The topological susceptibility slope χ' of large-N Yang–Mills theories

Claudio Bonanno

Instituto de Física Teórica (IFT)

UAM/CSIC Madrid

✉ claudio.bonanno@csic.es

2nd LatticeNET Workshop on Challenges in Lattice Field Theory

30/03–05/04/2025 — Centro de Ciencias Pedro Pascual, Benasque, Spain

Based on:

- Work in Progress
- The topological susceptibility slope χ' of the pure-gauge SU(3) Yang–Mills theory
  CB, *JHEP* **01** (2024) 116 [arXiv:2311.06646]
- Lattice determination of the topological susceptibility slope χ' of $2d$ $\mathrm{CP}^{\mathrm{N}-1}$ models at large N
  CB, *PRD* **107** (2023) 1, 014514 [arXiv:2212.02330]

The Topological Susceptibility Slope χ'

This talk is about the calculation of the next-to-leading-order coefficient of the **momentum expansion** of the topological charge density 2-point correlator.

$$\widetilde{C}(p^2) = \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}p\cdot x}\, C(x) = \sum_{n=0}^{\infty} (-1)^n \chi^{(n)} p^{2n} = \chi - \chi' p^2 + \mathcal{O}(p^4)$$

$$Q = \frac{1}{16\pi^2} \int \mathrm{d}^4x\, \mathrm{Tr}\left[G_{\mu\nu}(x)\widetilde{G}_{\mu\nu}(x)\right] = \int \mathrm{d}^4x\, q(x) \in \mathbb{Z} \qquad \text{[Topological Charge]}$$

$$C(x) = \langle q(x) q(0) \rangle \qquad \text{[Top. Charge Density Correlator]}$$

- $\chi = \widetilde{C}(0) = \int \mathrm{d}^4x\, C(x) = \lim_{V\to\infty} \frac{\langle Q^2 \rangle}{V}$ $\qquad$ $\mathcal{O}(p^0)$: **Top. Susceptibility**
- $\chi' = -\left.\frac{\mathrm{d}\widetilde{C}(p^2)}{\mathrm{d}p^2}\right|_{p^2=0} = \frac{1}{8}\int \mathrm{d}^4x\, |x|^2 C(x)$ $\qquad$ $\mathcal{O}(p^2)$: **Top. Susc. Slope**

Motivations

- **Theory** [Witten NPB **156** 289 (1979) — Veneziano NPB **159** 213 (1979)]

Witten–Veneziano equation:

$$\chi_{\mathrm{YM}} = \frac{m_{\eta'}^2 F_\pi^2}{2N_{\mathrm{f}}} \simeq (180\ \mathrm{MeV})^4 \qquad \text{(Large-}N\text{ limit)}$$

Consistency condition for this relation to hold:

[Veneziano NPB **159** 213 (1979) — Narison PLB **255** (1991) 101]

$$\widetilde{C}(p^2 = m_{\eta'}^2) \simeq \widetilde{C}(0) \qquad \Longrightarrow \qquad |\chi'_{\mathrm{YM}}| m_{\eta'}^2 \ll \chi_{\mathrm{YM}} \qquad \text{(Large-}N\text{ limit)}$$

since $\qquad \widetilde{C}(p^2 = m_{\eta'}^2) \simeq \chi - \chi' m_{\eta'}^2 \qquad \widetilde{C}(0) = \chi$

- **Phenomenology** [Shore, Veneziano PLB **244** (1990) 75-82]

The flavour-singlet nucleon axial charge can be measured in experiments (e.g., EMC experiment). Shore–Veneziano related it to χ' in QCD (chiral limit):

$$2m_{\mathrm{N}} g_{\mathrm{A}}^{(0)} = \lim_{\Delta p \to 0} \langle N(p) | \, \partial_\mu J_5^\mu \, | N(p + \Delta p) \rangle = \sqrt{|\chi'_{\mathrm{QCD}}|}\, g_{\eta_0 \mathrm{NN}}$$

Status

• Non-lattice

1. **QCD Sum Rule** [Narison (2006) hep-ph/0601066 — Narison NPA **1020** (2022) 122393]

   $\chi'_{\mathrm{YM}}(N=3) \approx [7(3)\ \mathrm{MeV}]^2$ $\qquad$ $\chi'_{\mathrm{QCD}}(m=0) \approx -[24.3(3.4)\ \mathrm{MeV}]^2$

   QCD Sum Rule however underestimates SU(3) pure-gauge top. susceptibility:

   $\chi_{\mathrm{YM}}(N=3) \approx [114\ \mathrm{MeV}]^4$ $\qquad$ (QCD Sum Rule)

   $\chi_{\mathrm{YM}}(N=3) \approx [200\ \mathrm{MeV}]^4$ $\qquad$ (Lattice)

2. **Chiral Pertubation Theory** [Leutwyler (2000) hep-ph/0008124]

$$\chi'_{\mathrm{QCD}} = -\frac{F_\pi^2}{2}\left(\frac{1}{m_{\mathrm{u}}^2}+\frac{1}{m_{\mathrm{d}}^2}+\frac{1}{m_{\mathrm{s}}^2}\right)\left(\frac{1}{m_{\mathrm{u}}}+\frac{1}{m_{\mathrm{d}}}+\frac{1}{m_{\mathrm{s}}}\right)^{-2}$$

   $\chi'_{\mathrm{QCD}}(m=0) = -\frac{1}{6}F_0^2 = -[32.8(2.4)\ \mathrm{MeV}]^2$ $\qquad$ $(m_{\mathrm{u}} = m_{\mathrm{d}} = m_{\mathrm{s}} \equiv m \to 0)$

3. **NJL model** gives $\chi' \approx -(20\ \mathrm{MeV})^2$ [Fukushima et al. PLB **514** (2001) 200-203]

• Lattice (so far)

Only few preliminary attempts in the 90s for SU(2) and SU(3), but no reliable calculation nor conclusive result yet.

[Di Giacomo NPB Proc. Suppl **23** (1991) 191 – Briganti, Di Giacomo, Panagopoulos PLB **253** (1991) 427]

My goal is to pursue a systematic research program targeted at the non-perturbative lattice investigation of χ'.

I proposed a novel strategy to compute this quantity on the lattice. Other ingredient: Parallel Tempering on Boundary Conditions.

- 2023: Lattice calculation of χ' in $2d$ CP^{N-1} models at large N.
  [CB PRD **107** (2023) 1, 014514 — arXiv:2212.02330]
  Lattice reproduces analytic predictions up to NLO in $1/N$
  [Campostrini, Rossi PLB **272** (1991) 305]
- 2024: Lattice calculation of χ' in $4d$ SU(3) Yang–Mills theory.
  [CB JHEP **01** (2024) 116 — arXiv:2311.06646]
- **2025 (this talk)**: work in progress about the investigation of χ' in large-N SU(N) Yang–Mills theories.

Parallel Tempering on Boundary Conditions

Topological freezing: autocorrelation of Q diverges severely with $1/a$ and N

[Allés et al. PLB **389** (1996) 107-111 — Del Debbio et al. PLB **594** (2004) 315-323]

$\longrightarrow$ at large-N Q is frozen even on coarse lattices

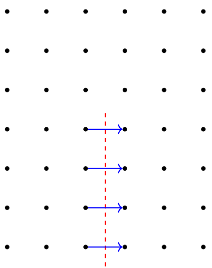
Algorithm adopted here: **Parallel Tempering on Boundary Conditions**

First proposed by M. Hasenbusch [PRD 96 (2017) 054504] in $2d$ CP^{N-1} models.
I have implemented and extensively used it in $4d$ gauge theories:

- θ-dep. of vacuum energy up to NLO in θ and $1/N$
  CB, Bonati, D'Elia JHEP 03 (2021) 111 [2012.14000]
- Impact of topological freezing on glueball mass computations
  CB, D'Elia, Lucini, Vadacchino PLB 833 (2022) 137281 [2205.06190]
- θ-dep. of deconfinement temperature up to NLO in θ and $1/N$
  CB, D'Elia, Verzichelli JHEP 02 (2024) 156 [2312.12202]
- θ-dep. of mass gap and string tension up to NLO in θ and $1/N$
  CB, Bonati, Papace, Vadacchino JHEP 05 (2024) 163 [2402.03096]
- Impact of topological freezing on renormalized strong coupling via gradient flow + step scaling
  CB, Dasilva Golán, D'Elia, García Pérez, Giorgieri EPJC 84 (2024) 9, 916 [2403.13607]
- Topological susceptibility in full QCD with dynamical quarks at physical point
  CB, Clemente, D'Elia, Maio, Parente JHEP 08 (2024) 236 [2404.14151]

The algorithm

- Consider a collection of N_{r} lattice replicas
- Replicas differ for boundary conditions on **small** sub-region: *the defect*
- Each replica is updated with standard methods
- After the updates, propose conf swaps among replicas via Metropolis test



- Links crossing the defect: $\beta \to \beta \cdot c(r)$.

- Periodic: $c = 1$. Open: $c = 0$.
Interpolating replicas: $0 < c(r) < 1$.

- Tempering parameters $c(r)$ tuned through short test runs to have uniform swap acceptances.

- Configuration *random walks* through the replicas $\Longrightarrow$ small autocorrelations of Q in open replica [Lüscher, Schaefer JHEP **1107** (2011) 036] transferred to periodic one.

- Observables are computed on periodic replica $\Longrightarrow$ Q well-defined, no boundary effects on correlators.

Simulation details

- $N = 3, 4, 5, 6$

- Wilson plaquette action

- Same range of lattice spacings across all N values

- Scale setting with string tension [Athenodorou, Teper JHEP **12** (2021) 082]

$$0.21 \lesssim a\sqrt{\sigma} \lesssim 0.13 \quad \longrightarrow \quad 0.09 \text{ fm} \lesssim a \lesssim 0.05 \text{ fm}$$

- ℓ^4 lattices with $\ell\sqrt{\sigma} \approx 3.6 \quad \longrightarrow \quad \ell \approx 1.5$ fm

- For all N-values: **2 finer lattice spacings** with respect to previous study of χ at large N with Open Boundaries [Cè, García Vera, Giusti, Schaefer PLB **762** (2016) 232-236]

Lattice observables

Gluonic top. charge on the lattice → discretization of $G\widetilde{G}$ + smoothing.
E.g., cooling, gradient flow, stout smearing.

Smoothing: kills short-scale fluctuations below $\dfrac{R_s}{a} \propto \sqrt{\text{amount of smoothing}}$

$$\chi_L(R_s) = \frac{1}{V}\langle Q_L^2(R_s)\rangle \qquad\qquad Q_L(R_s) = \sum_x q_L(x, R_s)$$

$$\chi'_L(R_s) = \frac{1}{8}\left\langle \sum_x d^2(x,0) q_L(x,R_s) q_L(0,R_s)\right\rangle$$

- $d(x,y)$ = shortest distance between lattice sites x and y in a periodic box
- $q_L(x,R_s)$ = clover discr. of $G\widetilde{G}$ after smoothing at smooth. rad. R_s
- $\dfrac{R_s}{a} = \dfrac{\sqrt{8t}}{a} = \sqrt{\dfrac{8}{3}n_{\text{cool}}}$ [Bonati, D'Elia PRD **89** (2014) 10, 105005]

I will use cooling for its cheapness: χ' required $\sim \mathcal{O}(10\text{M})$ trajectories.

- Minimum smoothing radius: $R_s \gtrsim 2a$.
- Maximum smoothing radius: $R_s \lesssim \frac{1}{2}\ell$.

Strategy

Smoothing alters short-distance behavior of the top. charge density correlator.

In the continuum: $C(x, R_s) = C(x) + \mathcal{O}(R_s^2)$

[Cè et al. PRD **92** (2015) 7, 074502; PLB **762** (2016) 232-236 — Altenkort et al. PRD **103** (2021) 114513]

- $$\chi = \int \mathrm{d}^4x\, C(x) = \frac{\langle Q^2 \rangle}{V}$$

χ depends on global top. charge $\implies$ should be insensitive to UV scale R_s

1. Continuum limit $a \to 0$ of χ_L at fixed R_s should be R_s-independent

2. Witten–Veneziano requires a finite large-N limit of χ_{YM}

- $$\chi' = \frac{1}{8} \int \mathrm{d}^4x\, |x|^2\, C(x)$$

This quantity will receive $\mathcal{O}(R_s^2)$ corrections in the continuum

1. Continuum limit $a \to 0$ of χ'_L at fixed R_s

2. Zero-smoothing limit $R_s \to 0$ of continuum results with $\mathcal{O}(R_s^2)$ corrections

3. What is expected N-scaling of χ'_{YM}?

A large-N argument to estimate χ'_{YM}

In [CB JHEP **01** (2024) 116] I have discussed a simple large-N argument to estimate χ'_{YM}.

Large-N expansion of topological charge density correlator (chiral limit):

$$C_{\mathrm{QCD}}(p^2) = \int \mathrm{d}^4x\, \mathrm{e}^{ip\cdot x}\, \langle q(x)q(0)\rangle_{\mathrm{QCD}} = C_{\mathrm{YM}}(p^2) - \frac{|A_{\eta'}|^2}{p^2 + m_{\eta'}^2} + \mathcal{O}\left(\frac{1}{N}\right)$$

- $N = \infty$: pure Yang–Mills contribution + η' propagator
- $|A_{\eta'}|^2 = |\langle 0|\, q(0)\, |\eta'\rangle|^2 = \frac{1}{6}F_\pi^2 m_{\eta'}^4$ [Veneziano NPB **159** 213 (1979)]

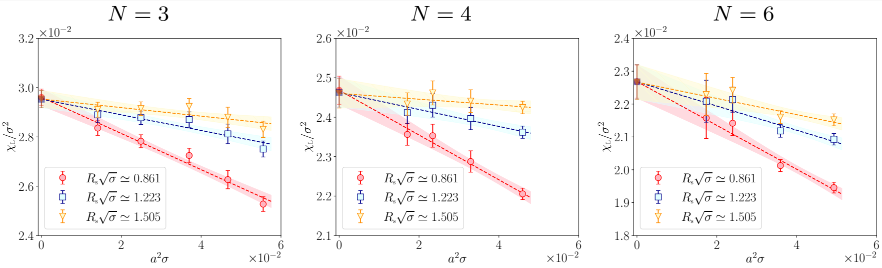
$$p^2 = 0 \quad \longrightarrow \quad \chi_{\mathrm{QCD}} = \chi_{\mathrm{YM}} - \frac{|A_{\eta'}|^2}{m_{\eta'}^2} = \chi_{\mathrm{YM}} - \frac{1}{6}m_{\eta'}^2 F_\pi^2 = 0$$

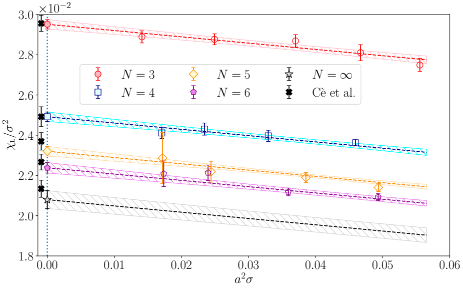
Witten–Veneziano follows since $\chi_{\mathrm{QCD}} = 0$ in the chiral limit

$$-\frac{\mathrm{d}}{\mathrm{d}p^2}\bigg|_{p^2=0} \longrightarrow \quad \chi'_{\mathrm{QCD}} = \chi'_{\mathrm{YM}} - \frac{|A_{\eta'}|^2}{(p^2 + m_{\eta'}^2)^2}\bigg|_{p^2=0} = \chi'_{\mathrm{YM}} - \frac{1}{6}F_\pi^2$$

- $\chi'_{\mathrm{QCD}} \sim \mathcal{O}(N)$ [Leutwyler (2000) hep-ph/0008124] and $F_\pi^2 \sim \mathcal{O}(N)$ $\implies$ $\chi'_{\mathrm{YM}} \sim \mathcal{O}(N)$
- χ'_{QCD} is non-zero in the chiral limit according to Chiral Pert. Theory

Results for χ





$R_s\sqrt{\sigma} \simeq 1.223 \longrightarrow R_s \simeq 0.5$ fm

$N = 5, 6$ still running (lower statistics)

Fixed R_s: lattice artifacts $\sim$ N-independent also observed in [Cè et al. PLB **762** (2016) 232]

Same combo fit of [Cè et al.]

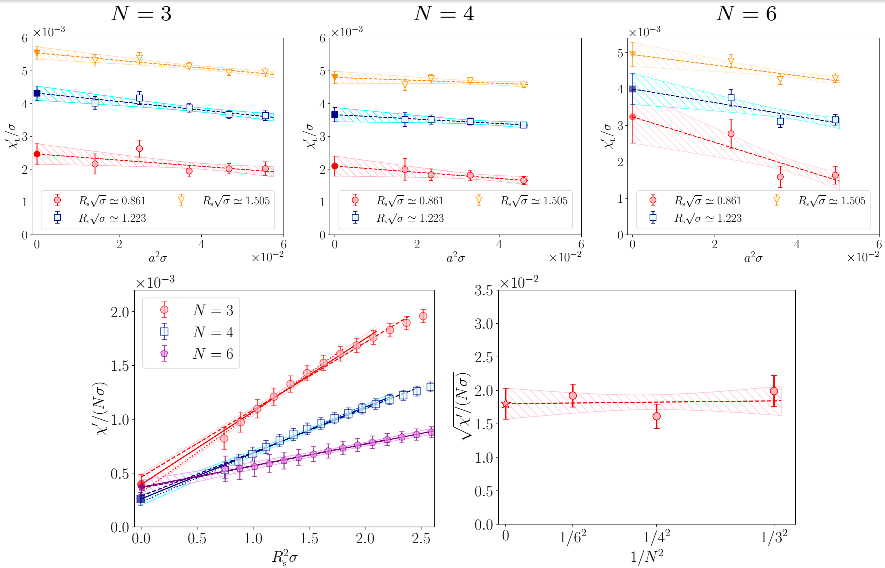
$$\frac{\chi_L}{\sigma^2}(a, N) = \frac{\chi_\infty}{\sigma^2} + \frac{A_1}{N^2} + \frac{A_2}{N^4} + Ca^2\sigma$$

gives reduced chi-squared of ~ 0.64

$$\frac{\chi}{\sigma^2}(N) = 0.02080(44) + \frac{0.050(12)}{N^2} + \frac{0.258(86)}{N^4}$$

Results for all N-values agree with Open Boundaries determinations [Cè et al.]

Results for χ'



- χ' (after continuum limit) shows non-trivial dep. on R_s compatible with $\mathcal{O}(R_s^2)$ corrections
  - $\lim_{N\to\infty} \chi'(N)/N = [0.00180(23)]^2 \times [1 + \mathcal{O}(1/N^2)]$

Conclusive discussion — I

1. Topological Susceptibility

$$\sqrt{8t_0\sigma}\Big|_{N=\infty} = 1.207(5) \quad \text{[combining Giusti and Teper results]}$$

$$\sqrt{8t_0} = 0.5 \text{ fm} \qquad \Longrightarrow \qquad \sqrt{\sigma}\Big|_{N=\infty} = 476 \text{ MeV}$$

- $\chi(N=3) = [197.29(39) \text{ MeV}]^4$
- $\chi(N=\infty) = [181.65(91) \text{ MeV}]^4$

Agrees with Witten–Veneziano prediction $\chi_{\rm YM}(N=\infty) \simeq (180 \text{ MeV})^4$

2. Topological Susceptibility Slope

- $\chi'(N=3) = [(16.4 \pm 1.9) \text{ MeV}]^2$
- $\displaystyle\lim_{N\to\infty} \frac{\chi'}{N} = [(8.6 \pm 1.1) \text{ MeV}]^2$

QCD Sum Rule: $\chi'_{\rm YM}(N=3) \simeq [7(3) \text{ MeV}]^2$. Underestimated but same ballpark.

QCD Sum Rule equally underestimates top. susc. : $\chi_{\rm YM}(N=3) \approx (114 \text{ MeV})^4$.

3. Recent pheno estimate

A few months ago, new study of role of anomaly in (spin-polarised) Deep Inelastic Scattering appeared (worldline effective action formalism).

[Tarasov, Venugopalan (2025) — arXiv:2501.10519]

The authors refined Shore–Veneziano phenomenological relation, obtaining finite-quark-mass corrections at leading order in $1/N$.

Their result involves $\chi'_{\rm YM}$ $\Longrightarrow$ experimental result for $g_{\rm A}^{(0)}$ used to obtain estimate for $\chi'_{\rm YM}(N=3)$. Same ballpark as my lattice result.

- $\chi'_{\rm YM}(N=3) \approx (24\ \mathrm{MeV})^2$ [Pheno]

[Tarasov, Venugopalan (2025) — arXiv:2501.10519]

- $\chi'_{\rm YM}(N=3) = [(16.4 \pm 1.9)\ \mathrm{MeV}]^2$ [Lattice]

[CB JHEP **01** (2024) 116]

4. Comparison of lattice result with large-N argument

Large-N estimate from the argument previously presented:

$$\frac{\chi'_{\mathrm{YM}}}{N} = \frac{F_\pi^2}{N} + \frac{\chi'_{\mathrm{QCD}}}{N} \approx [(12.0 \pm 1.5)\ \mathrm{MeV}]^2$$

$\frac{1}{\sqrt{N}} F_\pi \to$ TEK [García Pérez et al. JHEP **04** (2021) 230]

$\frac{1}{N}\chi'_{\mathrm{QCD}} \approx \frac{1}{3}\chi'_{\mathrm{ChPT}}$

Good agreement with lattice result $\displaystyle \lim_{N\to\infty} \frac{\chi'_{\mathrm{YM}}}{N} = [(8.6 \pm 1.1)\ \mathrm{MeV}]^2$

5. Consistency condition at large-N

$$\chi' m_{\eta'}^2 \ll \chi$$

- $N m_{\eta'}^2 = \mathcal{O}(N^0)$ estimated from $N = 3$ experimental value
- $\chi = \mathcal{O}(N^0)$ and $\chi'/N = \mathcal{O}(N^0)$ from my lattice results

$$\implies \qquad \left(\frac{\chi'}{N}\right)(N m_{\eta'}^2)\,\frac{1}{\chi} \approx 0.185$$

Future outlooks

1. Publish these results

2. Lattice calculation of χ' in full QCD? (hard)

3. Intermediate step towards full QCD: combining Wilson flow and multi-level to improve signal-to-noise ratio of χ' (especially at smaller R_{s})

[García Vera, Schaefer PRD **93** (2016) 074502]

4. Extract large-N limit of the mass of lightest 0^{-+} glueball from topological charge density correlator

Back-up

Reflection positivity: $\langle \Theta[\mathcal{O}(\Theta x)]\mathcal{O}(x)\rangle > 0$ for any operator $\mathcal{O}$
Θ = Euclidean time reflection + complex conjugation

- $q(x)$ **parity odd** $\implies C(x) < 0$ for $|x| = r > 0$.

- $C(x) \underset{r \ll 1}{\sim} -A^2/(r^8 \log^2 r)$ (Vicari (1998) hep-lat/9901008)

- $C(x) \underset{r \gg 1}{\sim} -B^2 \exp\{-mr\}$ (Chowdhury et al. (2015) 1409.6459 – Fukaya et al. (2015) 1509.00944)

But $\int \mathrm{d}^4x\, C(x) = \chi = \frac{1}{V}\langle Q^2\rangle \geq 0$. How to reconcile with reflection positivity?

$C(x)$ has positive singular **contact term** in $x = 0$ with peculiar features:
- positive divergent part to cancel negative singularity at short distances
- positive finite part to overcome negative finite contribution to χ at large r

$\chi' = \frac{1}{8}\int \mathrm{d}^4x\, C(x)|x|^2$. The $|x|^2$ promotes long-distances and depresses short ones:

1. χ' can be either positive or negative

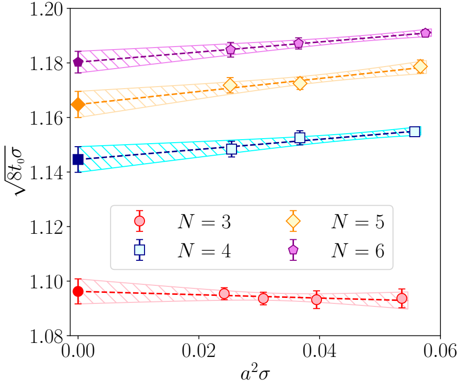
2. χ' does not need to vanish in chiral limit (unlike χ)

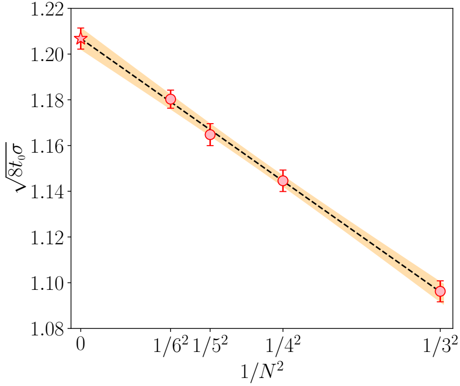
3. If $\chi'_{\mathrm{YM}} > 0$, plausible that $\chi'_{\mathrm{QCD}} < 0$ due to suppression of χ for $m \to 0$

$\sqrt{8t_0}$ [Cè, García Vera, Giusti, Schaefer PLB **762** (2016) 232-236]

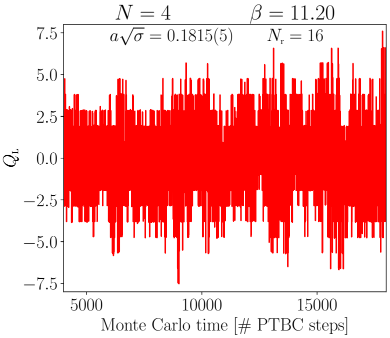
$\sqrt{\sigma}$ [Athenodorou, Teper JHEP **12** (2021) 082]

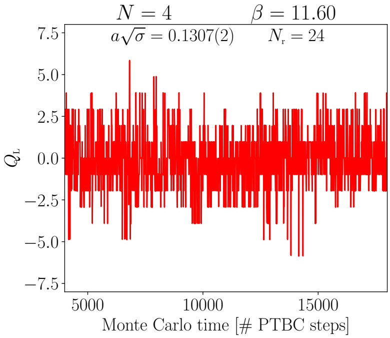
From these data we build $\sqrt{8t_0\sigma}$. We perform continuum limits at fixed $N = 3, 4, 5, 6$ and extrapolate the results towards $N \to \infty$.

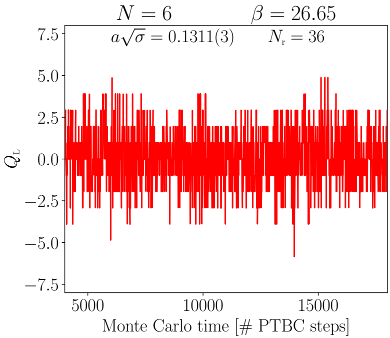




$$\sqrt{8t_0\sigma}(N) = 1.2068(46) - \frac{0.997(69)}{N^2} \qquad (N \geq 3)$$







- $\ell_d \to$ defect length
- $N_r \to$ number of replicas
- $\ell_d\sqrt{\sigma} \approx 0.5 \to \ell_d \approx 0.2$ fm
- $N_r = 12 - 36$ to get $\sim 20\%$ swap acc
- $N_r(a) \underset{a \to 0}{\sim} (\ell_d/a)^{1.5}$ (fixed N and ℓ_d)
- $N_r(N) \underset{N \to \infty}{\sim} N$ (fixed a and ℓ_d)