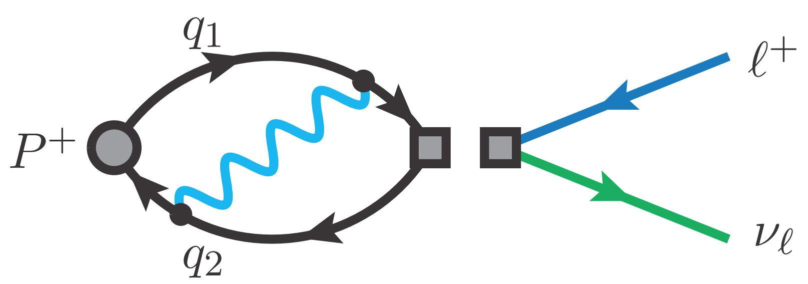
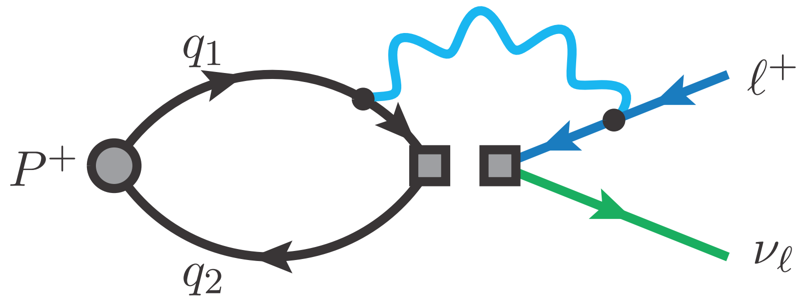
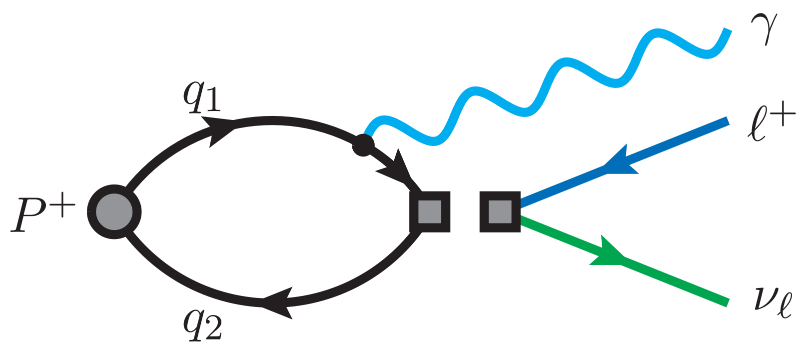
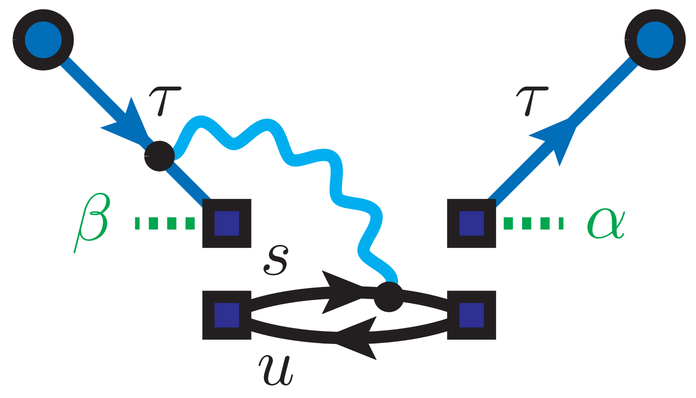
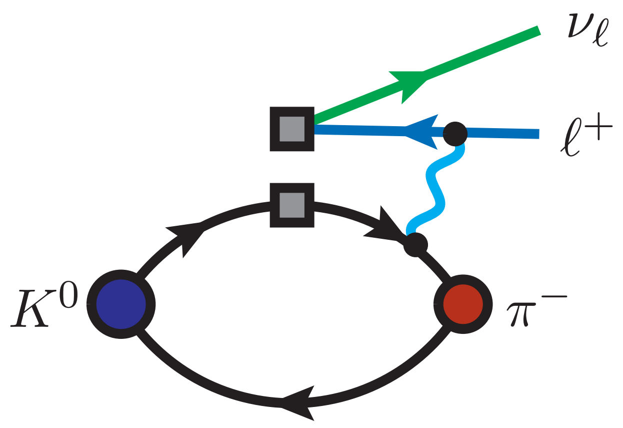










Isospin-breaking corrections to weak decays

Current status and future prospects

Matteo Di Carlo

3rd April 2025

CERN

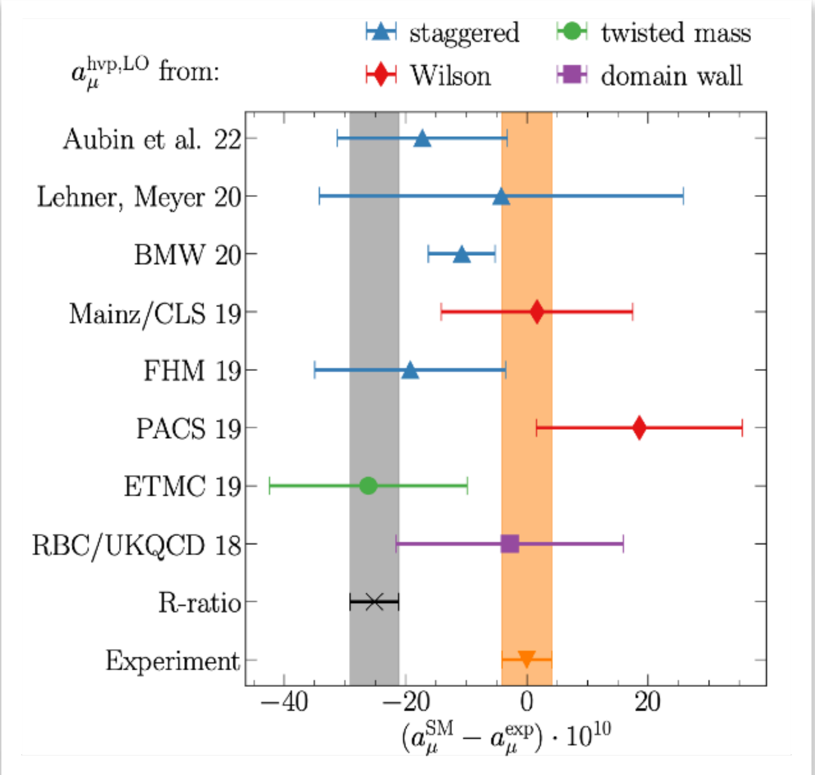
Funded by the European Union

Second LatticeNET workshop on challenges in Lattice field theory

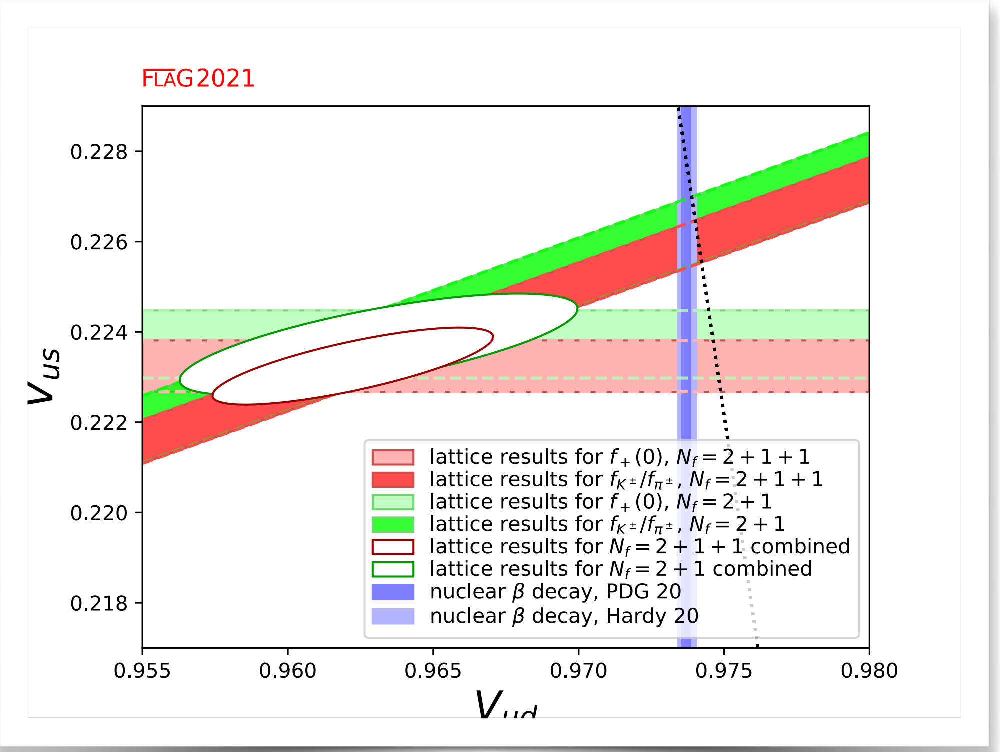
CENTRO DE CIENCIAS DE BENASQUE PEDRO PASCUAL

Motivations

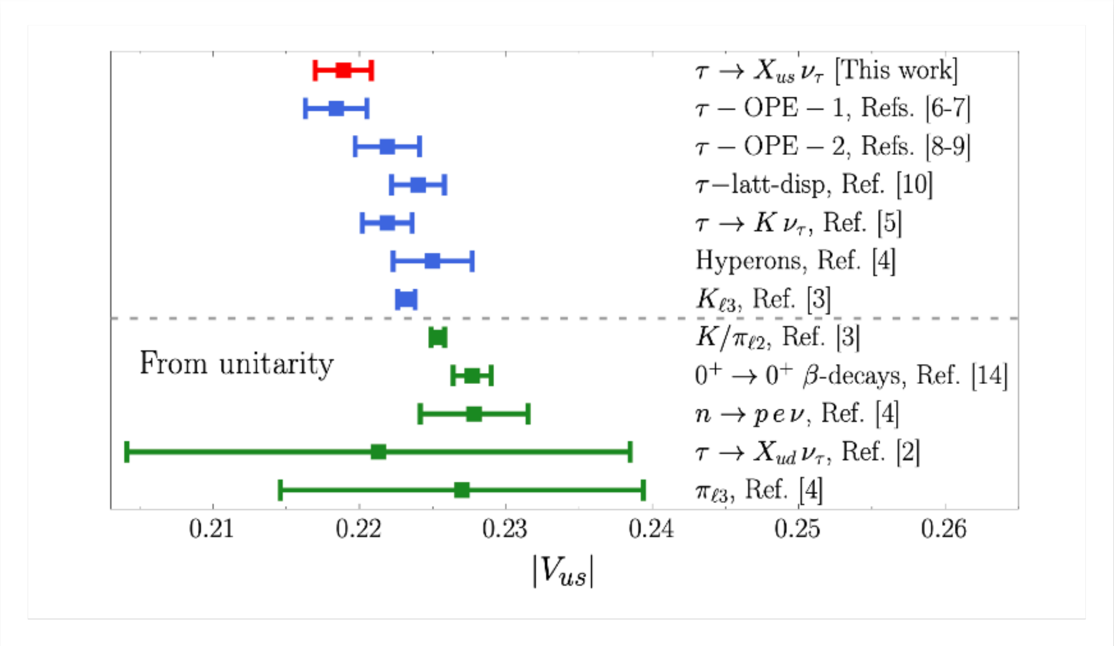
Indirect searches of new physics › high precision › isospin-breaking corrections



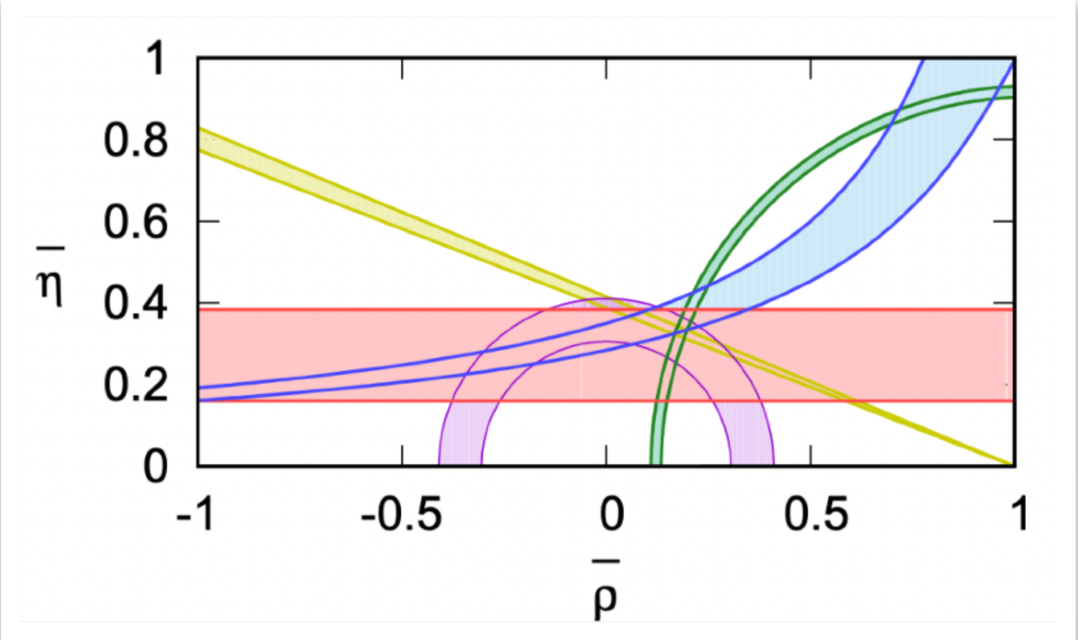
HVP muon g-2



(semi)leptonic decays



Inclusive τ decays



CP violation parameters

Flavour physics

Flavour physics offers opportunities to test the Standard Model and probe new physics effects

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in the Standard Model:

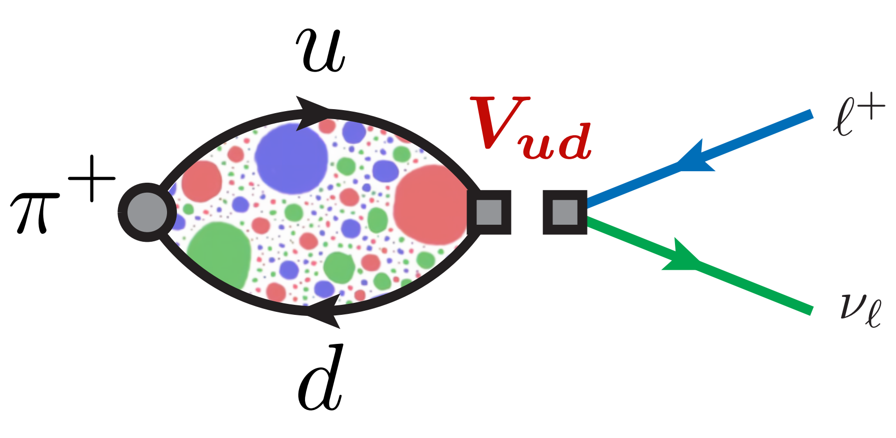
3 mixing angles + 1 CPV phase

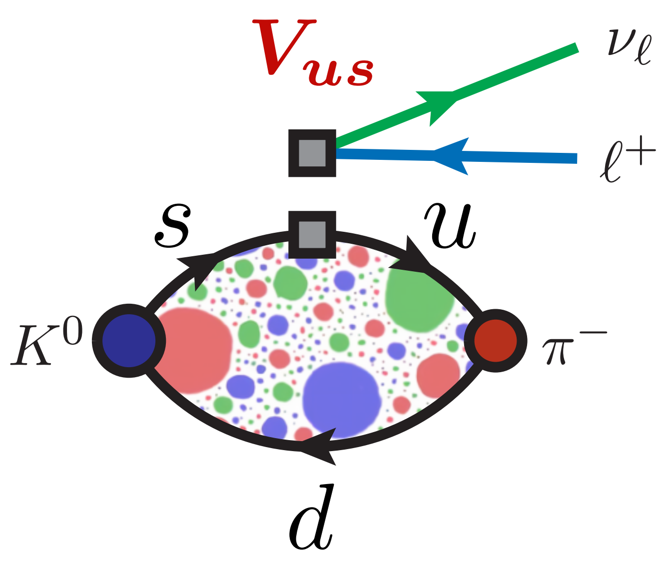
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Matrix elements can be extracted e.g. from leptonic and semileptonic decays of hadrons

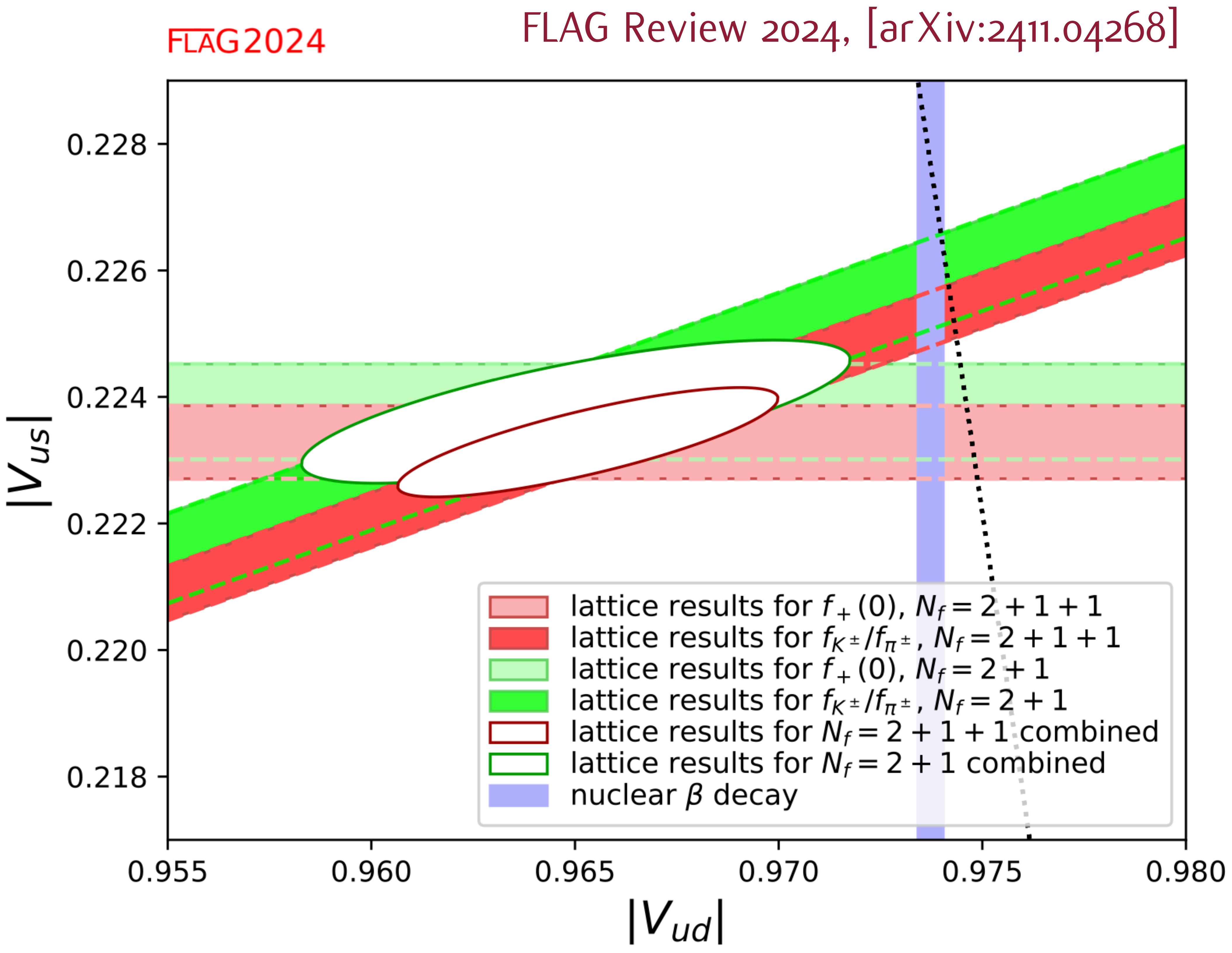
$$\underbrace{\frac{\Gamma\left[K \to \ell\nu_\ell(\gamma)\right]}{\Gamma\left[\pi \to \ell\nu_\ell(\gamma)\right]}}_{\text{experiments}} \propto \left|\frac{V_{us}}{V_{ud}}\right|^2 \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

$$\underbrace{\Gamma\left[K \to \pi\ell\nu_\ell(\gamma)\right]}_{\text{experiments}} \propto |V_{us}|^2 \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$





The Cabibbo anomaly



$$\frac{|\boldsymbol{V_{us}}|}{|\boldsymbol{V_{ud}}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(41)$$

$$|\boldsymbol{V_{us}}|\,|f_+^{K^0\pi^-}(0)| = 0.21654(41)$$

M.Moulson, PoS CKM2016 (2017)
PDG, PTET **2022** (2022)

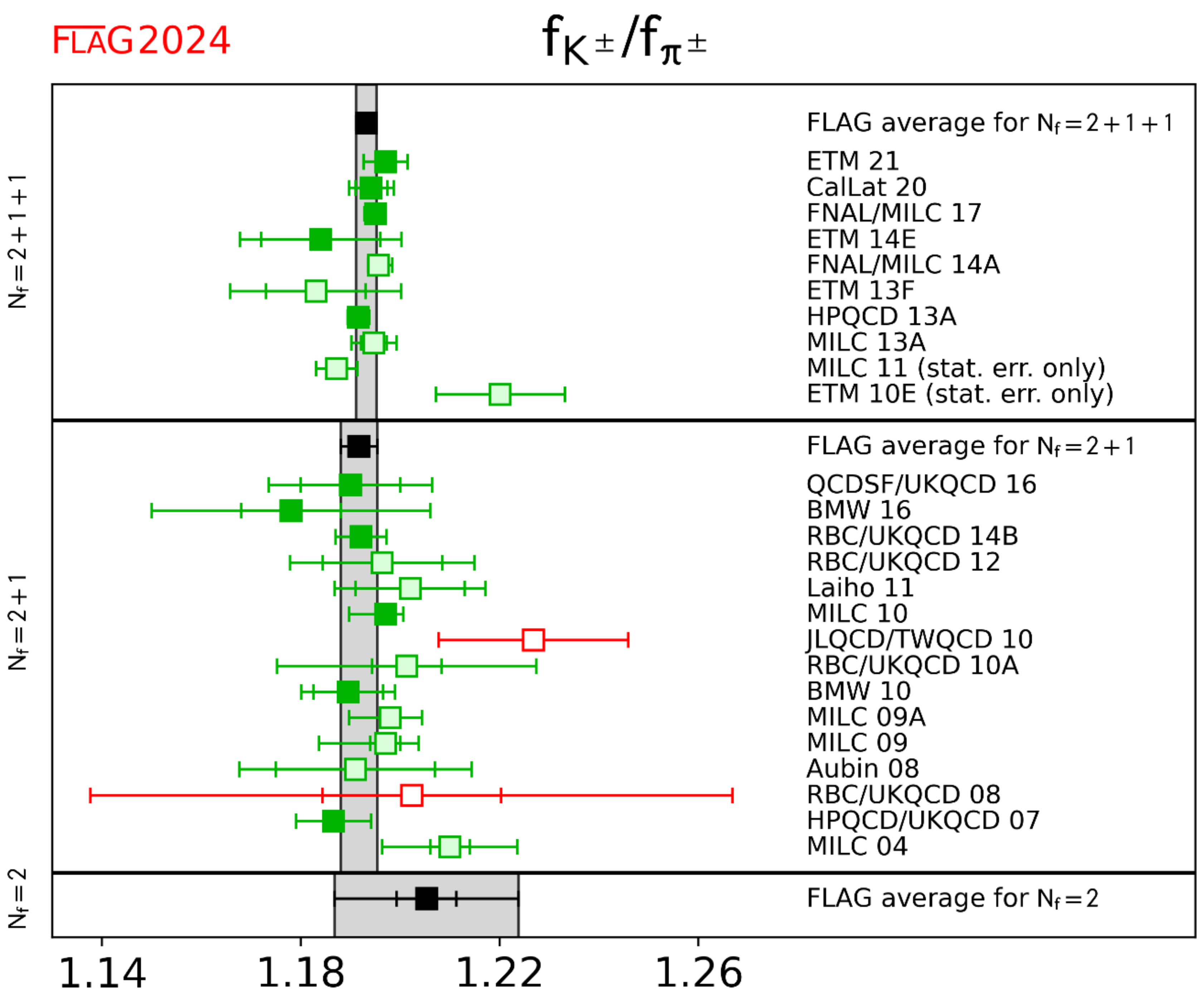
Different **tensions** in the V_{us}-V_{ud} plane:

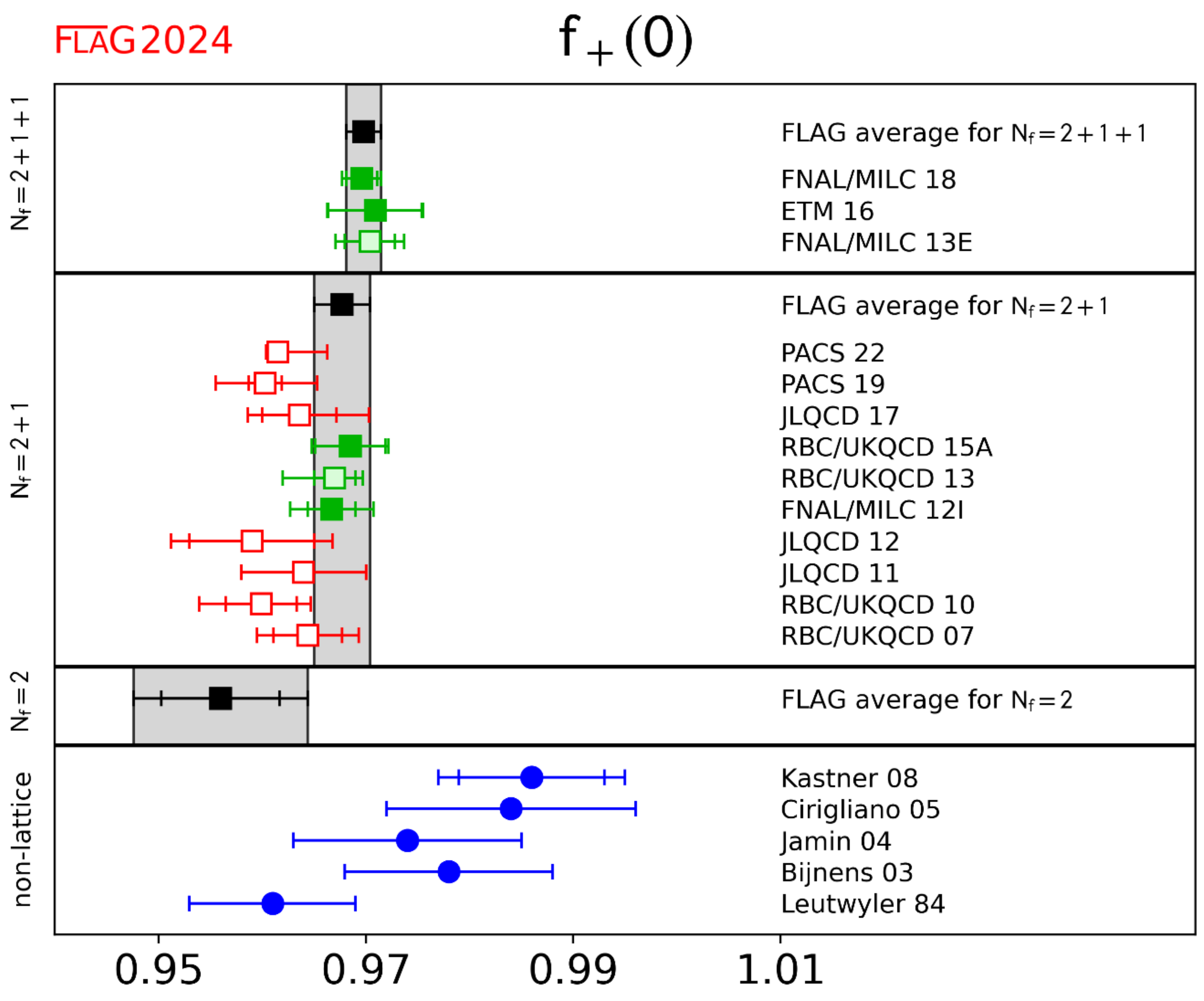
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and **theoretical** control of these quantities is of crucial importance to solve the issue

Lattice QCD inputs





$$f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19) \qquad f_+^{K\pi}(0) = 0.9698\,(17)$$

FLAG 2024 Flavour Lattice Averaging Group

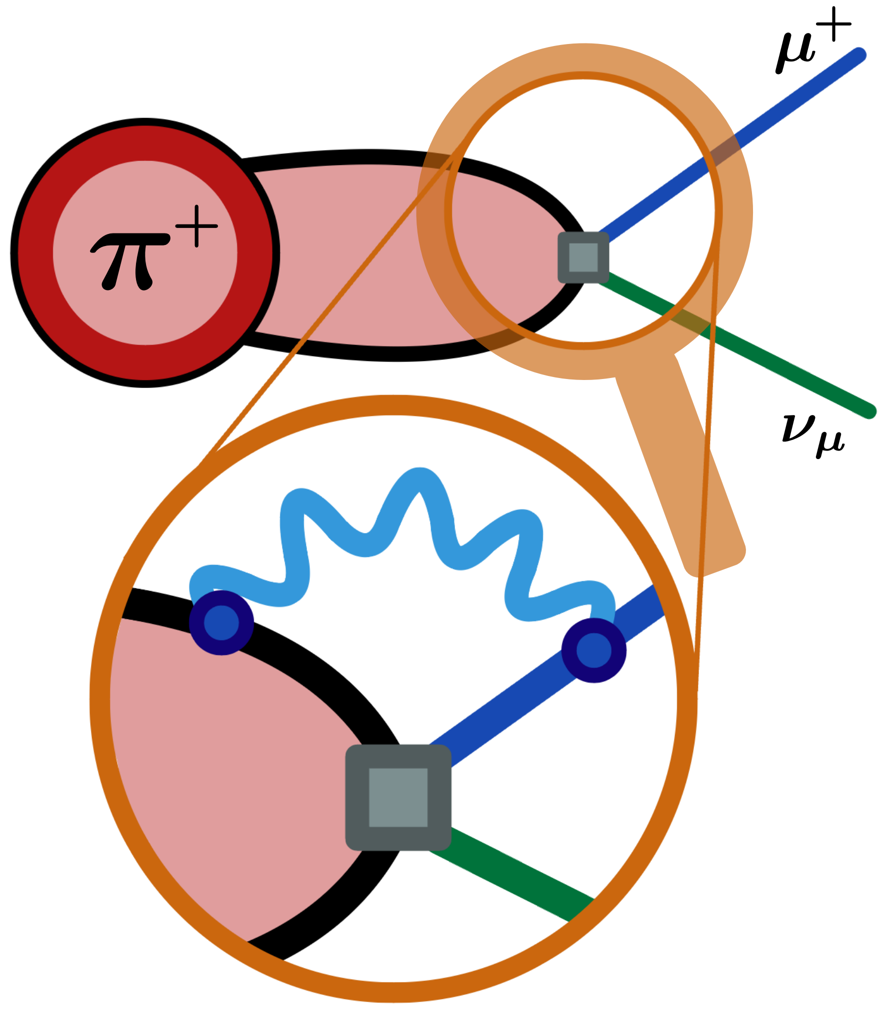
f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with **sub percent precision!**

QED and isospin-breaking effects

Current level of precision requires the inclusion of isospin-breaking corrections due to

- strong effects $[m_u - m_d]_{\mathrm{QCD}} \neq 0$
- electromagnetic effects $\alpha \neq 0$

$\sim \mathcal{O}(1\%)$



$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|\boldsymbol{V_{us}}|^2}{|\boldsymbol{V_{ud}}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |\boldsymbol{V_{us}}|^2 \, |f_+^{K\pi}(0)|^2 \, \mathcal{I}_{K\pi}^\ell \, (1 + \delta R_{K\pi}^\ell)$$

- results currently quoted in the PDG come from χPT — V.Cirigliano & H.Neufeld, PLB **700** (2011)
- fully **non-perturbative** (structure dependent) quantities
- first-principle **lattice calculations** are possible!

Lattice QCD + QED

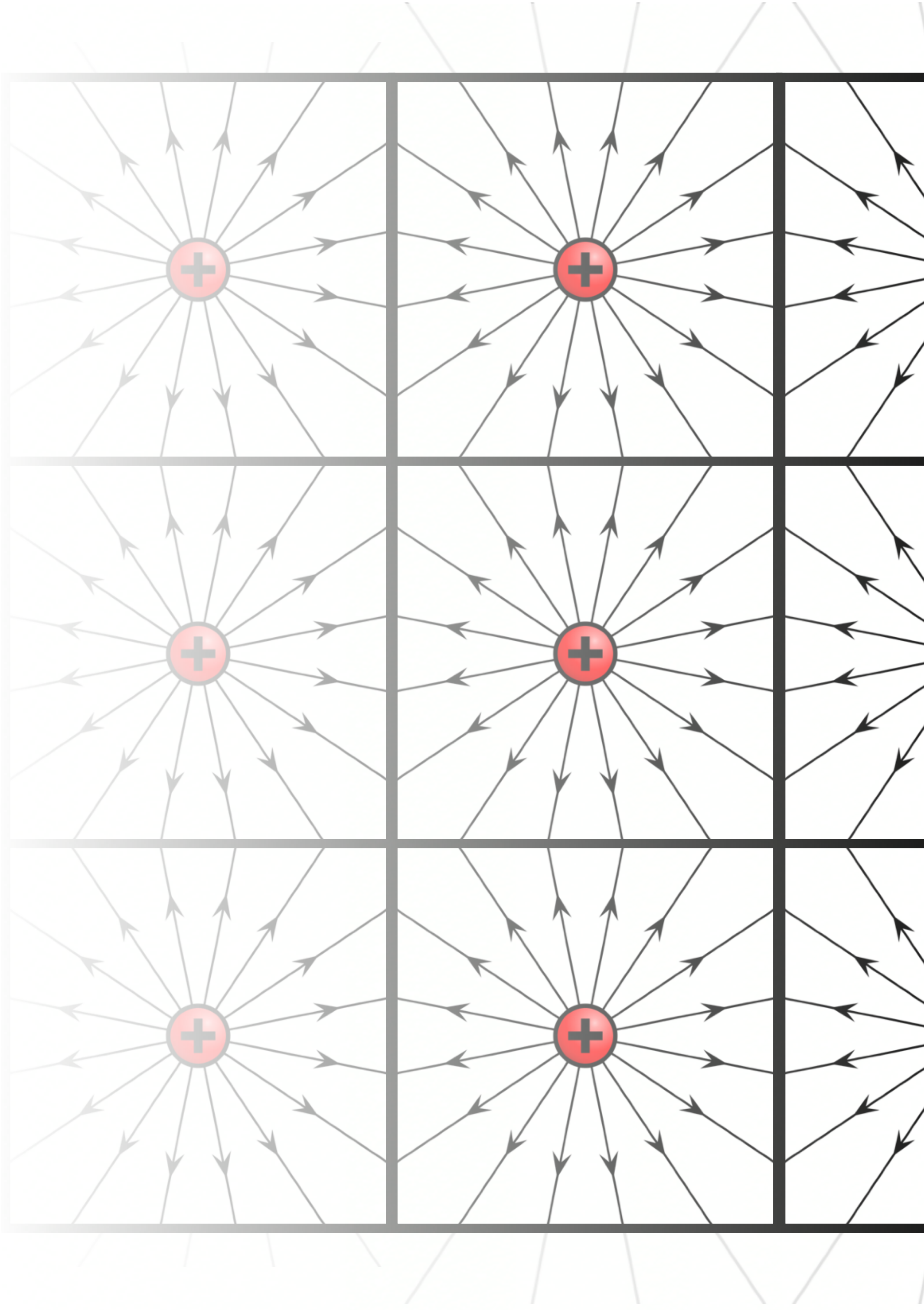
Computing **QED corrections on a** **finite-sized** **lattice** is challenging:

- **long-range interactions** don't like **finite volumes** with **periodic boundary conditions**
- **finite-volume effects** can be sizeable and power-like

  M.Hayakawa & S.Uno, PTP **120** (2008) / Z.Davoudi & M.Savage, PRD **90** (2014) / S.Borsanyi et al., Science **347** (2015)

- logarithmic **infrared divergences** arise in virtual/real decay rates

  V.Lubicz et al., PRD **95** (2017)



There are also recent proposals to compute radiative corrections as **convolutions** of hadronic correlators with **infinite-volume** QED kernels

N.Asmussen et al., [1609.08454] / T.Blum et al., PRD **96** (2017) / X.Feng & L.Jin, PRD **100** (2019) / N.Christ et al., [2304.08026] / J.Parrino et al., [2501.03192]

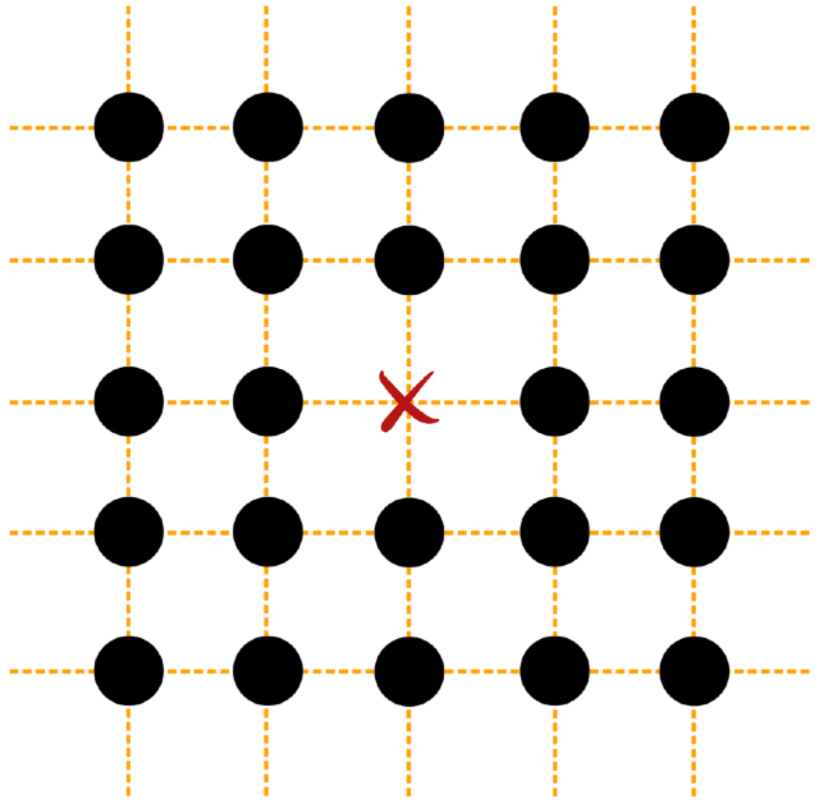
Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Possible solutions currently employed:

$\mathrm{QED_L}$

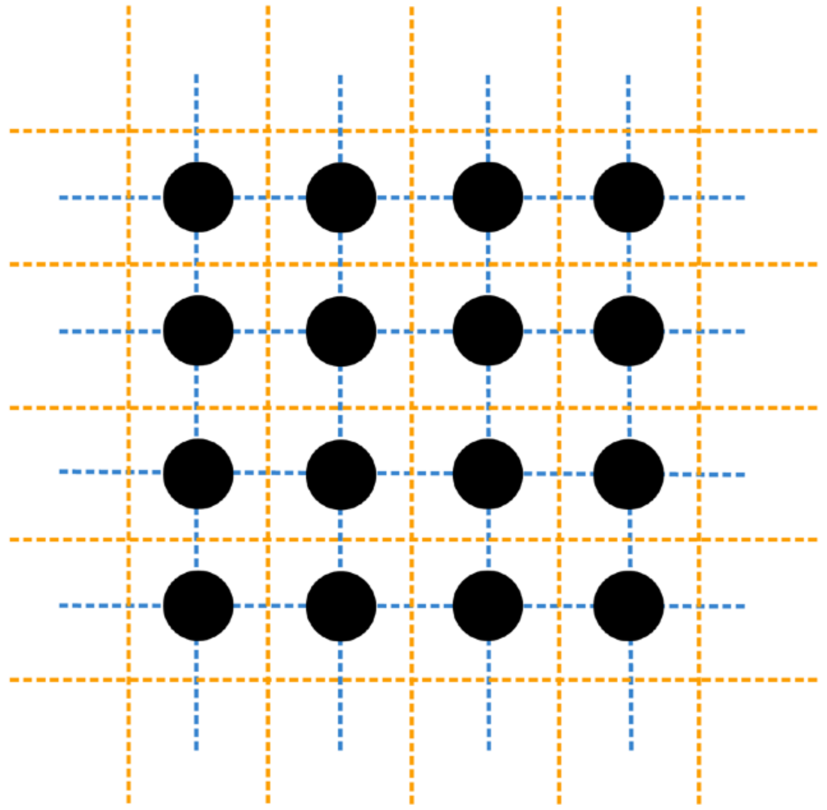


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

$\mathrm{QED_{C^*}}$

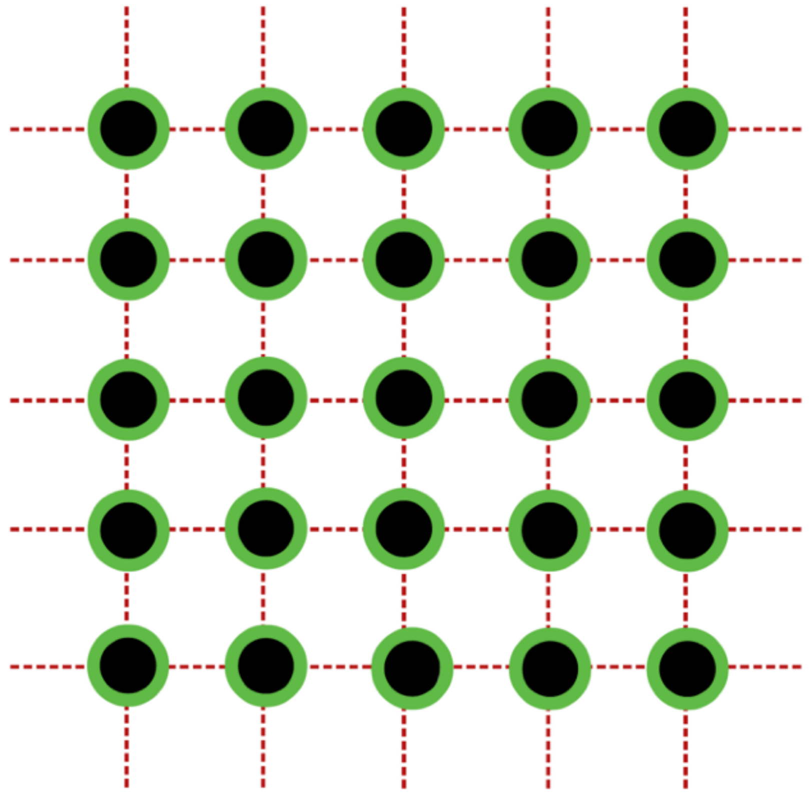


$\Omega_3 = 2\pi\mathbb{Z}^3/L \qquad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathrm{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB **357** (1991)
B.Lucini et al., JHEP **02** (2016)

$\mathrm{QED_m}$

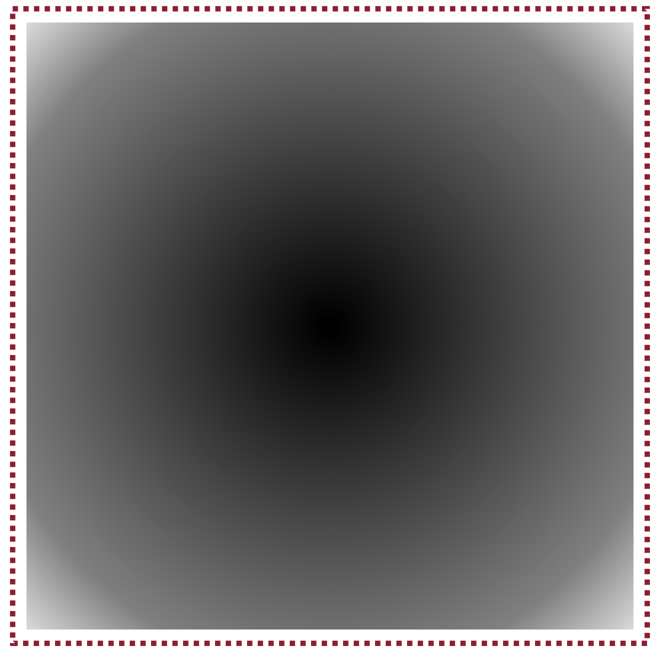


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

$\mathrm{QED_\infty}$



$\Omega_4 = \mathbb{R}^4$

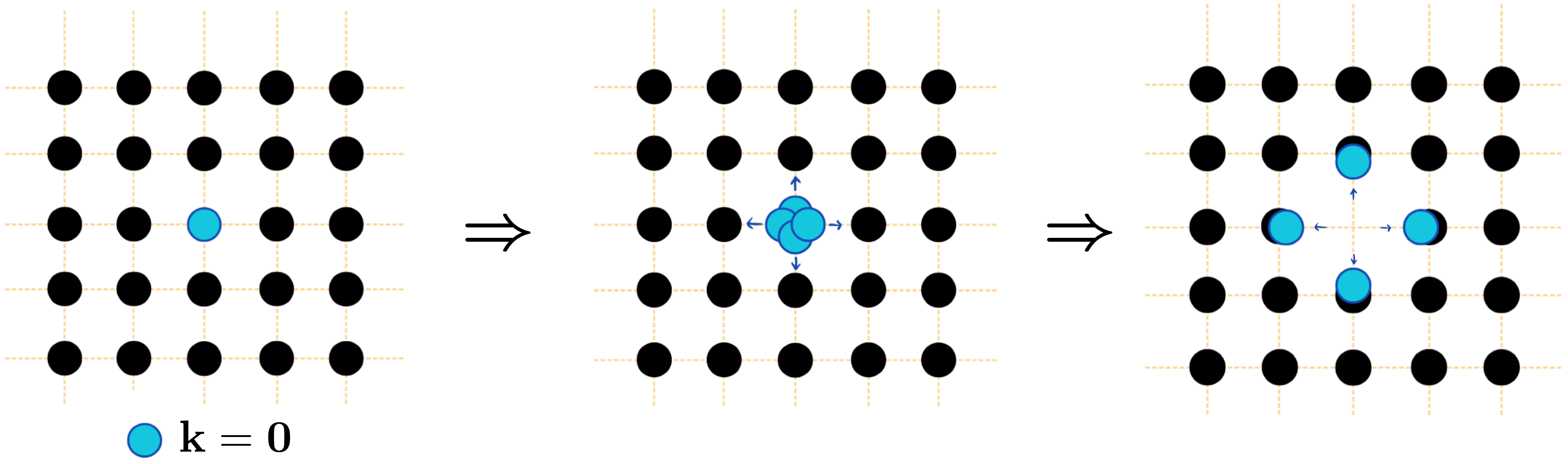
infinite-volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

QEDr regularization

Special case of "IR-improvement"

Z.Davoudi et al., PRD **99** (2019)
MDC, PoS LATTICE2023 (2024) [2401.07666]
MDC et al., [2501.07936]

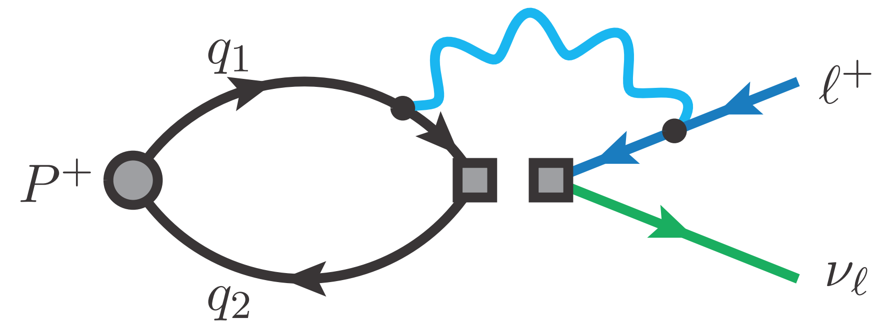


The spatial zero mode is not removed but **redistributed over the neighbouring modes** on a shell of radius $|\mathbf{p}| = \frac{2\pi}{L}|\mathbf{r}| \quad (\mathbf{r}\in\mathbb{Z}^3)$

QEDL: $D_L^{\mu\nu}(k_0,\mathbf{k}) = \delta^{\mu\nu}\frac{1-\delta_{\mathbf{k},\mathbf{0}}}{k_0^2+\mathbf{k}^2}$ $\Rightarrow$ **QEDr:** $D_{\mathbf{p}}^{\mu\nu}(k_0,\mathbf{k}) = \delta^{\mu\nu}\frac{1-\delta_{\mathbf{k},\mathbf{0}}}{k_0^2+\mathbf{k}^2} + \frac{\delta_{\mathbf{k}^2,\mathbf{p}^2}}{n(\mathbf{p}^2)}\frac{\delta^{\mu\nu}}{k_0^2+\mathbf{p}^2}$

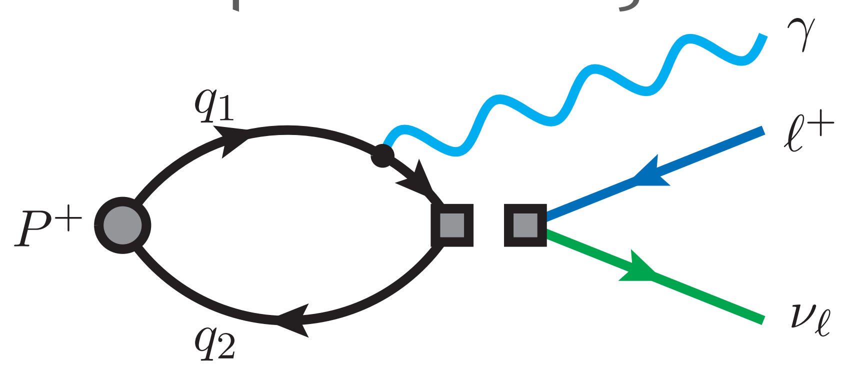
Weak decays — some recent works

leptonic decays



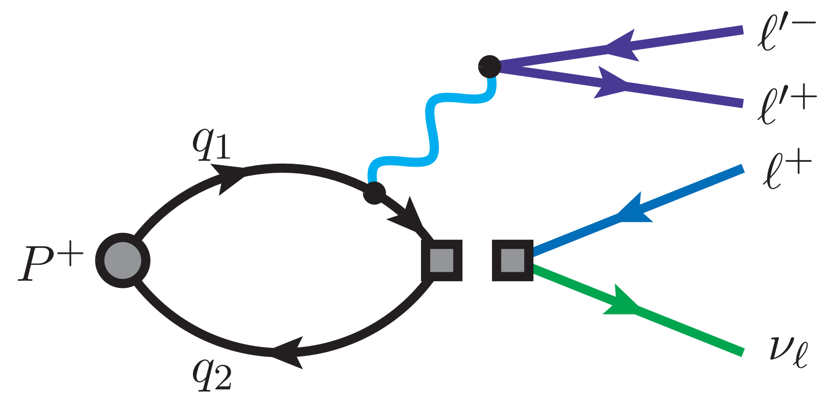
N.Carrasco et al., PRD 91 (2015)
V.Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199v2]
D.Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)
MDC et al., PRD 105 (2022)
P.Boyle, MDC et al., JHEP 02 (2023)
N.Christ et al., [2304.08026]

real photon emission
leptonic decays



G.M.de Divitiis et al., [1908.10160]
C.Kane et al., [1907.00279 & 2110.13196]
R.Frezzotti et al., PRD 103 (2021)
A.Desiderio et al., PRD 102 (2021)
D.Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

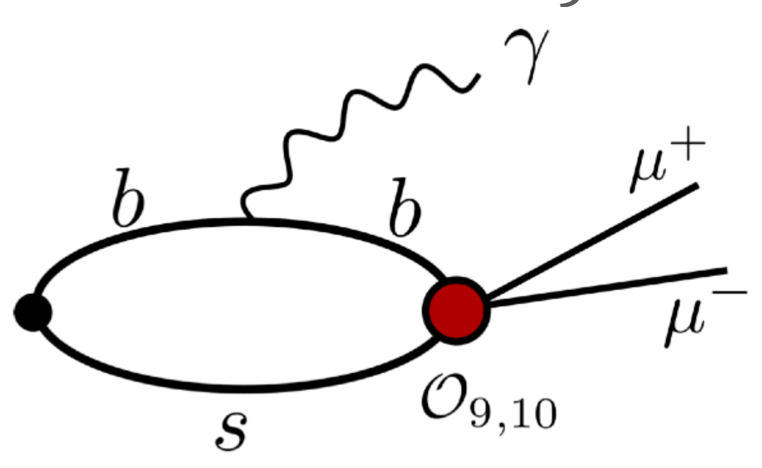
virtual photon emission
leptonic decays



G.Gagliardi et al., Phys. Rev. D 105 (2022)
R.Frezzotti et al., [2306.07228]

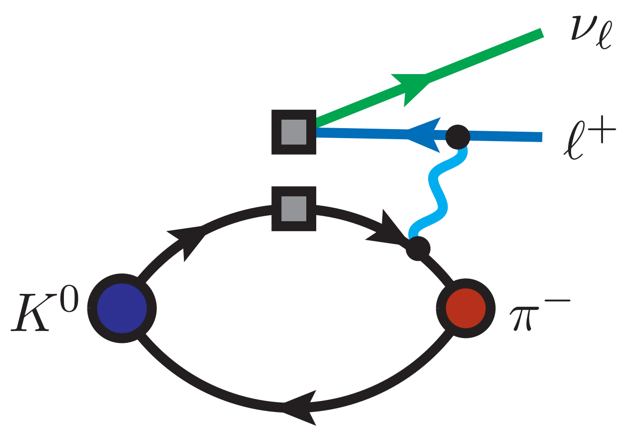
R.Frezzotti et al., [2402.03262]

rare FCNC decays



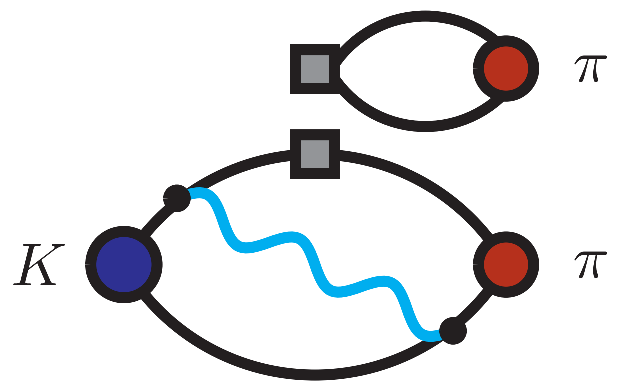
C.Sachrajda et al., [1910.07342]
N.Christ et al., PRD 108 (2023)
N.Christ et al., [2402.08915]

semileptonic decays

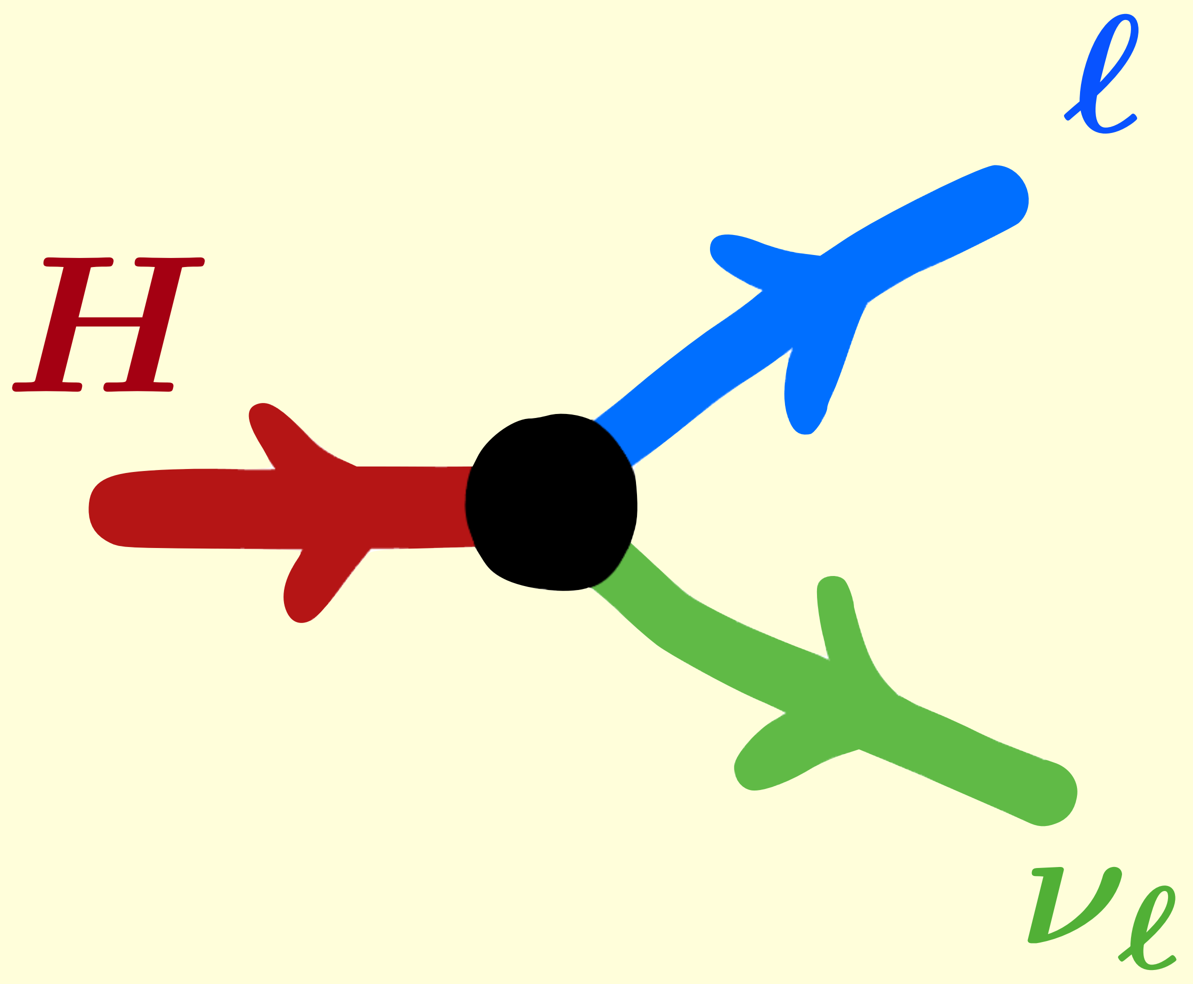


R.Abbott et al., PRD 102 (2020)
Z.Bai et al., PRL 115 (2015)
N.Christ et al., PRD 106 (2022)
N.Christ & X.Feng, EPJ Web Conf. 175 (2018)
Y.Cai & Z.Davoudi, [1812.11015]

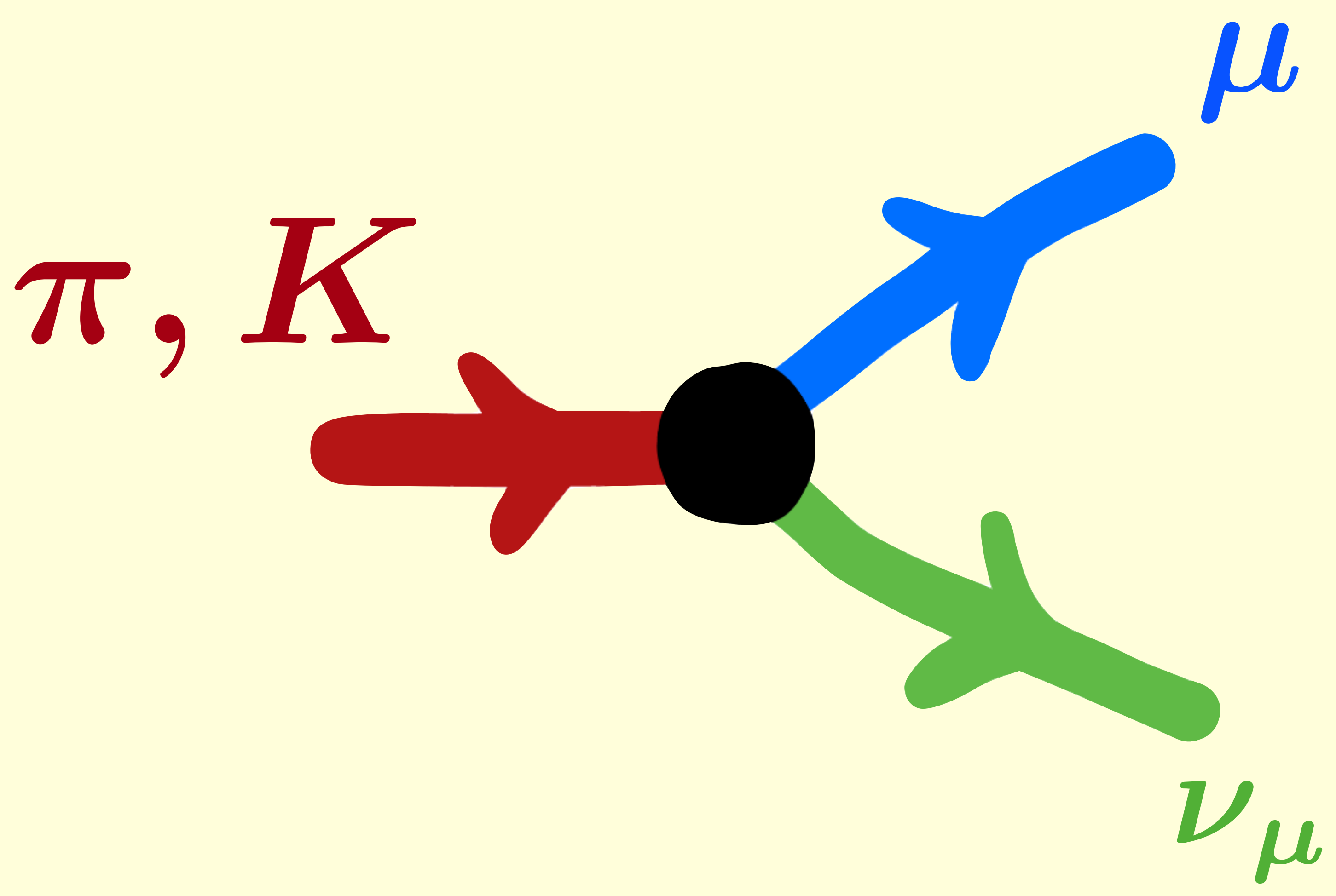
hadronic decays



Leptonic decays of hadrons



Leptonic decays of hadrons



Leptonic decays of pseudoscalar mesons

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by

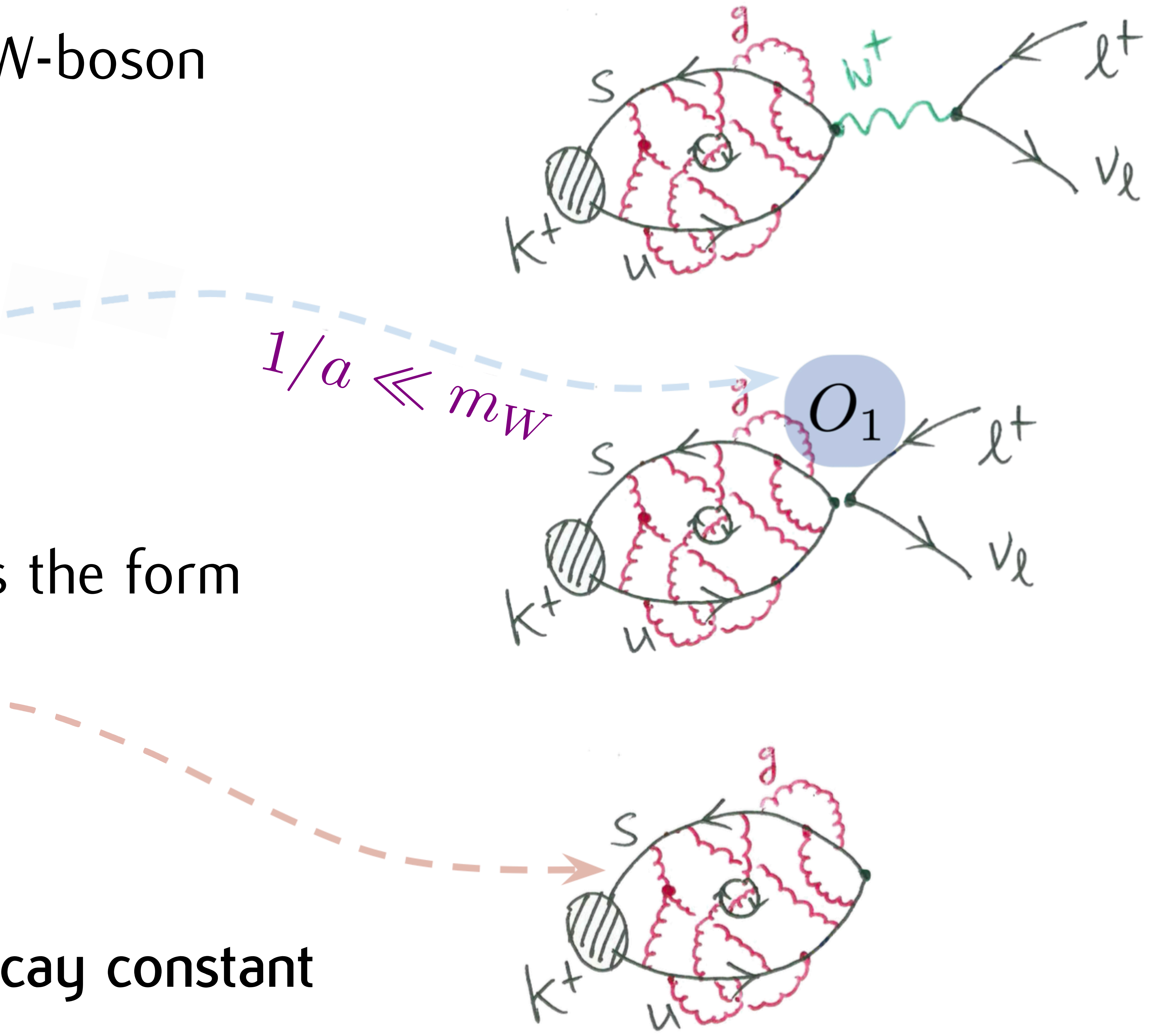
$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$

In the **PDG convention**, the tree-level decay rate takes the form

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$

with the non-perturbative dynamic encoded in the **decay constant**

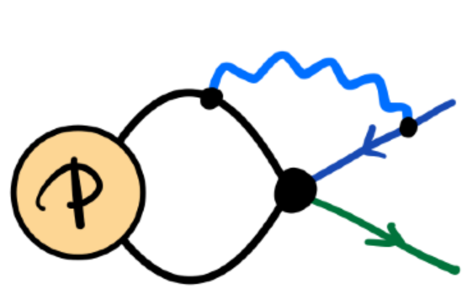
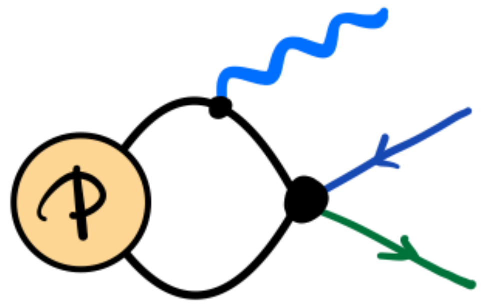
$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$



Leptonic decay rate at $\mathcal{O}(\alpha)$

- The **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity
- **IR divergences** appear in intermediate steps of the calculation

F. Bloch & A. Nordsieck, PR 52 (1937) 54

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Big\{$$  $$+$$  $$\Big\}$$

IR finite — IR divergent — IR divergent

- **UV divergences**: need to include QED corrections to the renormalization of the weak Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(1 + \frac{\alpha_{\rm em}}{\pi} \ln\left(\frac{M_Z}{M_W}\right)\right) \underbrace{[\,\bar{q}_2\,\gamma_\mu (1-\gamma_5)\, q_1\,]\,[\,\bar{\nu}_\ell\,\gamma^\mu (1-\gamma_5)\,\ell\,]}_{O_1^{\text{W-reg}}(M_W)}$$

A.Sirlin, NPB **196** (1982)

E.Braaten & C.S.Li, PRD **42** (1990)

$$O_1^{\text{W-reg}}(M_W) = Z^{\text{W-S}}\left(\frac{M_W}{\mu}, \alpha_s(\mu), \alpha_{\rm em}\right) O_1^{\rm S}(\mu)$$

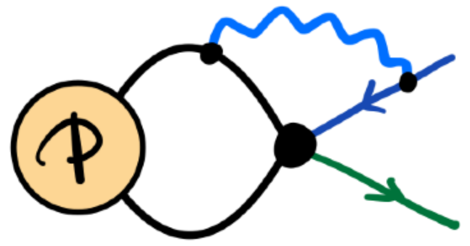
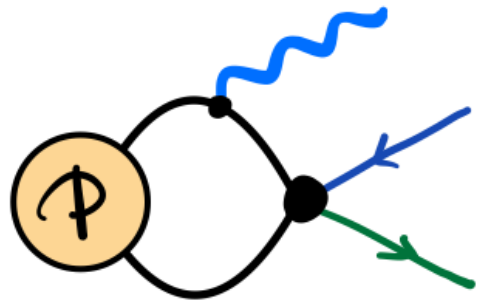
- perturbative @ 2 loops in QCD+QED
- non-perturbative in lattice QCD+QED

MDC et al., PRD **100** (2019)

Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}} \to 0} \Big\{$$  $$+$$  $$\Big\}$$

IR finite — IR divergent — IR divergent

Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

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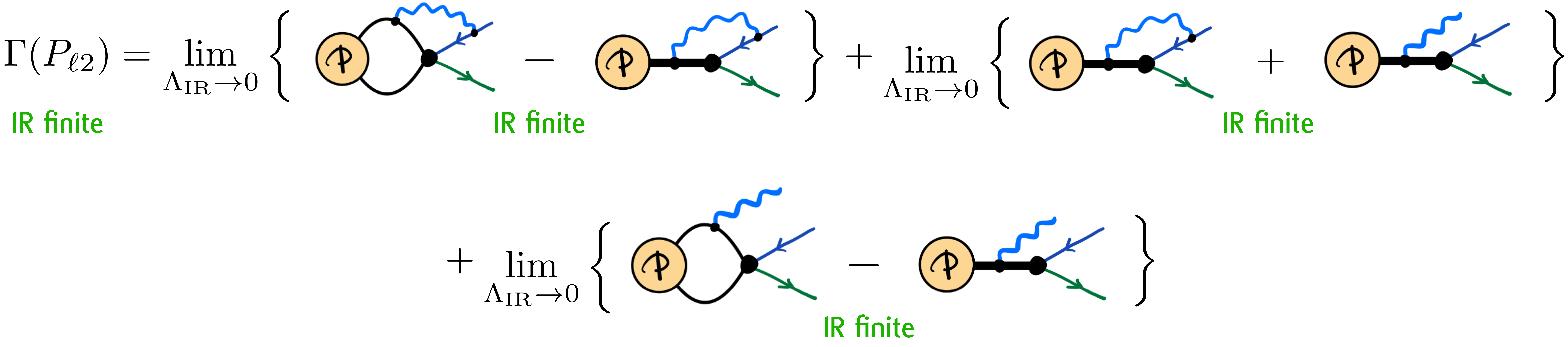
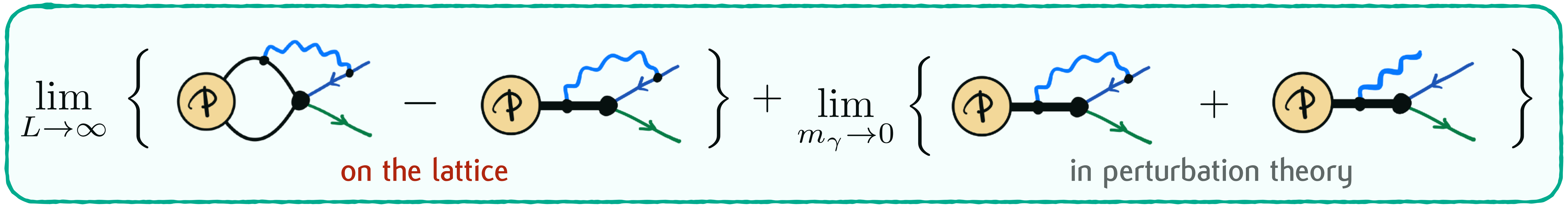


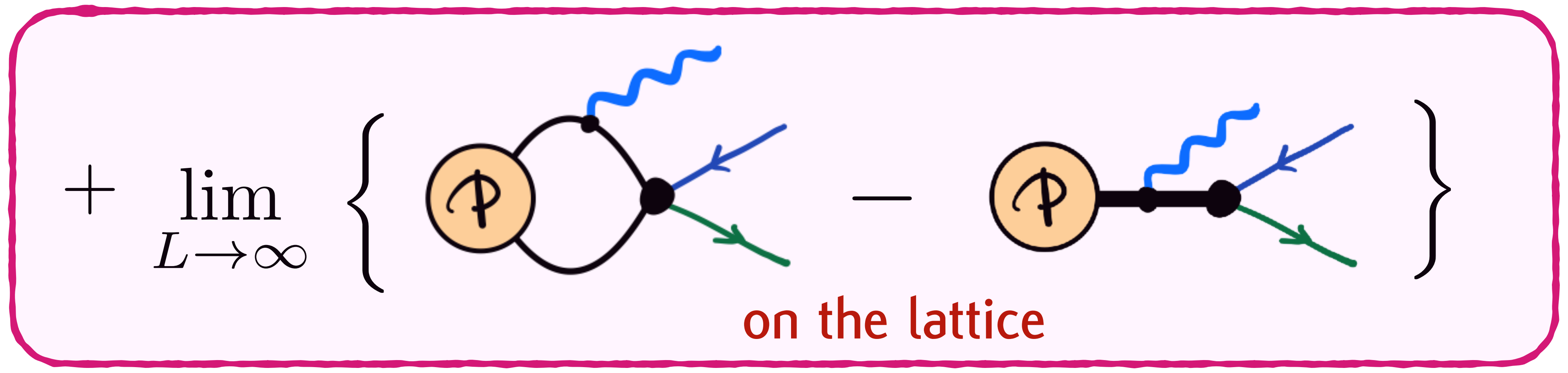
Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) =$





enough for $K_{\mu 2}$ and $\pi_{\mu 2}$

leading **finite-volume scaling** well studied

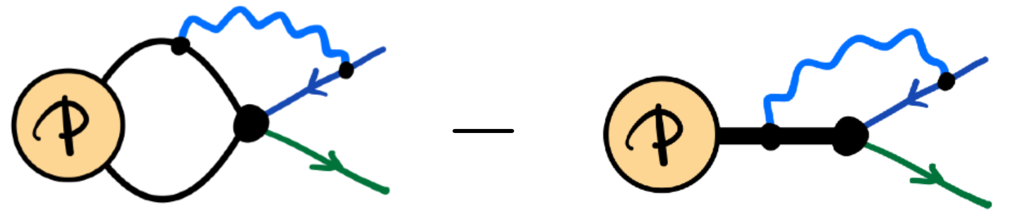
V.Lubicz et al., PRD **95** (2017) N.Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

relevant for K_{e2} and π_{e2}
& decays of heavier mesons

G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD **103** (2021)
A. Desiderio et al., PRD **102** (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

Leptonic decay rate at $\mathcal{O}(\alpha)$

Virtual decay rate

$\lim_{L\to\infty}$ {  }

on the lattice

$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}(1+\delta R_P) \quad \blacktriangleright \quad \Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1-\frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2 \quad \blacktriangleright \quad \delta R_P = 2\left(\frac{\delta\mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta\mathcal{Z}}{\mathcal{Z}_0}\right)$$

PDG convention

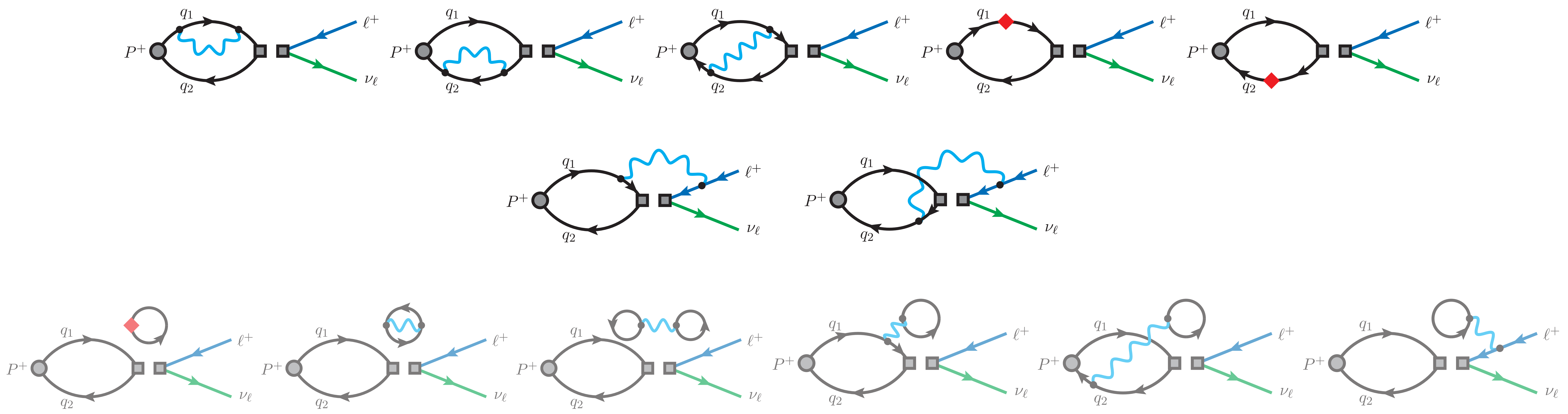
- $\boldsymbol{\delta\mathcal{A}_P}$ from the correction to the (bare) matrix element $\mathcal{M}_P^{rs}(\mathbf{p}_\ell) = \langle \ell^+, r, \mathbf{p}_\ell; \nu_\ell, s, \mathbf{p}_\nu | O_W | P^+, \mathbf{0}\rangle$
- $\boldsymbol{\delta m_P}$ correction to the meson mass
- $\boldsymbol{\delta\mathcal{Z}}$ correction to the renormalization of the weak operator O_W MDC et al., PRD **100** (2019)

$$\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})} \rightarrow \delta R_{K\pi} = 2\left(\frac{\delta\mathcal{A}_K}{\mathcal{A}_{K,0}} - \frac{\delta m_K}{m_{K,0}}\right) - 2\left(\frac{\delta\mathcal{A}_\pi}{\mathcal{A}_{\pi,0}} - \frac{\delta m_\pi}{m_{\pi,0}}\right)$$

IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

Correlation functions in RM123 approach



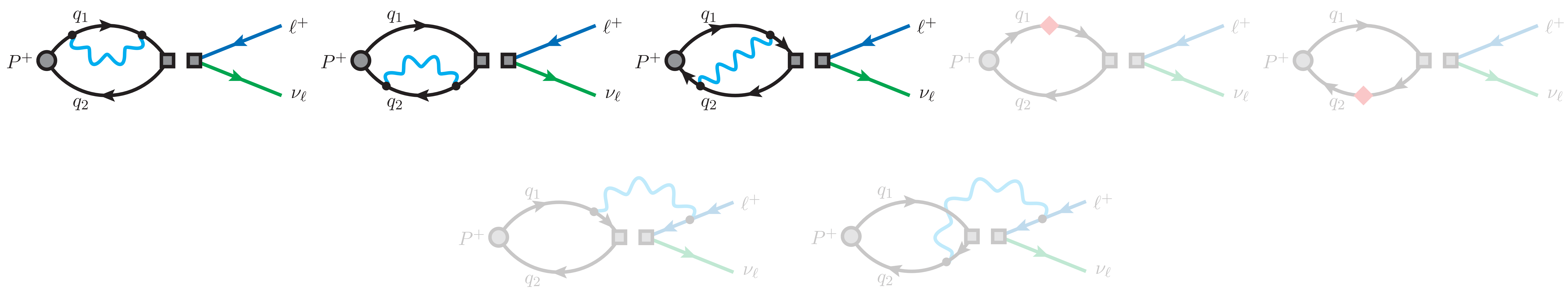
Current calculations have been performed in the **electro-quenched approximation (sea quarks electrically neutral)**.

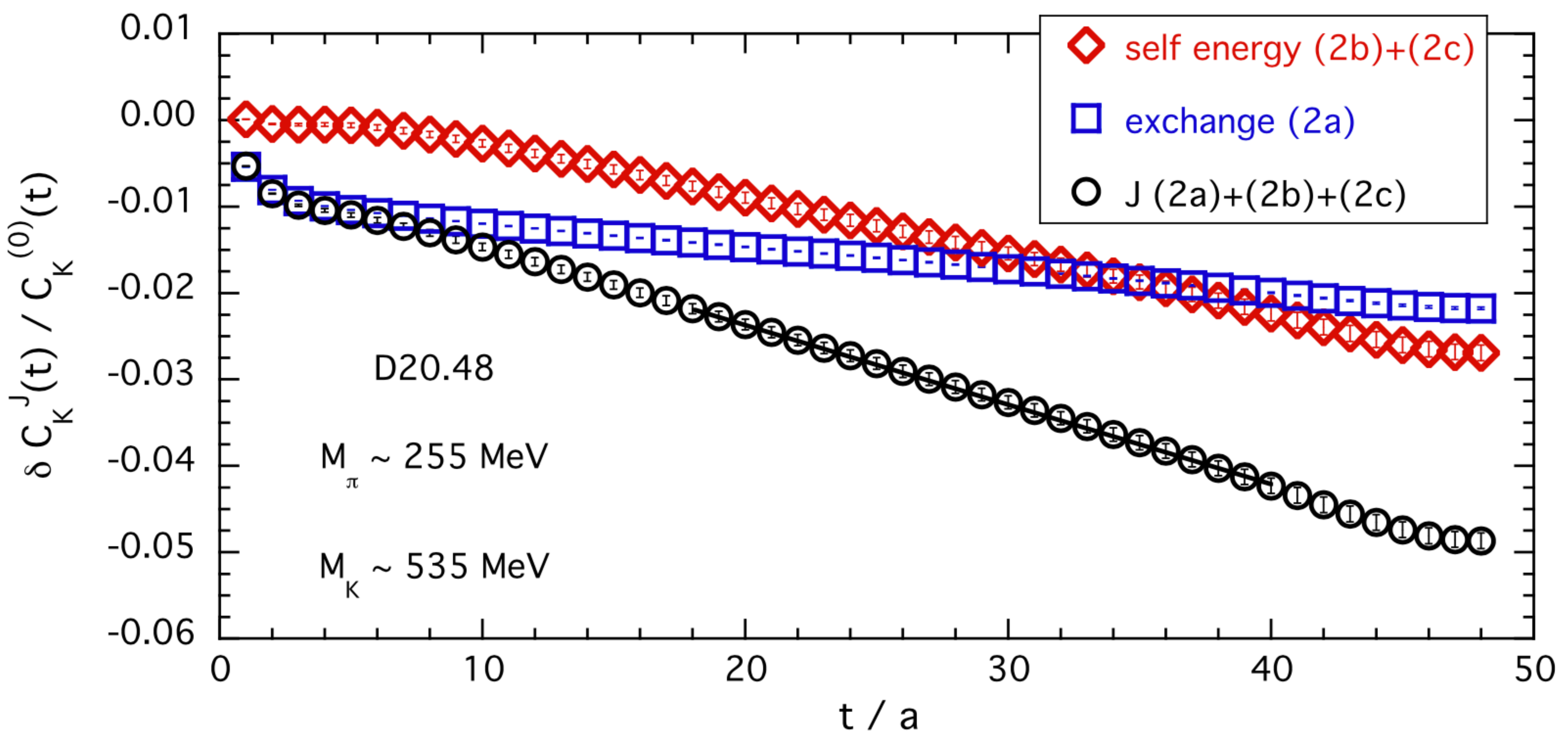
Work in progress to compute the remaining diagrams by different collaborations.

IB corrections to the decay amplitude

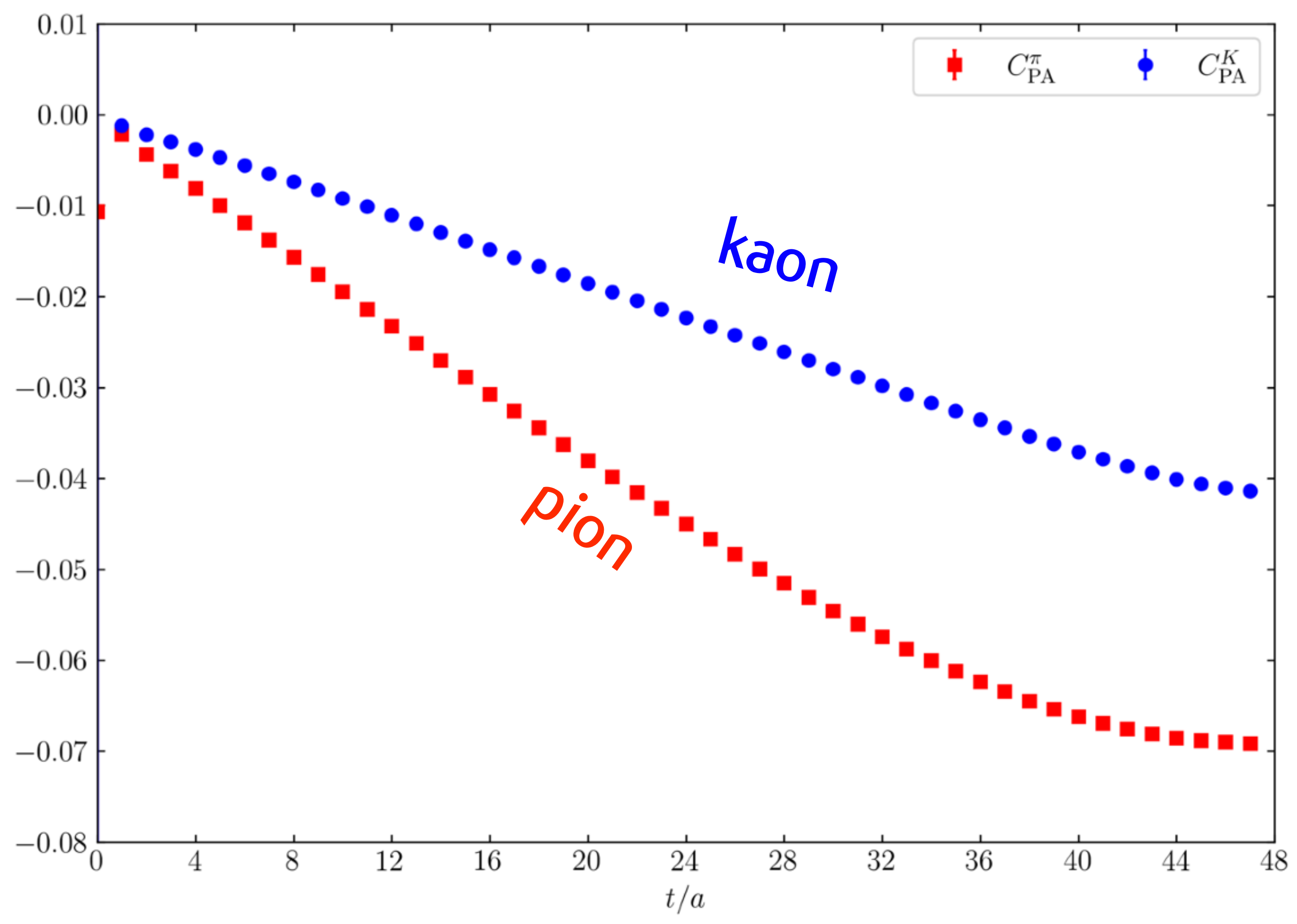
G.M.de Divitiis et al. [RM123], PRD 87 (2013)

Correlation functions in RM123 approach





MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

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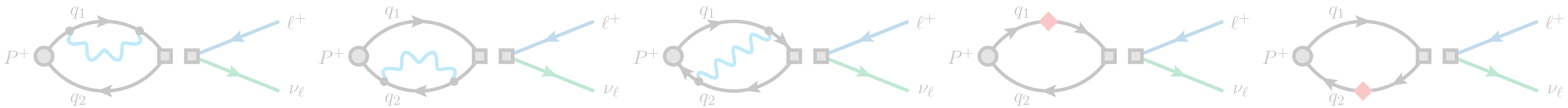
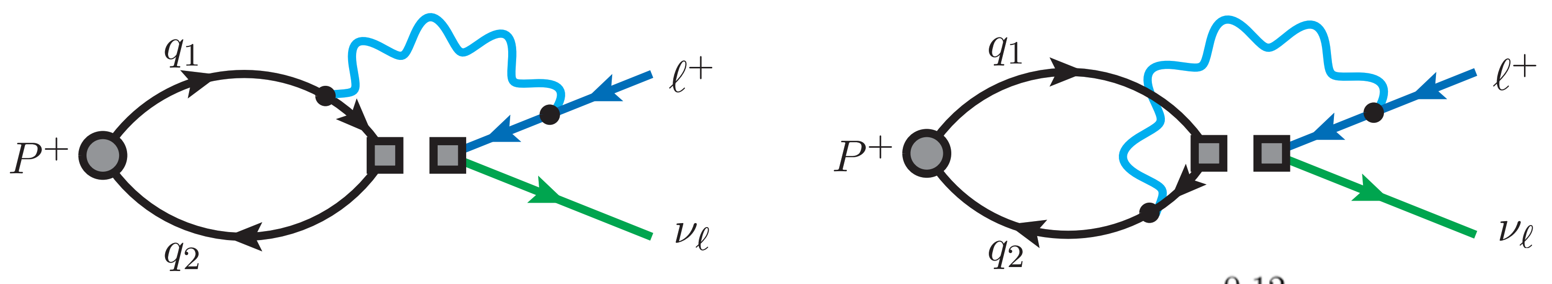
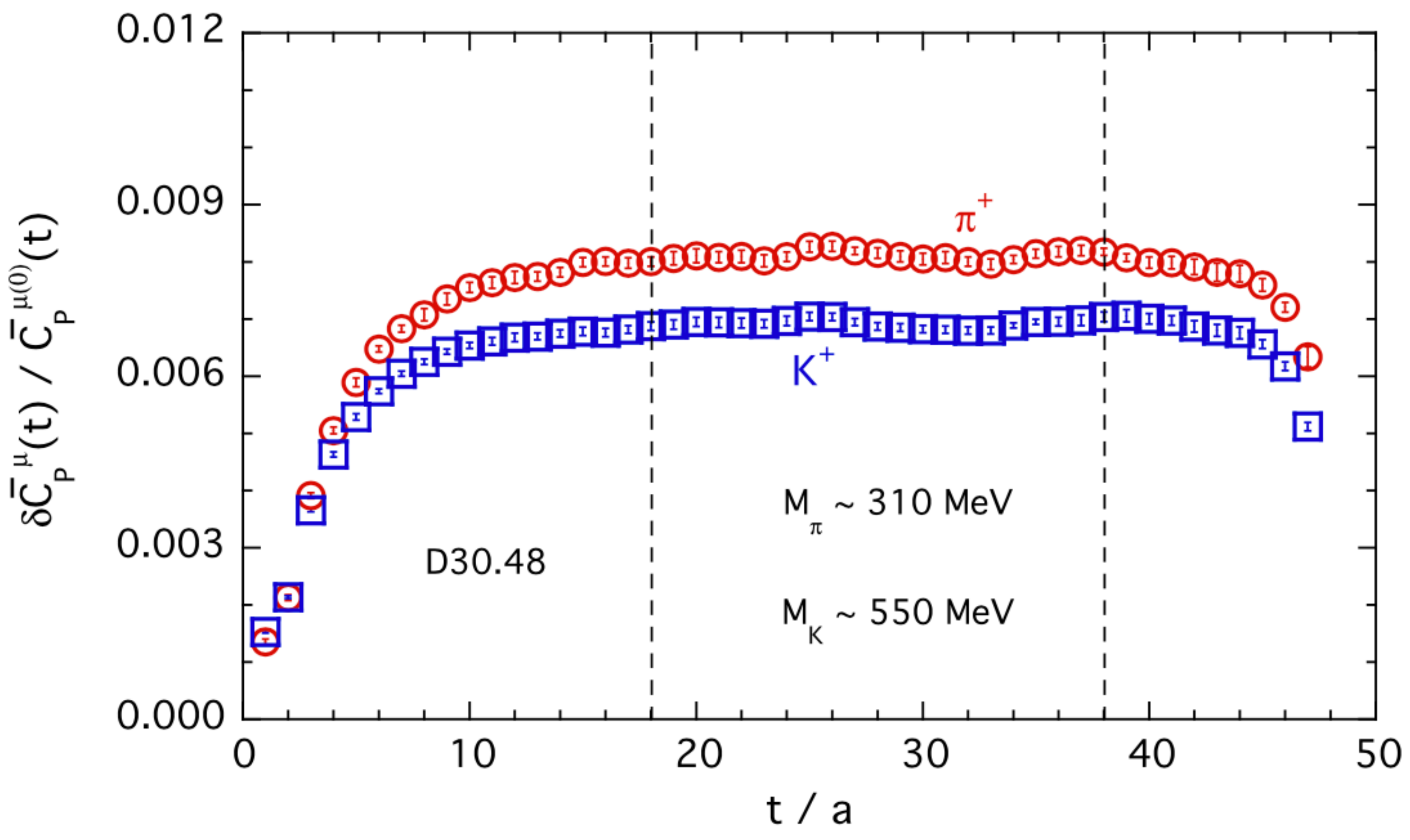
G.M.de Divitiis et al. [RM123], PRD 87 (2013)

Correlation functions in RM123 approach

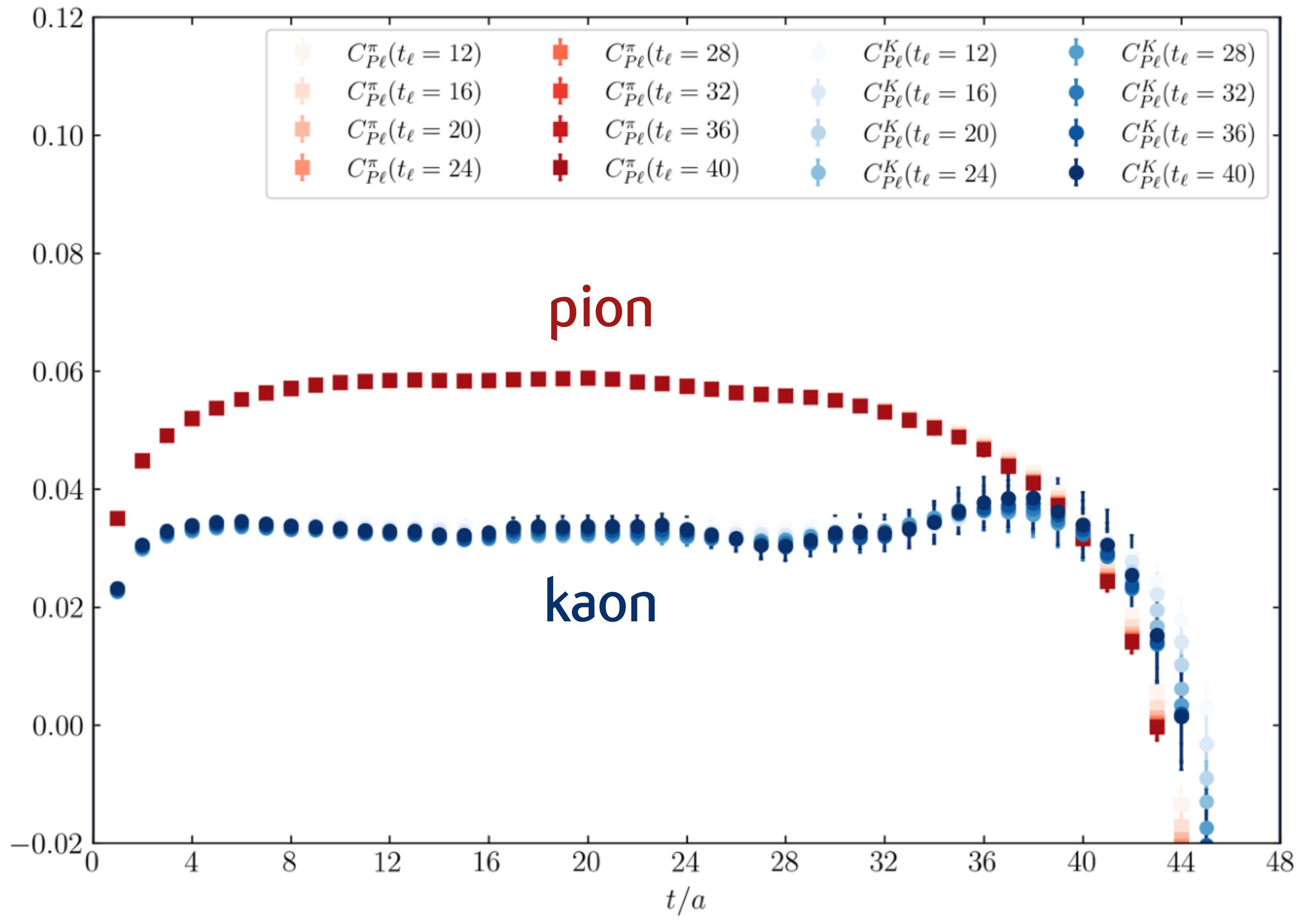




$$\boldsymbol{p_\ell} \propto (1,1,1)$$



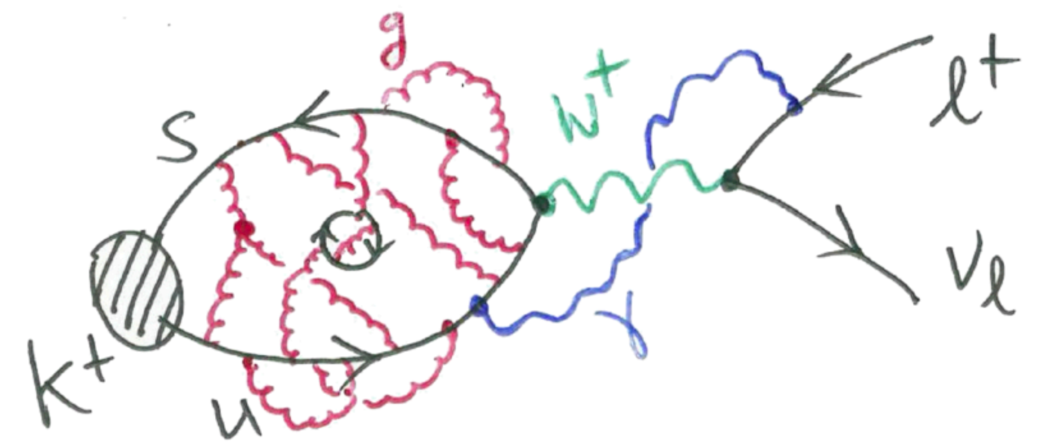
MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

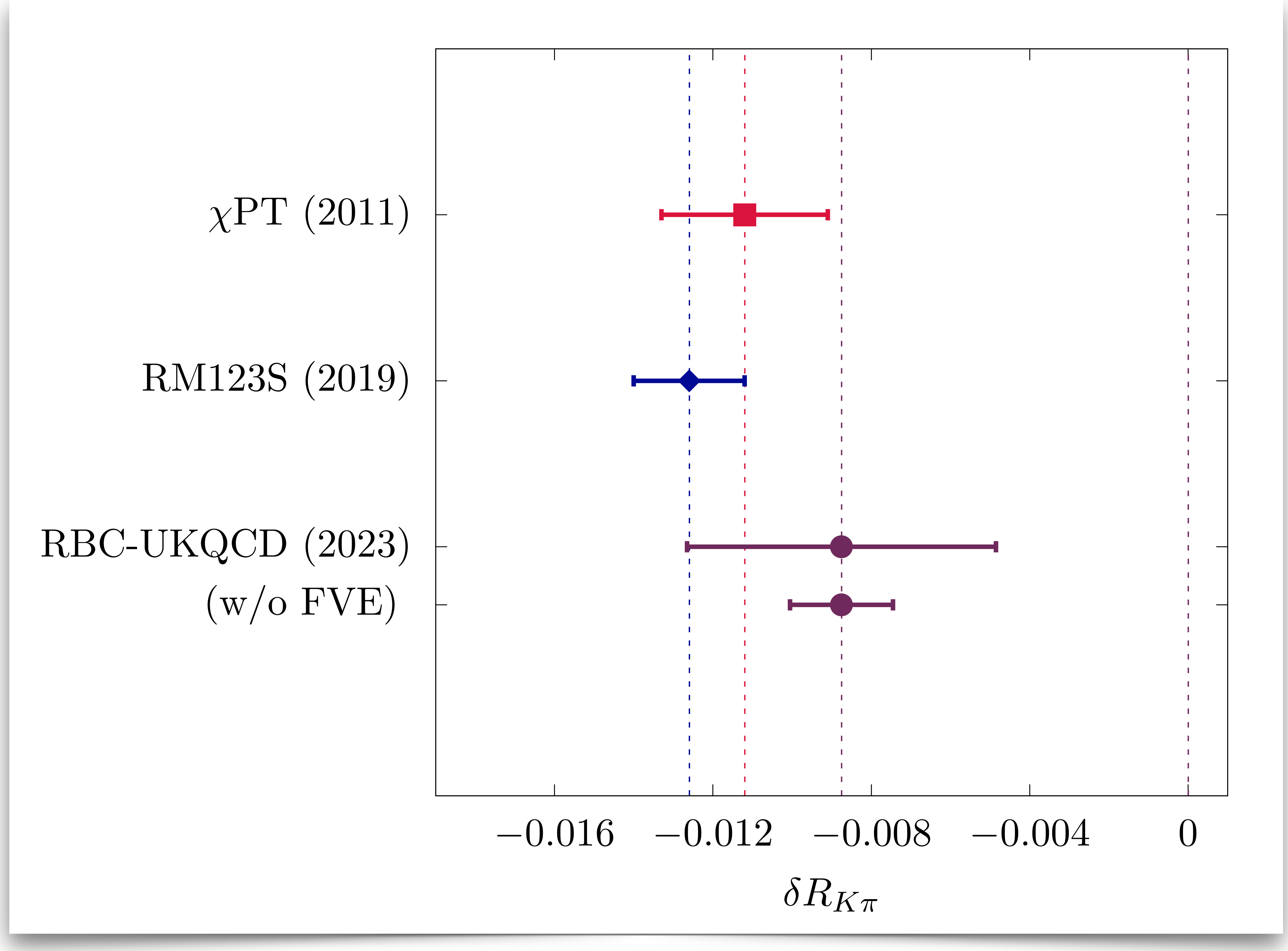
Results for $\delta R_{K\pi}$

- $\delta R_{K\pi} = -0.0112\,(21)$
- $\delta R_{K\pi} = -0.0126\,(14)$
- $\delta R_{K\pi} = -0.0086\,(13)(39)_{\mathrm{vol.}}$



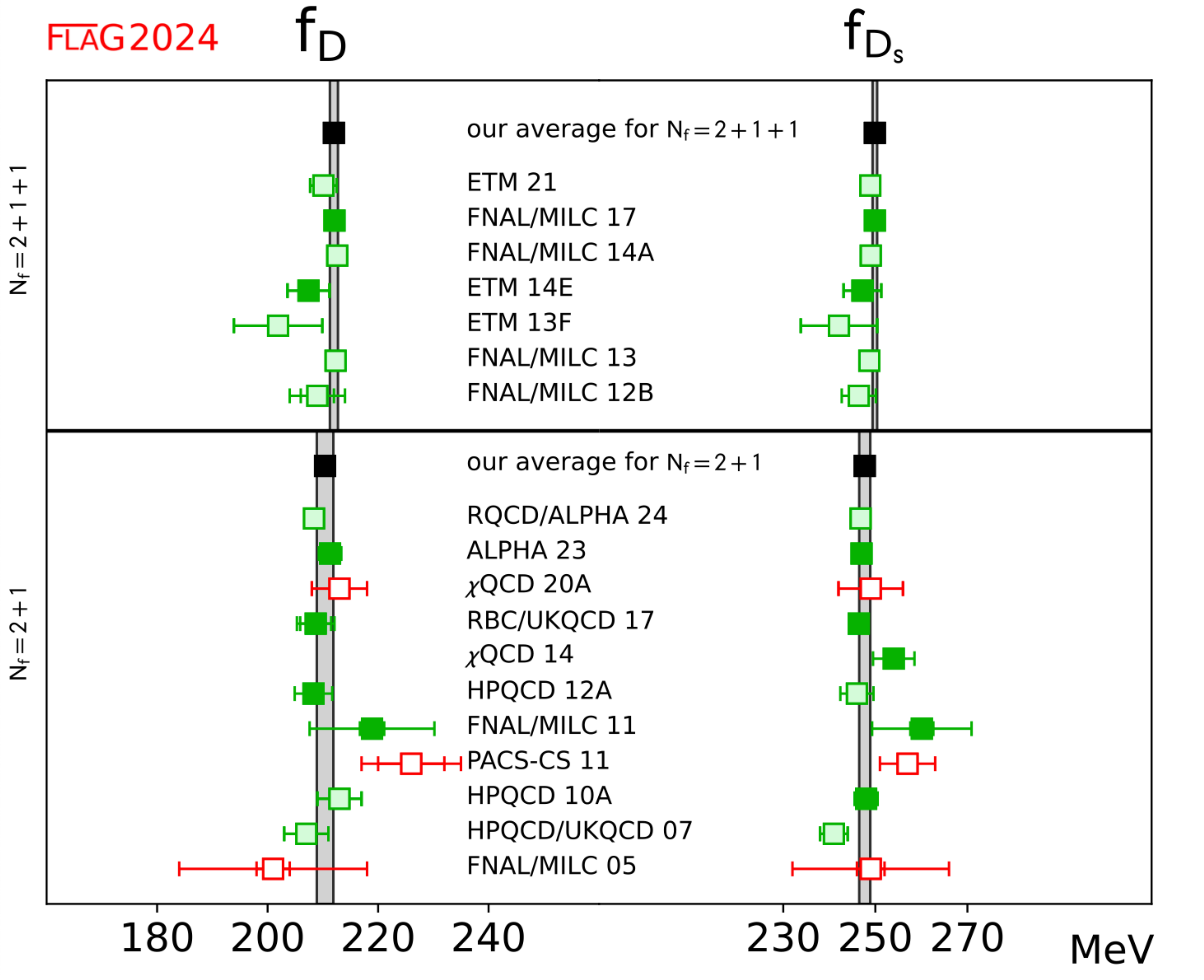
V.Cirigliano et al., PLB **700** (2011)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

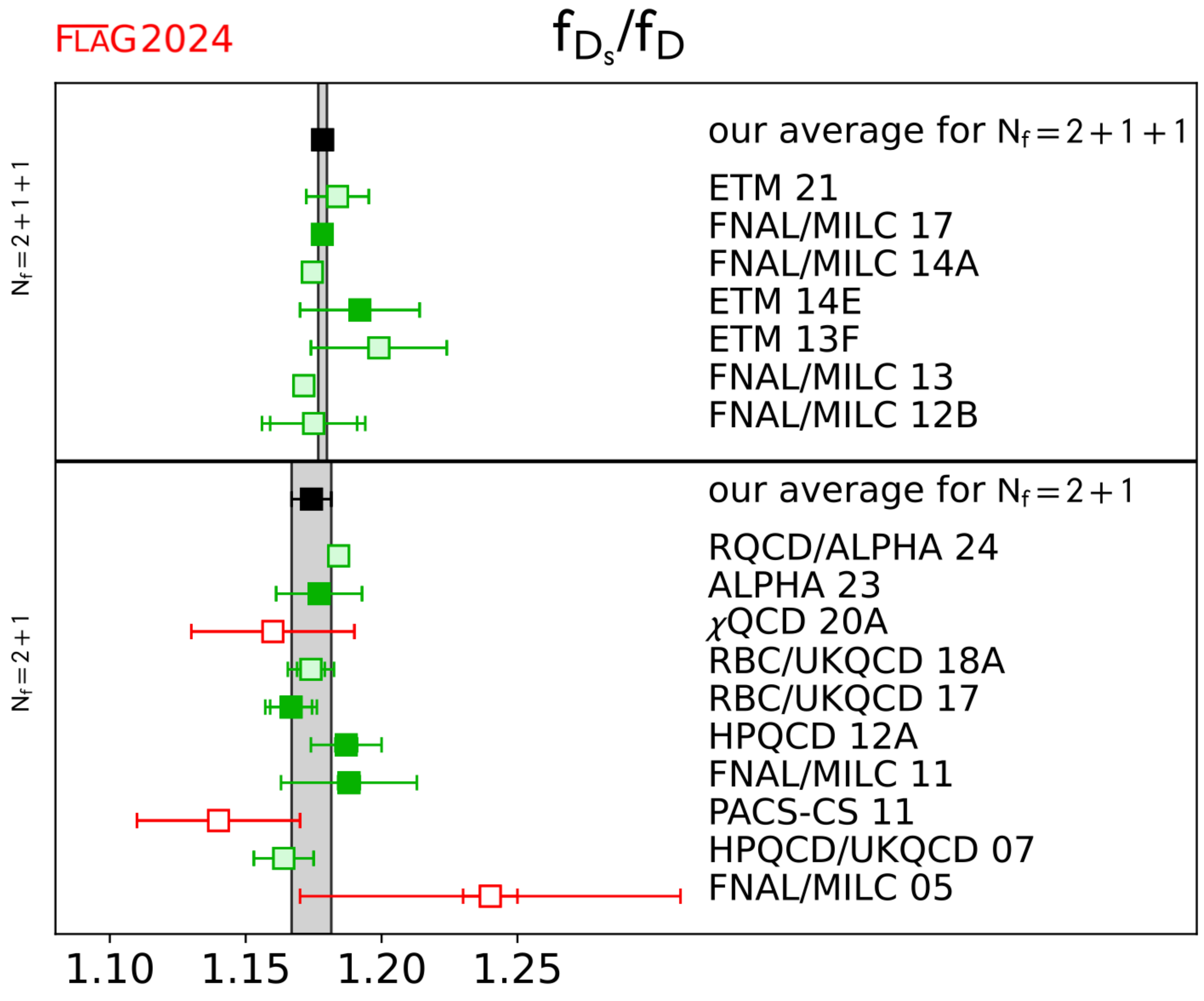
$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$



- Good evidence that $\delta R_{K\pi}$ **can be computed from first principles non-perturbatively on the lattice!**
- **RBC-UKQCD error** dominated by a large systematic uncertainty related to **finite-volume effects** (!)
  Work in progress to improve the result.
- **Errors on** $|V_{us}|/|V_{ud}|$ from **theoretical** inputs could become comparable with those from **experiments**

Charmed QCD decay constants

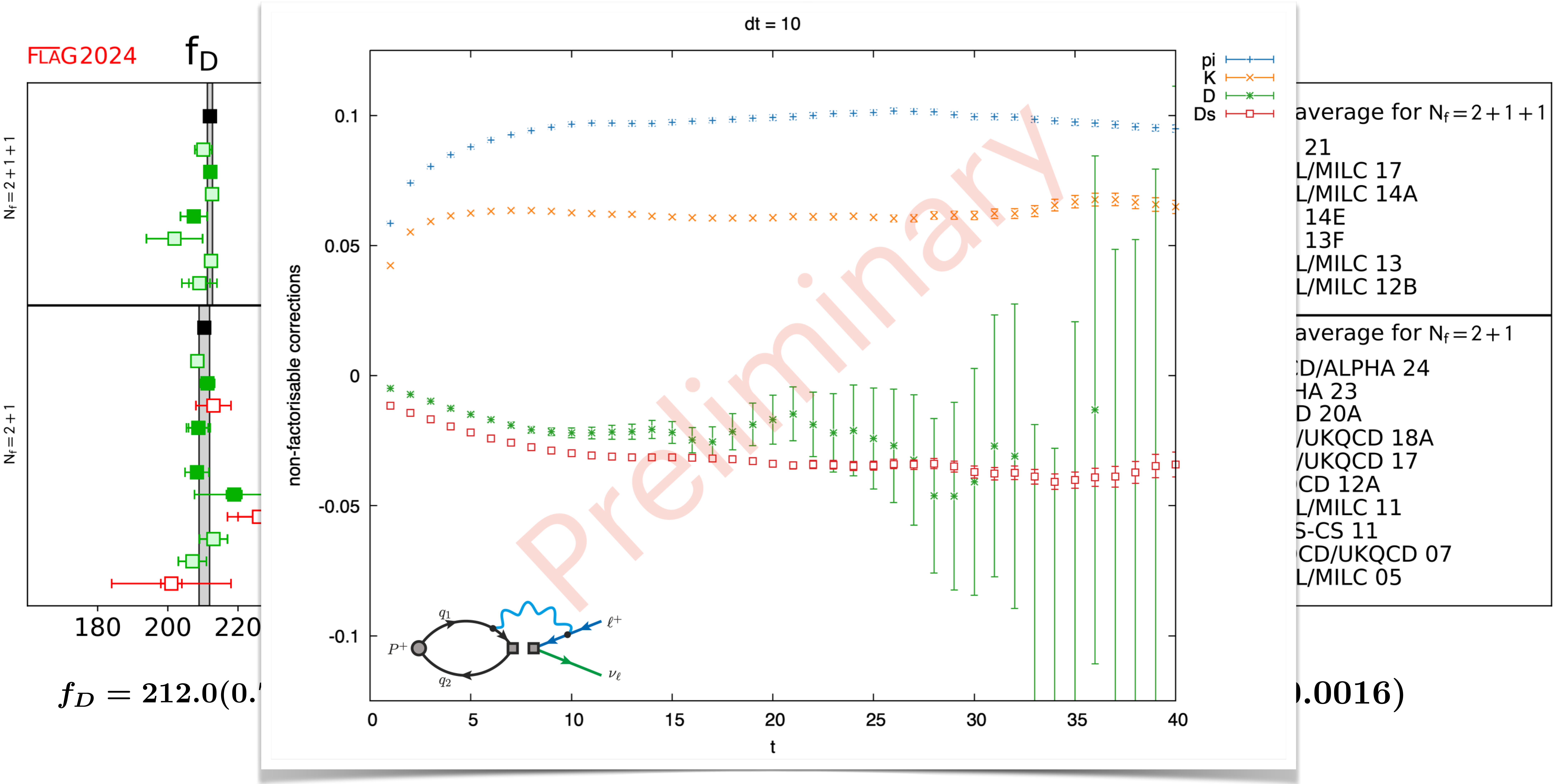




$f_D = 212.0(0.7)$ MeV $\quad f_{D_s} = 249.9(0.5)$ MeV

$f_{D_s}/f_D = 1.1783(0.0016)$

Charmed QCD decay constants



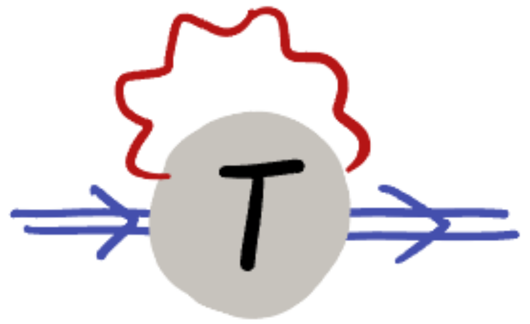
$f_D = 212.0(0.$ … $.0016)$

QED finite-volume effects

Hadron masses

using the notation of
B.Lucini et al., JHEP 1602 (2016)

1. Cottingham formula:



$$m_P = m_P^{(0)} + \frac{\mathrm{i}\, e^2}{4m_P} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{T_{\mu\nu}(k_0, \mathbf{k}) g^{\mu\nu}}{k_0^2 - \mathbf{k}^2 + \mathrm{i}\epsilon}$$

$$T_\mu{}^\mu(k_0, \mathbf{k}) = \mathrm{i} \int \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}kx} \left\langle P(\mathbf{0}) | \mathrm{T}\{ J_\mu(x) J^\mu(0) \} | P(\mathbf{0}) \right\rangle_{\mathrm{c}}$$

QED finite-volume effects

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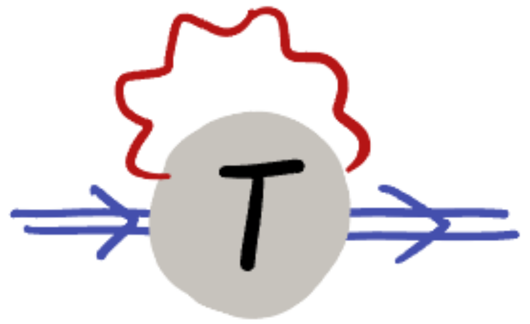

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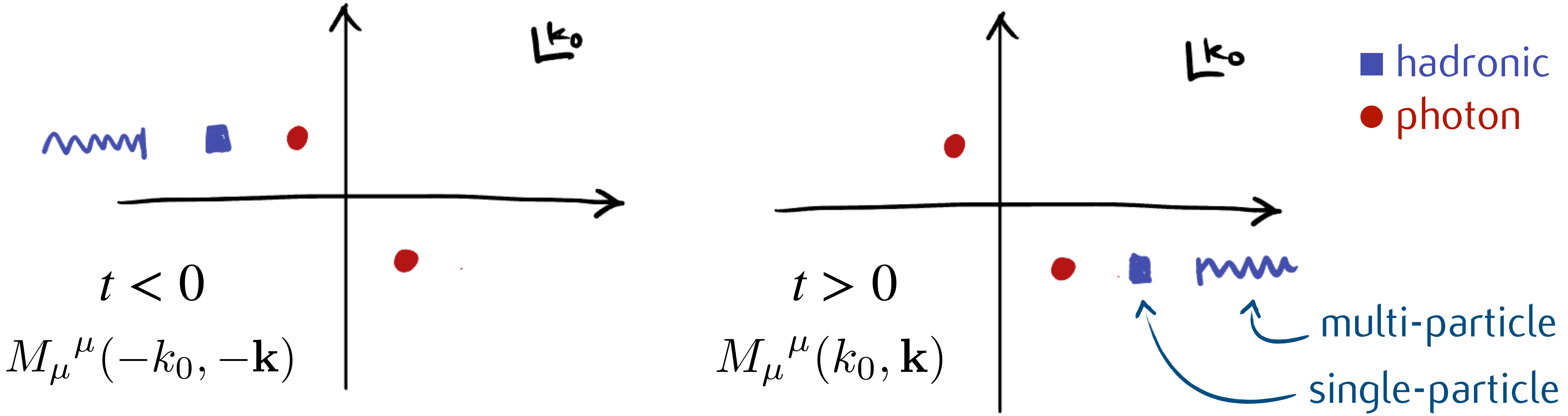

QED finite-volume effects

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B.Lucini et al., JHEP 1602 (2016)

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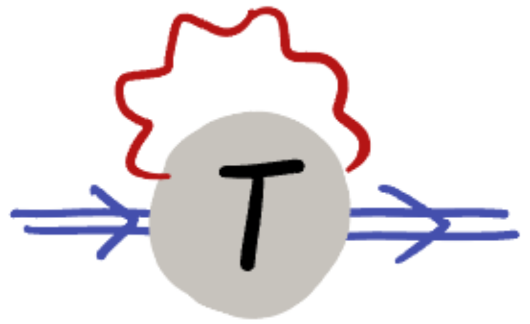

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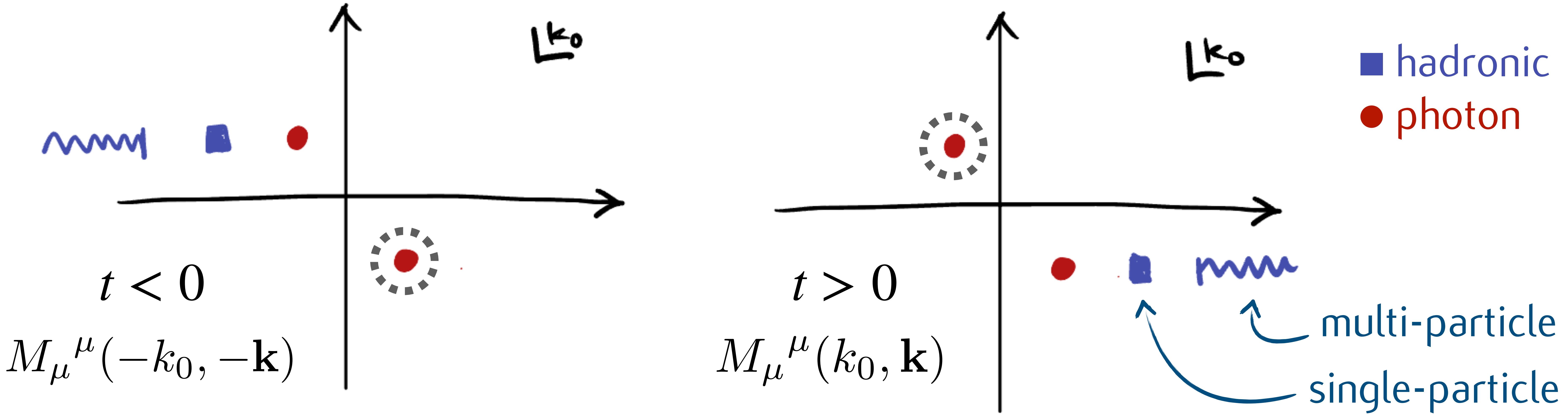

QED finite-volume effects

Hadron masses

3. $$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\left[\frac{1}{L^3}\sum_{\mathbf{k}\in\Pi_{\boldsymbol{\theta}}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right]\frac{{M_\mu}^\mu(-|\mathbf{k}|,\mathbf{k})}{|\mathbf{k}|}$$

$${M_\mu}^\mu(-|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

QED finite-volume effects

Hadron masses

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$$M_\mu{}^\mu(-|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|) \qquad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}$$

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

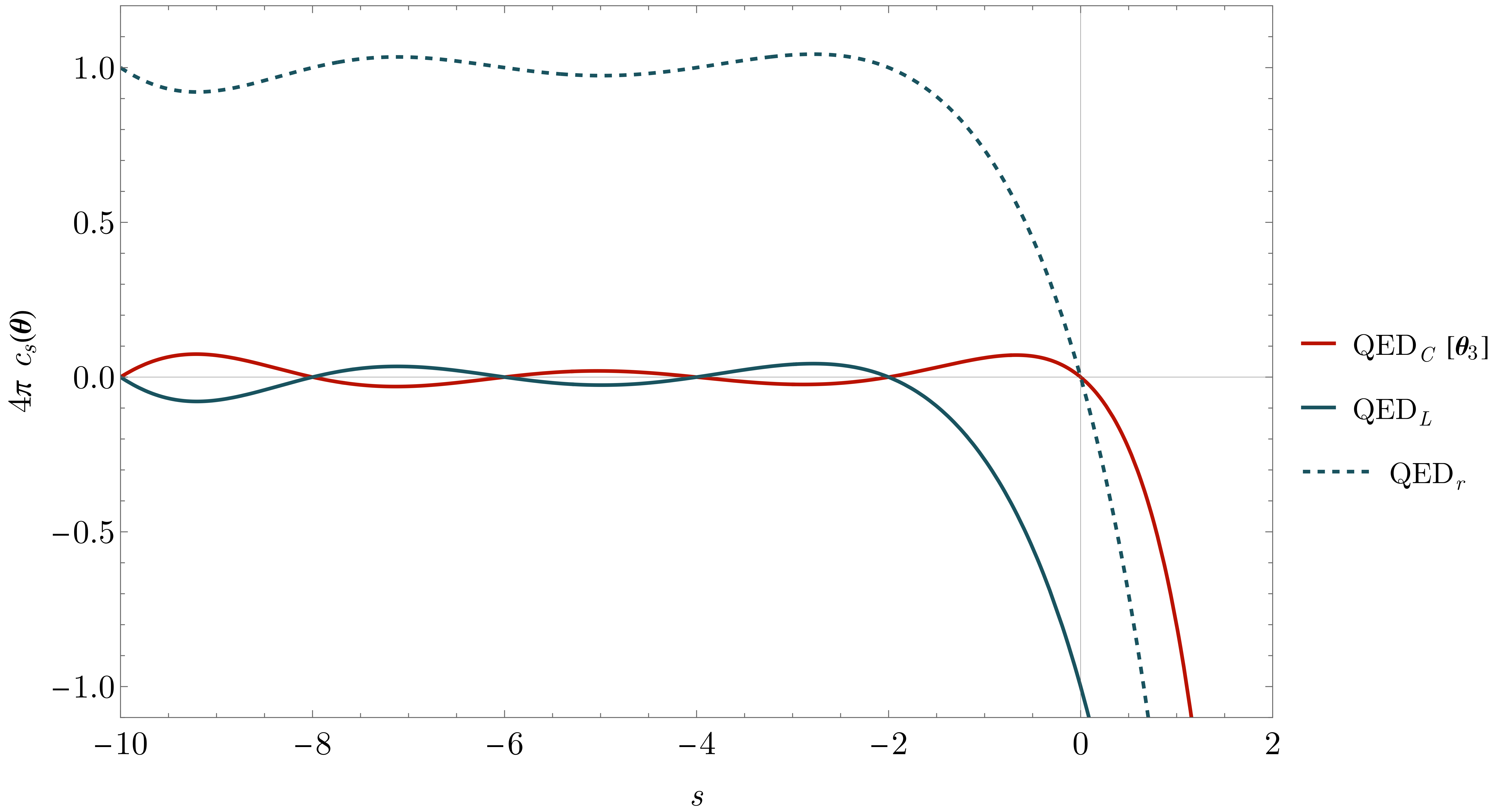
universal terms fixed by Ward identities

structure + multi-particle dependence

QED finite-volume effects

Hadron masses

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$



QED finite-volume effects

Hadron masses

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However... this series is potentially divergent and should rather be interpreted as an asymptotic series!

In **QED$_\mathrm{L}$** or **QED$_\mathrm{C}$**, the series can be rewritten as

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \frac{1}{L^2}\sum_{\ell=0}^{\infty} b_\ell \frac{1}{(m_P L)^{2\ell}}\right]$$

with $|b_\ell| \propto |(2\ell)!\,\bar{c}_{2+2\ell}(\boldsymbol{\theta})\,a_\ell|$ and $T_\mu{}^\mu(|\boldsymbol{k}|,\boldsymbol{k}) = \sum_{\ell=0}^{\infty} a_\ell\left(|\boldsymbol{k}|/m_P\right)^{2\ell}$

QED finite-volume effects

Asymptotic series: an instructive example

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{e^t}{t}\mathrm{d}t = \frac{e^z}{z}\left[1 + \frac{1}{z} + \frac{2}{z^2} + \frac{6}{z^3} + \dots\right] \approx \frac{e^z}{z}\sum_{\ell=0}^{\infty}\frac{\ell!}{z^\ell}$$

This series is divergent, due to the term $\ell!$ growing fast.

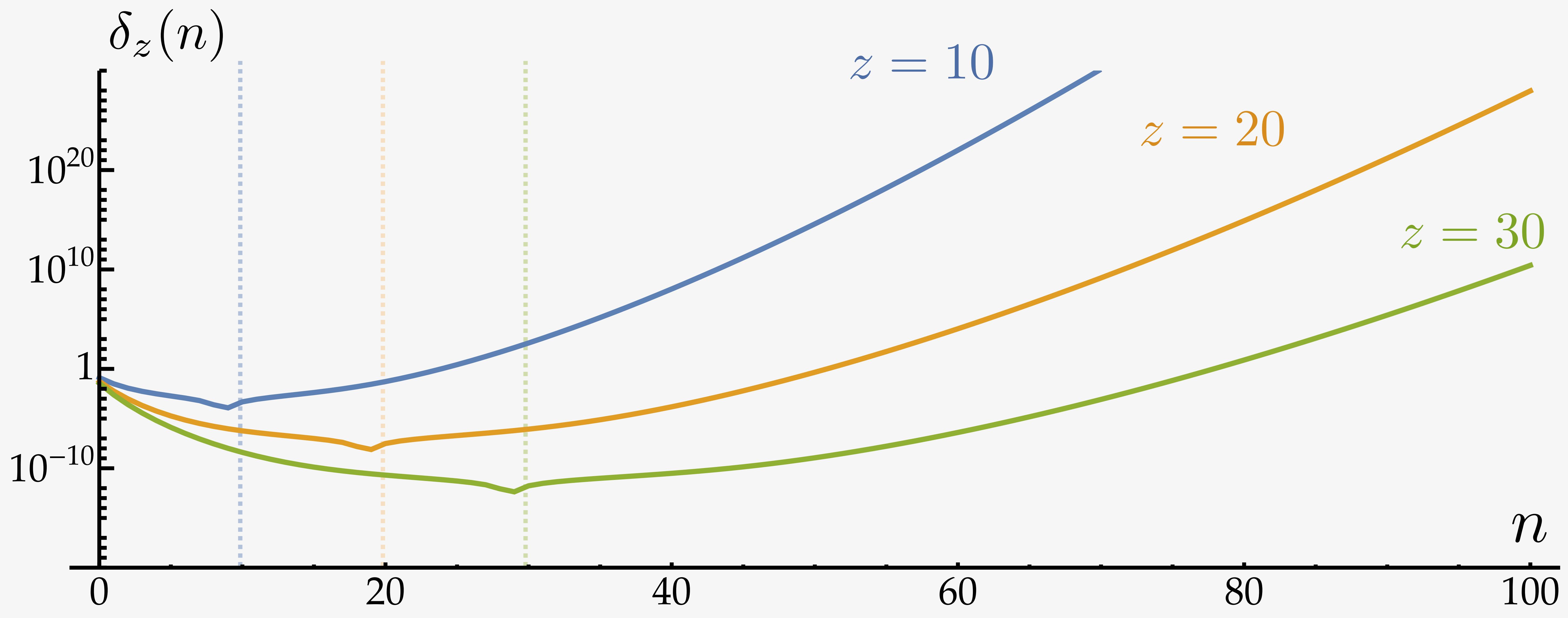
However, when **truncated** it can give a good approximation of $\mathrm{Ei}(z)e^{-z}z$:

$$\delta_z(n) = \left|\mathrm{Ei}(z)e^{-z}z - \sum_{\ell=0}^{n}\frac{\ell!}{z^\ell}\right|$$

There exist an optimal truncation:

$$n^\star \sim |z|$$

$$\delta_z(n^\star) \leq \sqrt{2\pi}|z|^{-1/2}e^{-|z|}$$



QED finite-volume effects

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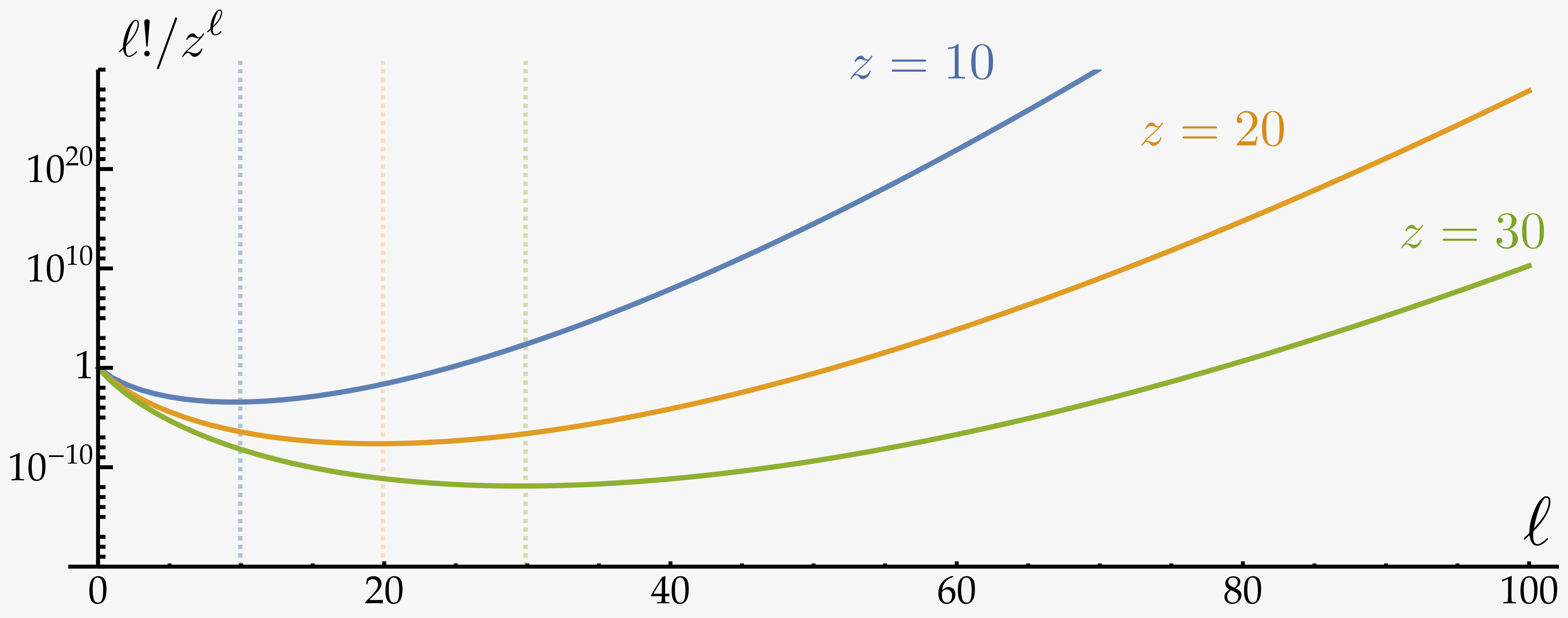


QED finite-volume effects

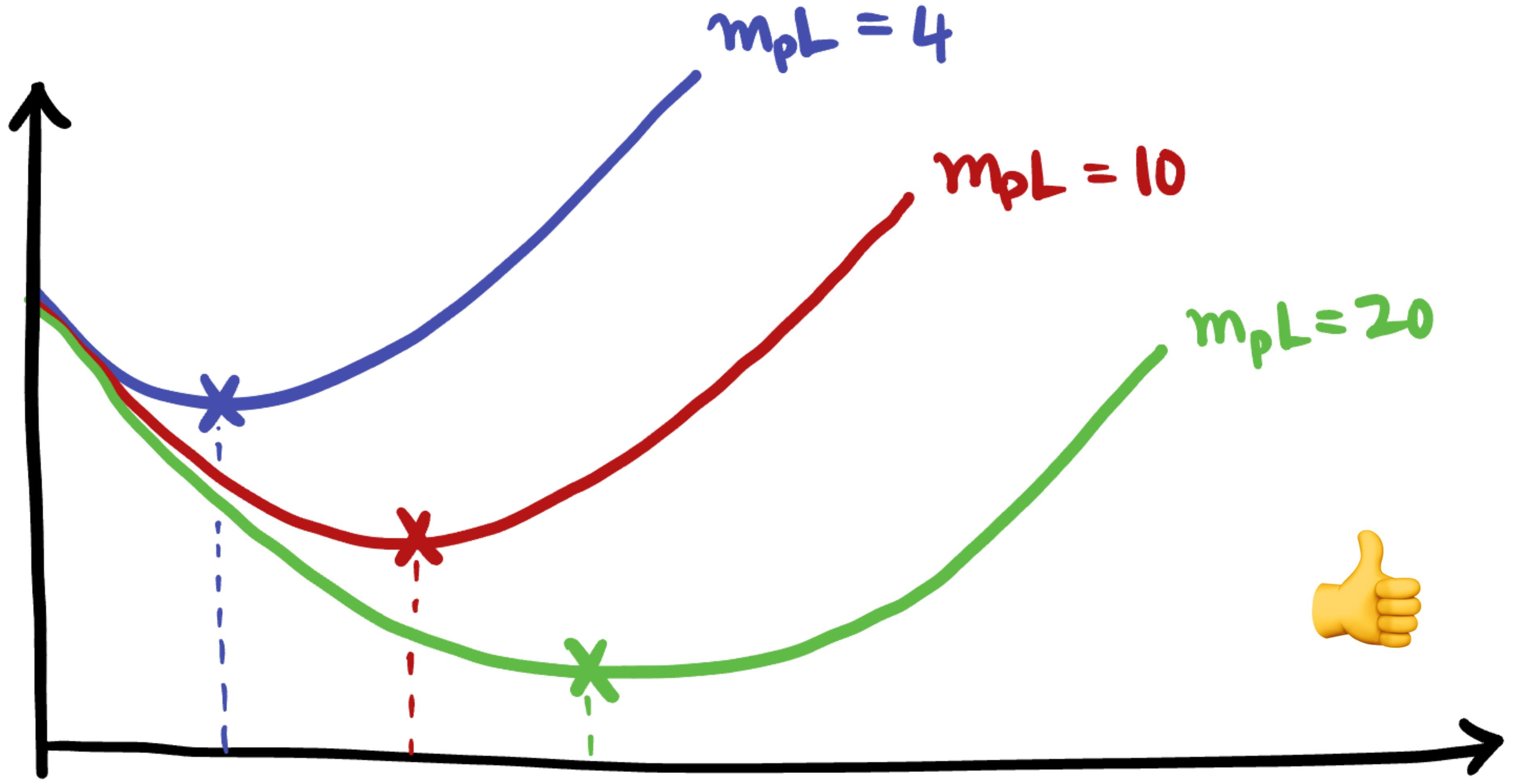
Hadron masses

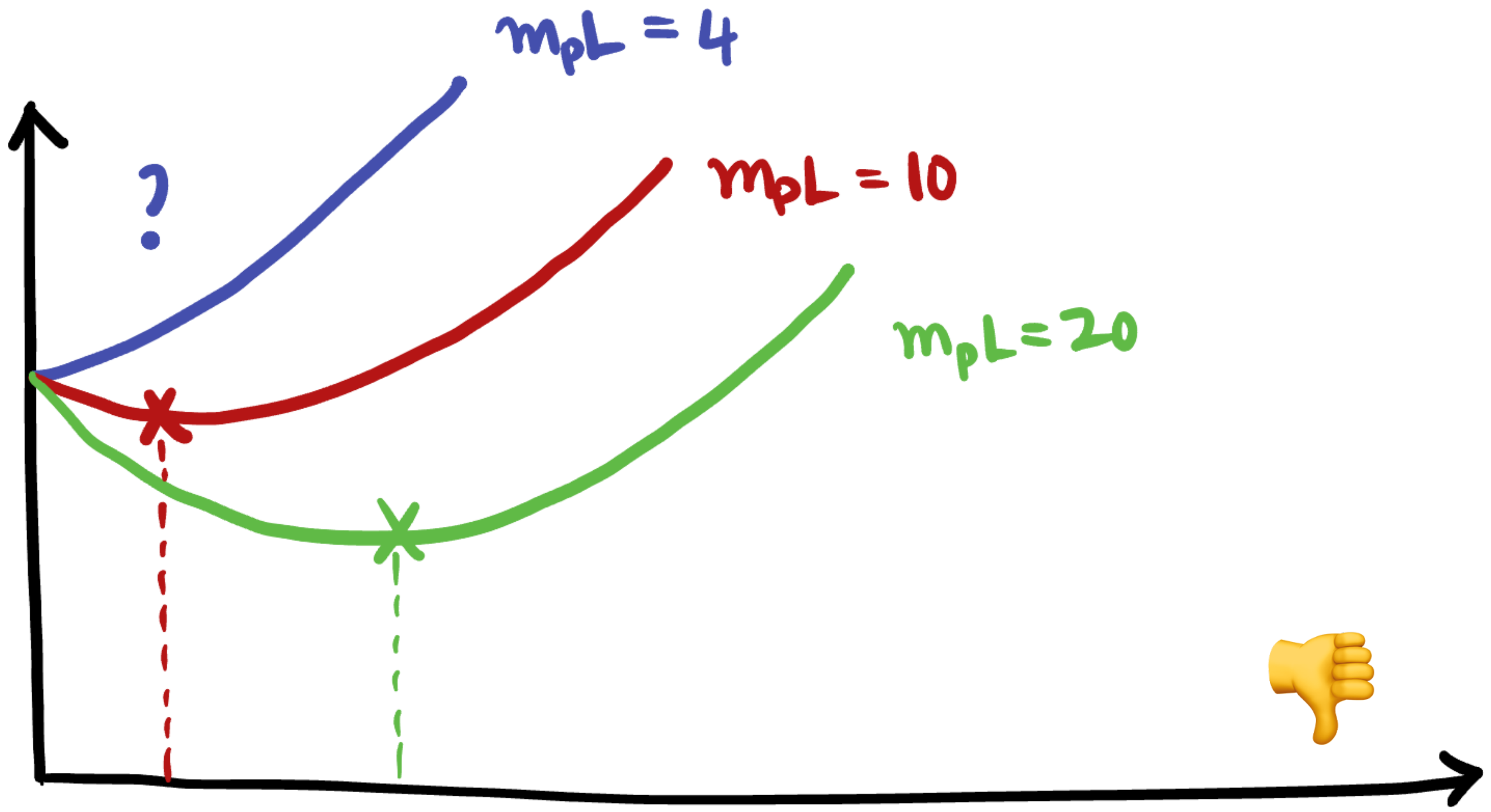
$$\frac{1}{L^2}\sum_{\ell=0}^{\infty} b_\ell \frac{1}{(m_P L)^{2\ell}} \qquad |b_\ell| \propto |(2\ell)!\, \bar{c}_{2+2\ell}(\boldsymbol{\theta})\, a_\ell|$$

...is there an optimal truncation here?

$$n^\star \overset{?}{\sim} (m_P L)/2$$

Possible scenarios:





Ongoing study in QED$_L$, QED$_C$ & QED$_r$

QED finite-volume effects

V. Lubicz et al., PRD **95** (2017)

Leptonic decays

Finite-volume calculation more tricky due to appearance of **infrared divergences** & dependence on **external lepton momentum** p_ℓ

$$\Gamma_P = \mathcal{K}_P\, f_P^2 (1 + e^2\, \delta R_P^{\text{virt}} + e^2\, \delta R_P^{\text{real}}) \qquad \delta R_P^{\text{virt}}(L) = \frac{Y(L)}{8\pi^2}$$

$$Y(L) = \lim_{\varepsilon\to 0} Y_\varepsilon(L) \equiv \lim_{\varepsilon\to 0} \left\{ Y_\varepsilon(\infty) + \Delta Y_\varepsilon(L) \right\} = Y^{\text{SD}}(\infty) + \lim_{\varepsilon\to 0} \left\{ Y_\varepsilon^{\text{uni}}(\infty) + \Delta Y_\varepsilon(L) \right\}$$

target of our lattice calculation ($Y^{\text{SD}}(\infty)$)

finite-volume effects $\Delta Y(L)$:
- point-like decay rate with massive photon ($Y_\varepsilon^{\text{uni}}(\infty)$)
- sum-integral differences at finite photon mass ($\Delta Y_\varepsilon(L)$)

QED finite-volume effects

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]
MDC et al., [2501.07936]

Leptonic decays

$$\Delta Y(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$

$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$

$$+ \cdots$$

$r_\ell = m_\ell / m_P$

$$C_\ell^{(2)} = \frac{32\pi^2}{f_P\, m_P^2\,(1-r_\ell^4)}\left[F_V - F_A + 2\, r_\ell^2 \frac{\partial F_A}{\partial x_\gamma}\right]$$

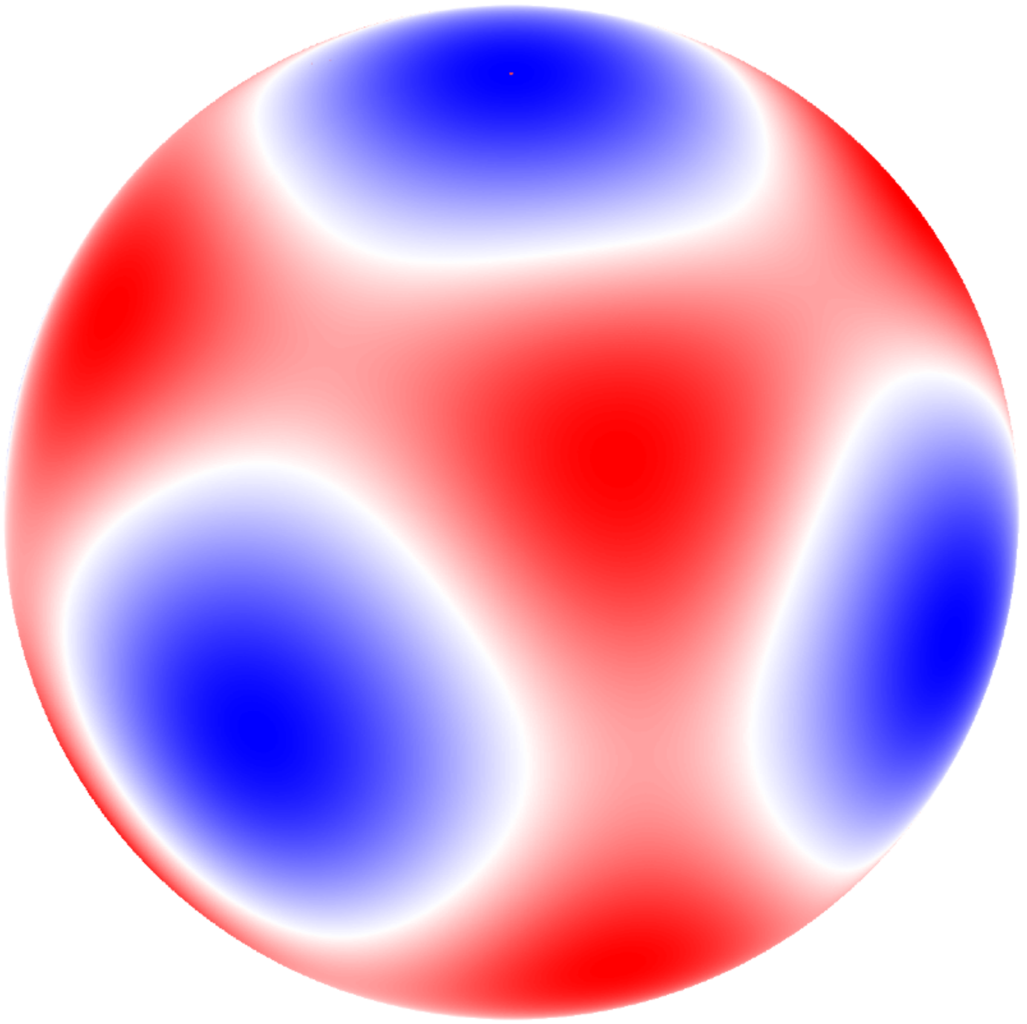
with new finite-volume coefficients $\quad c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq \mathbf{0}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$

Collinear divergences

In **infinite volume** they appear as $\log(m_\ell/m_P)$, arising from photons parallel to the lepton $\mathbf{v} \parallel \mathbf{k}$

In **finite volume** they also appear, but **rotational symmetry breaking** induces a dependence on $\hat{\mathbf{v}}$

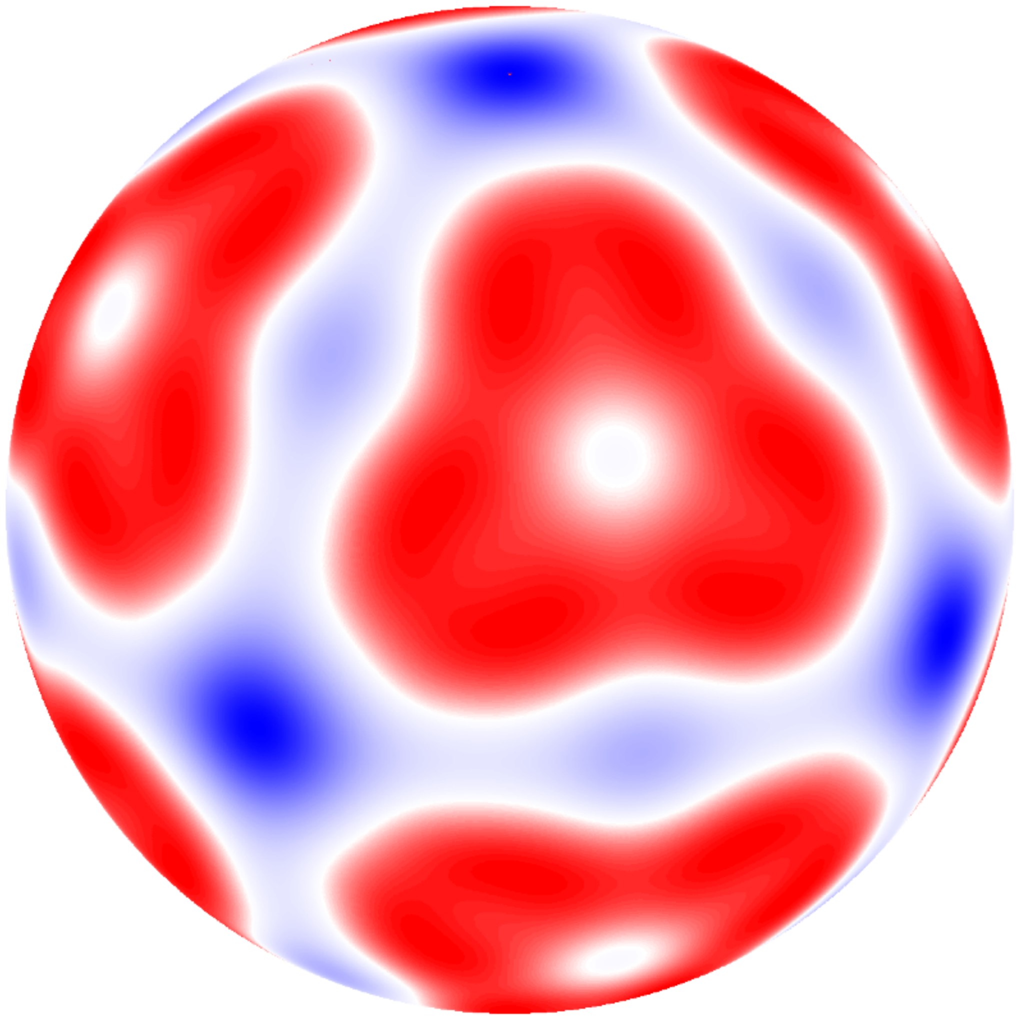
|v| = 0.40



$\max \bar{c}_0(\mathbf{v}) = 0.0171$

$\min \bar{c}_0(\mathbf{v}) = -0.0114$

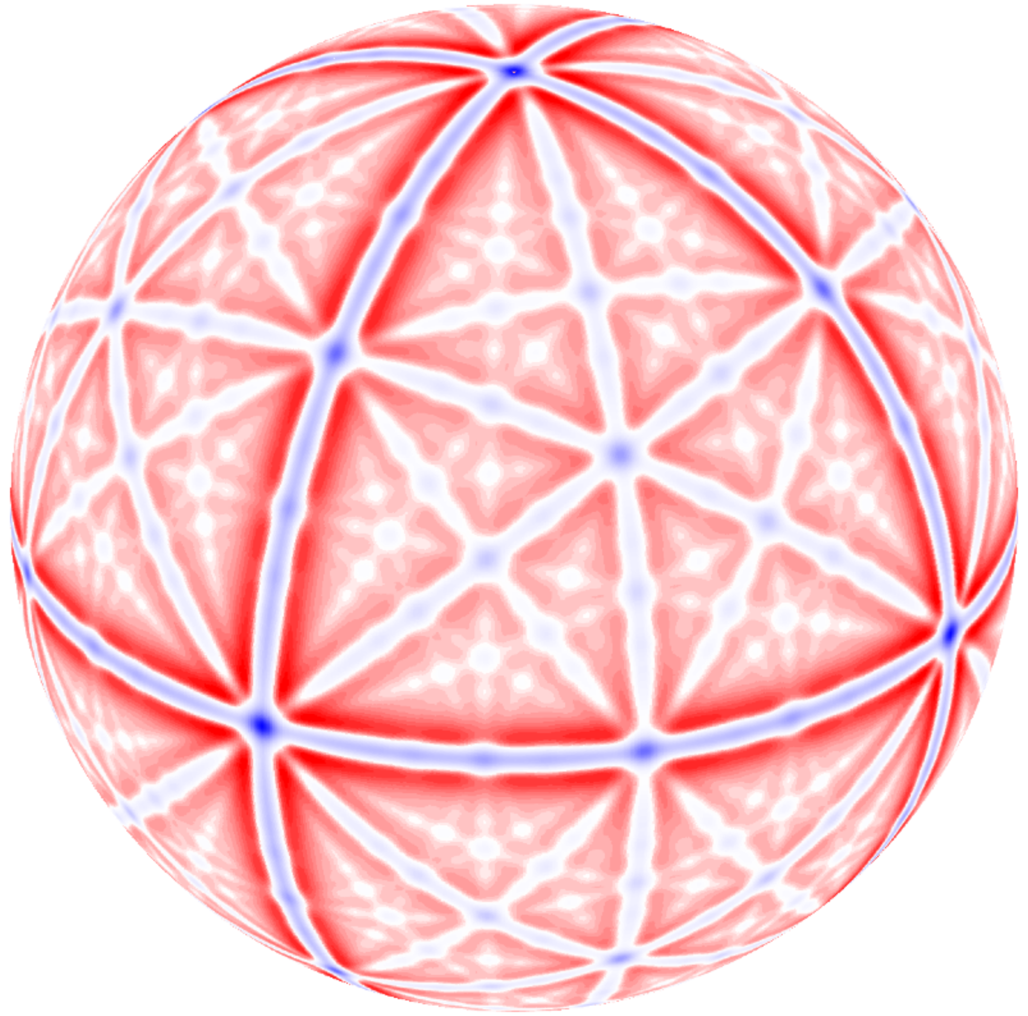
|v| = 0.95



$\max \bar{c}_0(\mathbf{v}) = 15.2832$

$\min \bar{c}_0(\mathbf{v}) = -2.8258$

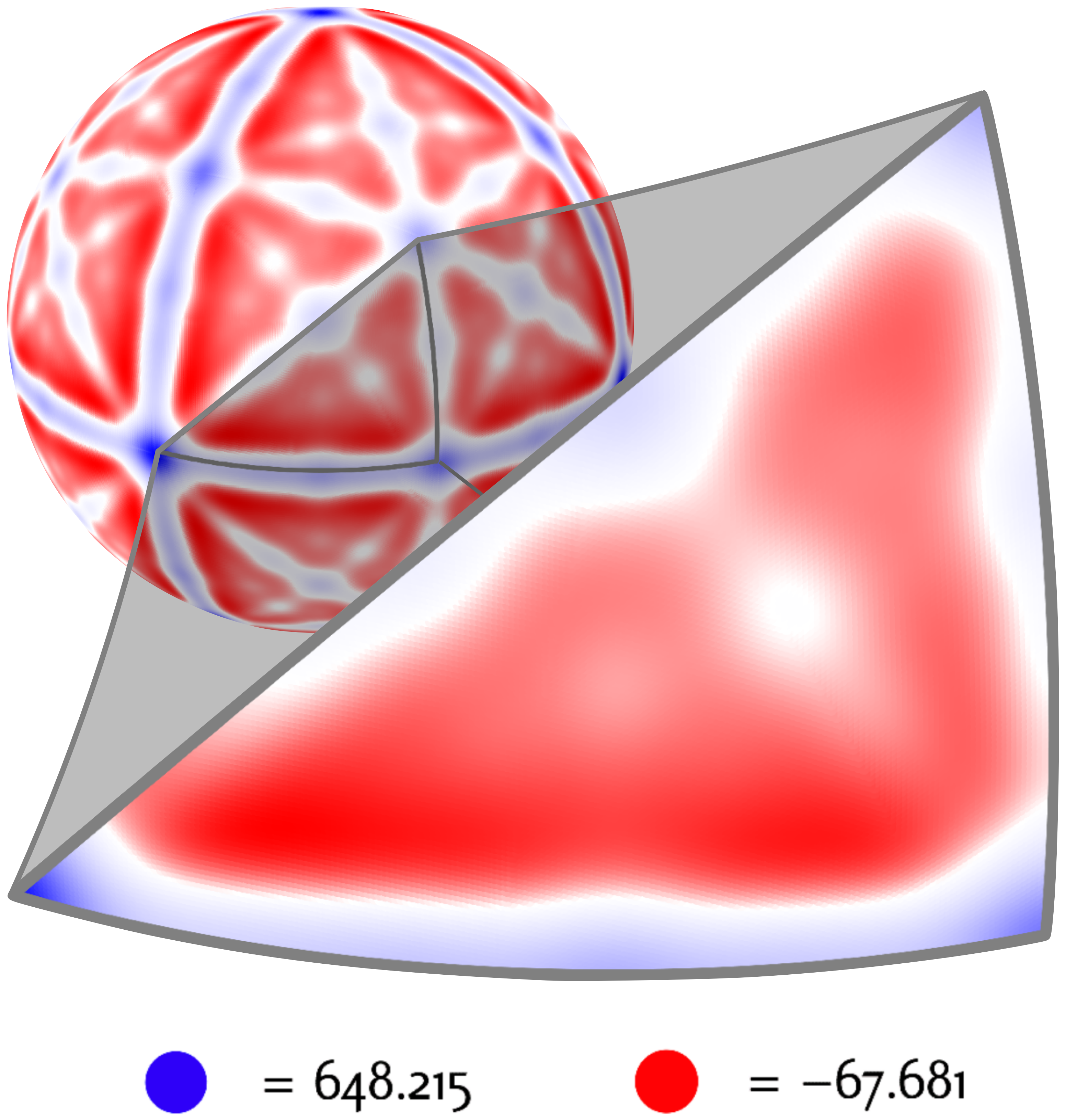
|v| = 0.999

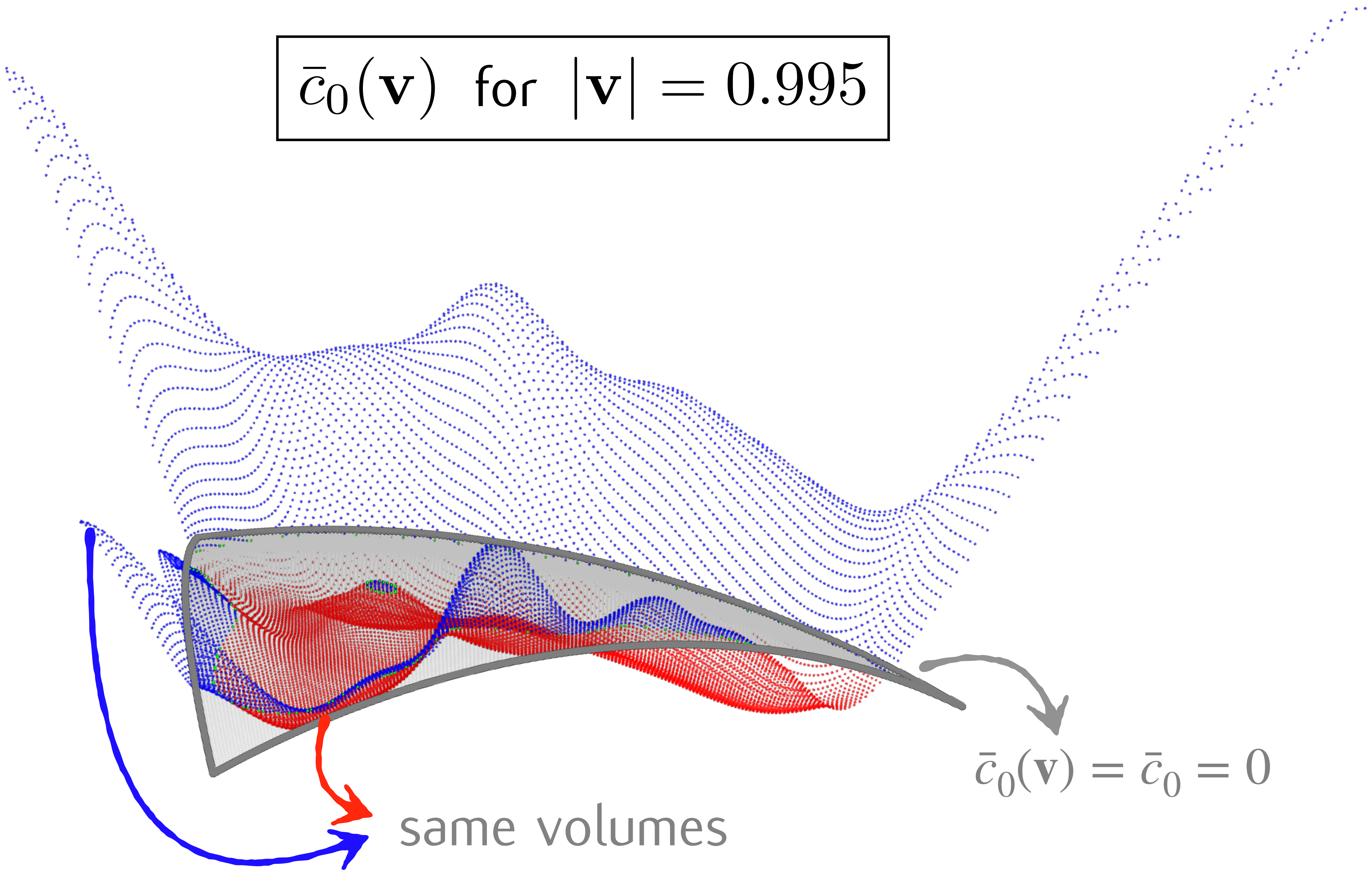


$\max \bar{c}_0(\mathbf{v}) = 9002.2317$

$\min \bar{c}_0(\mathbf{v}) = -807.4018$

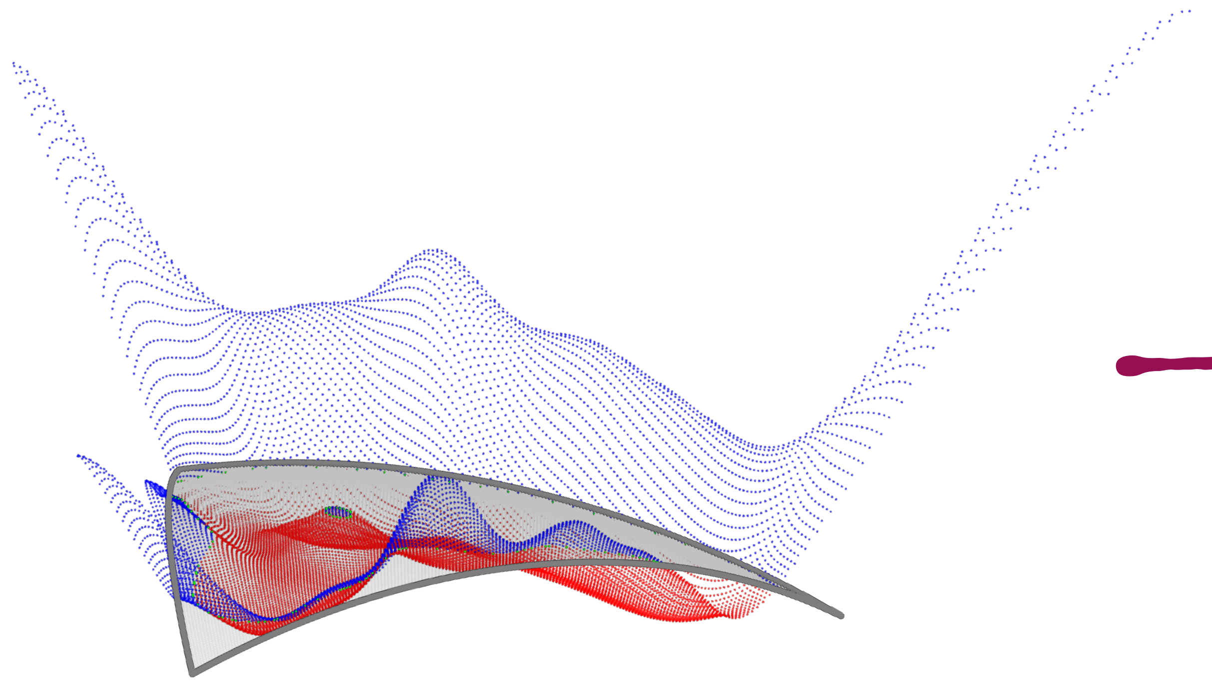
Velocity-dependent coefficients in QED_r



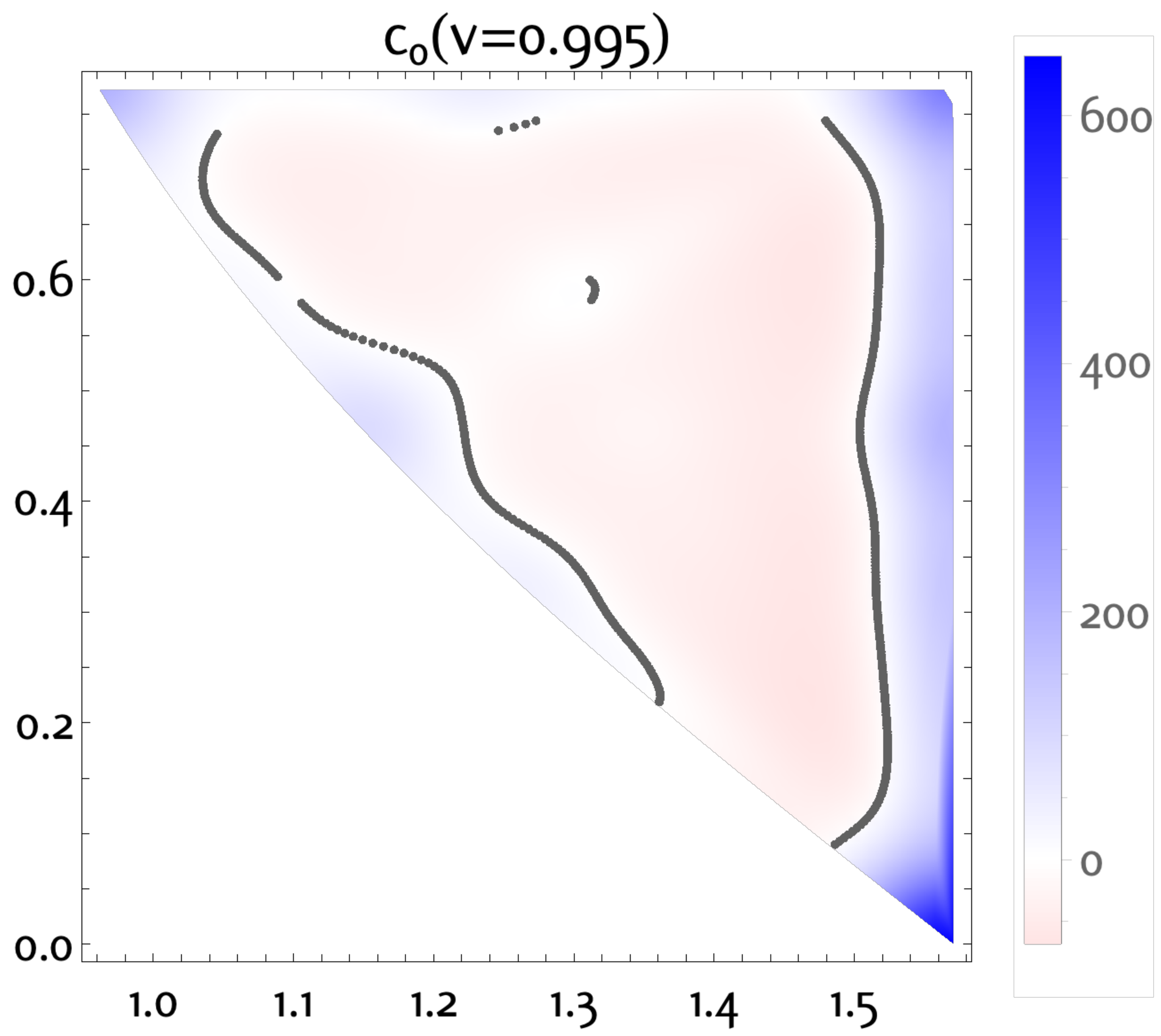


Velocity-dependent coefficients in QED$_r$

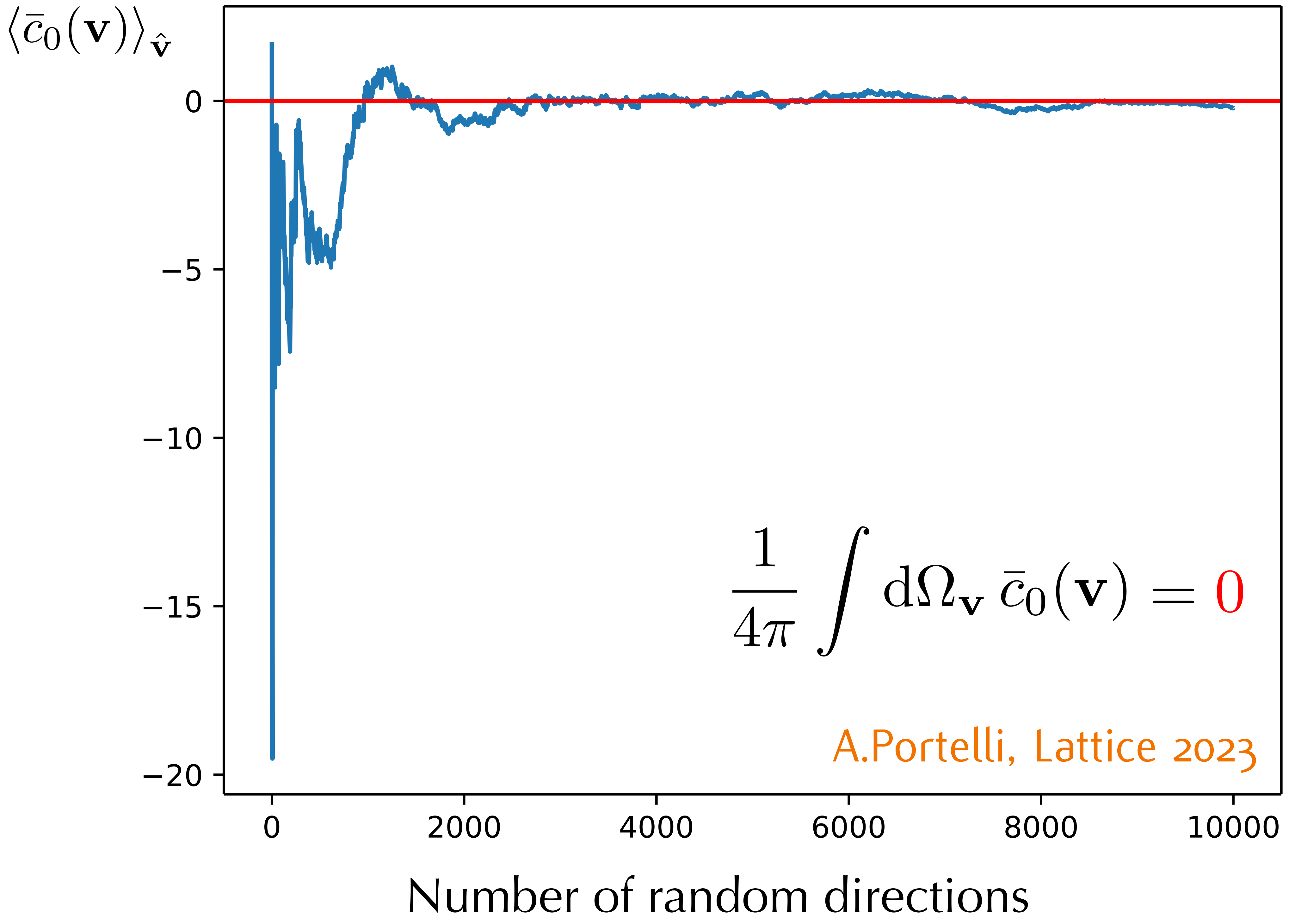
MDC et al., [2501.07936]



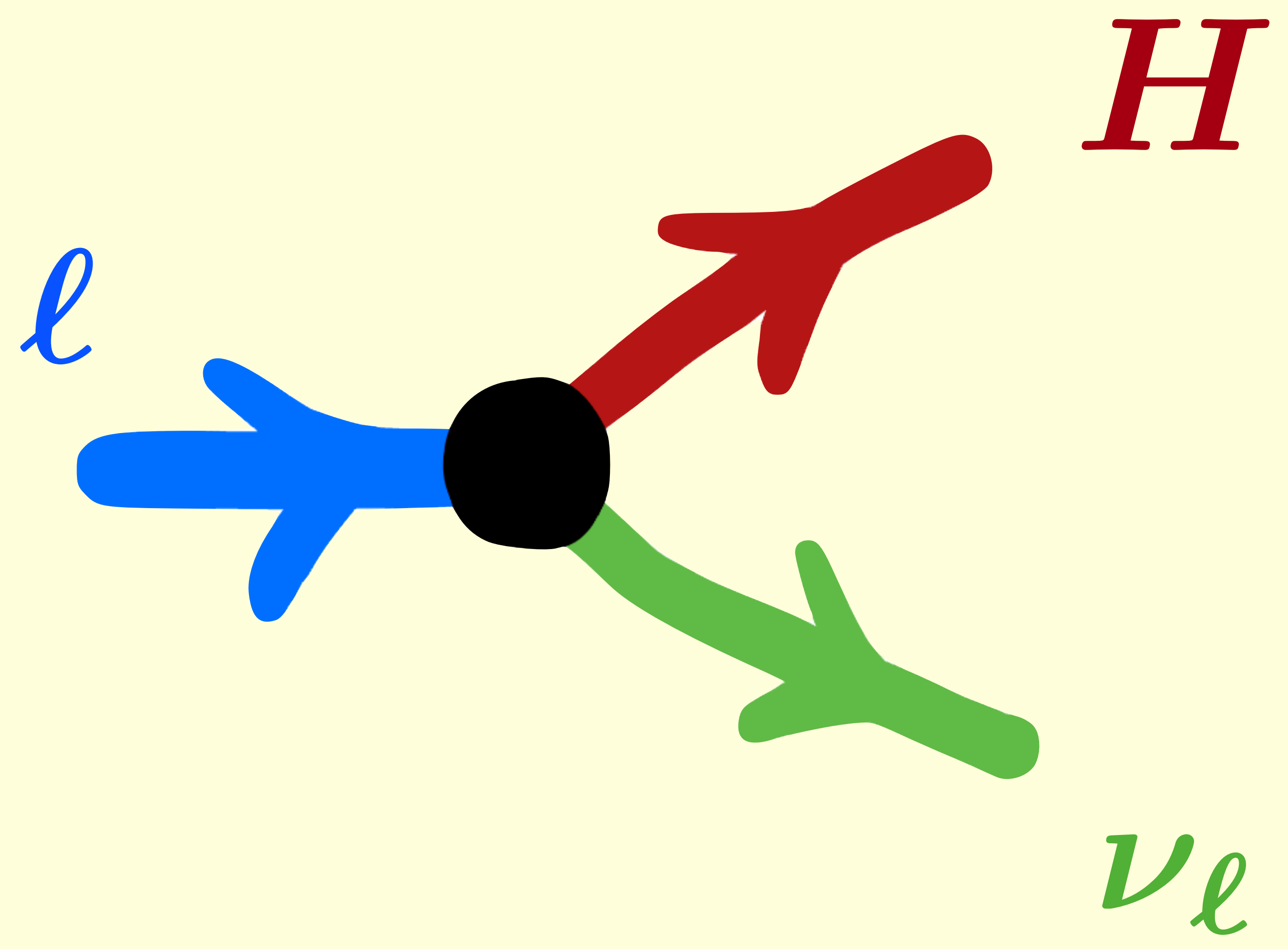
"magic angles"



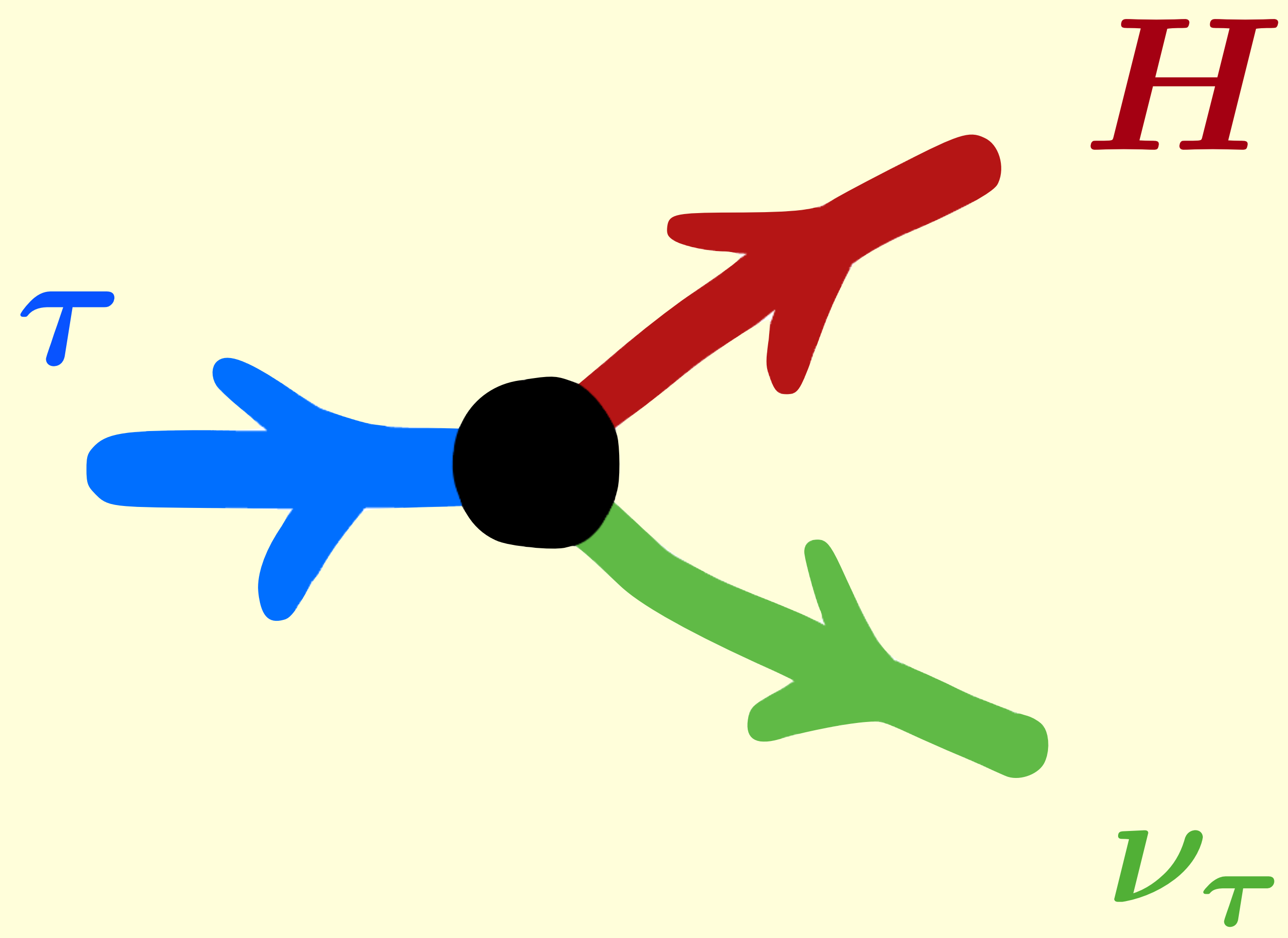
Stochastic direction average



Hadronic decays of leptons

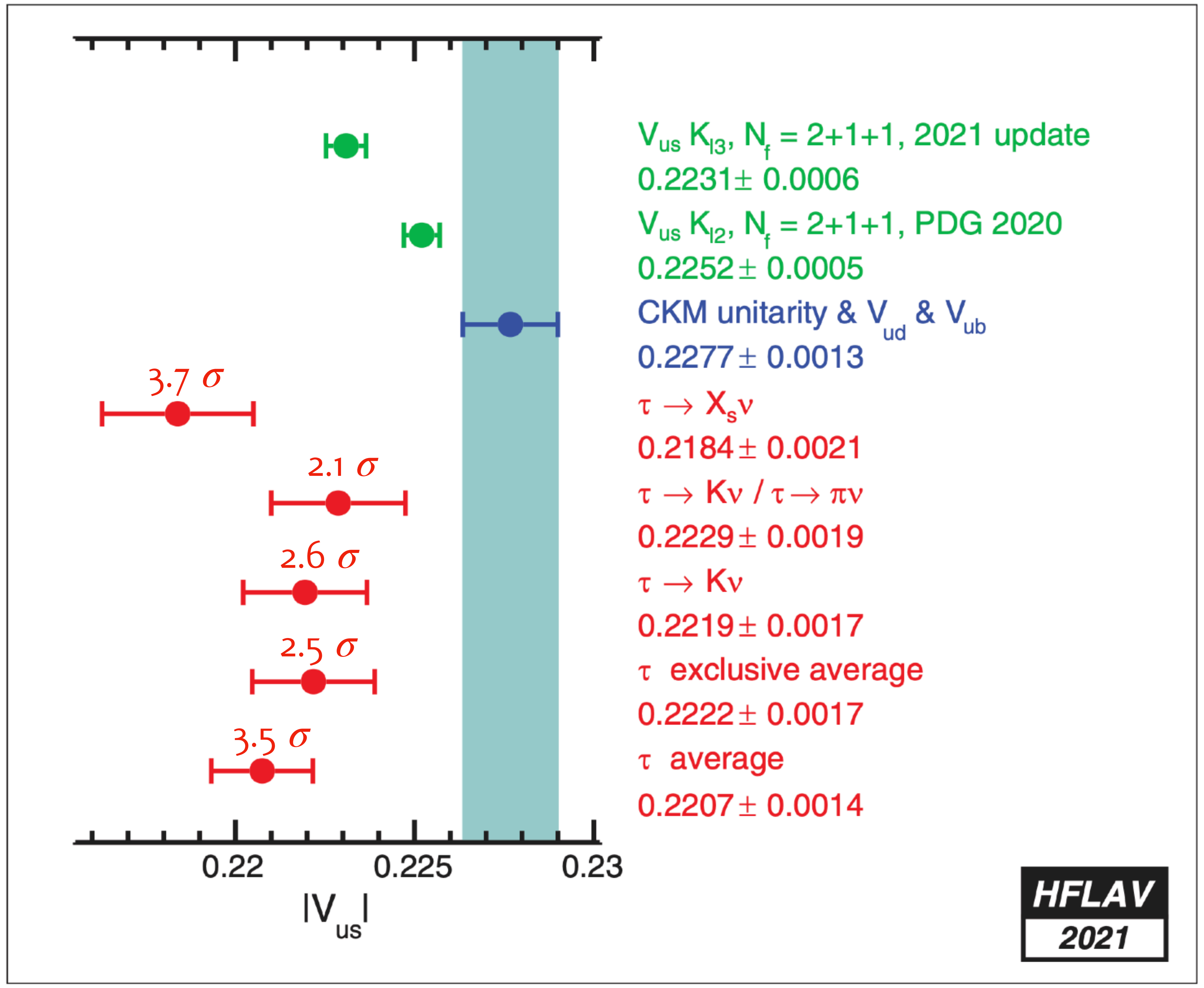


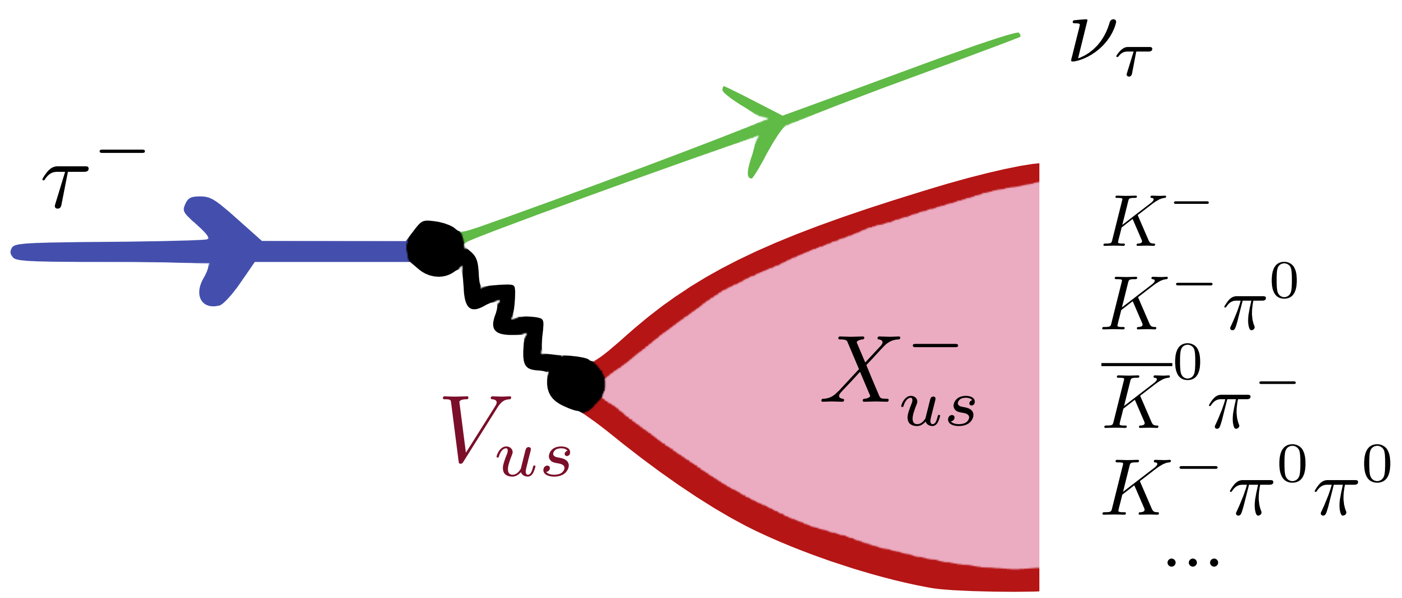
Hadronic decays of leptons



Inclusive hadronic τ decays

Alternative determinations of $|V_{us}|$ can be obtained from inclusive **hadronic τ decays**





- Yet another **puzzle**: lower value of $|V_{us}|_{\tau-\text{incl.}}$
- Inclusive $\tau \to X_{us}\nu_\tau$ result in HFLAV plot obtained using truncated OPE
- Exclusive channels give results larger than $|V_{us}|_{\tau-\text{incl.}}$ but smaller than that obtained imposing CKM unitarity

Inclusive hadronic τ decays

A.Evangelista et al. (ETMC), PRD **108** (2023)
C.Alexandrou et al. (ETMC), PRL **132** (2024)

Isosymmetric QCD

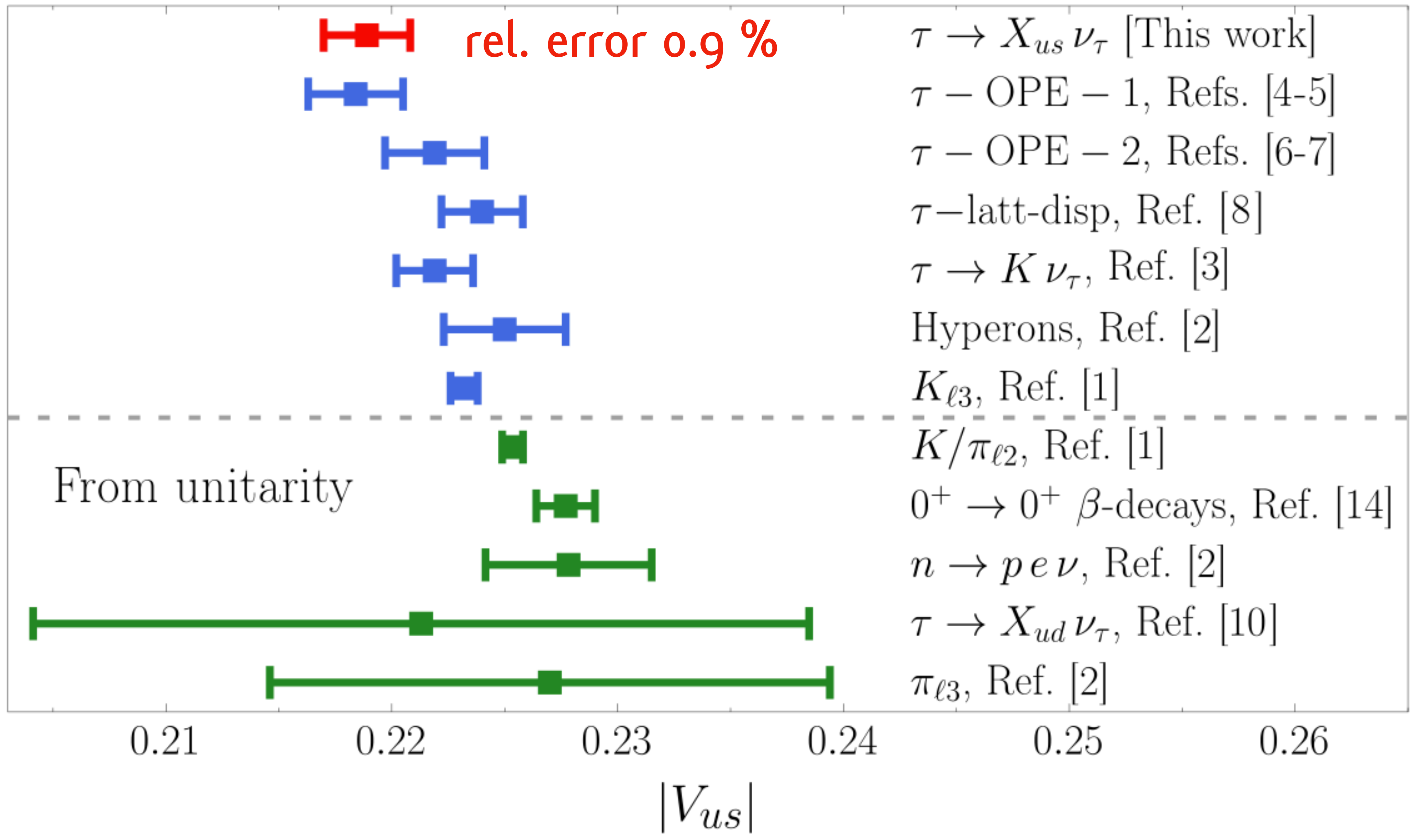
Recent calculation obtains **inclusive decay rate** using **smeared spectral densities** reconstructed from finite-volume **Euclidean lattice correlators** M.Hansen, A.Lupo & N.Tantalo, PRD **99** (2019)

$$\rho(q) = \langle \tau^- | \mathcal{H}_{\mathrm{w}}^{us} \, (2\pi)^4 \delta^4(\mathbb{P} - q) \, \mathcal{H}_{\mathrm{w}}^{us} | \tau^- \rangle$$

$$\hat{\rho}_L(E,\epsilon) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \Delta_\epsilon(E,\omega) \, \rho_L(\omega, \mathbf{0})$$

$$= \sum_{t=1}^{T} g_t(E,\epsilon) \, C_L(t)$$

$$\Rightarrow \quad \Gamma(\tau \to X_{us}\nu_\tau) = \lim_{\epsilon\to 0} \lim_{L\to\infty} \frac{\hat{\rho}_L(m_\tau,\epsilon)}{2m_\tau}$$



Result confirms $\sim 3\sigma$ **tension** between τ-inclusive and purely hadronic determinations

...underestimated exp. uncertainties? ...missing isospin-breaking effects?

Inclusive hadronic τ decays

Isospin breaking effects

Now we can consider the rate in full QCD+QED

$$\Gamma(\tau \to X_{us}\nu_\tau) = \frac{\rho(m_\tau, \mathbf{0})}{2m_\tau} \qquad \rho(q) = \langle \tau^- | \mathcal{H}_{\mathrm{w}}^{us}\,(2\pi)^4\delta^4(\mathbb{P}-q)\,\mathcal{H}_{\mathrm{w}}^{us} | \tau^- \rangle$$

with X_{us} being inclusive in hadrons + photons:

$$\Gamma = \Gamma_{\text{lep}} + \Gamma_{\text{fact}} + \Gamma_{\text{non-fact}}$$

- separately **infrared finite**
- computable to **all orders in** α_{em}

Building on **previous works:**

- current-current correlators already computed in other projects
- same $\mathcal{H}_{\mathrm{w}}$ as in $K_{\ell 2}$: similar non-perturbative QCD+QED renormalization

Università di Roma Tor Vergata
Antonio Evangelista
Roberto Frezzotti
Lorenzo Maio
Francesca Margari
Nazario Tantalo

Università di Roma Tre
Francesco D'Angelo
Giuseppe Gagliardi
Vittorio Lubicz
Francesco Sanfilippo
Silvano Simula

Humboldt Universität
Eric Bäske
Lukas Holan
Agostino Patella

Cyprus Institute
Simone Bacchio

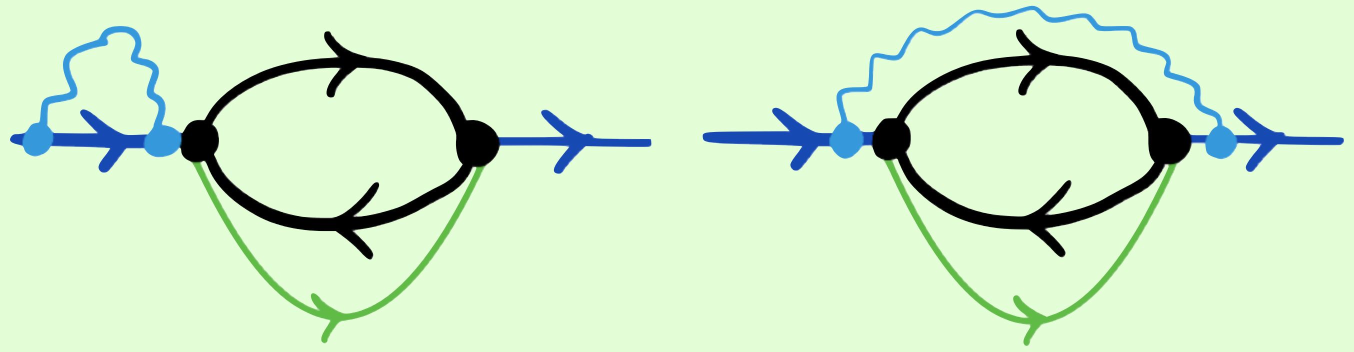
Helmholtz-Institut Mainz
Alessandro De Santis

CERN
Matteo Di Carlo

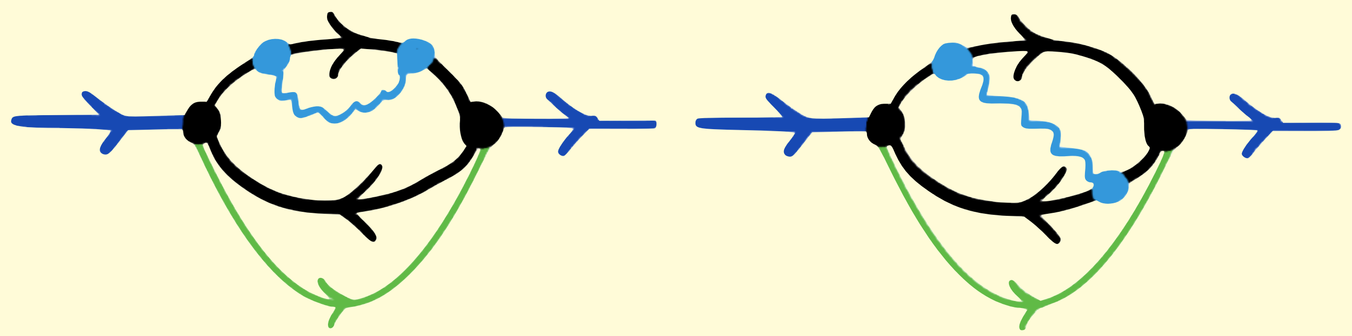
Inclusive hadronic τ decays

RM123 approach

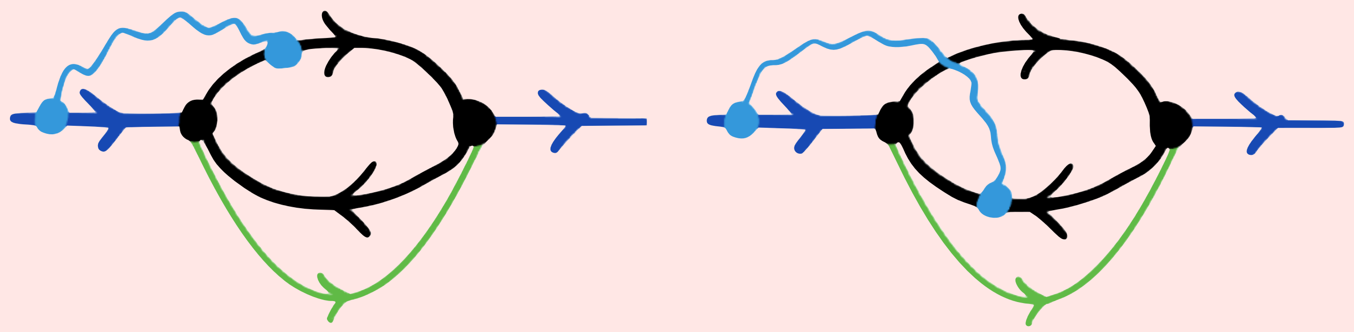
$$\Gamma = \Gamma_{\text{lep}} + \Gamma_{\text{fact}} + \Gamma_{\text{non-fact}}$$



$$\Gamma_{\text{lep}} = \frac{G_F^2\, m_\tau^5}{(4\pi)^4} \int_0^\infty ds\, \left[\delta\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}(s) + \delta\mathcal{K}_{\text{L}}(s)\rho_{\text{L}}(s)\right]$$



$$\Gamma_{\text{fact}} = \frac{G_F^2\, m_\tau^5}{32\pi^2} \int_0^\infty ds\, \left[\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}^{\text{full}}(s) + \mathcal{K}_{\text{L}}(s)\rho_{\text{L}}^{\text{full}}(s)\right]$$



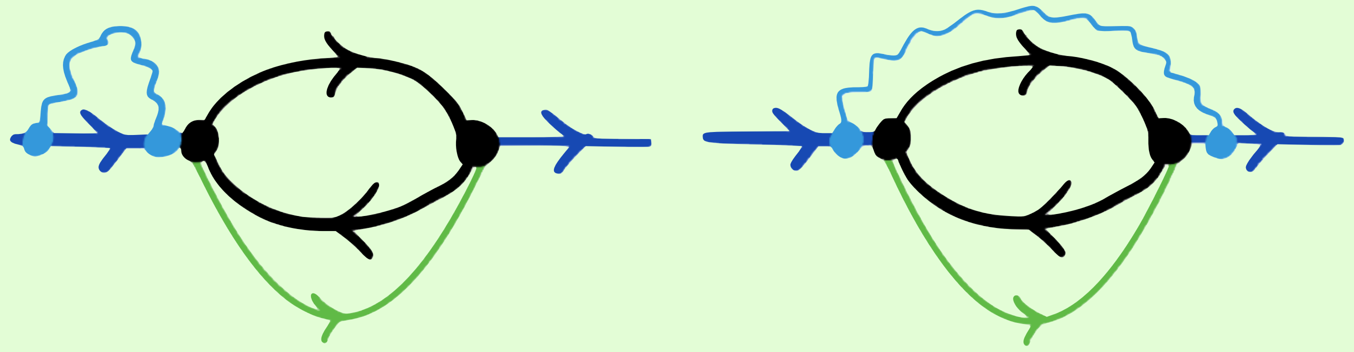
$$\Gamma_{\text{non-fact}} = \frac{G_F^2\, m_\tau^5}{64\pi^2} \int_0^\infty ds\, \mathcal{K}(s)\, \delta\rho(s)$$

Inclusive hadronic τ decays

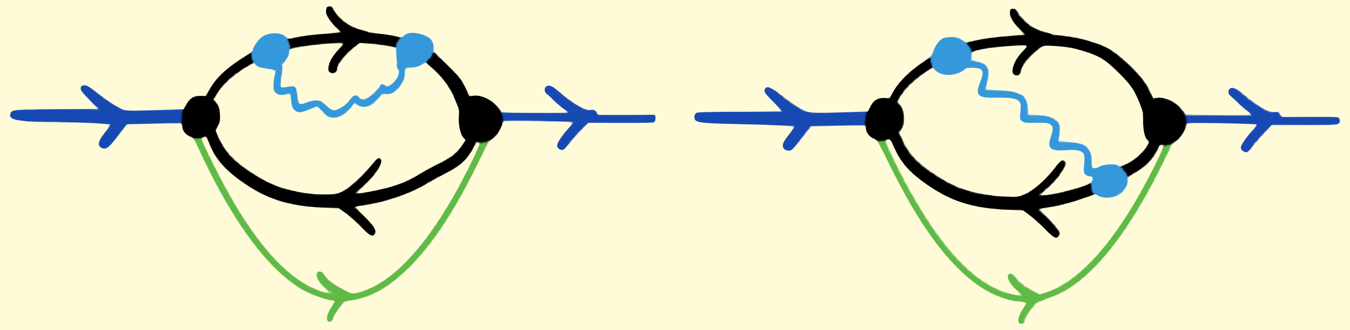
RM123 approach

$$\Gamma = \Gamma_{\text{lep}} + \Gamma_{\text{fact}} + \Gamma_{\text{non-fact}}$$

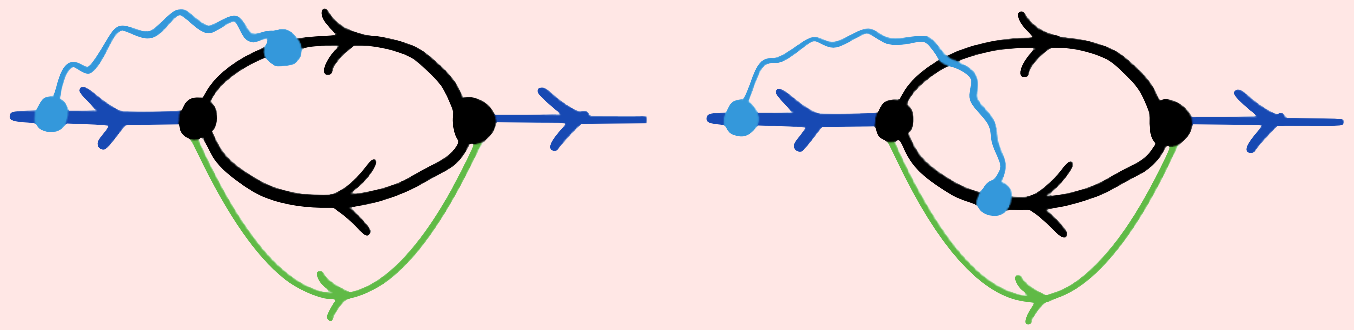
preliminary data look promising!



$$\Gamma_{\text{lep}} = \frac{G_F^2\, m_\tau^5}{(4\pi)^4} \int_0^\infty ds\, \left[\delta\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}(s) + \delta\mathcal{K}_{\text{L}}(s)\rho_{\text{L}}(s)\right]$$



$$\Gamma_{\text{fact}} = \frac{G_F^2\, m_\tau^5}{32\pi^2} \int_0^\infty ds\, \left[\mathcal{K}_{\text{T}}(s)\rho_{\text{T}}^{\text{full}}(s) + \mathcal{K}_{\text{L}}(s)\rho_{\text{L}}^{\text{full}}(s)\right]$$



$$\Gamma_{\text{non-fact}} = \frac{G_F^2\, m_\tau^5}{64\pi^2} \int_0^\infty ds\, \mathcal{K}(s)\, \delta\rho(s)$$

Other applications of spectral reconstruction methods

Huge **theoretical** and **computational progress** in the recent years. see A.Patella's talk on Friday

Spectral reconstruction methods applied to a **variety of processes**:

R-ratio, inclusive heavy meson decay rates, radiative leptonic decays, properties of quark-gluon plasma, scattering amplitudes, etc...

Other applications of spectral reconstruction methods

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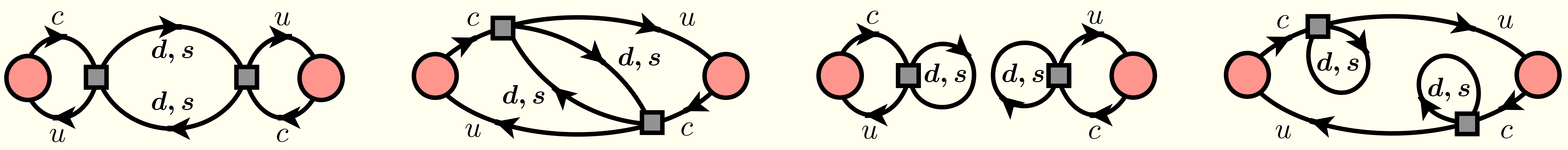
R-ratio, inclusive heavy meson decay rates, radiative leptonic decays, properties of quark-gluon plasma, scattering amplitudes, etc...

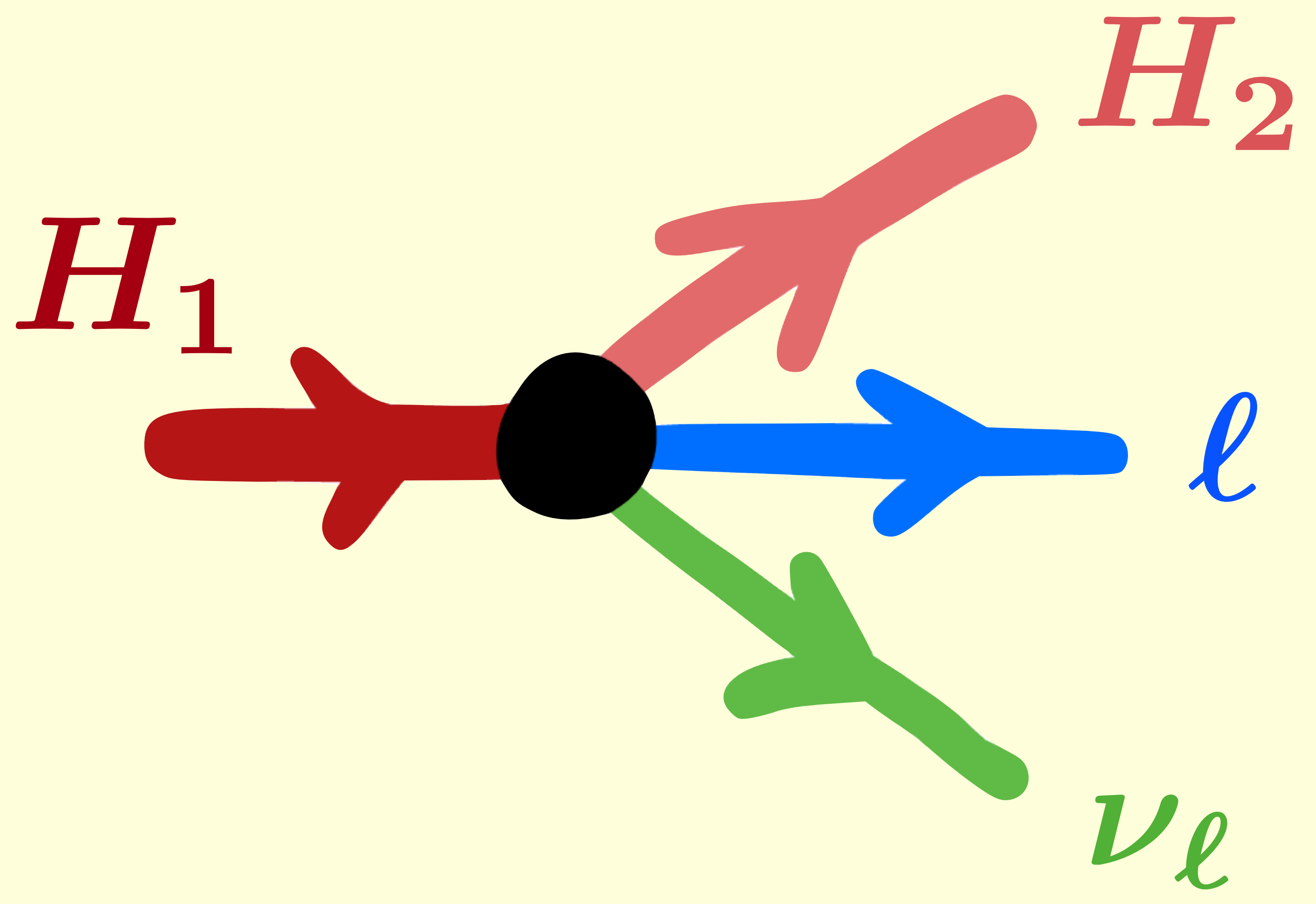
Novel application: long-distance contributions to neutral D-meson mixing [soon on arXiv!]

with **Felix Erben** (CERN) & **Max Hansen** (Edinburgh)

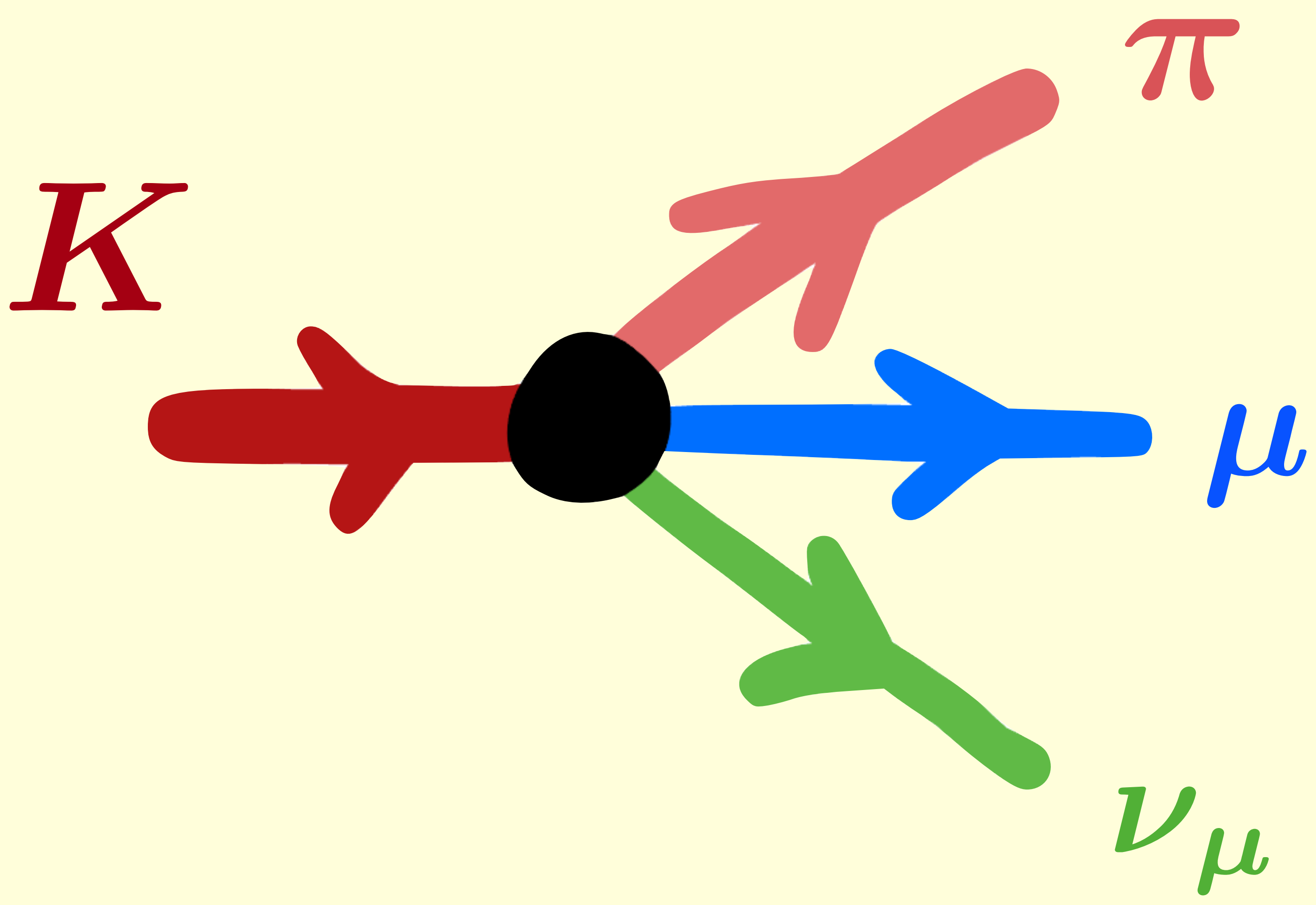
$$\mathcal{M}_{D^0\to\bar{D}^0} = -\langle\bar{D}^0,\boldsymbol{p}_D|\mathcal{H}_\mathrm{w}(0)|D^0,\boldsymbol{p}_D\rangle + \frac{\mathrm{i}}{2}\int \mathrm{d}^4x\,\langle\bar{D}^0,\boldsymbol{p}_D|\mathrm{T}\{\mathcal{H}_\mathrm{w}(x)\mathcal{H}_\mathrm{w}(0)\}|D^0,\boldsymbol{p}_D\rangle$$



Semi-leptonic decays of hadrons

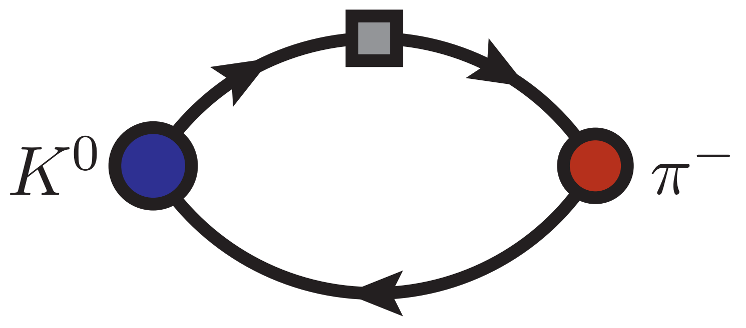


Semi-leptonic decays of hadrons



QED corrections to semileptonic decays

- Without QED corrections:



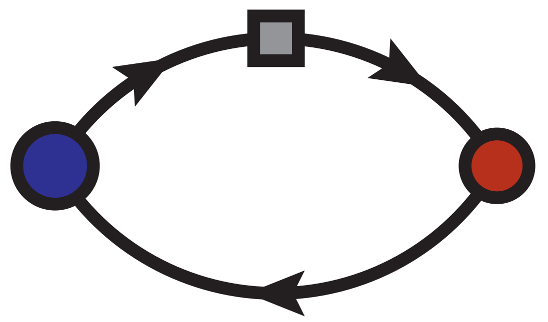
$$\langle \pi(p_\pi)|\bar{s}\gamma^\mu u|K(p_K)\rangle = \boldsymbol{f_+(q^2)}\left[(p_\pi + p_K)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu\right] + \boldsymbol{f_0(q^2)}\,\frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu$$

An appropriate observable to study is the **differential decay rate:** $s_{\pi\ell} = (p_\pi + p_\ell)^2,\; q^2 = (p_K - p_\pi)^2$

$$\frac{\mathrm{d}^2\Gamma^{(0)}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = G_F^2|V_{us}|^2\left[a_1(q^2, s_{\pi\ell})\,|\boldsymbol{f_+(q^2)}|^2 + a_2(q^2, s_{\pi\ell})\,\boldsymbol{f_+(q^2)f_0(q^2)} + a_3(q^2, s_{\pi\ell})\,|\boldsymbol{f_0(q^2)}|^2\right]$$

QED corrections to semileptonic decays

- **Without QED corrections:**



$$\langle \pi(p_\pi)|\bar{s}\gamma^\mu u|K(p_K)\rangle = f_+(q^2)\left[(p_\pi + p_K)^\mu - \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu\right] + f_0(q^2)\, \frac{m_K^2 - m_\pi^2}{q^2}\, q^\mu$$

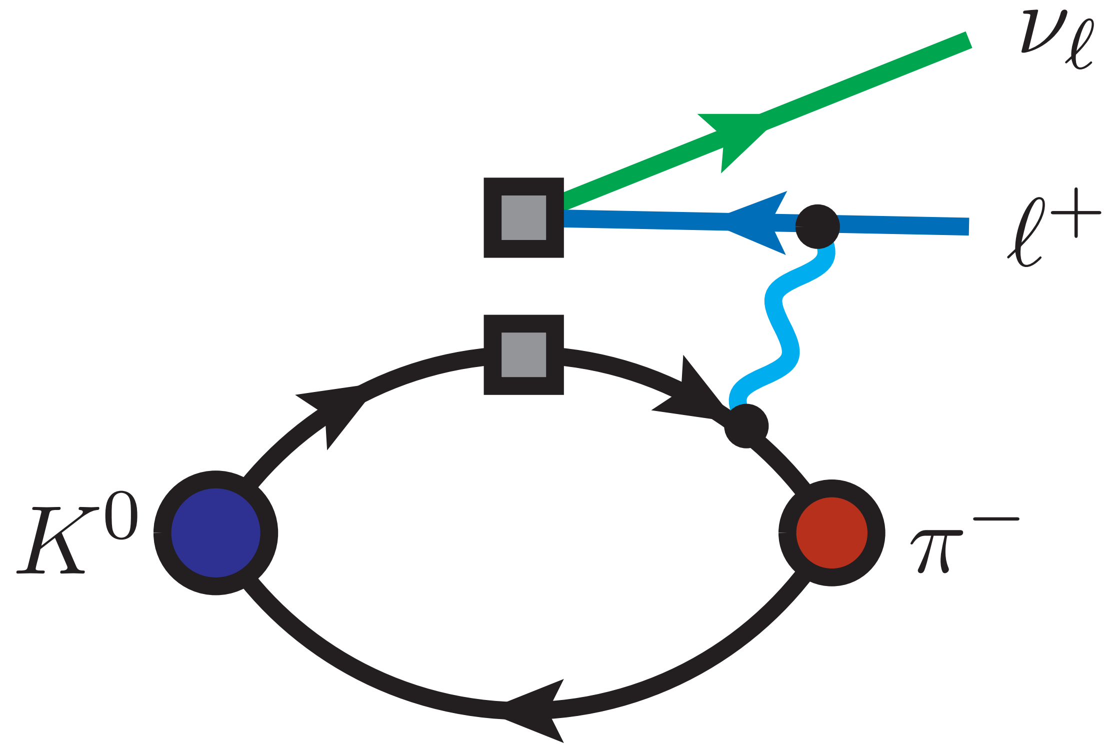
An appropriate observable to study is the **differential decay rate:** $s_{\pi\ell} = (p_\pi + p_\ell)^2,\ q^2 = (p_K - p_\pi)^2$

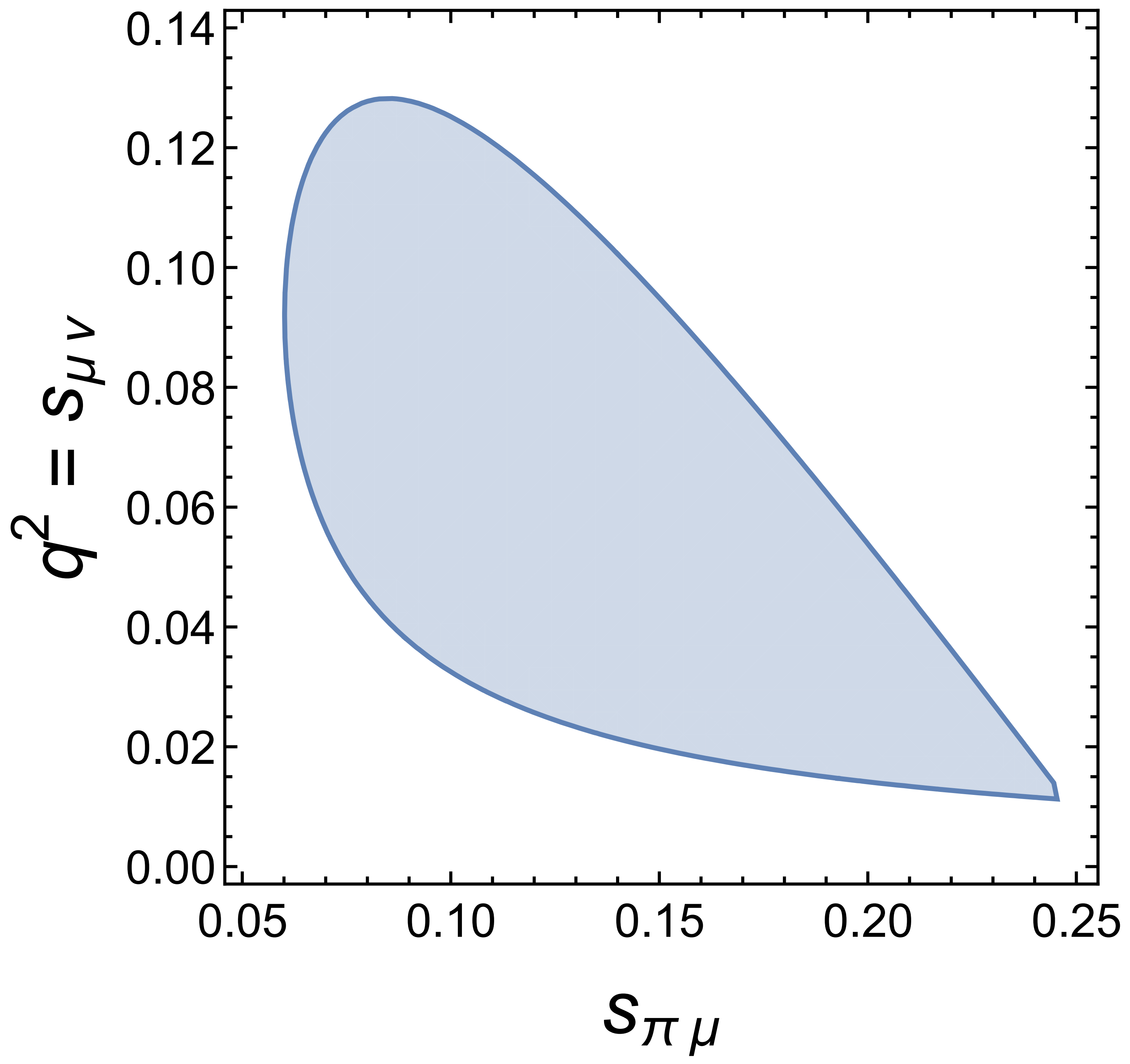
$$\frac{\mathrm{d}^2\Gamma^{(0)}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = G_F^2 |V_{us}|^2 \left[a_1(q^2, s_{\pi\ell})\,|f_+(q^2)|^2 + a_2(q^2, s_{\pi\ell})\, f_+(q^2) f_0(q^2) + a_3(q^2, s_{\pi\ell})\,|f_0(q^2)|^2\right]$$

- **Including QED**, we can treat IR divergences using the **RM123S method**: C.Sachrajda et al., [1910.07342]

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} = \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} - \frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right] + \lim_{\Lambda_{\mathrm{IR}}\to 0}\left[\frac{\mathrm{d}^2\Gamma_0^{\mathrm{pt}}}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}} + \frac{\mathrm{d}^2\Gamma_1}{\mathrm{d}q^2\mathrm{d}s_{\pi\ell}}\right]$$

QED corrections to semileptonic decays





Although the RM123+Soton method could in principle be applied, additional **difficulties** arise compared to **leptonic decays**:

- integration over **three-body phase-space**
- problems of **analytical continuation** when intermediate on-shell states are lighter than external ones

$$\{\omega_\pi(\boldsymbol{p_\pi} + \boldsymbol{k}) + \omega_\ell(\boldsymbol{p_\ell} - \boldsymbol{k})\} - \{\omega_\pi(\boldsymbol{p_\pi}) + \omega_\ell(\boldsymbol{p_\ell})\} < 0$$

A proper **finite-volume formalism is still missing...**

› Lellouch-Lüscher + QED?

› Spectral reconstruction techniques?

Conclusions and outlooks

- Current **tensions** in CKM unitarity require a combined effort of **theory** and **experiments**
- **Two lattice calculations** of IB and QED corrections to **light-meson leptonic decay rates**
- **Finite volume QED effects** have to be investigated to reach high precision on $|V_{us}/V_{ud}|$

✦ **QED$_r$ regularisation** could help removing unknown $1/L^3$ structure-dependent contributions

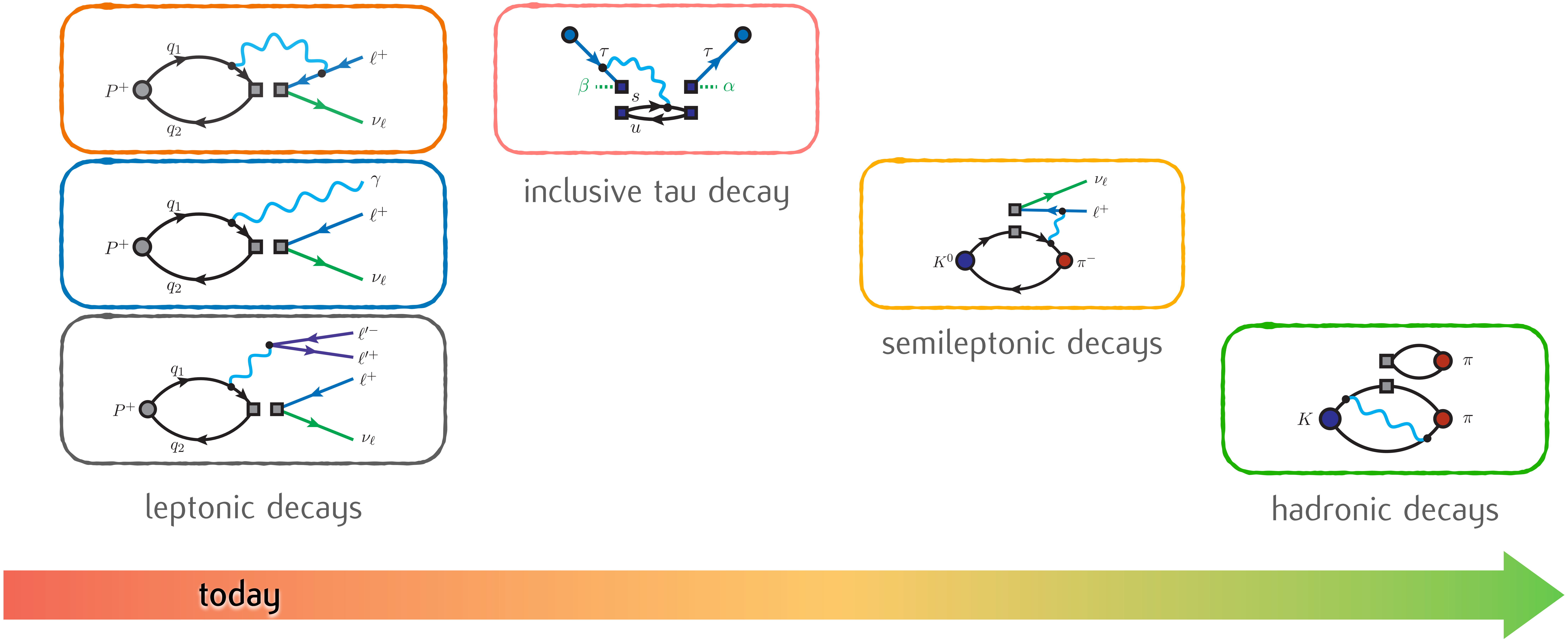
✦ Extension of the calculation to **multiple lattice spacings and volumes** is crucial

✦ Next important step: going **beyond electro-quenched approximation**

▸ **Spectral reconstruction methods** paves the way to the calculation IB effects in new processes:
inclusive tau decay, semileptonic decays, ... ?

Conclusions and outlooks

An interesting future ahead

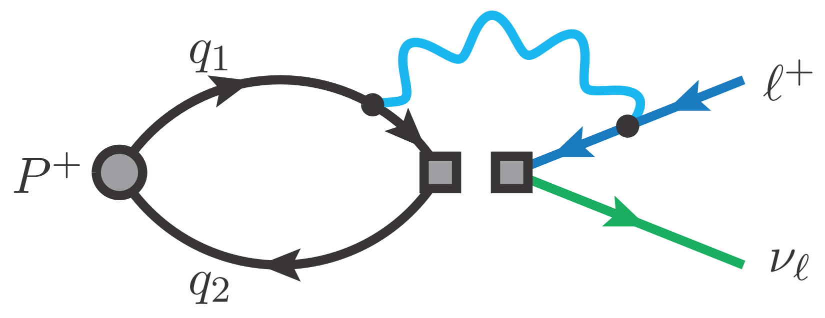


Thank you

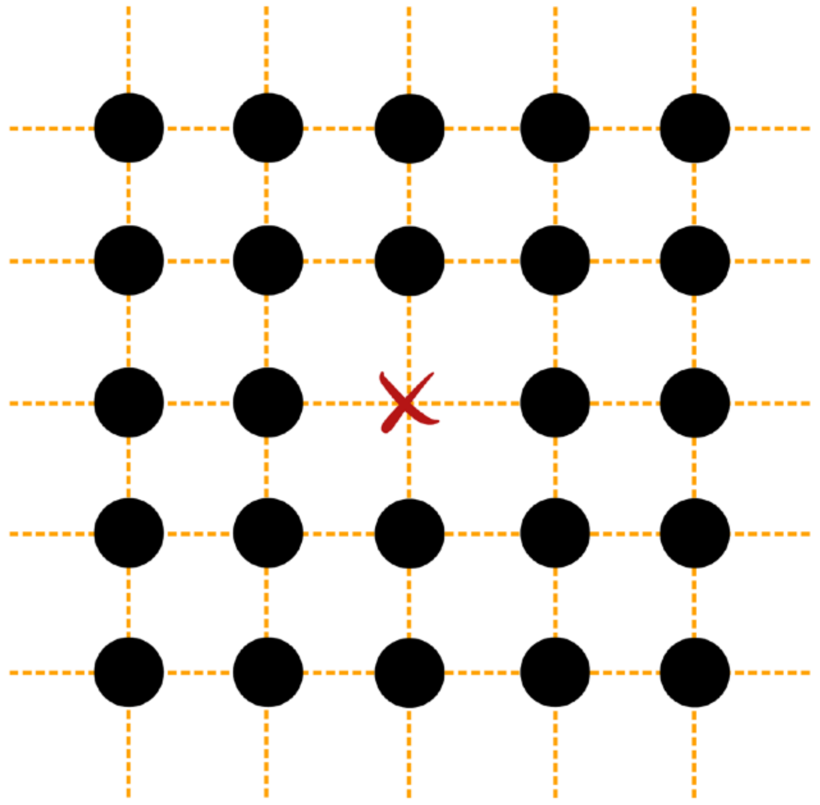
This work has received funding from the European Union's Horizon Europe research and innovation programme under the Marie Sklodowska-Curie grant agreement No 101108006

Backup slides

Removing the zero mode

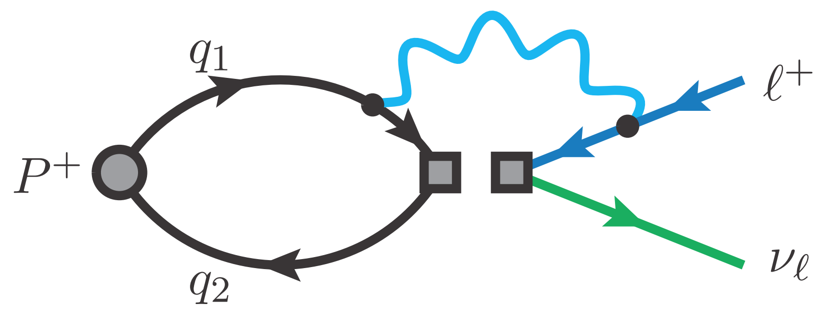


$$= \frac{1}{L^3}\sum_{\mathbf{k}}\int^{1/a}\frac{\mathrm{d}k_0}{2\pi}\,\frac{1-\boldsymbol{\delta_{\mathbf{k},0}}}{k^2}\,H^{\mu\nu}(k)L_{\mu\nu}(k)$$



$\Omega_3 = 2\pi\mathbb{Z}^3/L$

$$j^\mu(x)j^\nu_{\mathrm{w}}(0) \sim \frac{O^{\mu\nu}(0)}{x^3} \qquad H^{\mu\nu}(k) = \int \mathrm{d}^4x\, e^{ikx}\,\mathrm{T}\langle 0|j^\mu(x)j^\nu_{\mathrm{w}}(0)|P(p)\rangle \sim \frac{1}{k}$$



$$\sim \frac{1}{L^3}\sum_{\mathbf{k}}\int^{1/a}\frac{\mathrm{d}k_0}{2\pi}\,\frac{1-\boldsymbol{\delta_{\mathbf{k},0}}}{k^4}$$

The effect of quenching the zero mode will amount to

$$\frac{1}{L^3}\int^{1/a}\frac{\mathrm{d}k_0}{2\pi}\,\frac{1}{k_0^4} \sim \frac{a^3}{L^3}$$

-> **no new UV divergences** expected, but some interplay of cut-off and finite-volume effects

RM123S: lattice setup

Ensemble	β	V/a^4	N_{cfg}	$a\mu_{sea} = a\mu_{ud}$	$a\mu_\sigma$	$a\mu_\delta$	$a\mu_s$	M_π (MeV)	M_K (MeV)	$M_\pi L$
$A40.40$	1.90	$40^3 \times 80$	100	0.0040	0.15	0.19	0.02363	317 (12)	576 (22)	5.7
$A30.32$		$32^3 \times 64$	150	0.0030				275 (10)	568 (22)	3.9
$A40.32$			100	0.0040				316 (12)	578 (22)	4.5
$A50.32$			150	0.0050				350 (13)	586 (22)	5.0
$A40.24$		$24^3 \times 48$	150	0.0040				322 (13)	582 (23)	3.5
$A60.24$			150	0.0060				386 (15)	599 (23)	4.2
$A80.24$			150	0.0080				442 (17)	618 (14)	4.8
$A100.24$			150	0.0100				495 (19)	639 (24)	5.3
$A40.20$		$20^3 \times 48$	150	0.0040				330 (13)	586 (23)	3.0
$B25.32$	1.95	$32^3 \times 64$	150	0.0025	0.135	0.170	0.02094	259 (9)	546 (19)	3.4
$B35.32$			150	0.0035				302 (10)	555 (19)	4.0
$B55.32$			150	0.0055				375 (13)	578 (20)	5.0
$B75.32$			80	0.0075				436 (15)	599 (21)	5.8
$B85.24$		$24^3 \times 48$	150	0.0085				468 (16)	613 (21)	4.6
$D15.48$	2.10	$48^3 \times 96$	100	0.0015	0.1200	0.1385	0.01612	223 (6)	529 (14)	3.4
$D20.48$			100	0.0020				256 (7)	535 (14)	3.9
$D30.48$			100	0.0030				312 (8)	550 (14)	4.7

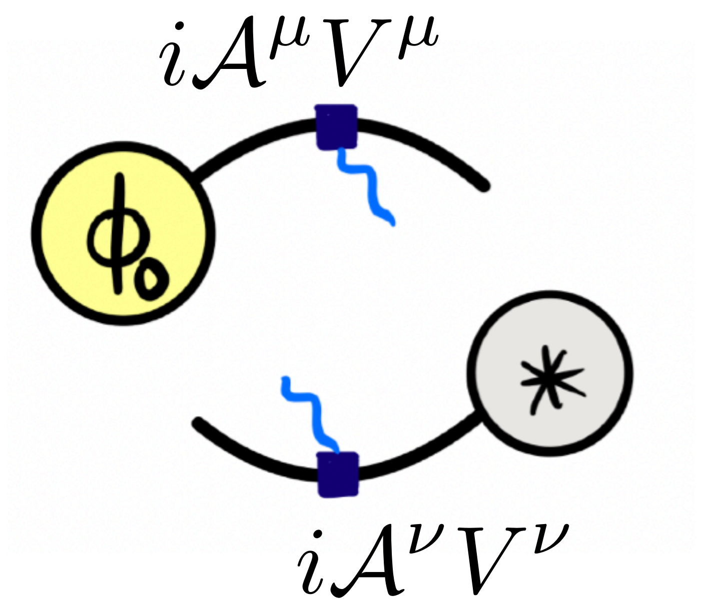
a = { **0.0885**(36), **0.0815**(30), **0.0619**(18) } fm at β = { 1.90, 1.95, 2.10 }

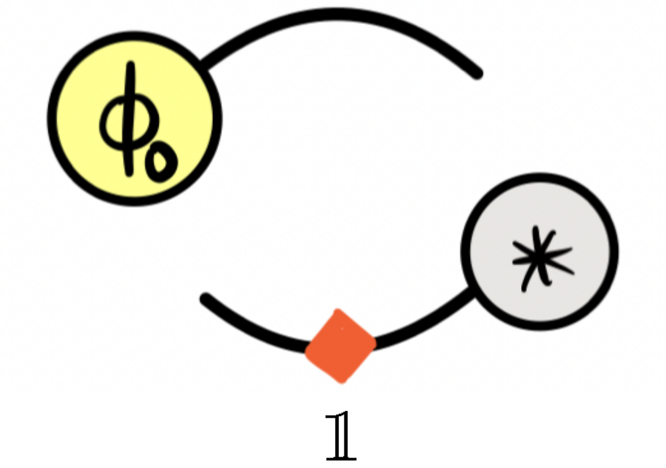
RBC/UKQCD: lattice setup

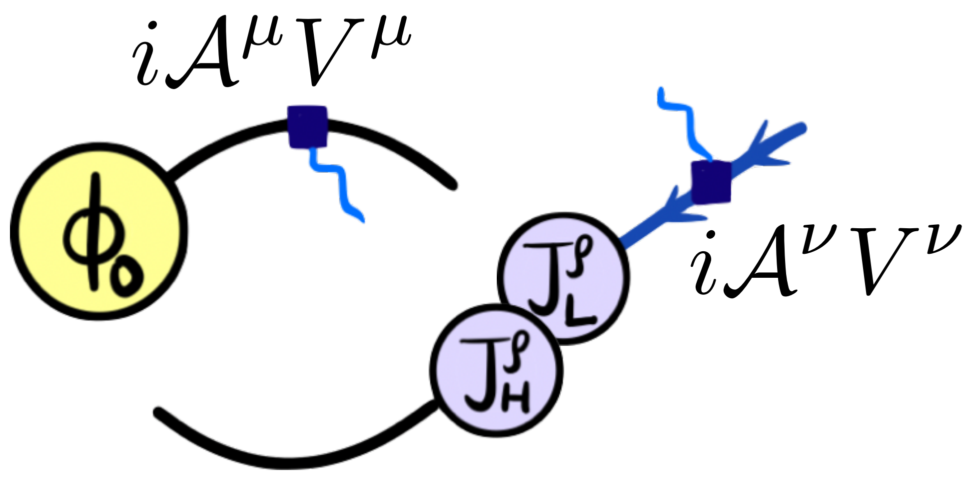
RBC/UKQCD Collaboration, PRD 93 (2016) 074505

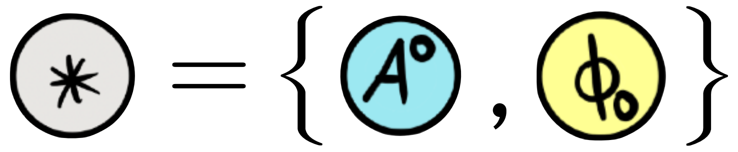
- Physical point **Möbius domain wall fermion** ensemble [$M_\pi = 139.15\,(36)$ MeV]
- $\mathrm{N_f} = 2 + 1$ flavours
- **Lattice geometry:** $48^3 \times 96\ (\times 24)_{L_s}$, $a^{-1} \simeq 1.730$ GeV (0.11 fm)
- **Valence light quarks:** zMöbius domain wall fermion action ($L_s = 10$)
  Charged lepton: free domain wall fermion action
- 60 configurations

Numerical implementation of correlators



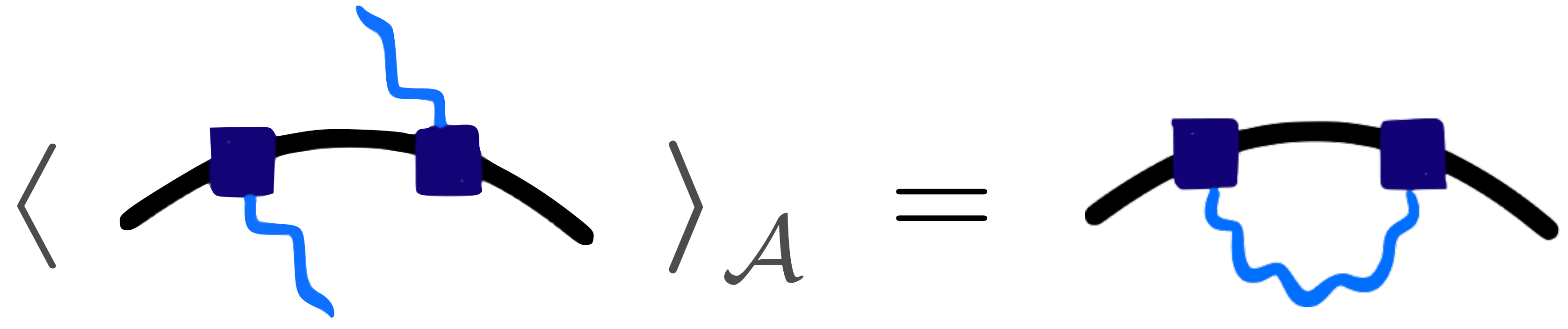






- Correlators created using **sequential propagators**
- **Muon momentum** $\mathbf{p}_\ell \propto \{1,1,1\}$ fixed by energy conservation & injected via twisted boundary conditions
- **Photon fields** sampled from Gaussian distribution ($\mathrm{QED_L}$)

$$P(\tilde{A})\mathrm{d}\tilde{A} \propto \mathrm{e}^{-S_\gamma[\tilde{A}]} \qquad S_\gamma^{\text{Feyn.}}[\tilde{A}] = \frac{1}{2V}\sum_{k_0,\mathbf{k}\neq\mathbf{0}} \hat{k}^2 \sum_\mu |\tilde{A}_\mu(k)|^2$$



- **Sources** ϕ_0 : point (RM123S) / Coulomb gauge-fixed wall (RBC-UKQCD)
- **Electromagnetic current:** conserved (RM123S) / local (RBC-UKQCD)

A general comparison of the calculations

	RBC/UKQCD	RM123+Soton
physical masses	✓ physical point simulations	extrapolation needed
chiral symmetry	✓ at finite lattice spacing	recovered in the continuum
fermionic action	Domain Wall	Twisted Mass
continuum limit	single lattice spacing	✓ continuum limit (3)
infinite volume limit	single volume	✓ multiple volumes
QED prescription	QED_L	QED_L
sea effects	electro-quenching	electro-quenching
IB scheme	BMW [a]	GRS [b]

[a] BMW, PRL 111 (2013); BMW, PRL 117 (2016)

[b] Gasser, Rusetsky & Scimemi, EPJC 32 (2003); RM123, PRD 87 (2013)

Defining the iso-symmetric theory

BMW, PRL 111 (2013)
BMW, PRL 117 (2016)

RBC/UKQCD (2023): BMW scheme with $N_f=2+1$

QCD+QED

$(\hat{m}_{ud}^{\phi}, \delta\hat{m}^{\phi}, \hat{m}_s^{\phi}|g, \alpha^{\phi})$

$$\left(\frac{\hat{m}_{\pi^+}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{m}_{K^+}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{m}_{K^0}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\phi}} = \left(\frac{m_{\pi^+}^2}{m_{\Omega^-}^2}, \frac{m_{K^+}^2}{m_{\Omega^-}^2}, \frac{m_{K^0}^2}{m_{\Omega^-}^2}\right)_{\mathrm{PDG}}$$

QCD

$(\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}}|g, 0)$

$$\left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\Delta\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\mathrm{QCD}}} = \left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\Delta\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\phi}}$$

iso-QCD

$(\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)}|g, 0)$

$$\left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\Delta\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{(0)}} = \left(\frac{\hat{M}_{\mathrm{ud}}^2}{\hat{m}_{\Omega^-}^2}, 0, \frac{\hat{M}_{K\chi}^2}{\hat{m}_{\Omega^-}^2}\right)_{\boldsymbol{\sigma}^{\phi}}$$

BMW mesons: $M_{\mathrm{ud}}^2 = \frac{1}{2}\left(M_{\bar{\mathrm{u}}\mathrm{u}}^2 + M_{\bar{\mathrm{d}}\mathrm{d}}^2\right)$ $M_{K\chi}^2 = \frac{1}{2}\left(M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2\right)$ $\Delta M_{\mathrm{ud}}^2 = M_{\bar{\mathrm{u}}\mathrm{u}}^2 - M_{\bar{\mathrm{d}}\mathrm{d}}^2$

Defining the iso-symmetric theory

Gasser, Rusetsky & Scimemi, EPJC 32 (2003)
RM123, PRD 87 (2013)

RM123S (2019): "GRS" scheme (electroquenched) with N_f=2+1+1

QCD+QED

$(\hat{m}_{ud}^{\phi}, \delta\hat{m}^{\phi}, \hat{m}_s^{\phi}, \hat{m}_c^{\phi} | g, \alpha^{\phi})$

$$\left(\frac{\hat{m}^2_{\pi^0}}{\hat{\mathcal{F}}^2_\pi}, \frac{\hat{m}^2_{K^0}}{\hat{\mathcal{F}}^2_\pi}, \frac{\hat{m}^2_{D_s}}{\hat{\mathcal{F}}^2_\pi}, \frac{\hat{m}^2_{K^+} - \hat{m}^2_{K^0}}{\hat{\mathcal{F}}^2_\pi} \right)_{\boldsymbol{\sigma}^\phi} = \left(\frac{\hat{m}^2_{\pi^0}}{\hat{\mathcal{F}}^2_\pi}, \frac{\hat{m}^2_{K^0}}{\hat{\mathcal{F}}^2_\pi}, \frac{\hat{m}^2_{D_s}}{\hat{\mathcal{F}}^2_\pi}, \frac{\hat{m}^2_{K^+} - \hat{m}^2_{K^0}}{\hat{\mathcal{F}}^2_\pi} \right)_{\mathrm{PDG}}$$

QCD

$(\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}}, \hat{m}_c^{\mathrm{QCD}} | g_0, 0)$

$$a^{\mathrm{QCD}} = a^{\phi} \qquad m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{\mathrm{QCD}} \equiv m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{\phi}$$

$$f = \{u, d, s, c\}$$

iso-QCD

$(\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)}, \hat{m}_c^{(0)} | g_0, 0)$

$$a^{(0)} = a^{\phi} \qquad m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{(0)} \equiv m_f^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})^{\phi}$$

$$f = \{ud, s, c\}$$

In practice, the renormalization condition on the strong coupling $g^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV}) \equiv g_0^{\mathrm{R}}(\overline{\mathrm{MS}}, 2\ \mathrm{GeV})$ is neglected in the "**electroquenched approximation**"

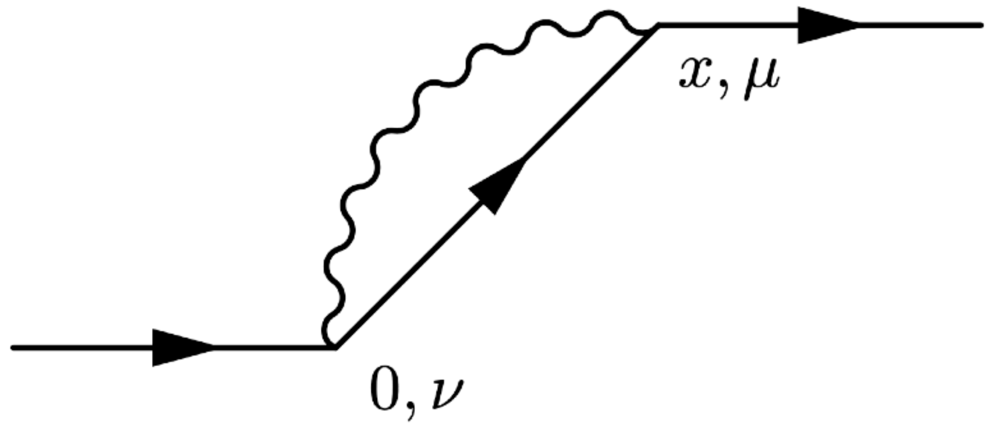
Infinite volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)

QED$_\infty$

Alternative approach: radiative corrections as a **convolution** of **hadronic correlators** with **infinite-volume QED kernels**

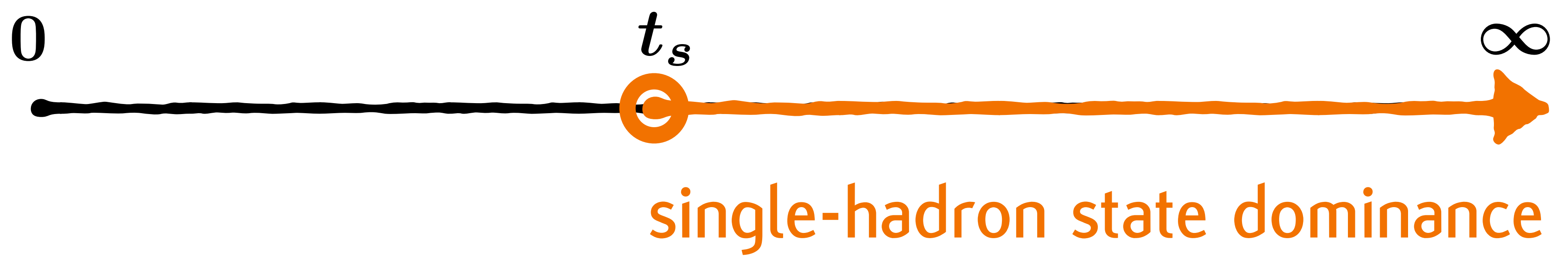
$$\Delta\mathcal{O} = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\; \mathcal{H}(t,\mathbf{x})\, f_{\mathrm{QED}}(t,\mathbf{x}) = \Delta\mathcal{O}^{(s)} + \Delta\mathcal{O}^{(l)}$$



Separate correlator into **short** and **long** distance parts:

$$\Delta\mathcal{O}^{(s)} \approx \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int_{L^3} \mathrm{d}^3\mathbf{x}\; \mathcal{H}^L(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x})$$

$$\Delta\mathcal{O}^{(l)} \approx \int_{L^3} \mathrm{d}^3\mathbf{x}\; \mathcal{H}^L(t_s,\mathbf{x})\, \mathcal{F}_{\mathrm{QED}}(t_s,\mathbf{x})$$



—› **Exponentially suppressed** finite-volume effects ... systematics under control?

Application to **leptonic decays** under study by RBC N.Christ et al., PRD **108** (2023)