}}^2z_{23}^2}{{z_{24}}^2z_{13}^2}\Big]$$
$$\times\,[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\mathrm{tr}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\} - (z_4\to z_3)]$$
$$+\,\frac{z_{12}^2z_{34}^2}{z_{13}^2{z_{24}}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2{z_{24}}^2}{{z_{14}}^2z_{23}^2}\Big]$$
$$\times\,[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\mathrm{tr}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_4}\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\hat{U}_{z_4}\hat{U}^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$b = \frac{11}{3}N_c - \frac{2}{3}n_f$ I. Balitsky and G.A.C (2007&2009)

$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

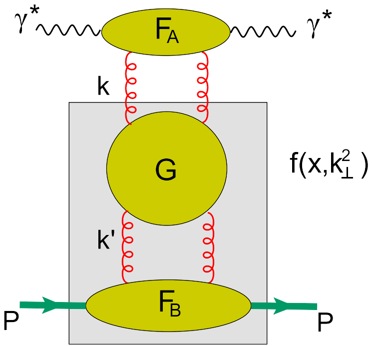
Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).



$$I^{\mathrm{LO}}\hat{\mathcal{U}}(x_\perp, y_\perp)$$
$$I^{\mathrm{NLO}}\Big\{\hat{\mathcal{U}}(x,z)+\hat{\mathcal{U}}(z,y)-\hat{\mathcal{U}}(x,y)\Big\}$$

where $\mathcal{U}(x,y) = 1 - \frac{1}{N_c}\mathrm{tr}\{U_x U_y^\dagger\}$ and we neglected the non-linear term $\hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)$

NLO DIS in the k_T factorization form



- $f(x,k_\perp^2) \propto \int \frac{d^2k'}{k'^2_\perp} F_B(k'^2_\perp)\ k_\perp^2 G(x,k_\perp,k'_\perp)$
- $\mathcal{V}(z) \equiv z^{-2}\mathcal{U}(z)$

$$2a\frac{d}{da}\mathcal{V}_a(z) \;=\; \frac{\alpha_s N_c}{2\pi^2}\int d^2z'\,\Big[\frac{2\mathcal{V}_a(z')}{(z-z')^2} - \frac{z^2\mathcal{V}_a(z)}{z'^2(z-z')^2}\Big]$$

$$\int d^4x\ e^{iqx}\langle p|T\{\hat{j}_\mu(x)\hat{j}_\nu(0)\}|p\rangle \;=\; \frac{s}{2}\int\frac{d^2k_\perp}{k_\perp^2}\ I_{\mu\nu}(q,k_\perp)\mathcal{V}_{a_m=x_B}(k_\perp)$$

NLO Evolution of the unintegrated gluon distribution

$$
\begin{aligned}
2a\frac{d}{da}\mathcal{V}_a(k) \;=\; & \frac{\alpha_s N_c}{\pi^2}\int d^2k'\Big\{\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2} - \frac{(k,k')\mathcal{V}_a(k)}{k'^2(k-k')^2}\Big] \\
& \times\Big(1+\;\frac{\alpha_s b}{4\pi}\Big[\ln\frac{\mu^2}{k^2} + \frac{N_c}{b}\Big(\frac{67}{9} - \frac{\pi^2}{3} - \;\frac{10n_f}{9N_c}\Big)\Big]\Big) - \;\frac{b\alpha_s}{4\pi} \\
& \times\Big[\frac{\mathcal{V}_a(k')}{(k-k')^2}\ln\frac{(k-k')^2}{k'^2} - \frac{k^2\mathcal{V}_a(k)}{k'^2(k-k')^2}\ln\frac{(k-k')^2}{k^2}\Big] \\
& + \frac{\alpha_s N_c}{4\pi}\Big[-\frac{\ln^2(k^2/k'^2)}{(k-k')^2} + \;F(k,k') + \Phi(k,k')\Big]\;\mathcal{V}_a(k')\Big\} \\
& + 3\frac{\alpha_s^2 N_c^2}{2\pi^2}\zeta(3)\mathcal{V}_a(k)
\end{aligned}
$$

I. Balitsky and G.A.C. (2013)

Unintegrated Dipole Gluon Distribution

$$\mathcal{V}_{x_B}(z_\perp,\mu) \;=\; \frac{4\pi^2 x_B}{N_c}\alpha_s(\mu)\mathcal{D}(x_B,z_\perp,\mu)$$

where

$$\mathcal{D}(x_B,z_\perp,\mu) \equiv \frac{4}{s^2}\int\frac{dz_*}{\pi x_B}\langle p|\mathrm{Tr}\Big\{\big[\infty p_1+z_\perp,\frac{2}{s}z_*p_1+z_\perp\big]$$
$$\hat{F}_{\bullet\xi}\big(\frac{2}{s}z_*p_1+z_\perp\big)\big[\frac{2}{s}x_*p_1+z_\perp,-\infty p_1+z_\perp\big][-\infty p_1,0]\hat{F}_\bullet{}^{\xi}(0)[0,\infty p_1]\Big\}|p\rangle^{a_m=x_B}$$

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form

I. Balitsky and G.A.C. (2013)

$$
\begin{aligned}
& I^{\mu\nu}(q,k_\perp) \\
&= \frac{N_c}{32}\int\frac{d\nu}{\pi\nu}\frac{\sinh\pi\nu}{(1+\nu^2)\cosh^2\pi\nu}\left(\frac{k_\perp^2}{Q^2}\right)^{\frac{1}{2}-i\nu}\Big\{\Big[\Big(\frac{9}{4}+\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_1(\nu)\Big)P_1^{\mu\nu} \\
&+\Big(\frac{11}{4}+3\nu^2\Big)\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_2(\nu)\Big)P_2^{\mu\nu}\Big] \\
&+\frac{\frac{1}{4}+\nu^2}{2k_\perp^2}\Big(1+\frac{\alpha_s}{\pi}+\frac{\alpha_s N_c}{2\pi}\mathcal{F}_3(\nu)\Big)\big[\tilde{P}^{\mu\nu}\bar{k}^2+\bar{P}^{\mu\nu}\tilde{k}^2\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
P_1^{\mu\nu} &= g^{\mu\nu}-\frac{q_\mu q_\nu}{q^2} \\
P_2^{\mu\nu} &= \frac{1}{q^2}\Big(q^\mu-\frac{p_2^\mu q^2}{q\cdot p_2}\Big)\Big(q^\nu-\frac{p_2^\nu q^2}{q\cdot p_2}\Big) \\
\bar{P}^{\mu\nu} &= \Big(g^{\mu 1}-ig^{\mu 2}-p_2^\mu\frac{\bar{q}}{q\cdot p_2}\Big)\Big(g^{\nu 1}-ig^{\nu 2}-p_2^\nu\frac{\bar{q}}{q\cdot p_2}\Big) \\
\tilde{P}^{\mu\nu} &= \Big(g^{\mu 1}+ig^{\mu 2}-p_2^\mu\frac{\tilde{q}}{q\cdot p_2}\Big)\Big(g^{\nu 1}+ig^{\nu 2}-p_2^\nu\frac{\tilde{q}}{q\cdot p_2}\Big)
\end{aligned}
$$

Photon Impact Factor for BFKL pomeron in momentum space

k_T-factorization form

I. Balitsky and G.A.C. (2013)

$$
\begin{aligned}
\mathcal{F}_{1(2)}(\nu) &= \Phi_{1(2)}(\nu) + \chi_\gamma \Psi(\nu), \quad \mathcal{F}_3(\nu) = F_6(\nu) + \Big(\chi_\gamma - \frac{1}{\bar\gamma\gamma}\Big)\Psi(\nu), \\
\Psi(\nu) &\equiv \psi(\bar\gamma) + 2\psi(2-\gamma) - 2\psi(4-2\gamma) - \psi(2+\gamma), \\
F_6(\gamma) &= F(\gamma) - \frac{2C}{\bar\gamma\gamma} - 1 - \frac{2}{\gamma^2} - \frac{2}{\bar\gamma^2} - 3\frac{1+\chi_\gamma - \frac{1}{\gamma\bar\gamma}}{2+\bar\gamma\gamma}, \\
\Phi_1(\nu) &= F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{25}{18(2-\gamma)} + \frac{1}{2\bar\gamma} - \frac{1}{2\gamma} - \frac{7}{18(1+\gamma)} + \frac{10}{3(1+\gamma)^2} \\
\Phi_2(\nu) &= F(\gamma) + \frac{3\chi_\gamma}{2+\bar\gamma\gamma} + 1 + \frac{1}{2\bar\gamma\gamma} - \frac{7}{2(2+3\bar\gamma\gamma)} + \frac{\chi_\gamma}{1+\gamma} + \frac{\chi_\gamma(1+3\gamma)}{2+3\bar\gamma\gamma}, \\
F(\gamma) &= \frac{2\pi^2}{3} - \frac{2\pi^2}{\sin^2\pi\gamma} - 2C\chi_\gamma + \frac{\chi_\gamma - 2}{\bar\gamma\gamma}
\end{aligned}
$$

where $\gamma = \frac{1}{2} + i\nu$

No symmetry $\gamma \to 1-\gamma$

Comparing with NLO BFKL

two gluons in the dipole $\mathcal{U}(k)$ come with extra g^2 factor $\Rightarrow$

$$\mathcal{L}(k) \;=\; \frac{1}{g^2(k)}\mathcal{V}(k)$$

$$\begin{aligned}
2a\frac{d}{da}\mathcal{L}_a(k) \;=\; & \frac{\alpha_s(k^2)N_c}{\pi^2}\int d^2k'\Big\{\Big[\frac{\mathcal{L}_a(k')}{(k-k')^2} \\
& - \frac{(k,k')\mathcal{L}_a(k)}{k'^2(k-k')^2}\Big]\Big[1+\frac{\alpha_s N_c}{4\pi}\Big(\frac{67}{9}-\frac{\pi^2}{3}-\frac{10n_f}{9N_c}\Big)\Big] - \\
& - \frac{b\alpha_s}{4\pi}\Big[\frac{\mathcal{L}_a(k')}{(k-k')^2}-\frac{k^2\mathcal{L}_a(k)}{k'^2(k-k')^2}\Big]\ln\frac{(k-k')^2}{k^2} \\
& + \frac{\alpha_s N_c}{4\pi}\Big[-\frac{\ln^2(k^2/k'^2)}{(k-k')^2}+F(k,k')+\Phi(k,k')\Big]\mathcal{L}_a(k')\Big\} \\
& + 3\frac{\alpha_s^2N_c^2}{2\pi^2}\zeta(3)\mathcal{L}_a(k)
\end{aligned}$$

Eigenvalues (LO eigenfunctions) coincide with NLO BFKL eq.

“evolution” of an operator

$$\mathcal{U}(x,y) \equiv 1 - \frac{1}{N_c}\,\mathrm{Tr}\{U(x_\perp, y_\perp)\}$$

$$\mathcal{U}(x,y) \to \mathcal{U}^{conp}(x,y) \to \mathcal{V}(x,y) \to \mathcal{L}(x,y)$$

BFKL equation in the $\mathcal{N}$=4 SYM case

- In $\mathcal{N} = 4$ SYM theory the coupling constant does not run.
- $\Rightarrow (k^2)^{-\frac{1}{2}+i\nu}$ are eigenfunctions at any order.

$$K(q,k) = \alpha_{\rm SYM} K^{\rm LO}(q,k) + \alpha^2_{\rm SYM} K^{\rm NLO}(q,k) + \dots$$

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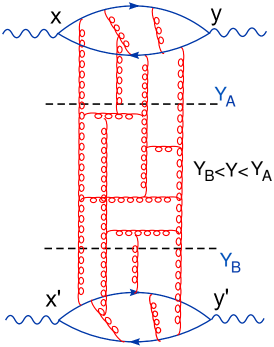
$$K(q,k) = \alpha_{\rm SYM} K^{\rm LO}(q,k) + \alpha^2_{\rm SYM} K^{\rm NLO}(q,k) + \dots$$

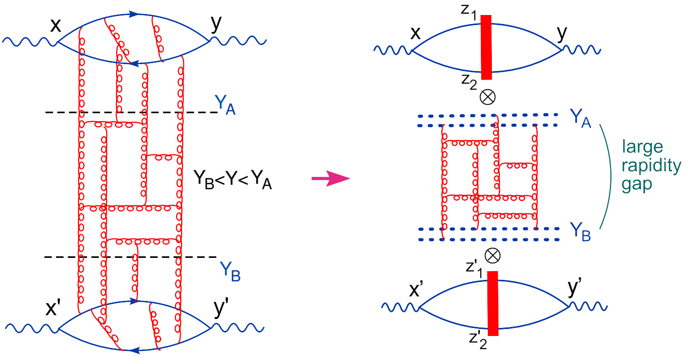
$$\int d^2q\, K(q,k)\,(q^2)^{-\frac{1}{2}+i\nu} = [\alpha_{\rm SYM}\chi_0(\nu) + \alpha^2_{\rm SYM}\chi_1(\nu)\dots]\,(k^2)^{-\frac{1}{2}+i\nu}$$

$$G(k,k',Y) = \int \frac{d\nu}{2\pi^2 k\, k'}\, e^{[\alpha_{\rm SYM}\chi_0(\nu)+\alpha^2_{\rm SYM}\chi_1(\nu)\dots]} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

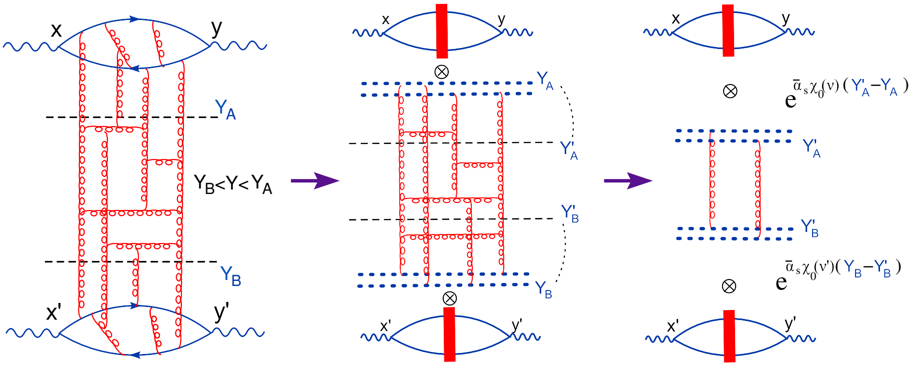
- The eigenvalues $\bar\alpha_\mu\,\chi_0(\nu) + \bar\alpha^2_\mu\,\chi_1^{\rm SYM}(\nu) + \dots$ are real and symmetric for $\nu \leftrightarrow -\nu$.
- NLO Impact Factor in $\mathcal{N}$=4 is symmetric for $\nu \leftrightarrow -\nu$.

$\gamma^*\gamma^*$ scattering cross-section at LO





$$\langle j^\alpha(x)j^\beta(y)j^\rho(x')j^\lambda(y')\rangle \propto I_A^{\alpha\beta}(x,y;z_1,z_2)\, I_B^{\rho\lambda}(x',y';z'_1,z'_2)$$
$$\otimes\langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle$$



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$$\otimes\langle \mathrm{tr}\{U_{z_1}U^\dagger_{z_2}\}^{Y_A}\rangle_A\langle \mathrm{tr}\{U_{z_3}U^\dagger_{z_4}\}^{Y_B}\rangle_A$$

$\gamma^*\gamma^*$ scattering cross-section at LO

$$U_x = \mathrm{Pexp}\left\{ ig\int_{-\infty}^{+\infty} dx^+ A^-(x^+ + x_\perp)\right\}$$

$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1,q_2) \propto i\frac{\alpha_s^2}{Q_1Q_2}\int d\nu\, I_{\mathrm{LO}}^{\alpha\beta}(\nu)\, I_{\mathrm{LO}}^{\rho\lambda}(\nu)\left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar{\alpha}_\mu\,\chi_0(\nu)\,(Y_A - Y_B)}$$

$$Y_A = \frac{1}{2}\ln\frac{s}{Q_1^2}, \qquad Y_B = -\frac{1}{2}\ln\frac{s}{Q_2^2}, \qquad s = (q_1+q_2)^2$$

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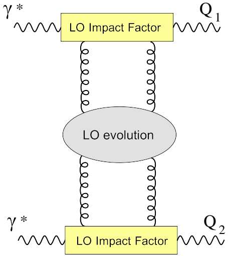
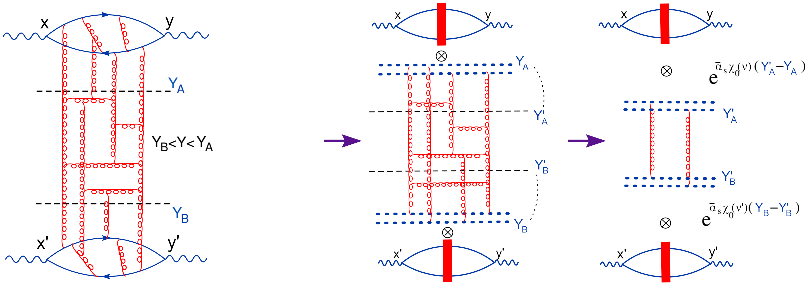
$$\mathcal{A}^{\alpha\beta\,\rho\lambda}(q_1, q_2) \propto i\frac{\alpha_s^2}{Q_1 Q_2} \int d\nu\, I^{\alpha\beta}_{\mathrm{LO}}(\nu)\, I^{\rho\lambda}_{\mathrm{LO}}(\nu) \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu} e^{\bar{\alpha}_\mu\, \chi_0(\nu)\, \ln\frac{s}{Q_1 Q_2}}$$

This is Ian's PhD thesis result

First “evolution” of the diagram





What about NLO?

- Can we repit the steps performed at LO also at NLO?
- Problems to be solved:
  - Solve NLO BFKL equation   G.A.C and Yu. Kovchegov (2013)
  - Calculate NLO Impact Factor   I. Balitsky and G.A.C. (2011 & 2012)
  - NLO Impact Factor has to be conformal invariant;
  - ⇒ Energy dependence of NLO Impact Factor needs to be eliminated;
  - ⇒ Composite Wilson line operators   I. Balisky and G.A.C. (2009)

BFKL equation at NLO in QCD

$$K^{\mathrm{LO+NLO}}(k,q) \equiv \bar{\alpha}_\mu\, K^{\mathrm{LO}}(k,q) + \bar{\alpha}_\mu^2\, K^{\mathrm{NLO}}(k,q)$$

$$\int d^2q\, K^{\mathrm{LO+NLO}}(k,q)\; q^{2\gamma-2} = \left[\bar{\alpha}_\mu\, \chi_0(\gamma) - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2\, \frac{\delta(\gamma)}{4}\right] k^{2\gamma-2}$$

$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \qquad \beta_2 = \frac{11\,N_c - 2\,N_f}{12\,N_c}$$

BFKL equation at NLO in QCD

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$$\int d^2q\, K^{\mathrm{LO+NLO}}(k,q)\; q^{2\gamma-2} = \left[\bar{\alpha}_\mu\, \chi_0(\gamma) - \bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2} + \bar{\alpha}_\mu^2\, \frac{\delta(\gamma)}{4}\right] k^{2\gamma-2}$$

$$\bar{\alpha}_\mu = \frac{\alpha_\mu N_c}{\pi}, \qquad \beta_2 = \frac{11\, N_c - 2\, N_f}{12\, N_c}$$

- $-\bar{\alpha}_\mu^2\, \beta_2\, \chi_0(\gamma)\, \ln\frac{k^2}{\mu^2}$ 1-loop running coupling.
- $\delta(\gamma) = -2\,\beta_2\, \chi_0'(\gamma) + 4\,\chi_1(\gamma)$ Fadin-Lipatov (1998)
- $\chi_1(\gamma)$ Real and symmetric in $\gamma \leftrightarrow 1-\gamma$ $\gamma = \frac{1}{2} + i\nu$.
- $\frac{d}{d\gamma}\chi_0(\gamma) \equiv \chi_0'(\gamma)$ imaginary and NOT symmetric in $\gamma \leftrightarrow 1-\gamma$.

Solution of NLO BFKL equation

- NLO eigenfunctions: perturbation around the conformal LO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 + \bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2} + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right]$$

- NLO eigenvalues $\quad \Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\,\chi_1(\nu)$

Solution of NLO BFKL equation

- NLO eigenfunctions: perturbation around the conformal LO eigenfunctions

$$H_{\frac{1}{2}+i\nu}(k) = k^{-1+2i\nu}\left[1 + \bar{\alpha}_\mu\,\beta_2\left(i\,\frac{\chi_0(\nu)}{2\,\chi_0'(\nu)}\,\ln^2\frac{k^2}{\mu^2} + \frac{1}{2}\left(\frac{\partial}{\partial\nu}\frac{\chi_0(\nu)}{\chi_0'(\nu)}\right)\ln\frac{k^2}{\mu^2}\right)\right]$$

- NLO eigenvalues $\Delta(\nu) = \bar{\alpha}_\mu\,\chi_0(\nu) + \bar{\alpha}_\mu^2\,\chi_1(\nu)$

Solution of NLO BFKL equation **G.A.C. and Yu. Kovchegov**

$$G(k,k',Y) = \int_{-\infty}^{\infty}\frac{d\nu}{2\pi^2}\,e^{\left[\bar{\alpha}_\mu\,\chi_0(\nu)+\bar{\alpha}_\mu^2\,\chi_1(\nu)\right]Y}\,H_{\frac{1}{2}+i\nu}(k)\,\left[H_{\frac{1}{2}+i\nu}(k')\right]^*$$

- The perturbative expansion is in both the exponent and in the eigenfunctions (contrary to DGLAP case and $\mathcal{N}$=4 BFKL).

- The $H_\gamma(k)$ eigenfunctions diagonalize the LO+NLO BFKL kernel

$$\bar{\alpha}_\mu \, K^{\rm LO}(k,q) + \bar{\alpha}_\mu^2 \, K^{\rm NLO}(k,q) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\gamma}{2\,\pi^2\, i} \, \Delta(\gamma) \, H_\gamma(k) H^*_\gamma(q)$$

- LO+NLO BFKL kernel is μ-independent up to $\mathcal{O}(\alpha_\mu^3) \Rightarrow$
- So is its diagonalization through $H_\gamma(k)$ eigenfunctions.

μ-independence of the NLO solution

$$
\begin{aligned}
G(k,k',Y) &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\,k\,k'}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\chi_1(\nu)\right]Y} H_\gamma(k)H_\gamma^*(q) \\
&= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\,k\,k'}\, e^{\left[\bar\alpha_\mu\,\chi_0(\nu)+\bar\alpha_\mu^2\,\chi_1(\nu)\right]Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1-\bar\alpha_\mu^2\,\beta_2\,\chi_0(\nu)\,Y\,\ln\frac{kk'}{\mu^2}\right)
\end{aligned}
$$

- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar\alpha_\mu\, \chi_0(\nu)+\bar\alpha_\mu^2\, \chi_1(\nu)\right]\, Y}\, H_\gamma(k) H^*_\gamma(q)$$

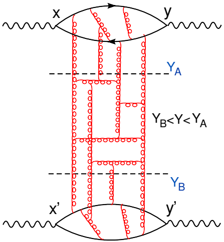
$$= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar\alpha_\mu\, \chi_0(\nu)+\bar\alpha_\mu^2\, \chi_1(\nu)\right]\, Y} \left(\frac{k^2}{k'^2}\right)^{i\nu} \left(1 - \bar\alpha_\mu^2\, \beta_2\, \chi_0(\nu)\, Y\, \ln\frac{kk'}{\mu^2}\right)$$

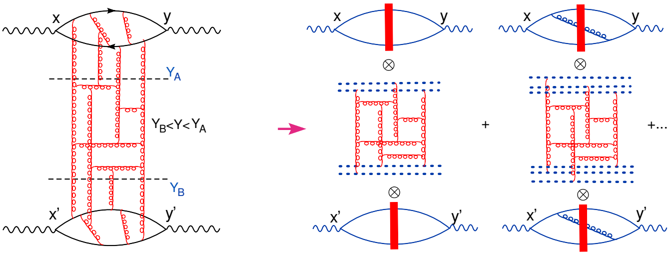
- $\Rightarrow G(k,k',Y)$ is μ-independent up to order $\mathcal{O}(\alpha_\mu^3)$.
- At NLO we may write the solution as (the structure is the same as NLO $\mathcal{N}$=4 SYM)

$$G(k,k',Y) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi^2\, k\, k'}\, e^{\left[\bar\alpha_s(k\,k')\, \chi_0(\nu)+\bar\alpha_s^2(k\,k')\, \chi_1(\nu)\right]\, Y} \left(\frac{k^2}{k'^2}\right)^{i\nu}$$

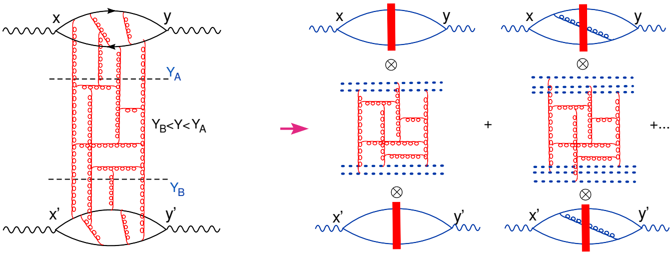
- At this order the scale $\bar\alpha_s^\lambda(k^2)\ \bar\alpha_s^\lambda(k'^2)\bar\alpha_s^{1-2\lambda}(k\,k')$ (for real λ) works as well.

$\gamma^*\gamma^*$ scattering cross-section at NLO





Using high-energy Operator Product Expansion in composite Wilson line operators we get NLO Impact Factor that does not scale with energy.



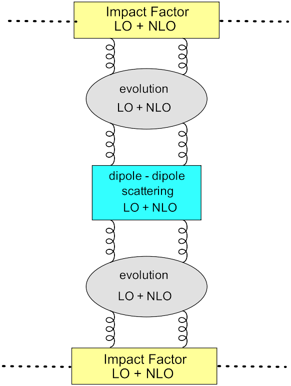
Composite Wilson line operators

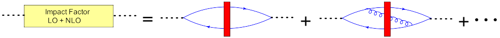
I. Balitsky and G.A.C.

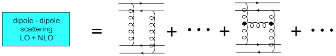
$$[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]^{\mathrm{conf}} = \mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\frac{\alpha_s}{4\pi^2}\int d^2z_3\,\frac{z_{12}^2}{z_{13}^2 z_{23}^2}[\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat{U}^\eta_{z_3}\hat{U}^{\dagger\eta}_{z_2}\} - N_c\mathrm{tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}]\ln\frac{az_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)$$

Factorization of scattering amplitude at NLO in $\mathcal{N}$=4

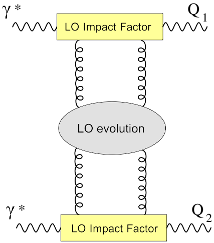
I. Balitsky and G.A.C.

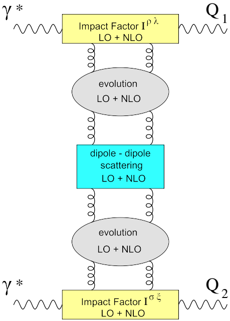


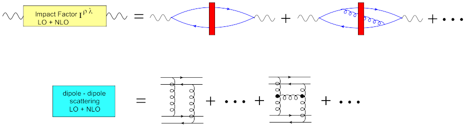


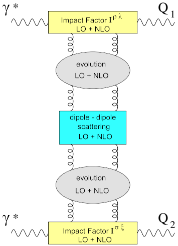


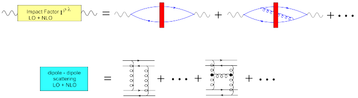
Second “evolution of the diagram











$$\sigma^{\gamma^*\gamma^*}_{\rm LO+NLO}(TT) = \frac{1}{4}\sum_{\lambda_1,\lambda_2=\pm1} 32\pi^6 \varepsilon^{\lambda_1 *}_{\rho_1}(q_1)\varepsilon^{\lambda_1}_{\sigma_1}(q_1)\varepsilon^{\lambda_2 *}_{\rho_2}(q_2)\varepsilon^{\lambda_2}_{\sigma_2}(q_2)\frac{N_c^2-1}{N_c^2}\frac{\alpha_s(Q_1^2)\alpha_s(Q_2^2)}{Q_1Q_2}$$
$$\times\int_{-\infty}^{\infty} d\nu \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\left(\frac{s}{Q_1Q_2}\right)^{\bar\alpha_s(Q_1Q_2)\,\chi_0(\nu)+\bar\alpha_s^2(Q_1Q_2)\chi_1(\nu)} \tilde I^{\rho_1\sigma_1}_{\rm LO+NLO}(q_1,\nu)\tilde I^{\rho_2\sigma_2}_{\rm LO+NLO}(q_2,-\nu)$$
$$\times\Big[1+\bar\alpha_s(Q_1Q_2){\rm Re}[F(\nu)]\Big]$$

${\rm Re}[F(\nu)]$ is the NLO dipole-dipole scattering projected on the LO eigenfunctions.

$\gamma^*\gamma^*$ scattering cross-section at NLO

$$\sigma^{\gamma^*\gamma^*}_{\mathrm{LO+NLO}}(TT) = \frac{1}{4}\sum_{\lambda_1,\lambda_2=\pm1} 32\pi^6 \varepsilon^{\lambda_1 *}_{\rho_1}(q_1)\varepsilon^{\lambda_1}_{\sigma_1}(q_1)\varepsilon^{\lambda_2 *}_{\rho_2}(q_2)\varepsilon^{\lambda_2}_{\sigma_2}(q_2)\frac{N_c^2-1}{N_c^2}\frac{\alpha_s(Q_1^2)\alpha_s(Q_2^2)}{Q_1Q_2}$$
$$\times\int_{-\infty}^{\infty} d\nu \left(\frac{Q_1^2}{Q_2^2}\right)^{i\nu}\left(\frac{s}{Q_1Q_2}\right)^{\bar\alpha_s(Q_1Q_2)\,\chi_0(\nu)+\bar\alpha_s^2(Q_1Q_2)\chi_1(\nu)}$$
$$\times \tilde I^{\rho_1\sigma_1}_{\mathrm{LO+NLO}}(q_1,\nu)\tilde I^{\rho_2\sigma_2}_{\mathrm{LO+NLO}}(q_2,-\nu)\left[1+\bar\alpha_s(Q_1Q_2)\mathrm{Re}[F(\nu)]\right]$$

- NLO Impact factor is not symmetric in $\gamma \to 1-\gamma$
  BUT the full amplitude is!

G.A.C. and Yu. Kovchegov (2014)

- NLO BFKL: predicted the small-x asymptotic of the NNLO anomalous dimension of leading-twist operator

- It is mysterious procedure $\Rightarrow$ Light-ray operator

  - Analytic continuation of twist-2 operator to non physical point $j = 1$

$$F^a_{\xi +}(x)\nabla_+^{j-2}F^{a\,\xi}_{+}(x)\Big|_{x=0} = \frac{\Gamma(2-j)}{2\pi i}\int_0^{+\infty} du\ u^{1-j}\, F^a_{\xi +}(0)[0,un]^{ab}F^{b\,\xi}_{+}(un)$$

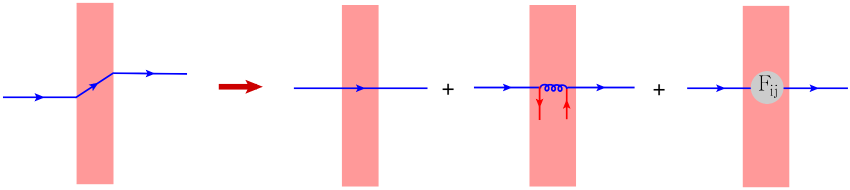
Extend High-energy OPE up to sub-eikonal corrections

- Calculate the quark and gluon propagator up to sub-eikonal corrections
- Apply the High-energy OPE to the T-product of two electromagnetic currents
- Isolate the impact factor(s) from the resulting operators
- Calculate the evolution of the new operators

G.A.C. JHEP 01 (2019) 118 arXiv: 1807.11435 [hep-ph]

G.A.C. JHEP 06 (2021) 096 arXiv: 2101.12744 [hep-ph]

Quark propagator with sub-eikonal corrections



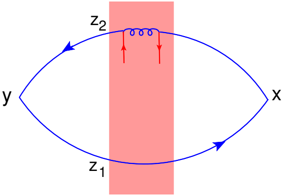
Quark propagator for g_1 structure function

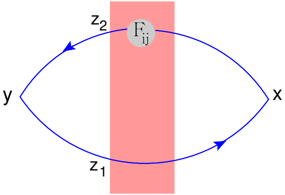
$$\langle \mathrm{T}\{\psi(x)\bar{\psi}(y)\}\rangle_{A,\psi,\bar{\psi}} = -\frac{1}{2\pi^3 x_*^2 y_*^2}\int \frac{d^2 z}{(\mathcal{Z}+i\epsilon)^3}\left(\frac{2}{s}x_*\not{p}_1 + \not{X}_\perp\right)$$
$$\times\left\{i\not{p}_2 U(z_\perp) + \frac{\mathcal{Z}}{8}\left(\frac{1}{s}\not{p}_2\gamma^5\mathcal{F}(z_\perp) + \gamma_\perp^\mu Q(z_\perp)\gamma_\mu^\perp\right)\right\}\left(\frac{2}{s}y_*\not{p}_1 + \not{Y}_\perp\right)$$

$$\mathcal{Z} \equiv \frac{(x-z)_\perp^2}{x_*} - \frac{(y-z)_\perp^2}{y_*} - \frac{4}{s}(x_\bullet - y_\bullet)$$

$x_* = \sqrt{\frac{s}{2}}x^+ \qquad x_\bullet = \sqrt{\frac{s}{2}}x^-$

OPE with sub-eikonal corrections





$$
\begin{aligned}
&\mathrm{T}\{\hat{\bar{\psi}}(x)\gamma^\mu\psi(x)\hat{\bar{\psi}}(y)\gamma^\nu\hat{\psi}(y)\} \\
&= \int dz_1 dz_2\, \mathcal{I}^{\mu\nu}_{LO}(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} + \frac{\mathcal{Z}_2}{8\,s}\Big(\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{\mathcal{Q}}_{1z_2}\} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{\mathcal{Q}}^\dagger_{1z_2}\}\Big)\Big] \\
&+\frac{1}{s}\int d^2z_1 d^2z_2\, \mathcal{I}^{\mu\nu}_5(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{(\hat{\mathcal{Q}}_{5z_2} + \hat{\mathcal{F}}_{z_2})\hat{U}^\dagger_{z_1}\} + \mathrm{Tr}\{(\hat{\mathcal{Q}}^\dagger_{5z_2} + \hat{\mathcal{F}}^\dagger_{z_2})\hat{U}_{z_1}\}\Big] \\
&+\mathcal{O}(\alpha_s) + \mathcal{O}(\lambda^{-2})
\end{aligned}
$$

Sub-eikonal Impact Factors

$$\mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2) = \frac{\mathcal{Z}_2}{8}\mathcal{I}_{LO}^{\mu\nu} = \frac{1}{4\pi^6 x_*^4 y_*^4}\frac{I_1^{\mu\nu}(x,y;z_1,z_2)}{[\mathcal{Z}_1+i\epsilon]^3[\mathcal{Z}_2+i\epsilon]^2}$$

$$\mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2) = -\frac{1}{4\pi^6 x_*^4 y_*^4}\frac{I_5^{\mu\nu}(x,y;z_1,z_2)}{[\mathcal{Z}_1+i\epsilon]^3[\mathcal{Z}_2+i\epsilon]^2}$$

$$I_1^{\mu\nu}(x,y;z_1,z_2) = \frac{1}{2}x_*^2 y_*^2 \frac{\partial^2}{\partial x_\mu \partial y_\nu}\left(\mathcal{Z}_1\mathcal{Z}_2 - z_{12\perp}^2\frac{(x-y)^2}{x_* y_*}\right)$$

$$I_5^{\mu\nu}(x,y;z_1,z_2) = \left(x_*\partial_x^\mu - p_2^\mu\right)\left(y_*\partial_y^\nu - p_2^\nu\right)\left[(\vec{Y}_1\times\vec{Y}_2)X_1\cdot X_2 - (\vec{X}_1\times\vec{X}_2)Y_1\cdot Y_2\right]$$

$X_i^\mu = \frac{2}{s}x_* p_1^\mu + X_{i\perp}^\mu \qquad X_{i\perp}^\mu = x_\perp^\mu - z_{i\perp}^\mu \qquad i = 1,2$

$\vec{x}\times\vec{y} = \epsilon^{ij}x_i y_j$

Symmetry of the sub-eikonal Impact Factors

Sub-eikonal Impact Factors are electromagnetic gauge invariant

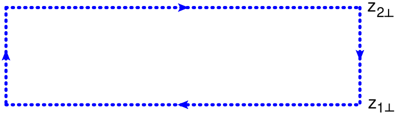
$$\partial_\mu \mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2) = 0$$
$$\partial_\mu \mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2) = 0$$

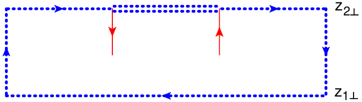
and $SL(2,C)$ Möbius invariant (inv. $x^\mu \to \frac{x^\mu}{x^2}$)

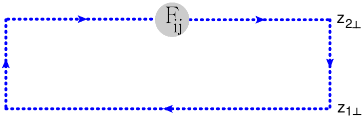
$$\int d^2z_2 d^2z_2\, \mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2) \stackrel{\text{inv.}}{=} \int d^2z_2 d^2z_2\, \mathcal{I}_1^{\mu\nu}(x,y;z_1,z_2)$$
$$\int d^2z_2 d^2z_2\, \mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2) \stackrel{\text{inv.}}{=} \int d^2z_2 d^2z_2\, \mathcal{I}_5^{\mu\nu}(x,y;z_1,z_2)$$

High-Energy OPE with sub-eikonal corrections

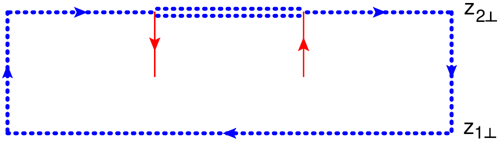
$$
\begin{aligned}
&\mathrm{T}\{\bar{\hat{\psi}}(x)\gamma^{\mu}\psi(x)\bar{\hat{\psi}}(y)\gamma^{\nu}\hat{\psi}(y)\}\\
&= \int dz_1 dz_2\, \mathcal{I}^{\mu\nu}_{LO}(z_{1\perp}, z_{2\perp}; x, y)\left[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^{\dagger}_{z_2}\} + \frac{\mathcal{Z}_2}{8\,s}\Big(\mathrm{Tr}\{\hat{U}^{\dagger}_{z_1}\hat{\mathcal{Q}}_{1z_2}\} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{\mathcal{Q}}^{\dagger}_{1z_2}\}\Big)\right]\\
&+\frac{1}{s}\int d^2z_1 d^2z_2\, \mathcal{I}^{\mu\nu}_{5}(z_{1\perp}, z_{2\perp}; x, y)\left[\mathrm{Tr}\{\big(\hat{\mathcal{Q}}_{5z_2} + \hat{\mathcal{F}}_{z_2}\big)\hat{U}^{\dagger}_{z_1}\} + \mathrm{Tr}\{\big(\hat{\mathcal{Q}}^{\dagger}_{5z_2} + \hat{\mathcal{F}}^{\dagger}_{z_2}\big)\hat{U}_{z_1}\}\right]\\
&+\mathcal{O}(\alpha_s) + \mathcal{O}(\lambda^{-2})
\end{aligned}
$$



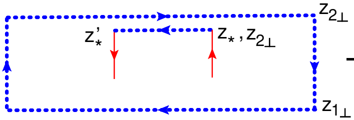




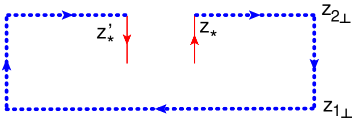
Fierz identity



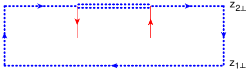
$= \frac{1}{2}$



$- \frac{1}{2N_c}$

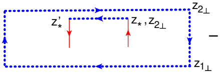


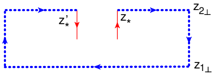
Fierz identity



$$\mathrm{Tr}\{\hat{\mathcal{Q}}_{1x}\hat{U}_y^\dagger\} = \frac{1}{2}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{U}_x\}\hat{Q}_{1\,x} - \frac{1}{2N_c}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{\tilde{Q}}_{1x}\}$$

$$\mathrm{Tr}\{\hat{\mathcal{Q}}_{5x}\hat{U}_y^\dagger\} = \frac{1}{2}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{U}_x\}\hat{Q}_{5\,x} - \frac{1}{2N_c}\,\mathrm{Tr}\{\hat{U}_y^\dagger\hat{\tilde{Q}}_{5x}\}$$

$= \frac{1}{2}$ 

$- \frac{1}{2N_c}$ 

$$Q_{1x} = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\,\bar{\psi}(z'^+,x_\perp)\,i\,\gamma^-[z'^+,z^+]_x\psi(z^+,x_\perp)$$

$$Q_{5x} \equiv -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\,\bar{\psi}(z'^+,x_\perp)\,\gamma^5\gamma^-[z'^+,z^+]_x\psi(z^+,x_\perp)$$

$$\tilde{Q}_{1ij}(x_\perp) \equiv g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\left([\infty p_1,z^+]_x\mathrm{tr}\{\psi(z^+,x_\perp)\bar{\psi}(z'^+,x_\perp)\,i\,\gamma^-\}[z'^+,-\infty p_1]\right)_{ij}$$

$$\tilde{Q}_{5ij}(x_\perp) \equiv g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty}dz^+\int_{-\infty}^{z^+}dz'^+\left([\infty p_1,z^+]_x\mathrm{tr}\{\psi(z^+,x_\perp)\bar{\psi}(z'^+,x_\perp)\,\gamma^5\gamma^-\}[z'^+,-\infty p_1]\right)_{ij}$$

High-Energy OPE with sub-eikonal corrections

$$
\begin{aligned}
&\mathrm{T}\{\bar{\hat{\psi}}(x)\gamma^\mu\psi(x)\bar{\hat{\psi}}(y)\gamma^\nu\hat{\psi}(y)\}\\
&= \int dz_1 dz_2\, \mathcal{I}^{\mu\nu}_{LO}(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{U}^\dagger_{z_2}\}\\
&+\frac{1}{s}\frac{\mathcal{Z}_2}{16}\Big(\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{U}_{z_2}\}\hat{Q}_{1z_2} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\hat{Q}^\dagger_{1z_2} - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{\tilde{Q}}_{1z_2}\} - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}_{z_1}\hat{\tilde{Q}}^\dagger_{1z_2}\}\Big)\Big]\\
&+\frac{1}{s}\int d^2z_1 d^2z_2\, \mathcal{I}^{\mu\nu}_5(z_{1\perp}, z_{2\perp}; x, y)\Big[\mathrm{Tr}\{\hat{U}^\dagger_{z_1}\hat{U}_{z_2}\}\hat{Q}_{5z_2} + \mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\hat{Q}^\dagger_{5z_2}\\
&-\frac{1}{N_c}\mathrm{Tr}\{\hat{U}^\dagger_{z_1}(\hat{\tilde{Q}}_{5z_2} - 2N_c\hat{\mathcal{F}}_{z_2})\} - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}_{z_1}(\hat{\tilde{Q}}^\dagger_{5z_2} - 2N_c\hat{\mathcal{F}}^\dagger_{z_2})\}\Big]\\
&+\mathcal{O}(\alpha_s) + \mathcal{O}(\lambda^{-2})
\end{aligned}
$$

Eikonal term of the high-energy OPE: I. Balitsky (1995)

Sub-eikonal terms of the high-energy OPE: G.A.C. (2021)

Sub-eikonal corrections for gluon propagator: F^{-i} components

- Applied to the single inclusive gluon production cross section at central rapidities and the light-front helicity asymmetry, in pA collisions.
  - T. Altinoluk, N. Armesto, G. Beuf, M. Martínez and C. A. Salgado (2014)
  - T. Altinoluk,a N. Armesto,a G. Beuff and A. Moscoso (2015)

- Study rapidity evolution of gluon transverse momentum dependent distribution (TMD) changes from nonlinear evolution at small $x_B \ll 1$ to linear evolution at moderate $x_B \sim 1$.
  - I. Balitsky and A. Tarasov (2015-2016)

Sub-eikonal corrections for gluon propagator: F^{-i} components

Light-cone gauge

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_A = \left[-\int_0^{+\infty}\frac{d\!\!{}^-\alpha}{2\alpha}\theta(x^+-y^+) + \int_{-\infty}^0\frac{d\!\!{}^-p^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^--y^-)}$$
$$\times\langle x_\perp|e^{-i\frac{\hat p_\perp^2}{2p^+}x^+}\left(\delta^\xi_\mu-\frac{n_{2\mu}}{p^+}p^\xi\right)\mathcal{O}_\alpha(x^+,y^+)\left(g_{\xi\nu}-p_\xi\frac{n_{2\nu}}{p^+}\right)e^{i\frac{\hat p_\perp^2}{2p^+}y^+}|y_\perp\rangle^{ab} + i\langle x|\frac{n_{2\mu}n_{2\nu}}{{p^+}^2}|y\rangle^{ab}$$

$$\mathcal{O}_\alpha(x^+,y^+) \equiv \; [x^+,y^+] + \frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big(\{p^i,[x^+,\omega^+]\,\omega^+\,F_i{}^-(\omega^+)\,[\omega^+,y^+]\}$$
$$+g\int_{\omega^+}^{x^+}d\frac{2}{s}\,\omega'^+(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\;F_i{}^-[\omega^+,y^+]\Big)$$

I. Balitsky and A. Tarasov (2015-2016)

In the background-Feynamn gauge see paper G.A.C. 2019

Sub-eikonal corrections for gluon propagator

G.A.C JHEP 01 (2019)

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_A = \left[-\int_0^{+\infty}\frac{đ\alpha}{2\alpha}\theta(x^+-y^+)+\int_{-\infty}^0\frac{đp^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^--y^-)}$$

$$\times\langle x_\perp|e^{-i\frac{\hat{p}^2_\perp}{2p^+}x^+}\left(\delta^\xi_\mu-\frac{n_{2\mu}}{p^+}p^\xi\right)\mathcal{O}_\alpha(x^+,y^+)\left(g_{\xi\nu}-p_\xi\frac{n_{2\nu}}{p^+}\right)e^{i\frac{\hat{p}^2_\perp}{2p^+}y^+}|y_\perp\rangle^{ab}+i\langle x|\frac{n_{2\mu}n_{2\nu}}{p^{+2}}|y\rangle^{ab}$$

$$+\left[-\int_0^{+\infty}\frac{đp^+}{2p^+}\theta(x^+-y^+)+\int_{-\infty}^0\frac{đp^+}{2p^+}\theta(y^+-x^+)\right]e^{-ip^+(x^--y^-)}\langle x_\perp|e^{-i\frac{\hat{p}^2_\perp}{2p^+}x^+}$$

$$\times\left[\mathfrak{G}^{ab}_{1\mu\nu}(x^+,y^+;p_\perp)+\mathfrak{G}^{ab}_{2\mu\nu}(x^+,y^+;p_\perp)+\mathfrak{G}^{ab}_{3\mu\nu}(x^+,y^+;p_\perp)+\mathfrak{G}^{ab}_{4\mu\nu}(x^+,y^+;p_\perp)\right]$$

$$\times e^{i\frac{\hat{p}^2_\perp}{2p^+}y^+}|y_\perp\rangle+O(\lambda^{-2})$$

$$\mathcal{O}_\alpha(x^+,y^+)\equiv\;[x^+,y^+]+\frac{ig}{2p^+}\int_{y^+}^{x^+}d\omega^+\Big(\{p^i,[x^+,\omega^+]\,\omega^+\,F_i{}^-(\omega^+)\,[\omega^+,y^+]\}$$

$$+g\int_{\omega^+}^{x^+}d\omega'^+(\omega^+-\omega'^+)[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]\;F_i{}^-[\omega^+,y^+]\Big)$$

Sub-eikonal corrections for gluon propagator

G.A.C JHEP 01 (2019)

$$\mathfrak{G}^{ab}_{1\mu\nu}(x^+,y^+;p_\perp) = -\frac{g\,n_{2\mu}n_{2\nu}}{4(p^+)^3}\int_{y^+}^{x^+}d\omega^+\Big[4p^i[x^+,\omega^+]F_{ij}[\omega^+,y^+]p^j$$

$$+ig\int_{\omega'^+}^{x^+}d\omega'^+\,(\omega'^+-\omega^+)[x^+,\omega'^+]iD^iF_i{}^-[\omega'^+,\omega^+]iD^jF_j{}^-[\omega^+,y^+]\Big]^{ab},$$

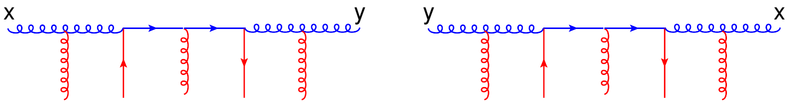
$$\mathfrak{G}^{ab}_{2\mu\nu}(x^+,y^+;p_\perp) = -\frac{g}{p^+}\delta^i_\mu\delta^j_\nu\int_{y^+}^{x^+}d\omega^+\big([x^+,\omega^+]F_{ij}[\omega^+,y^+]\big)^{ab},$$

$$\mathfrak{G}^{ab}_{3\mu\nu}(x^+,y^+;p_\perp) = \frac{g}{2(p^+)^2}\Big(\delta^j_\mu n_{2\nu}+\delta^j_\nu n_{2\mu}\Big)\int_{y^+}^{x^+}d\omega^+\,\big([x^+,\omega^+]iD^iF_{ij}[\omega^+,y^+]\big)^{ab},$$

$$\mathfrak{G}^{ab}_{4\mu\nu}(x^+,y^+;p_\perp) = -\frac{g^2}{(p^+)^2}\int_{y^+}^{x^+}d\omega^+\int_{\omega^+}^{x^+}d\omega'^+\Big(\delta^j_\nu n_{2\mu}[x^+,\omega'^+]F^{i-}[\omega'^+,\omega^+]F_{ij}[\omega^+,y^+]$$

$$+\delta^j_\mu n_{2\nu}[x^+,\omega'^+]F_{ij}[\omega'^+,\omega^+]F^{i-}[\omega^+,y^+]\Big)^{ab}$$

Gluon propagator in the background of quark fields



$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_{\psi,\bar\psi} = \Big[-\int_0^{+\infty}\frac{d\!\!{}^-\alpha}{2\alpha}\theta(x^+-y^+) + \int_{-\infty}^0\frac{d\!\!{}^-p^+}{2p^+}\theta(y^+-x^+)\Big]e^{-ip^+(x^--y^-)}$$

$$\times g^2\int_{y^+}^{x^+}dz_1^+\int_{y^+}^{z_1^+}dz_2^+\,\frac{1}{4p^+}\Bigg[\langle x_\perp|\,e^{-i\frac{\hat p_\perp^2}{2p^+}x^+}\Big(g^\xi_{\perp\mu}-\frac{n_{2\mu}}{p^+}p^\xi_\perp\Big)$$

$$\times\bar\psi(z_1^+)\gamma^\perp_\xi\,\gamma^-\,[z_1^+,x^+]t^a[x^+,y^+]t^b[y^+,z_2^+]\gamma^\sigma_\perp\psi(z_2^+)\Big(g^\perp_{\sigma\nu}-p^\perp_\sigma\frac{n_{2\nu}}{p^+}\Big)e^{i\frac{\hat p_\perp^2}{2p^+}y^+}|y_\perp\rangle$$

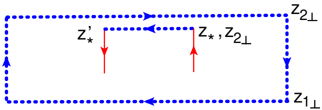
$$+\langle y_\perp|\,e^{-i\frac{\hat p_\perp^2}{2p^+}y^+}\Big(g^\xi_{\perp\nu}-\frac{n_{2\nu}}{p^+}p^\xi_\perp\Big)\bar\psi(z_2^+)\gamma^\perp_\xi\,\gamma^-[z_2^+,y^+]t^b[y^+,x^+]t^a[x^+,z_1^+]\gamma^\sigma_\perp\psi(z_1^+)$$

$$\times\Big(g^\perp_{\sigma\mu}-p^\perp_\sigma\frac{n_{2\mu}}{p^+}\Big)e^{i\frac{\hat p_\perp^2}{2p^+}x^+}|x_\perp\rangle\Bigg]+O(\lambda^{-2})$$

G.A.C JHEP 01 (2019)

Evolution equation of sub-eikonal corrections

$$\langle \mathrm{Tr}\{U^{\dagger}_{z_1} U_{z_2}\} Q_{1 z_2} \rangle$$

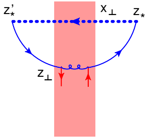


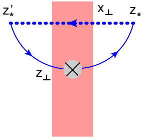
Diagrams at one loop: quantum quark field

Figure: Diagrams with $\hat{Q}_{1x}$ and $\hat{Q}_{5x}$ quantum.

Evolution equation of sub-eikonal corrections

$$\langle \mathrm{Tr}\{U^\dagger_{z_1} U_{z_2}\} Q_{1z_2}\rangle$$





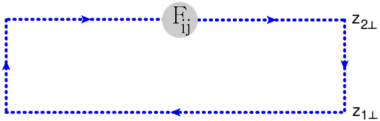
$$\frac{d}{d\eta}\mathrm{Tr}\{U^\dagger_y U_x\}Q_{1x} = \frac{\alpha_s}{4\pi^2}\int d^2z \frac{\mathrm{Tr}\{U^\dagger_y U_x\}}{(x-z)^2_\perp}\left[\mathrm{Tr}\{U^\dagger_x U_z\}\, Q_{1z} - \frac{1}{N_c}\mathrm{Tr}\{U^\dagger_x \tilde{Q}_{1z}\}\right]$$

and

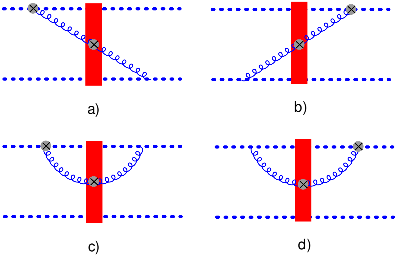
$$\frac{d}{d\eta}\mathrm{Tr}\{U^\dagger_y U_x\}Q_{5x} = \frac{\alpha_s}{4\pi^2}\int d^2z \frac{\mathrm{Tr}\{U^\dagger_y U_x\}}{(x-z)^2_\perp} \times\left[\mathrm{Tr}\{U^\dagger_x U_z\}\, Q_{5z} - \frac{1}{N_c}\mathrm{Tr}\{U^\dagger_x \big(\tilde{Q}_{5z} - 2N_c\mathcal{F}_z\big)\}\right]$$

Sanity check: operators of different parity do not mix

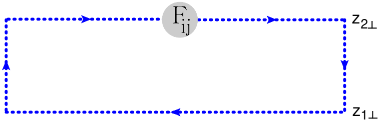
Evolution equation of sub-eikonal corrections



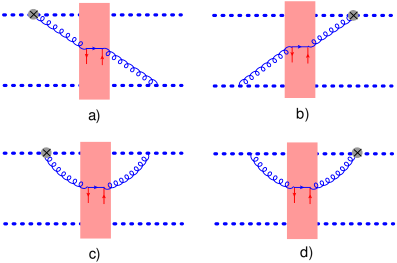
Diagrams with F_{ij} quantum



Evolution equation of sub-eikonal corrections



Diagrams with F_{ij} quantum



Evolution equation of sub-eikonal corrections

Diagrams with F_{ij} quantum

$$\frac{d}{d\eta}\mathrm{Tr}\{\mathcal{F}_x\, U^\dagger_y\} = -\frac{\alpha_s}{\pi^2}\,\mathrm{Tr}\{U_x t^a U^\dagger_y t^b\}\int d^2z\Bigg[\frac{(\vec{x}-\vec{z})\times(\vec{z}-\vec{y})}{(x-z)^2_\perp\,(y-z)^2_\perp}\left(\mathcal{Q}^{ba}_{1z}-\mathcal{Q}^{ba\,\dagger}_{1z}\right)$$
$$-\left(\frac{(x-z,z-y)}{(x-z)^2_\perp\,(y-z)^2_\perp}+\frac{1}{(x-z)^2_\perp}\right)\left(\mathcal{Q}^{ba}_{5z}+\mathcal{Q}^{ba\,\dagger}_{5z}+\mathcal{F}^{ba}_z\right)$$
$$-4\pi^2\int\frac{d\!\!{}^-{}^2q}{q^2_\perp}\left(e^{i(q,y-z)}-e^{i(q,x-z)}\right)\delta^{(2)}(z-x)\mathcal{F}^{ba}_z\Bigg]$$

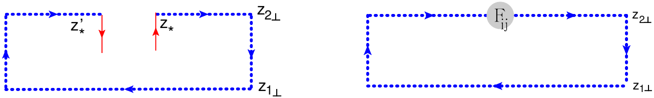
$$\mathcal{Q}^{ab}_5(z_\perp)\equiv g^2\int_{-\infty}^{+\infty}dz_{1*}\int_{-\infty}^{z_{1*}}dz_{2*}\bar{\psi}(z_{1*},z_\perp)\,\gamma^5\not{p}_1\,[z_{1*},\infty p_1]_z\,t^aU_z\,t^b\,[-\infty p_1,z_{2*}]_z\,\psi(z_{2*},z_\perp)$$

$$\mathcal{Q}^{ab}_1(z_\perp)\equiv g^2\int_{-\infty}^{+\infty}dz_{1*}\int_{-\infty}^{z_{1*}}dz_{2*}\bar{\psi}(z_{1*},z_\perp)\,i\not{p}_1\,[z_{1*},\infty p_1]_z\,t^aU_z\,t^b\,[-\infty p_1,z_{2*}]_z\,\psi(z_{2*},z_\perp)$$

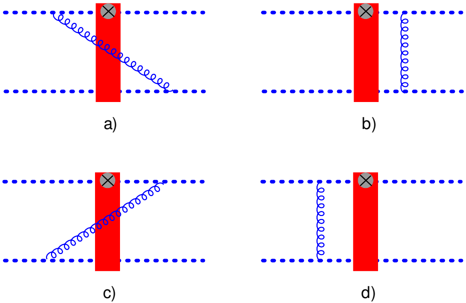
Evolution equation of sub-eikonal corrections

Diagrams with F_{ij} quantum

$$
\begin{aligned}
\frac{d}{d\eta}\mathrm{Tr}\{\mathcal{F}_x\, U_y^\dagger\} &= \frac{\alpha_s}{2\pi^2}\int d^2z \\
&\times\Bigg\{\frac{1}{2}\frac{(\vec{x}-\vec{z})\times(\vec{z}-\vec{y})}{(x-z)_\perp^2\,(y-z)_\perp^2}\Big[\mathrm{Tr}\{U_y^\dagger\tilde{Q}_{1\,z}\}\mathrm{Tr}\{U_z^\dagger U_x\} - \mathrm{Tr}\{U_x\tilde{Q}_{1z}^\dagger\}\mathrm{Tr}\{U_y^\dagger U_z\} \\
&+\frac{1}{N_c}\Big(\mathrm{Tr}\{U_xU_y^\dagger\tilde{Q}_{1z}U_z^\dagger\} + \mathrm{Tr}\{U_y^\dagger U_xU_z^\dagger\tilde{Q}_{1z}\} - \mathrm{Tr}\{U_xU_y^\dagger U_z\tilde{Q}_{1z}^\dagger\} - \mathrm{Tr}\{U_y^\dagger U_x\tilde{Q}_{1z}^\dagger U_z\}\Big) \\
&+\frac{1}{N_c^2}\mathrm{Tr}\{U_y^\dagger U_x\}\Big(Q_{1z}^\dagger - Q_{1z}\Big)\Big] - \frac{1}{2}\Big[\frac{(x-z,z-y)}{(x-z)_\perp^2\,(y-z)_\perp^2} + \frac{1}{(x-z)_\perp^2}\Big] \\
&\times\Big[\mathrm{Tr}\{U_y^\dagger(\tilde{Q}_{5z}-2\mathcal{F}_z)\}\mathrm{Tr}\{U_z^\dagger U_x\} + \mathrm{Tr}\{U_x(\tilde{Q}_{5z}^\dagger - 2\mathcal{F}_z^\dagger)\}\mathrm{Tr}\{U_y^\dagger U_z\} \\
&-\frac{1}{N_c}\Big(\mathrm{Tr}\{U_xU_y^\dagger U_z\tilde{Q}_{5z}^\dagger\} + \mathrm{Tr}\{U_y^\dagger U_x\tilde{Q}_{5z}^\dagger U_z\} + \mathrm{Tr}\{U_xU_y^\dagger\tilde{Q}_{5z}U_z^\dagger\} + \mathrm{Tr}\{U_y^\dagger U_xU_z^\dagger\tilde{Q}_{5z}\}\Big) \\
&+\frac{1}{N_c^2}\mathrm{Tr}\{U_y^\dagger U_x\}\Big(Q_{5z} + Q_{5z}^\dagger\Big)\Big]\Bigg\}
\end{aligned}
$$



Diagrams with F_{ij} or $\tilde{Q}_1$ (and $\tilde{Q}_5$) classical: BK-type diagrams



+ self-energy diagrams

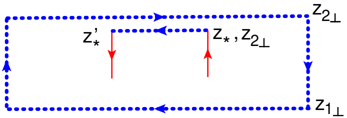
Evolution equation of sub-eikonal corrections

Diagrams with F_{ij} or $\tilde{Q}_1$ (and $\tilde{Q}_5$) classical: BK-type diagrams

$$\frac{d}{d\eta}\mathrm{Tr}\{\tilde{Q}_{1x}\, U^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\, \frac{(x-y)^2_\perp}{(x-z)^2_\perp (y-z)^2_\perp} \times\Big[\mathrm{Tr}\{U^\dagger_z \tilde{Q}_{1x}\}\,\mathrm{Tr}\{U^\dagger_y U_z\} - N_c\,\mathrm{Tr}\{U^\dagger_y \tilde{Q}_{1x}\}\Big]$$

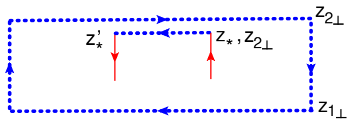
$$\frac{d}{d\eta}\mathrm{Tr}\{\tilde{Q}_{5x}\, U^\dagger_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z\, \frac{(x-y)^2_\perp}{(x-z)^2_\perp (y-z)^2_\perp} \times\Big[\mathrm{Tr}\{U^\dagger_z \tilde{Q}_{5x}\}\,\mathrm{Tr}\{U^\dagger_y U_z\} - N_c\,\mathrm{Tr}\{U^\dagger_y \tilde{Q}_{5x}\}\Big]$$

$$\langle \mathrm{Tr}\{\mathcal{F}_x\, U^\dagger_y\}\rangle = \frac{\alpha_s}{2\pi^2}\int_0^{+\infty}\frac{d\alpha}{\alpha}\int d^2z\, \frac{(x-y)^2_\perp}{(x-z)^2_\perp (y-z)^2_\perp} \times\Big[\mathrm{Tr}\{U^\dagger_z \mathcal{F}_x\}\,\mathrm{Tr}\{U^\dagger_y U_z\} - N_c\,\mathrm{Tr}\{U^\dagger_y \mathcal{F}_x\}\Big]$$

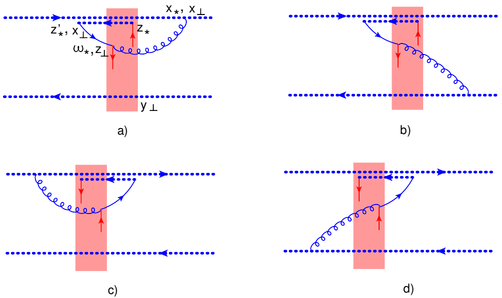


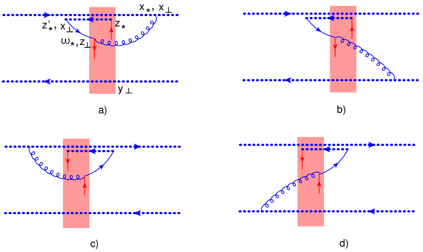
quark-to-gluon diagrams

Evolution equation of sub-eikonal corrections



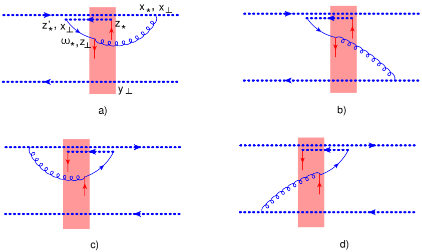
quark-to-gluon diagrams





$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\}Q_{1x} = \frac{\alpha_s}{4\pi^2}\int d^2z\Bigg\{\frac{1}{(x-z)_\perp^2}\Bigg[\mathrm{Tr}\left\{U_xU_y^\dagger U_z\mathcal{X}_{1zx}^\dagger\right\}$$
$$+\mathrm{Tr}\{U_zU_y^\dagger U_x\mathcal{X}_{1xz}^\dagger\} + \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big(\mathcal{H}_{1xz}^- + \mathcal{H}_{1zx}^+\Big)\Bigg] + \frac{(x-z,z-y)_\perp}{(y-z)_\perp^2(x-z)_\perp^2}$$
$$\times\Bigg[\mathrm{Tr}\{U_xU_y^\dagger U_z\mathcal{X}_{1zx}^\dagger\} + \mathrm{Tr}\left\{U_zU_y^\dagger U_x\mathcal{X}_{1xz}^\dagger\right\} + \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big(\mathcal{H}_{1xz}^- + \mathcal{H}_{1zx}^+\Big)\Bigg]$$
$$+\frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(x-z)_\perp^2}\Bigg[\mathrm{Tr}\{U_xU_y^\dagger U_z\mathcal{X}_{5zx}^\dagger\} - \mathrm{Tr}\left\{U_zU_y^\dagger U_x\mathcal{X}_{5xz}^\dagger\right\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big(\mathcal{H}_{5xz}^- - \mathcal{H}_{5zx}^+\Big)\Bigg]\Bigg\}.$$

Evolution equation of sub-eikonal corrections

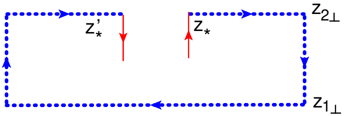


$$\frac{d}{d\eta}\mathrm{Tr}\{U_xU_y^\dagger\}Q_{5x} = -\frac{\alpha_s}{4\pi^2}\int d^2z\left\{\frac{1}{(x-z)_\perp^2}\left[\mathrm{Tr}\left\{U_xU_y^\dagger U_z\mathcal{X}_{5zx}^\dagger\right\}\right.\right.$$

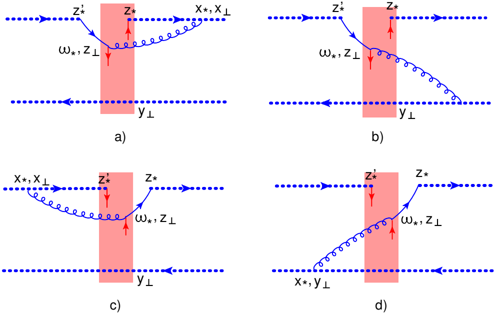
$$\left.+\mathrm{Tr}\left\{U_zU_y^\dagger U_x\mathcal{X}_{5xz}^\dagger\right\}-\frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger U_x\}\Big(\mathcal{H}_{5xz}^- + \mathcal{H}_{5zx}^+\Big)\right] + \frac{(x-z,z-y)_\perp}{(y-z)_\perp^2(x-z)_\perp^2}$$

$$\times\left[\mathrm{Tr}\{U_xU_y^\dagger U_z\mathcal{X}_{5zx}^\dagger\} + \mathrm{Tr}\left\{U_zU_y^\dagger U_x\mathcal{X}_{5xz}^\dagger\right\} - \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big(\mathcal{H}_{5xz}^- + \mathcal{H}_{5zx}^+\Big)\right]$$

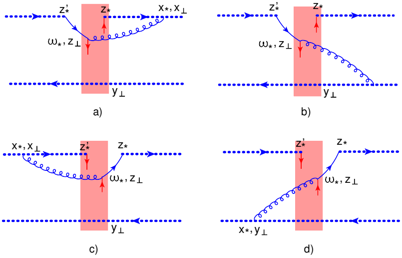
$$\left.+\frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(x-z)_\perp^2}\left[\mathrm{Tr}\{U_zU_y^\dagger U_x\mathcal{X}_{1xz}^\dagger\} - \mathrm{Tr}\{U_xU_y^\dagger U_z\mathcal{X}_{1zx}^\dagger\} + \frac{1}{N_c}\mathrm{Tr}\{U_xU_y^\dagger\}\Big(\mathcal{H}_{1zx}^+ - \mathcal{H}_{1xz}^-\Big)\right]\right\}.$$



quark-to-gluon diagrams

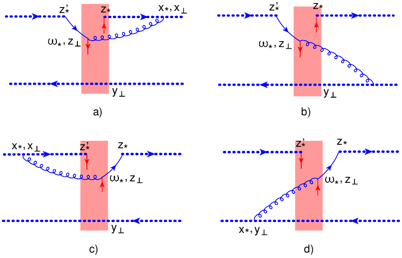


Evolution equation of sub-eikonal corrections



$$
\begin{aligned}
\frac{d}{d\eta}\,\mathrm{Tr}\{U_y^\dagger \tilde{Q}_{1x}\} = & -\frac{\alpha_s}{4\pi^2}\int d^2z \\
& \times\Bigg\{\frac{1}{(x-z)_\perp^2}\Big[\mathrm{Tr}\left\{U_z U_y^\dagger\right\}\Big(\mathcal{H}^+_{1xz}+\mathcal{H}^-_{1zx}\Big)-\frac{1}{N_c}\mathrm{Tr}\left\{U_y^\dagger\left(\mathcal{X}_{1xz}+\mathcal{X}_{1zx}\right)\right\}\Big] \\
& +\frac{(x-z,z-y)}{(y-z)_\perp^2(z-x)_\perp^2}\Big[\mathrm{Tr}\{U_z U_y^\dagger\}\Big(\mathcal{H}^+_{1xz}+\mathcal{H}^-_{1zx}\Big)-\frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger(\mathcal{X}_{1xz}+\mathcal{X}_{1zx})\}\Big] \\
& +\frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(z-x)_\perp^2}\Big[\mathrm{Tr}\{U_z U_y^\dagger\}\Big(\mathcal{H}^-_{5zx}-\mathcal{H}^+_{5xz}\Big)+\frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger(\mathcal{X}_{5xz}-\mathcal{X}_{5zx})\}\Big]\Bigg\}
\end{aligned}
$$

Evolution equation of sub-eikonal corrections



$$
\begin{aligned}
\frac{d}{d\eta}\,\mathrm{Tr}\{U_y^\dagger \tilde{Q}_{5x}\} = -\frac{\alpha_s}{4\pi^2}\int d^2z \\
\times\Bigg\{ \frac{1}{(x-z)_\perp^2}\left[\mathrm{Tr}\left\{U_z U_y^\dagger\right\}\left(\mathcal{H}^+_{5xz} + \mathcal{H}^-_{5zx}\right) - \frac{1}{N_c}\,\mathrm{Tr}\left\{U_y^\dagger\left(\mathcal{X}_{5xz} + \mathcal{X}_{5zx}\right)\right\}\right] \\
+\frac{(x-z,z-y)}{(y-z)_\perp^2(z-x)_\perp^2}\left[\mathrm{Tr}\{U_z U_y^\dagger\}\left(\mathcal{H}^+_{5xz} + \mathcal{H}^-_{5zx}\right) - \frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger\left(\mathcal{X}_{5xz} + \mathcal{X}_{5zx}\right)\}\right)\right] \\
+\frac{(\vec{x}-\vec{z})\times(\vec{y}-\vec{z})}{(y-z)_\perp^2(z-x)_\perp^2}\left[\mathrm{Tr}\{U_z U_y^\dagger\}\left(\mathcal{H}^+_{1xz} - \mathcal{H}^-_{1zx}\right) + \frac{1}{N_c}\mathrm{Tr}\{U_y^\dagger(\mathcal{X}_{1zx} - \mathcal{X}_{1xz})\}\right]\Bigg\}
\end{aligned}
$$

$\mathcal{X}$-operators

$$\mathcal{X}_1(x_\perp, y_\perp) = -\sqrt{\frac{s^3}{8}}g^2\int_{-\infty}^{+\infty} dz^+ d\omega^+ \bar{\psi}(z^+, y_\perp)[z^+, -\infty p_1]_y\, i\gamma^-\, [\infty n_1, \omega^+]_x \psi(\omega^+, x_\perp)$$

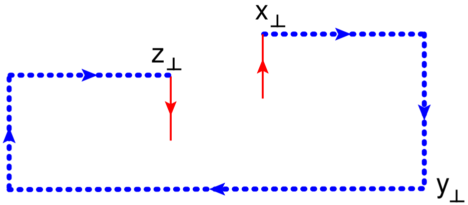
$$\mathcal{X}_1^\dagger(x_\perp, y_\perp) = g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+\, \bar{\psi}(\omega^+, x_\perp)[\omega^+, \infty n_1]_x\, i\gamma^-\, [-\infty n_1, z^+]_y \psi(z^+, y_\perp)$$

$$\mathcal{X}_5(x_\perp, y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+ \bar{\psi}(z^+, y_\perp)[z^+, -\infty p_1]_y\, \gamma^5\gamma^-\, [\infty n_1, \omega^+]_x \psi(\omega^+, x_\perp)$$

$$\mathcal{X}_5^\dagger(x_\perp, y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+ \bar{\psi}(\omega^+, x_\perp)[\omega^+, \infty n_1]_x\, \gamma^5\gamma^-\, [-\infty n_1, z^+]_y \psi(z^+, y_\perp)$$

$\mathcal{X}$-dipole operators

$\mathrm{Tr}\{U_y^\dagger \mathcal{X}_{1xz}\}$ or $\mathrm{Tr}\{U_y^\dagger \mathcal{X}_{5xz}\}$



$\mathcal{H}$-operators

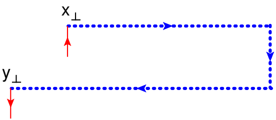
$$\mathcal{H}_1^+(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+ \, \bar{\psi}(\omega^+,y_\perp)[\omega^+,\infty n_1]_y \, i\gamma^- \; [\infty n_1, z^+]_x \psi(z^+,x_\perp)$$

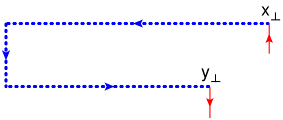
$$\mathcal{H}_5^+(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+ \, \bar{\psi}(\omega^+,y_\perp)[\omega^-,\infty n_1]_y \, \gamma^5\gamma^- \; [\infty n_1, z^+]_x \psi(z^+,x_\perp)$$

$$\mathcal{H}_1^-(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+ \, \bar{\psi}(\omega^+,y_\perp)[\omega^+,-\infty n_1]_y \, i\gamma^- \; [-\infty n_1, z^+]_x \psi(z^+,x_\perp)$$

$$\mathcal{H}_5^-(x_\perp,y_\perp) = -g^2\sqrt{\frac{s^3}{8}}\int_{-\infty}^{+\infty} dz^+ d\omega^+ \, \bar{\psi}(\omega^+,y_\perp)[\omega^+,-\infty n_1]_y \, \gamma^5\gamma^- \; [-\infty n_1, z^+]_x \psi(z^+,x_\perp)$$

TMD operators that usually appear in SIDIS and Drell-Yan processes.





$$g_1(Q^2,x) = -\sum_f \frac{e_f^2}{s^2}\frac{1}{4\pi^3}\int_0^1 dz\left(\frac{1}{z}-2\right)\int d^2z_1 d^2z_2\,[K_1(\bar{Q}|z_{12}|)]^2$$

$$\times\left(-N_c Q_{5z_1}\mathcal{U}_{z_1z_2} + \frac{1}{N_c}\left(\Psi_{5z_1z_2} - 2N_c\mathcal{F}_{z_1z_2}\right) + N_c\,\mathcal{U}_{z_2z_1}Q^\dagger_{5z_1} - \frac{1}{N_c}\left(\Psi^\dagger_{5z_1z_2} - 2N_c\mathcal{F}^\dagger_{z_1z_2}\right)\right)$$

where we defined

$$\Psi_{5z_1z_2} = \mathrm{Tr}\{\tilde{Q}_{5z_1}(U^\dagger_{z_1} - U^\dagger_{z_2})\} \qquad \Psi^\dagger_{5z_1z_2} = \mathrm{Tr}\{\tilde{Q}^\dagger_{5z_1}(U_{z_1} - U_{z_2})\}$$
$$\mathcal{F}_{z_1z_2} = \mathrm{Tr}\{\mathcal{F}_{z_1}U^\dagger_{z_2}\} \qquad \mathcal{F}^\dagger_{z_1z_2} = \mathrm{Tr}\{\mathcal{F}^\dagger_{z_1}U_{z_2}\}$$

and $\bar{Q} = z\bar{z}Q^2$

Notice that for $z_1 \to z_2$ we have $g_1 \to 0$

c.f. Yu. Kovchegov et al.

Upcoming event

Workshop at ECT* Trento, Italy

Bridging TMD Frameworks: Intersections, Tensions, and Applications

09 - 13 March 2026

Organizers:

Aleksandra Lelek (University of Antwerp)

Pia Zurita (Universidad Complutense de Madrid)

Alexey Vladimirov (Universidad Complutense de Madrid)

Giovanni A. Chirilli (University of Salento, Lecce)

Appendix

The Q^2 behavior of DIS structure function is obtained from the anomalous dimension of twist-two operators

$$\mu\frac{d}{\mu}F^a_{\xi +}\nabla_+^{n-2}F_+^{a\ \xi} = \gamma(\alpha_s, n)F^a_{\xi +}\nabla_+^{n-2}F_+^{a\ \xi}$$

Dipole DIS cross-section can be written as

$$\sigma^{\gamma^* p}(x_B, Q^2) = \int d\nu\, F(\nu)\, x_B^{-\aleph(\nu)-1}\left(\frac{Q^2}{P^2}\right)^{\frac{1}{2}+i\nu}$$

$-q^2 = Q^2 \gg P^2$, and $s = (P+q)^2 \gg Q^2$

$\aleph(\gamma)$ BFKL pomeron intercept.

The n-th moment of the structure function is

$$\int_0^1 dx_B\, x_B^{n-1}\sigma^{\gamma^* p}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma\, \frac{F(\gamma)}{n-1-\aleph(\gamma)}\left(\frac{Q^2}{P^2}\right)^{\gamma}$$

Integrating over γ-parameter we get the anomalous dimensions of the leading and higher twist operators at the *unphysical point* $n = 1$.

Analytic continuation

$$\int_0^1 dx_B\, x_B^{n-1} \sigma^{\gamma^* p}(x_B, Q^2) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma\, \frac{F(\gamma)}{\omega - \aleph(\gamma)} \left(\frac{Q^2}{P^2}\right)^{\gamma}$$

Analytic continuation: $n-1 \to \omega$ complex continuous variable

$\Rightarrow$ Residues $\omega = \aleph(\gamma)$; expand $\aleph(\gamma)$ for small γ and solve for γ

$$\gamma(\alpha_s, \omega) = \frac{\alpha_s N_c}{\pi\omega} + \mathcal{O}(\alpha_s^2), \quad F(\omega, Q^2) \sim \left(\frac{Q^2}{P^2}\right)^{\frac{\alpha_s N_c}{\pi\omega}}$$

Thus, we get the analytic continuation of anomalous dimension at the *unphysical point* $j \to 1$ of twist-2 gluon operator $F^a_{\xi+}\nabla^{-1}F^{\xi a}{}_{+}$

Light-ray operator

Analytic continuation of light-ray operators at $j = 1$

$$F^a_{\xi+}(x)\nabla_+^{j-2}F^{a\ \xi}_{+}(x)\Big|_{x=0} = \frac{\Gamma(2-j)}{2\pi i}\int_0^{+\infty} du\ u^{1-j}\, F^a_{\xi+}(0)[0,un]^{ab}F^{b\ \xi}_{+}(un)$$

OPE in light-ray operators in QCD (Balitsky, Braun (1989))

2-point function in BFKL limit (Balitsky; Balitsky, Kazakov, Sobkov (2013-2018))

2-point function in triple Regge limit (Balitsky 2018)

A lot of activity on light-ray operators in CFT (e.g. Kravchuk, Simmons-Duffin (2018))

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Super-multiplet of local operators in $\mathcal{N}$=4 SYM

$$\mathcal{O}^j_\phi(x_\perp) = \int du\, \bar{\phi}^a_{AB} \nabla^j_- \phi^{ABa}(up_1 + x_\perp)$$

$$\mathcal{O}^j_\lambda(x_\perp) = \int du\, i\bar{\lambda}^a_A \nabla^{j-1}_- \lambda^a_A(up_1 + x_\perp)$$

$$\mathcal{O}^j_g(x_\perp) = \int du\, F^{a+}{}_i \nabla^{j-2}_- F^{a+i}(up_1 + x_\perp)$$

Multiplicatively renormalizable operators

$$S^j_1 = \mathcal{O}^j_g + \frac{1}{4}\mathcal{O}^j_\lambda - \frac{1}{2}\mathcal{O}^j_\phi$$

$$S^j_2 = \mathcal{O}^j_g - \frac{1}{4(j-1)}\mathcal{O}^j_\lambda + \frac{j+1}{6(j-1)}\mathcal{O}^j_\phi$$

$$S^j_3 = \mathcal{O}^j_g - \frac{j+2}{2(j-1)}\mathcal{O}^j_\lambda - \frac{(j+1)(j+2)}{2j(j-1)}\mathcal{O}^j_\phi$$

with anomalous dimensions

$$\gamma^{S_1}_j = 4[\psi(j-1) + \gamma_E] + O(\alpha_s^2), \quad \gamma^{S_2}_j = \gamma_{j+2}, \quad \gamma^{S_3}_j = \gamma^{S_1}_{j+4}$$

A. V. Belitsky, *et al* (2004)

Analytical continuation of the local super-multiplet

$$\mathcal{F}^j(x_\perp) = \int_0^\infty du\, u^{1-j} \mathcal{F}(up_1 + x_\perp)\,,$$
$$\Lambda^j(x_\perp) = \int_0^\infty du\, u^{-j} \Lambda(up_1 + x_\perp)\,,$$
$$\Phi^j(x_\perp) = \int_0^\infty du\, u^{-1-j} \Phi(up_1 + x_\perp)$$

with

$$\mathcal{F}(up_1, x_\perp) = \int dv\, F^{a-}{}_\mu(up_1 + vp_1 + x_\perp)[u+v, v]_x^{ab} F^{b-\mu}(vp_1 + x_\perp)\,,$$
$$\Lambda(up_1, x_\perp) = \frac{i}{2}\int dv\Big(-\bar\lambda^a_A(up_1 + vp_1 + x_\perp)[u+v, v]_x^{ab}\sigma_- \lambda^b_A(vp_1 + x_\perp)$$
$$+\bar\lambda^a_A(vp_1 + x_\perp)[v, u+v]_x^{ab}\sigma_- \lambda^b_A(up_1 + vp_1 + x_\perp)\Big)\,,$$
$$\Phi(u, x_\perp) = \int dv\, \phi^a_I(up_1 + vp_1 + x_\perp)[u+v, v]_x^{ab} \phi^b_I(vp_1 + x_\perp)$$

I. Balitsky, V. Kazakov, and E. Sobko (2013)

Forward matrix elements

$$
\begin{aligned}
\mathcal{S}_1^j &= \mathcal{F}^j + \frac{j-1}{4}\Lambda^j - j(j-1)\frac{1}{2}\Phi^j\,,\\
\mathcal{S}_2^j &= \mathcal{F}^j - \frac{1}{4}\Lambda^j + \frac{j(j+1)}{6}\Phi^j\,,\\
\mathcal{S}_3^j &= \mathcal{F}^j - \frac{j+2}{2}\Lambda^j - \frac{(j+1)(j+2)}{2}\Phi^j\,.
\end{aligned}
$$

Notice the different coefficients between the S-operators and the $\mathcal{S}$-operators.

Correlation function in CFT at high-energy, $j \to 1$

$$
\langle \mathcal{F}^j(x_\perp)\mathcal{F}^{j'}(y_\perp)\rangle = \langle \mathcal{S}_1^j(x_\perp)\mathcal{S}_1^{j'}(y_\perp)\rangle \overset{\text{CFT}}{=} \delta(j-j')\frac{C(\Delta,j)s^{j-1}}{[(x-y)_\perp^2]^{\Delta-1}}\mu^{-2\gamma_a}
$$

Δ: canonical dimension d plus anomalous dim.
μ: normalization point.
$C(\Delta,j)$: unknown structure constant. Calculate it in the BFKL limit.

In the BFKL limit the two-point correlation function is UV divergent.

Regularization: point splitting ⇒

- Wilson frame Balitsky (2013, 2019), Balitsky, Kazhakov, Sobko (20013-2018)
  - Motivation: Give an example of actual calculation of correlation function; goal: understanding full dynamics of $\mathcal{N}=4$ SYM.
- quasi-pdf frame G.A.C. *Quark and Gluon quasi-pdf at low-x* (in preparation)
  - Motivation: check of the calculation comparing with expected CFT general result; goal: calculate the behavior of the quasi-pdf at small-x_B.