Dirac spectrum in the chirally symmetric phase of QCD

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Based on arXiv:2404.03546
and ongoing work

QCD in the chiral limit

QCD close to $N_f = 2$ chiral limit ($m_{u,d} \to 0$)

$$\mathrm{U}(2)_L \times \mathrm{U}(2)_R = \underbrace{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}_{\to \mathrm{SU}(2)_V \text{ @low T}} \times \underbrace{\mathrm{U}(1)_A}_{\text{anomalous}} \times \mathrm{U}(1)_V$$

Open questions:

1. nature of finite-temperature transition?
2. fate of anomalous $\mathrm{U}(1)_A$ in the symmetric phase?

Standard lore [Pisarski, Wilczek (1984)]: first order for $N_f > 2$, for $N_f = 2$

$\mathrm{U}(1)_A$ broken $\Rightarrow$ 2nd order, $\mathrm{O}(4)$ class

$\mathrm{U}(1)_A$ restored $\Rightarrow$ 1st order, or

2nd order, $\frac{\mathrm{U}(2)_L \times \mathrm{U}(2)_R}{\mathrm{U}(2)_V}$ class [Pelissetto, Vicari (2013)]

Yet unconfirmed; alternative scenarios:

[Cuteri, Philipsen, Sciarra (2021), Fejős (2022), Bernhardt, Fischer (2023), Fejős, Hatsuda (2024), Pisarski, Rennecke (2024), Giacosa *et al.* (2024)]

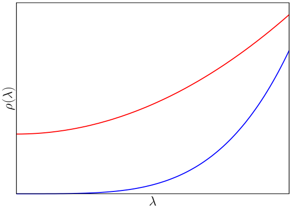
Eigenmodes of $\not{D}$ encode quark dynamics, good place to look for clues

Chiral symmetry restoration and the Dirac spectrum

$\mathrm{SU}(2)_A$ broken if $\rho(0^+;0) \neq 0$ [Banks, Casher (1980)]

$$\langle\bar{\psi}\psi\rangle = \int_0^\infty d\lambda\, \frac{2m}{\lambda^2+m^2}\rho(\lambda;m)$$

$$\rho(\lambda;m) = \lim_{\mathrm{V}\to\infty}\frac{\mathrm{T}}{\mathrm{V}}\left\langle \sum\nolimits_{n,\lambda_n\neq 0}\delta(\lambda-\lambda_n)\right\rangle$$



@ low T: $\langle\bar{\psi}\psi\rangle_{m\to 0}\neq 0$, expect $\rho(0^+;m)\neq 0$

(actually log divergence [Osborn, Toublan, Verbaarschot (1999)])

@ high T: $\langle\bar{\psi}\psi\rangle_{m\to 0} = 0$, expect $\rho(0^+;m) = 0$

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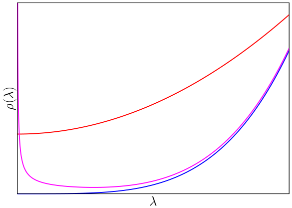

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...instead singular peak, fate as $m \to 0$ unclear

[Edwards *et al.* (1992), Cossu *et al.* (2013), Alexandru, Horváth (2015), Dick *et al.* (2015), Brandt *et al.* (2016), Tomiya *et al.* (2017), Ding *et al.* (2019), Aoki *et al.* (2021), Vig, Kovács (2021), Kaczmarek *et al.* (2021), Meng *et al.* (2023), Alexandru *et al.* (2024)]

- How does a singular peak fit with chiral symmetry restoration?
- What does it do to $\mathrm{U}(1)_A$?

Chiral symmetry restoration and the fate of $\mathrm{U}(1)_A$

How does $\mathrm{SU}(2)_A$ restoration affect $\mathrm{U}(1)_A$?

assumptions	conclusions
• observables analytic in m^2 ρ power series near $\lambda = 0$ (or $\sim \lambda^\alpha$, $\alpha > 0$)	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration [Cohen (1996,1998), Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]
• thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\not\Rightarrow$ $\mathrm{U}(1)_A$ restoration, $\mathrm{U}(1)_A$ broken by topological effects [Evans, Hsu, Schwetz (1996), Lee, Hatsuda (1996)]
• observables analytic in m^2 thermodynamic and chiral limit commute	$\mathrm{SU}(2)_A$ restoration $\Rightarrow$ $\mathrm{U}(1)_A$ restoration, unless $\rho \sim m^2\delta(\lambda)$ [Azcoiti (2023)]

What are the correct assumptions?

Symmetry restoration condition(s)

0. Local field theory: symmetry restored

   $\Rightarrow$ correlators of local operators related by symmetry become equal

1. No massless excitations, finite corr. length expected in restored phase
   $\Rightarrow$ susceptibilities related by symmetry become equal

   Does not apply at T_c for continuous transitions

   Symmetry may be manifest at the level of susceptibilities without being restored at the level of correlators, but that seems unlikely

2. Gauge fields unaffected by chiral transformations

   $\Rightarrow$ susceptibilities involving arbitrary functionals of gauge fields only (e.g., spectral density) become equal if related by symmetry

   Reasonable, but an additional assumption nonetheless

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Ginsparg-Wilson fermions

Ginsparg-Wilson Dirac operator D obeys [Ginsparg, Wilson (1982)]

$$\{D, \gamma_5\} = 2D\gamma_5 R D$$

Exact $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry [Lüscher (1998)]

Scalar and pseudoscalar bilinears

$$S = \bar{\psi}(1 - DR)\psi \qquad P = \bar{\psi}(1 - DR)\gamma_5\psi$$
$$\vec{P} = \bar{\psi}(1 - DR)\vec{\sigma}\gamma_5\psi \qquad \vec{S} = \bar{\psi}(1 - DR)\vec{\sigma}\psi$$

$O_V = \begin{pmatrix} S \\ i\vec{P} \end{pmatrix}$ and $O_W = \begin{pmatrix} iP \\ -\vec{S} \end{pmatrix}$ irreducible under chiral transformations

$$O_{V,W} \longrightarrow \mathcal{R}^T O_{V,W} \qquad \mathcal{R} \in \mathrm{SO}(4)$$

$\mathrm{SO}(4)$ doubly covered by $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$

Corresponding susceptibilities expressible in terms of the spectrum of D

What constraints result from symmetry restoration?

Generating function of scalar/pseudoscalar susceptibilities

Include source terms for bilinears in the partition function

$$\mathcal{Z}(V,W;m) = \int DU\, e^{-S_{\rm eff}(U)} \int D\psi D\bar{\psi}\, e^{-\bar{\psi}D_m(U)\psi - K(\psi,\bar{\psi},U;V,W)}$$

$S_{\rm eff}$ = gauge + massive-fermion contributions

$D_m = D + m(1 - DR)$

$$K = \underbrace{j_S S + i\vec{\jmath}_P \cdot \vec{P}}_{V\cdot O_V} + \underbrace{i j_P P - \vec{\jmath}_S \cdot \vec{S}}_{W\cdot O_W} \qquad V = \begin{pmatrix} j_S \\ \vec{\jmath}_P \end{pmatrix} \quad W = \begin{pmatrix} j_P \\ \vec{\jmath}_S \end{pmatrix}$$

Generating function of scalar and pseudoscalar susceptibilities

$$\mathcal{W}(V,W;m) = \lim_{\mathrm{V}\to\infty} \frac{\mathrm{T}}{\mathrm{V}} \ln \mathcal{Z}(V,W;m)$$

Formal power series in the sources, treat as a polynomial of arbitrary order
$\Rightarrow$ can exchange derivatives wrt sources with thermo/chiral limit, ∂_m, etc.

Symmetry restoration in the scalar/pseudoscalar sector

Under a chiral transformation

$$\mathcal{W}(V, W; m) \to \mathcal{W}(\mathcal{R}V, \mathcal{R}W; m)$$

Symmetry restoration condition (at the level of susceptibilities):

$$\lim_{m\to 0} \mathcal{W}(\mathcal{R}V, \mathcal{R}W; m) = \lim_{m\to 0} \mathcal{W}(V, W; m) \quad \forall \mathcal{R} \in \mathrm{SO}(4)$$

$\Rightarrow \mathcal{W}$ depends only on $\mathrm{SO}(4)$ invariants in the chiral limit

$\mathcal{Z}$ function of $j_S + m$ only, exactly massless theory is chirally symmetric $\Rightarrow$

$$\mathcal{Z}(V, W; m) = \mathcal{Z}(\tilde{V}, W; 0) = \mathcal{Z}(\mathcal{R}\tilde{V}, \mathcal{R}W; 0) \qquad \tilde{V} = \begin{pmatrix} j_S + m \\ \vec{j}_P \end{pmatrix}$$

$\Rightarrow \mathcal{W}$ is an $\mathrm{SO}(4)$ invariant function of $\tilde{V}$ and W

$$\mathcal{W}(V, W; m) = \hat{\mathcal{W}}(\tilde{V}^2, W^2, 2\tilde{V} \cdot W)$$

Symmetry restoration in the scalar/pseudoscalar sector II

$$\mathcal{W}(V,W;m) = \hat{\mathcal{W}}(\tilde{V}^2, W^2, 2\tilde{V}\cdot W)$$
$$= \hat{\mathcal{W}}(m^2 + \overbrace{2mj_S + V^2}^{u}, \overbrace{W^2}^{w}, \overbrace{2(mj_P + V\cdot W)}^{\tilde{u}})$$
$$= \sum_{n_u,n_w,n_{\tilde{u}}=0}^{\infty} \frac{u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}}}{n_u! n_w! n_{\tilde{u}}!} \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)$$

$\{\mathcal{A}_{n_u n_w n_{\tilde{u}}}\}$ equivalent to a subset of $\vec{P}$ and $\vec{S}$ susceptibilities

chiral symmetry restored
$\Longleftrightarrow$
$\mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)$ finite (non-divergent) in the chiral limit

Proof:

$\Longleftarrow$ $u^{n_u} w^{n_w} \tilde{u}^{n_{\tilde{u}}} \to (V^2)^{n_u} (W^2)^{n_w} (2V\cdot W)^{n_{\tilde{u}}}$

$\Longrightarrow$ $\partial_{j_S}^{\mathrm{expl}} \mathcal{W} \to 0 \Rightarrow m\partial_u \hat{\mathcal{W}} \to 0 \Rightarrow \partial_m \mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2) \to 0 \Rightarrow \mathcal{A}_{n_u n_w n_{\tilde{u}}}(0)$ finite

Symmetry restoration in the scalar/pseudoscalar sector III

Mass-squared derivative

$$\partial_{m^2}\mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2) = \mathcal{A}_{n_u+1\, n_w n_{\tilde{u}}}(m^2)$$

chiral symmetry restored

$\Longleftrightarrow$

$\mathcal{A}_{n_u n_w n_{\tilde{u}}}(m^2)$ infinitely differentiable at zero

$\Longleftrightarrow$

even susceptibilities infinitely differentiable in m^2

even susceptibilities = even number of isosinglet bilinears

If we assume χSR also for *nonlocal* gauge functionals

(or χSR for local operators in partially quenched theory)

$\rho(\lambda; m)$ infinitely differentiable in m^2

Spectrum of GW Dirac operator

Restrict to

$$R = \tfrac{1}{2}, \quad D^\dagger = \gamma_5 D \gamma_5 \quad \Longrightarrow \qquad D + D^\dagger = DD^\dagger$$

domain-wall [Kaplan (1992)]
overlap [Neuberger (1997)]

Spectrum of D on a circle

- pairs of complex conjugate modes $\mu_n \neq \mu_n^*$, spectral density

$$\rho(\lambda; m) = \frac{\mathrm{T}}{\mathrm{V}} \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle \qquad \lambda_n = |\mu_n| \, \mathrm{sgn}(\mathrm{Im}\mu_n)$$

- $N_\pm$ chiral zero-modes $\mu_n = 0$, $N'_\pm$ chiral "doubler" modes $\mu_n = 2$, topological charge $Q = N_+ - N_- = N'_- - N'_+ = -\frac{1}{2}\mathrm{tr}\, \gamma_5 D$

Density of zero modes $\lim_{\mathrm{V}\to\infty} \frac{\langle N_+ + N_- \rangle}{\mathrm{V}} = 0$ in the thermo limit, but higher cumulants, correlations with complex modes do not

Generating function from the Dirac spectrum

$$\mathcal{W}(V, W; m) - \mathcal{W}(0,0;m)$$

$$= \sum_{\vec{n}\neq 0} \frac{X_0^{n_1} X_0^{*n_2}}{n_1! n_2! n_3!} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \underbrace{s(n_1,k_1)s(n_2,k_2)}_{\substack{\text{Stirling numbers} \\ \text{of the first kind}}} I^{(n_3)}_{N_+^{k_1} N_-^{k_2}}[X,\ldots,X]$$

$$X_0 \equiv \frac{u - w + i\tilde{u}}{m^2}$$

$$X(\lambda) \equiv 2\left(f(\lambda;m)u + \tilde{f}(\lambda;m)w\right) + f(\lambda;m)^2\left((u-w)^2 + \tilde{u}^2\right)$$

$$f(\lambda;m) \equiv \frac{h(\lambda)}{\lambda^2 + m^2 h(\lambda)} \qquad \tilde{f}(\lambda;m) \equiv f(\lambda;m) - 2m^2 f(\lambda;m)^2$$

$$h(\lambda) \equiv 1 - \frac{\lambda^2}{4}$$

$$I^{(k)}_{N_+^{k_1} N_-^{k_2}}[g_1,\ldots,g_k] \equiv \left[\prod_{i=1}^{k} \int_0^2 d\lambda_i\, g_i(\lambda_i)\right] \underbrace{\rho^{(k)}_{N_+^{k_1} N_-^{k_2}\, c}(\lambda_1,\ldots,\lambda_k;m)}_{\text{connected eigenvalue correlation function}}$$

First-order constraints

$\mathcal{W}$ even in $\tilde{u}$ due to CP symmetry

$$\mathcal{W} = \mathcal{W}|_{j=0} + \frac{\chi_\pi}{2}u + \frac{\chi_\delta}{2}w + \frac{1}{2m^2}\left(\frac{\chi_\pi - \chi_\delta}{4} - \frac{\chi_t}{m^2}\right)\tilde{u}^2 + \ldots$$

$|\chi_\delta| \leq \chi_\pi$, requirements from chiral symmetry restoration reduce to

$$\lim_{m\to 0}\frac{\chi_\pi}{4} = \lim_{m\to 0} I^{(1)}[f] = \lim_{m\to 0}\int_0^2 d\lambda\, \frac{\rho(\lambda;m)h(\lambda)}{\lambda^2 + m^2 h(\lambda)} < \infty$$

$$\frac{\chi_\pi - \chi_\delta}{4} = 2m^2 I^{(1)}[f^2] = \frac{\chi_t}{m^2} + O(m^2)$$

$\mathrm{U}(1)_A$ order parameter

$$\Delta \equiv \lim_{m\to 0}\frac{\chi_\pi - \chi_\delta}{4} = \lim_{m\to 0}\frac{\chi_t}{m^2} < \infty$$

Not new, but *all* the direct constraints on ρ:
higher-order coefficients involve higher-point eigenvalue correlators

Fate of $\mathrm{U}(1)_A$ I

$\mathrm{SU}(2)_A$ restoration and $\mathrm{U}(1)_A$ breaking compatible at this stage, need further assumptions on ρ to make progress

1. $\rho(\lambda; m)$ admitting power-series expansion

$$\rho_{\mathrm{series}}(\lambda; m) = \sum_{n=0}^{\infty} \rho_n(m^2)\lambda^n \,,$$

- if $\rho_0 \propto |m|$ then $\mathrm{U}(1)_A$ is broken
- if ρ is C^∞ in m^2, $\mathrm{U}(1)_A$ restored ($\Delta = 0$) if $\mathrm{SU}(2)_A$ restored

Agrees with [Aoki, Fukaya, Taniguchi (2012), Kanazawa, Yamamoto (2016)]

and $\rho_0 = O(m^4)$, $\rho_{1,2,3} = O(m^2)$, $\rho_{n\geq 4} = O(m^0) \Rightarrow \rho(\lambda, 0) = O(\lambda^4)$

Disagrees with [Aoki, Fukaya, Taniguchi (2012)]

2. Possibly singular power-law behaviour

$$\rho(\lambda; m) = \sum_{i=1}^{s} C_i(m)\lambda^{\alpha_i(m)} + \bar{\rho}(\lambda; m), \qquad \alpha_i(m) > -1 \text{ for } m \neq 0$$

with $-1 \leq \alpha_1(0) < \alpha_2(0) < \ldots < \alpha_s(0) \leq 1$ and $\bar{\rho} = O(\lambda^{1+\zeta})$

- $\mathrm{SU}(2)_A$ restoration at the level of susceptibilities requires only

$$C_i(m) = \tfrac{2}{\pi}\cos\left(\alpha_i(m)\tfrac{\pi}{2}\right)|m|^{1-\alpha_i(0)}\hat{C}_i(m) \qquad |\hat{C}_i(0)| < \infty$$

$\Delta = \sum_i [1-\alpha_i(0)]\hat{C}_i(0)$, symmetry broken if some $\hat{C}_i(0) \neq 0$

$C_s(m) = \frac{\hat{C}_s(m)}{\ln(2/|m|)}$ if $\alpha_s(0) = 1$, no $\mathrm{U}(1)_A$ breaking

- if ρ is $C^\infty(m^2)$ then α_i, C_i are C^∞, only possibility to break $\mathrm{U}(1)_A$

$$\boxed{\alpha_1(0) = -1}$$

Singular peak

Singular peak compatible with χSR and $\mathrm{U}(1)_A$ breaking if

$$\rho_{\mathrm{peak}}(\lambda; m) \underset{m\to 0}{\to} \left[\frac{\Delta}{2} + O(m^2)\right] \frac{m^2\gamma(m)}{\lambda^{1-\gamma(m)}} \qquad \gamma > 0\,,\ \gamma = O(m^2)$$

Close relation with topology, $\mathrm{U}(1)_A$ broken if $n_{\mathrm{peak}} \propto m^2$ in the chiral limit

$$\lim_{m\to 0} \frac{n_{\mathrm{peak}}}{m^2} = \lim_{m\to 0} \frac{2}{m^2} \int_0^2 d\lambda\, \rho_{\mathrm{peak}}(\lambda; m) = \Delta = \lim_{m\to 0} \frac{\chi_t}{m^2}$$

Singular peak compatible with commutativity of thermo and chiral limits!
Necessary condition for commutativity [Azcoiti (2023)] satisfied by ρ_{peak}

$$\lim_{m\to 0} \frac{\rho(|m|z; m)}{|m|} = \Delta\delta(z) \qquad z \in [-1, 1]$$

Under general conditions, $\mathrm{U}(1)_A$ breaking requires ρ effectively developing $\sim m^2\delta(\lambda)$ in the chiral limit, a peak of some sort is required

Second-order constraints – two-point function

More constraints from second-order coefficients in the expansion of $\mathcal{W}$

Two constraints involving the two-point function

$$\rho_c^{(2)}(\lambda,\lambda';m) = \lim_{\mathrm{V}\to\infty}\frac{\mathrm{T}}{\mathrm{V}}\Big(\langle \textstyle\sum_{n\neq n'}\delta(\lambda-\lambda_n)\delta(\lambda'-\lambda_{n'})\rangle - \langle \textstyle\sum_n \delta(\lambda-\lambda_n)\rangle\langle\textstyle\sum_{n'}\delta(\lambda'-\lambda_{n'})\rangle\Big)$$

$$\boxed{\begin{aligned} 4m^2 I^{(2)}[f,f] &= \left[-\lim_{\mathrm{V}\to\infty}\frac{\mathrm{T}}{\mathrm{V}}\frac{\langle N_0\rangle^2}{m^2}\right] + O(m^2) \\ I^{(2)}[\hat{f},\hat{f}] &= O(m^0) \end{aligned}}$$

$$I^{(2)}[g_1,g_2] \equiv \int_0^2 d\lambda \int_0^2 d\lambda'\, g_1(\lambda)g_2(\lambda')\rho_c^{(2)}(\lambda,\lambda';m)$$

$$\hat{f}(\lambda;m) \equiv f(\lambda;m) - m^2 f(\lambda;m)^2$$

Assume $\rho_c^{(2)}$ ordinary function (no δs)

Two-point function finite at the origin

If $\rho_c^{(2)}(\lambda, \lambda'; m)$ finite at the origin

$$\rho_c^{(2)}(\lambda, \lambda'; m) = A(m) + B(\lambda, \lambda'; m)$$

$|B(\lambda, \lambda'; m)| \leq b(\lambda^2 + \lambda'^2)^{\frac{\beta}{2}}$ with $\beta < 1$

$$\lim_{m\to 0} 4m^2 I^{(2)}[f, f] = \pi^2 A(0) = -\mathrm{T} \lim_{m\to 0} \lim_{\mathrm{V}\to\infty} \frac{\langle N_0 \rangle^2}{m^2 \mathrm{V}}$$
$$\lim_{m\to 0} (4m)^2 I^{(2)}[\hat{f}, \hat{f}] = \pi^2 A(0) = 0$$

$\Rightarrow$ measure of $\frac{N_0}{m\sqrt{\mathrm{V}}}$ concentrated in zero in thermo and chiral limit

$\Rightarrow \lim_{m\to 0} \frac{\chi_t}{m^2} = 0$

$\Rightarrow \Delta = 0$

No $\mathrm{U}(1)_A$-breaking topological effects if $\rho_c^{(2)}(\lambda, \lambda'; m)$ finite at zero

Localised near-zero modes

Above T_c modes localised below "mobility edge" λ_c, obey Poisson statistics [MG, Kovács (2021)]

Purely Poisson spectrum: $\rho_c^{(2)}(\lambda,\lambda') \propto \rho(\lambda)\rho(\lambda')$ [Kanazawa, Yamamoto (2016)]

Expect $\rho_c^{(2)}(\lambda_{\rm loc},\lambda_{\rm deloc})/[\rho(\lambda_{\rm loc})\rho(\lambda_{\rm deloc})] \sim O(1)$ (and negative)

If near-zero modes are localised, expect

$$|\rho_c^{(2)}(\lambda,\lambda';m)| \le C\rho(\lambda;m)\rho(\lambda';m) \quad \text{if } \lambda,\lambda' < \lambda_c \text{ or } \lambda < \lambda_c < \lambda'$$

$$\text{If } \lambda_c \underset{m\to 0}{\not\to} 0 \Rightarrow \lim_{m\to 0}\lim_{\mathrm{V}\to\infty} \frac{\langle N_0\rangle^2}{m^2\mathrm{V}/\mathrm{T}} = -\lim_{m\to 0} 4m^2 I^{(2)}[f,f] = 0$$

$\mathrm{U}(1)_A$-breaking topological effects require

- either $\lambda_c \to 0$ as $m \to 0$
- or another $\lambda_c' < \lambda_c$ close to zero

Non-Poissonian repulsion [Ding *et al.* (2019)]

Near-zero modes delocalised, second mobility edge? [Meng *et al.* (2023)]

Second-order constraints – topology

One constraint on topological-charge distribution

$$\boxed{\frac{b_{Q^4} - \chi_t}{m^2} = O(m^2)}$$

First nontrivial cumulant

$$b_{Q^4} \equiv \lim_{\mathrm{V}\to\infty} \frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\mathrm{V}/\mathrm{T}}$$

$b_{Q^4} = \chi_t + O(m^4)$, same as gas of non-interacting (anti)instantons

$$Q = n_i - n_a \qquad P_{n_i}(n) = P_{n_a}(n) = e^{-\frac{\chi_t}{2}} \frac{1}{n!} \left(\frac{\chi_t}{2}\right)^n$$

Generalises to all cumulants (see also [Kanazawa, Yamamoto (2015)])

⇒ expect early onset of dilute-gas behaviour for physical masses, as in pure gauge [Bonati *et al.* (2013)]

⇒ does not necessarily mean there is an actual gas of actual instantons

Instanton-gas picture

$\mathrm{U}(1)_A$ breaking in symmetric phase possible but requires specific features

- singular spectral density $\rho \to \lambda^{-1}$ (if power-like near zero)
- singular two-point function, delocalised near-zero modes
- $\chi_t \propto m^2$

Natural features if an ideal instanton gas-like component is present

- satisfies ideal gas-like behaviour requirement
- mixing of localised instanton zero-modes can give near-zero peak of delocalised modes: exp. small matrix elements $\Rightarrow$ small eigenvalues, (nearly) degenerate $\Rightarrow$ mixing easy
- instanton gas model provides singular peak $\rho \propto \lambda^{\alpha(m)}$ [Kovács (2023)]
    - $\alpha \to -1$ as disorder (here $\sim 1/n_{\mathrm{inst}}$) increases in a similar cond-mat model [Evangelou and Katsanos (2003)]
    - delocalised near-zero modes, mobility edge found in another similar cond-mat model [García-García, Cuevas (2006)]
    - separation of topological modes (peak) from the bulk provided by the Polyakov-loop driven mobility edge $\Rightarrow \chi_t \propto m^2$

Viable mechanism for $\mathrm{U}(1)_A$ breaking – is it the correct one?

Summary and outlook

- Chiral symmetry is restored in the scalar/pseudoscalar sector in the $N_f = 2$ massless limit *if and only if* all susceptibilities are non-divergent ($\Rightarrow C^\infty$ property)
- $\mathrm{SU}(2)_A$ restoration compatible with singular ρ at $\lambda = 0$ for $m \neq 0$
- $\mathrm{U}(1)_A$ breaking compatible with $\mathrm{SU}(2)_A$ restoration but requires
  - singular near-zero spectral density $\rho \sim O(m^4)/\lambda$ as $m \to 0$
  - singular two-point function
  - near-zero modes *not* localised (near-zero mobility edge?)
- Features required for $\mathrm{U}(1)_A$ breaking occur naturally if gauge field topology includes ideal instanton gas contribution

Open issues:

- other sectors
- larger N_f
- test against numerical results

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