Partially quenched chiral perturbation theory with QED

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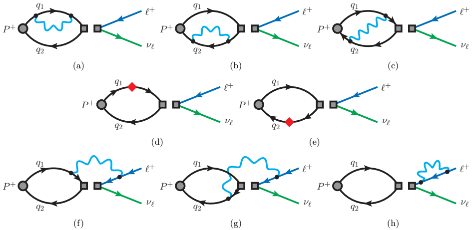
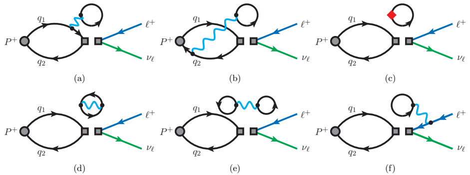
- We already heard quite a bit about isospin-breaking effects

$$\alpha \sim \frac{m_u - m_d}{\Lambda_{\mathrm{QCD}}} \sim 1\%$$

- Lattice QCD+QED: valence and sea quarks
- Consistent treatment: need of isospin-breaking effects with all quarks
- Many collaborations working to include this RBC/UKQCD (Ryan Hill)
- Today: Partially quenched chiral perturbation theory



Connected



Disconnected

Figs. from [RBC/UKQCD 23]

- Chiral symmetry: $SU(N_f)_L \times SU(N_f)_R$ for N_f light quarks
- Build Lagrangian $\mathcal{L}_\chi$ in terms of $N_f^2 - 1$ mesons and external fields
- Power counting: p^{2n}

$$\mathcal{L}_\chi = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots$$

- Quark masses in $\chi_i = 2\, B_0 m_i$ and mesons in

$$u = \exp\left\{\frac{i\Phi}{\sqrt{2}F_0}\right\} \qquad \Phi:\ N_f \times N_f$$

$$\Phi = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2\,\eta}{\sqrt{3}} \end{pmatrix}$$

- The leading-order Lagrangian:

$$\mathcal{L}_2 = \frac{F_0^2}{4} \langle u^\mu u_\mu + \chi_+ \rangle$$

- At a general order:

$$\mathcal{L}_{2n} = \sum_{i=1}^{N_{2n}} \mathcal{O}_i^{(2n)} c_i^{(2n)}$$

[Wess, Zumino 71; Witten 83; Gasser, Leutwyler 83/84; Bijnens, Colangelo, Ecker 99; Bijnens, Girlanda, Talavera 02; Bijnens, NHT, Wang 18; Bijnens, NHT, Ruiz-Vidal 23]

- Low-energy constants: Must be known!

	N_f	$N_f = 3$	$N_f = 2$
p^2	2	2	2
p^4	13	12	10
p^6	115 (+24)	94 (+23)	56 (+13)
p^8	1862 (+999)	1254 (+705)	475 (+211)

- Can add photons, leptons, resonances, ...

  [Ecker, Gasser, Pich, de Rafael et al. 89; Urech 95; Knecht et al. 00, ...]

- With dynamical photons: Combined expansion $e^{2m} \times p^{2n}$

- $Q = \mathrm{diag}(q_1, \ldots, q_{N_f})$ and A_μ

- Assuming only interested in e^2: [Urech 95; Bijnens, Danielsson 2007]

	N_f	$N_f = 3$
$e^0 p^2 + e^2 p^0$	2+1	2+1
$e^0 p^4 + e^2 p^2$	13+16	12+14
$e^0 p^6 + e^2 p^4$	115+274	94+205
$e^0 p^8 + e^2 p^6$	1862+7059	1254+4133

- NB: $e^2 p^4$ and $e^2 p^6$ preliminary numbers from [Bijnens, Ruiz-Vidal In prep.]

- Low-energy constants poorly known here [Bijnens, Ecker 14]

- Look at photon Lagrangians [Urech 95; Neufeld, Rupertsberger 95; Bijnens, Danielsson 07]

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 + \frac{F_0^2}{4} \langle u^\mu u_\mu + \chi_+ \rangle + e^2 C \langle \mathcal{Q}_L \mathcal{Q}_R \rangle$$

$$\mathcal{L}_4 \overset{N_f}{=} \sum_{i=0}^{12} L_i X_i + e^2 F_0^2 \left(\sum_{i=1}^{14} K_i^E Q_i^s + K_{18}^E Q_{18}^s + K_{19}^E Q_{19}^s \right)$$

- From these Lagrangians one finds e.g. mass shifts and decay constants

$$M_{\pi^\pm}^2 \overset{p^2+e^2p^0}{=} B_0 (m_u + m_d) + \frac{2e^2 C}{F_0^2}$$

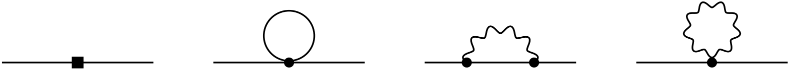
$$M_{K^\pm}^2 \overset{p^2+e^2p^0}{=} B_0 (m_u + m_s) + \frac{2e^2 C}{F_0^2}$$

- At order $p^4 + e^2 p^2$ with $m_u = m_d$ for N_f flavours [Hansen, NHT, Ungar MSc 25]

$$
\begin{aligned}
M^2_{\pi^\pm} \overset{p^4+e^2p^2}{=} & \; M^2_{\pi^\pm, p^4} - e^2 \frac{M^2_{\pi,2}}{16\pi^2}\left[3\ln\frac{M^2_{\pi,2}}{\mu^2} - 4\right] - e^2 \frac{1}{8\pi^2}\frac{C}{F_0^4}\left[2\,M^2_{\pi,2}\ln\frac{M^2_{\pi,2}}{\mu^2} + M^2_{K,2}\ln\frac{M^2_{K,2}}{\mu^2}\right] \\
& - 16 e^2 \frac{C}{F_0^4}\left[(M^2_{\pi,2} + 2\,M^2_{K,2})L^r_4 + M^2_{\pi,2}L^r_5\right] + 8e^2\,M^2_{K,2}K^r_8 - \frac{4}{9}\,e^2 M^2_{\pi,2}\Big[6\,K^r_1 + 6\,K^r_2 \\
& + 5\,K^r_5 + 5\,K^r_6 - 6\,K^r_7 - 15\,K^r_8 - 5\,K^r_9 - 23\,K^4_{10} - 18\,K^r_{11}\Big]
\end{aligned}
$$

- $p^6 + e^2 p^4$: isospin-breaking mass now known at 2 loops [Bijnens, Yu 24]

- ChPT: Predict finite-volume effects in QCD+QED at order $p^{2n} \times e^2$
- QED in finite volume needs prescription
- $\mathrm{QED_x}$: Exclude/redistribute photon zero-momentum mode [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
  - $\mathrm{QED_L}$: Exclude photon zero-mode [Hayakawa, Uno 2008]
  - $\mathrm{QED_r}$: Redistribute photon zero-mode [Di Carlo, Hansen, NHT, Portelli 2025]
- Photon loop $\implies \frac{1}{M_\pi L} + \frac{1}{(M_\pi L)^2} + \dots$
- Meson loops $\implies e^{-M_\pi L}$



- ChPT through order $p^4 \times e^2 p^2$ with $m_u = m_d$ [Hansen, NHT, Ungar MSc25]

$$\Delta^{\mathrm{X}} M_\pi^2(L) = \frac{M_{\pi,2}^2}{2\,F_0^2}\left[\Delta I_1(M_{\pi,2}^2, L) - \frac{1}{3}\,\Delta I_1(M_{\eta,2}^2, L)\right] + e^2\frac{C}{F_0^4}\left[-4\,\Delta I_1(M_{\pi,2}^2, L) - 2\,\Delta I_1(M_{K,2}^2, L)\right] - e^2\left[3\,\Delta^{\mathrm{X}} I_1^0(L) + 2\,\Delta^{\mathrm{X}} J_{11}(M_{\pi,2}^2, L) + 4 M_{\pi,2}^2\,\Delta^{\mathrm{X}} I_{11}(M_{\pi,2}^2, L)\right]$$

- Master integrals known in $\mathrm{QED_L}$ [Hayakawa, Uno 08]
- QCD master integral $\Delta I_1(M^2, L)$: exponential
- Master integrals with QED: Power law + exponentials
- All-order result in volume: Integrals extended to $\mathrm{QED_r}$
- NB: No powerlaw structure dependence at $p^4 \times e^2 p^2$

- Now I talked a lot about regular ChPT
- Partially quenched ChPT: Treat valence and sea quarks differently

  [Bernard, Golterman 94; Sharpe et al. 97...; Cohen, Kaplan, Nelson 99; Bijnens et al. 04...; Tiburzi, Walker-Loud 04; ...]
- n_{val} valence quarks, n_{sea} sea quarks
- Again $u = \exp\{\frac{i\Phi}{\sqrt{2}F_0}\}$, but now Φ is $(2n_{\mathrm{val}} + n_{\mathrm{sea}}) \times (2n_{\mathrm{val}} + n_{\mathrm{sea}})$
- Here we choose $n_{\mathrm{val}} = n_{\mathrm{sea}} = 3 \implies 9 \times 9$
- Formulated in terms of quark fields

$$\Phi = (\Phi_{ij})_{9\times 9} = (\bar{q}_i q_j)_{9\times 9}$$

- Can use N_f-flavour ChPT Lagrangians with supertrace

$$\mathcal{L}_2^{\mathrm{ChPT}} = \frac{F_0^2}{4}\,\langle u^\mu u_\mu + \chi_+\rangle \longrightarrow \frac{F_0^2}{4}\,\mathrm{Str}\left[u^\mu u_\mu + \chi_+\right] = \mathcal{L}_2^{\mathrm{PQChPT}}$$

$$\mathrm{Str}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \langle A\rangle - \langle D\rangle$$

- However, additional LECs in the N_f theory

- Relation to $N_f = 3$ LECs known [Bijnens, Danielsson, Lähde 06; Bijnens, Danielsson 07]

	N_f	$N_f = 3$
$e^0p^2 + e^2p^0$	2+1	2+1
$e^0p^4 + e^2p^2$	13+16	12+14
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- Meson $\Phi_{ij} = \bar{q}_i q_j$
- Indices: Valence $(k = 1, 2, 3)$, Sea $(k = 4, 5, 6)$
- Charges q_k and masses $\chi_k = 2B_0 m_k$
- Gives mass to given loop order [Bijnens, Danielsson 07]

$$(M^2_{\mathrm{phys}})_{ij} = (M^2_0)_{ij} + \frac{\delta^{(4)}_{ij}}{F_0^2} + \mathcal{O}(p^6, p^4\, e^2)\,.$$

- Also done for factorisable decay constant [Bijnens, Danielsson 07]

- Leading-order mass:

$$(M_0^2)_{ij} = \frac{1}{2}\,(\chi_i + \chi_j) + \frac{2e^2\,C}{F_0^4}\,(q_i - q_j)^2$$

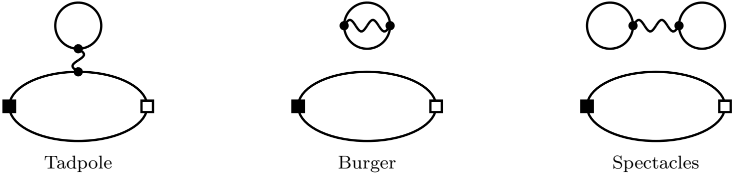
- $\delta_{ij}^{(4)}$ more complicated: LECs and sea contributions

$$\begin{aligned}
\delta^{(4)23} = {} & [48L_6^r - 24L_4^r]\,\bar{\chi}_1\chi_{13} + [16L_8^r - 8L_5^r]\,\chi_{13}^2 \\
& - 48e^2 C\,F_0^{-2} L_4^r q_{13}^2 \bar{\chi}_1 - 16e^2 C\,F_0^{-2} L_5^r q_{13}^2 \chi_{13} \\
& - e^2 F_0^2 \left[12K_1^{Er} + 12K_2^{Er} - 12K_7^{Er} - 12K_8^{Er}\right] \overline{Q}_2 \chi_{13} \\
& - e^2 F_0^2 \left[4K_5^{Er} + 4K_6^{Er}\right] q_p^2 \chi_{13} + e^2 F_0^2 \left[4K_9^{Er} + 4K_{10}^{Er}\right] q_p^2 \chi_p \\
& + 12e^2 F_0^2 K_8^{Er} q_{13}^2 \bar{\chi}_1 + 8e^2 F_0^2 \left[K_{10}^{Er} + K_{11}^{Er}\right] q_{13}^2 \chi_{13} \\
& - e^2 F_0^2 \left[8K_{18}^{Er} + 4K_{19}^{Er}\right] q_1 q_3 \chi_{13} \\
& - 1/3\bar{A}(\chi_m) R_{n13}^m \chi_{13} - 1/3\bar{A}(\chi_p) R_{q\pi\eta}^p \chi_{13} \\
& + e^2 F_0^2 \bar{A}(\chi_{13}) q_{13}^2 + 2e^2 C\,F_0^{-2} \bar{A}(\chi_{1s}) q_{1s} q_{13} \\
& - 2e^2 C\,F_0^{-2} \bar{A}(\chi_{3s}) q_{3s} q_{13} + 4e^2 F_0^2 \bar{B}(\chi_\gamma, \chi_{13}, \chi_{13}) q_{13}^2 \chi_{13} \\
& - 4e^2 F_0^2 \bar{B}_1(\chi_\gamma, \chi_{13}, \chi_{13}) q_{13}^2 \chi_{13}.
\end{aligned}$$

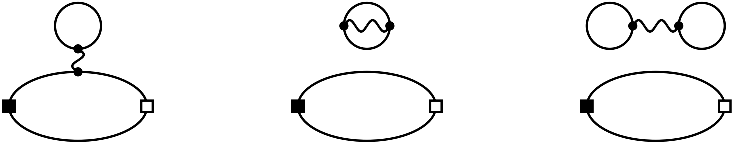
- Can separate quark-contributions to the mass as

$$(M^2_{\mathrm{phys}})_{ij} = (M^2_{\mathrm{val}})_{ij} + (M^2_{\mathrm{tad}})_{ij} + (M^2_{\mathrm{specs}})_{ij} + (M^2_{\mathrm{burger}})_{ij} + \mathrm{QCD\ only}$$

- Similar construction holds for (factorisable) decay constant



$$(M^2_{\mathrm{tad}})_{ij} = \sum_{s=4}^{6} q_s \left[q_i \frac{\partial^2 (M^2_{\mathrm{phys}})_{ij}}{\partial q_i \partial q_s} + q_j \frac{\partial^2 (M^2_{\mathrm{phys}})_{ij}}{\partial q_j \partial q_s} \right]$$



Tadpole Burger Spectacles

$$(M^2_{\rm tad})_{\pi^+} = 0\,, \qquad \longleftarrow \text{Suppressed}$$

$$(M^2_{\rm specs})_{\pi^+} + (M^2_{\rm burger})_{\pi^+} = -\frac{\chi_1}{3}\, e^2\, 12\, Y_1 \sum_{s=4}^{6} q_s^2\,,$$

$$(M^2_{\rm tad})_{K^+} = (q_1 - q_3)\,\frac{e^2\,C}{8\pi^2\,F_0^4}\sum_{s=4}^{6} q_s\left[\chi_{1s}\log\frac{\chi_{1s}}{\mu^2} - \chi_{3s}\log\frac{\chi_{3s}}{\mu^2}\right],$$

$$(M^2_{\rm specs})_{K^+} + (M^2_{\rm burger})_{K^+} = -\frac{\chi_{13}}{3}\, e^2\, 12\, Y_1 \sum_{s=4}^{6} q_s^2\,.$$

- Order e^2p^2 LEC only in disonnected: $Y_1 = K_1^{E,r} + K_2^{E,r} - K_7^{E,r} - K_8^{E,r}$
- Infinite-volume: $\sim 1\%$ shift on isospin breaking also seen in [Portelli et al. 12]

- Factorisable decay constant, infinite volume

$$(F_{\rm tad})_{\pi^+} = F_0 \, \frac{C\, e^2}{16\pi^2\, F_0^4} \sum_{s=4}^{6} \left\{ q_1\, q_s\, \left[1 + \log\frac{\chi_{1s}}{\mu^2}\right] + q_2\, q_s\, \left[1 + \log\frac{\chi_{2s}}{\mu^2}\right]\right\} \,,$$

$$(F_{\rm specs})_{\pi^+} + (F_{\rm burger})_{\pi^+} = -F_0\, \frac{C\, e^2}{16\pi^2\, F_0^4} \sum_{s=4}^{6} q_s^2 \left\{2 + \log\frac{\chi_{1s}}{\mu^2} + \log\frac{\chi_{2s}}{\mu^2}\right\} + \frac{1}{3}\, e^2\, 6\, Y_6 \sum_{s=4}^{6} q_s^2 \,,$$

$$(F_{\rm tad})_{K^+} = F_0 \, \frac{C\, e^2}{16\pi^2\, F_0^4} \sum_{s=4}^{6} \left\{ q_1\, q_s\, \left[1 + \log\frac{\chi_{1s}}{\mu^2}\right] + q_3\, q_s\, \left[1 + \log\frac{\chi_{3s}}{\mu^2}\right]\right\} \,,$$

$$(F_{\rm specs})_{K^+} + (F_{\rm burger})_{K^+} = -F_0\, \frac{C\, e^2}{16\pi^2\, F_0^4} \sum_{s=4}^{6} q_s^2 \left\{2 + \log\frac{\chi_{1s}}{\mu^2} + \log\frac{\chi_{3s}}{\mu^2}\right\} + \frac{1}{3}\, e^2\, 6\, Y_6 \sum_{s=4}^{6} q_s^2 \,.$$

- LEC is $Y_6 = K_1^{E,r} + K_2^{E,r}$
- No chiral suppression for pion tadpole

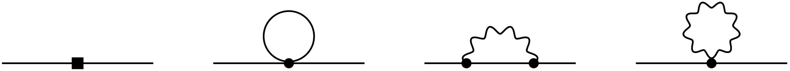
- Subtracting unquenched FVEs from purely connected data?
- Finite-volume mass studied in $\mathrm{QED_L}$ [Hayakawa, Uno 08]
- Split as $\Delta^{\mathrm{X}} M_{ij}^2(L) = \Delta^{\mathrm{X}} M_{\mathrm{QCD},ij}^2(L) + \Delta^{\mathrm{X}} M_{\mathrm{QED},ij}^2(L)$
- $\mathrm{QED_r}$ result is

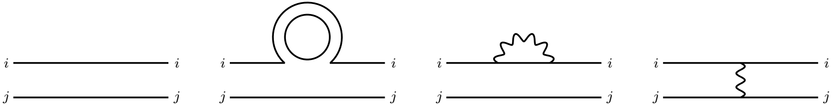
$$\Delta M_{\mathrm{QCD},ij}^2(L) = \frac{\chi_i + \chi_j}{2 n_{\mathrm{sea}} F_0^2} \left\{ {R_{ij}}^i \, \Delta I_1(\chi_i, L) + {R_{ij}}^j \, \Delta I_1(\chi_j, L) + \sum_x^{sm} {R_{ij}}^x \, \Delta I_1(\chi_x, L) \right\}$$

$$\Delta^{\mathrm{r}} M_{\mathrm{QED},ij}^2(L) = -\frac{2e^2 \, C}{F_0^4} (q_i - q_j) \sum_{s=4}^{6} \left\{ (q_i - q_s) \, \Delta I_1(\chi_{is}, L) + (q_s - q_j) \, \Delta I_1(\chi_{sj}, L) \right\}$$
$$+ (q_i - q_j)^2 \, e^2 \left[3 \, \Delta^{\mathrm{r}} I_0^1(L) + 2 \, \Delta^{\mathrm{r}} J_{11}(\chi_{ij}, L) + 4 \, \chi_{ij} \, \Delta^{\mathrm{r}} I_{11}(\chi_{ij}, L) \right]$$

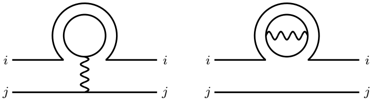
- $p^4 \times e^2 p^2$: No powerlaw correction for sea quarks!
- No powerlaw effects with structure
- Same is true for the (factorisable) decay constant

- Why is this the case? EM current flavour conserving
- At 1-loop order not enough flavours going around

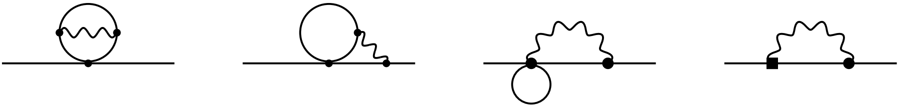




- Need to go to higher order, e.g.



$$\int_{k,\ell} \frac{q_s\, q_1}{[k^2+i\varepsilon]\,[\ell^2-\chi_{1s}+i\varepsilon]^2\,[(\ell+k)^2-\chi_{1s}+i\varepsilon]}$$



Conclusions and outlook

- Partially quenched ChPT can be used to study qualitative behaviour
- Of interest to see suppression in disconnected diagrams
- At leading order ($p^4 \times e^2 p^2$): Suppression
  - in disconnected infinite-volume predictions
  - in powerlaw FVEs
- Would be interesting to see how FVEs generalise to higher orders
- Some low-energy constants only appear in disconnected pieces

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Backup slides

- Extend integrals of [Hayakawa, Uno 08] to $\mathrm{QED_r}$: Fully closed form

$$
\Delta^{\mathrm{r}} I_1^0(L) = \Delta^{\mathrm{L}} I_1^0(L) + \frac{1}{4\pi L^2} ,
$$

$$
\Delta^{\mathrm{r}} J_{11}(m^2, L) = \Delta^{\mathrm{L}} J_{11}(m^2, L) + \frac{1}{2} \left\{ -\frac{1}{4\pi L^2} + \frac{1}{2L^3} \frac{1}{\sqrt{\left(\frac{2\pi}{L}\right)^2 + m^2}} \right\} ,
$$

$$
\Delta^{\mathrm{r}} I_{11}(m^2, L) = \Delta^{\mathrm{L}} I_{11}(m^2, L) - \frac{1}{4m} \left\{ \frac{1}{4\pi^2 L} + \frac{1}{L^3} \frac{1}{\sqrt{\left(\frac{2\pi}{L}\right)^2 + m^2} \left[m - \sqrt{\left(\frac{2\pi}{L}\right)^2 + m^2} \right]} \right\} .
$$

- Building blocks in Lagrangian

$$\begin{aligned} u_\mu &= i\Big[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\Big]\,, \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u\chi^\dagger u\,, \\ f_\pm^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu} u\,. \end{aligned}$$

- External fields in ℓ_μ, r_μ and $F^{\mu\nu}_{L,R}$, $Q = \mathrm{diag}(q_1, \ldots, q_{N_f})$
- External photon: $r_\mu = \ell_\mu = e\,Q\,A_\mu$
- Quark masses in $\chi_i = 2\,B_0 m_i$ and mesons in

$$u = \exp\{\frac{i\Phi}{\sqrt{2}F_0}\} \qquad \Phi : \; N_f \times N_f$$

$$\Phi = \begin{pmatrix} \pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{\eta}{\sqrt{3}} & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\bar{K}^0 & -\frac{2\,\eta}{\sqrt{3}} \end{pmatrix}$$

- Now I talked a lot about regular ChPT
- Partially quenched ChPT: Treat valence and sea quarks differently

  [Bernard, Golterman 94; Sharpe et al. 97...; Cohen, Kaplan, Nelson 99; Bijnens et al. 04...; Tiburzi, Walker-Loud 04; ...]

- New symmetry group: Graded see e.g. [Sharpe, Shoresh 01]

$$SU(n_{\mathrm{val}} + n_{\mathrm{sea}}|n_{\mathrm{val}})_L \times SU(n_{\mathrm{val}} + n_{\mathrm{sea}}|n_{\mathrm{val}})_R$$

- Again $u = \exp\{\frac{i\Phi}{\sqrt{2}F_0}\}$, but now Φ is $(2n_{\mathrm{val}} + n_{\mathrm{sea}}) \times (2n_{\mathrm{val}} + n_{\mathrm{sea}})$
- Formulated in terms of quark fields

$$\Phi = \left(\begin{array}{cc|c} (q_V\bar{q}_V)_{n_{\mathrm{val}}\times n_{\mathrm{val}}} & (q_V\bar{q}_S)_{n_{\mathrm{val}}\times n_{\mathrm{sea}}} & (q_V\bar{q}_B)_{n_{\mathrm{val}}\times n_{\mathrm{val}}} \\ (q_S\bar{q}_V)_{n_{\mathrm{sea}}\times n_{\mathrm{val}}} & (q_S\bar{q}_S)_{n_{\mathrm{sea}}\times n_{\mathrm{sea}}} & (q_S\bar{q}_B)_{n_{\mathrm{sea}}\times n_{\mathrm{val}}} \\ \hline (q_B\bar{q}_V)_{n_{\mathrm{val}}\times n_{\mathrm{val}}} & (q_B\bar{q}_S)_{n_{\mathrm{val}}\times n_{\mathrm{sea}}} & (q_B\bar{q}_B)_{n_{\mathrm{val}}\times n_{\mathrm{val}}} \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

- Here we choose $n_{\mathrm{val}} = n_{\mathrm{sea}} = 3 \implies 9 \times 9$

$$
\begin{aligned}
\mathcal{L}_{p^4} = {} & \hat{L}_0 \big\langle u^\mu u^\nu u_\mu u_\nu \big\rangle + \hat{L}_1 \big\langle u^\mu u_\mu \big\rangle^2 + \hat{L}_2 \big\langle u^\mu u^\nu \big\rangle \big\langle u_\mu u_\nu \big\rangle + \hat{L}_3 \big\langle (u^\mu u_\mu)^2 \big\rangle \\
& + \hat{L}_4 \big\langle u^\mu u_\mu \big\rangle \big\langle \chi_+ \big\rangle + \hat{L}_5 \big\langle u^\mu u_\mu \chi_+ \big\rangle + \hat{L}_6 \big\langle \chi_+ \big\rangle^2 + \hat{L}_7 \big\langle \chi_- \big\rangle^2 + \frac{\hat{L}_8}{2} \big\langle \chi_+^2 + \chi_-^2 \big\rangle \\
& - i \hat{L}_9 \big\langle f_+^{\mu\nu} u_\mu u_\nu \big\rangle + \frac{\hat{L}_{10}}{2} \big\langle f_+^2 - f_-^2 \big\rangle + \hat{H}_1 \big\langle F_L^2 + F_R^2 \big\rangle + \hat{H}_2 \big\langle \chi \chi^\dagger \big\rangle.
\end{aligned}
$$

$$
\begin{aligned}
Q_1^s &= \frac{1}{2}\langle \mathcal{Q}_L^2 + \mathcal{Q}_R^2\rangle\langle u_\mu u^\mu\rangle,\\
Q_2^s &= \langle \mathcal{Q}_L \mathcal{Q}_R\rangle\langle u_\mu u^\mu\rangle,\\
Q_3^s &= -\langle \mathcal{Q}_L u_\mu\rangle\langle \mathcal{Q}_L u^\mu\rangle - \langle \mathcal{Q}_R u_\mu\rangle\langle \mathcal{Q}_R u^\mu\rangle,\\
Q_4^s &= \langle \mathcal{Q}_L u_\mu\rangle\langle \mathcal{Q}_R u^\mu\rangle,\\
Q_5^s &= \langle (\mathcal{Q}_L^2 + \mathcal{Q}_R^2) u_\mu u^\mu\rangle,\\
Q_6^s &= \langle (\mathcal{Q}_L \mathcal{Q}_R + \mathcal{Q}_R \mathcal{Q}_L) u_\mu u^\mu\rangle,\\
Q_7^s &= \frac{1}{2}\langle \mathcal{Q}_L^2 + \mathcal{Q}_R^2\rangle\langle \chi_+\rangle,\\
Q_8^s &= \langle \mathcal{Q}_L \mathcal{Q}_R\rangle\langle \chi_+\rangle,\\
Q_9^s &= \langle (\mathcal{Q}_L^2 + \mathcal{Q}_R^2)\chi_+\rangle,\\
Q_{10}^s &= \langle (\mathcal{Q}_L \mathcal{Q}_R + \mathcal{Q}_R \mathcal{Q}_L)\chi_+\rangle,\\
Q_{11}^s &= \langle (\mathcal{Q}_R \mathcal{Q}_L + \mathcal{Q}_L \mathcal{Q}_R)\chi_-\rangle,\\
Q_{12}^s &= i\langle \left[\hat{\nabla}_\mu \mathcal{Q}_R, \mathcal{Q}_R\right] u^\mu - \left[\hat{\nabla}_\mu \mathcal{Q}_L, \mathcal{Q}_L\right] u^\mu\rangle,\\
Q_{13}^s &= \langle \hat{\nabla}_\mu \mathcal{Q}_L \hat{\nabla}^\mu \mathcal{Q}_R\rangle,\\
Q_{14}^s &= \langle \hat{\nabla}_\mu \mathcal{Q}_L \hat{\nabla}^\mu \mathcal{Q}_L + \hat{\nabla}_\mu \mathcal{Q}_R \hat{\nabla}^\mu \mathcal{Q}_R\rangle,\\
Q_{18}^s &= \langle \mathcal{Q}_L u_\mu \mathcal{Q}_L u^\mu + \mathcal{Q}_R u_\mu \mathcal{Q}_R u^\mu\rangle,\\
Q_{19}^s &= \langle \mathcal{Q}_L u_\mu \mathcal{Q}_R u^\mu\rangle.
\end{aligned}
$$

$$
\begin{aligned}
K_1 &= K_1^E + K_{18}^E, \\
K_2 &= K_2^E + \frac{1}{2} K_{19}^E, \\
K_3 &= K_1^E - K_{18}^E, \\
K_4 &= K_4^E + K_{19}^E, \\
K_5 &= K_5^E - 2K_{18}^E, \\
K_6 &= K_6^E - K_{19}^E, \\
K_i &= K_i^E \quad \text{for} \quad i = 7, \dots, 14.
\end{aligned}
$$

$$
\begin{aligned}
L_1^r &= \hat{L}_1^r + \frac{1}{2} \hat{L}_0^r, \\
L_2^r &= \hat{L}_2^r + \hat{L}_0^r, \\
L_3^r &= \hat{L}_3^r - 2\hat{L}_0, \\
L_i^r &= \hat{L}_i^r \quad \text{for} \quad i = 4, \dots, 12.
\end{aligned}
$$

$$\mathcal{L}_2^{2\phi} = \frac{1}{2}\langle(\partial_\mu\phi)^2\rangle - \frac{1}{4}[\langle\phi^2\chi\rangle + \langle\phi^2\chi^\dagger\rangle] + 2e^2\frac{C}{F_0^2}[\langle\phi Q\phi Q\rangle - \langle\phi^2 Q^2\rangle],$$

$$\mathcal{L}^{2\phi,1A_\mu} = ieA_\mu[\langle\phi Q(\partial_\mu\phi)\rangle - \langle\phi(\partial_\mu\phi)Q\rangle],$$

$$\mathcal{L}^{2\phi,2A_\mu} = e^2A^2[\langle\phi^2Q^2\rangle - \langle\phi Q\phi Q\rangle],$$

$$\mathcal{L}_2^{4\phi} = \frac{1}{6F_0^2}[\langle\phi(\partial_\mu\phi)\phi(\partial_\mu\phi)\rangle - \langle\phi^2(\partial_\mu\phi)^2\rangle] + \frac{1}{24F_0^2}[\langle\phi^4\chi\rangle + \langle\phi^4\chi^\dagger\rangle]$$
$$+ e^2\frac{C}{F_0^4}[\frac{1}{3}\langle\phi^4Q^2\rangle - \frac{4}{3}\langle\phi^3Q\phi Q\rangle + \langle\phi^2Q\phi^2Q\rangle],$$

$$\mathcal{L}_2^{4\phi,1A_\mu} = iA_\mu\frac{e}{F_0^2}[-\frac{1}{2}\langle\phi^2(\partial^\mu\phi)\phi Q\rangle + \frac{1}{6}\langle\phi^3(\partial^\mu\phi)Q\rangle - \frac{1}{6}\langle\phi^3Q(\partial^\mu\phi)\rangle + \frac{1}{2}\langle\phi^2Q\phi(\partial^\mu)\rangle],$$

$$\mathcal{L}_2^{4\phi,2A_\mu} = e^2A^2\frac{1}{F_0^2}[-\frac{1}{6}\langle\phi^4Q^2\rangle + \frac{2}{3}\langle\phi^3Q\phi Q\rangle - \frac{1}{2}\langle\phi^2Q\phi^2Q\rangle],$$