Recent progress in the study of the muon $g-2$ from lattice QCD

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Second LatticeNET workshop on challenges in Lattice field theory

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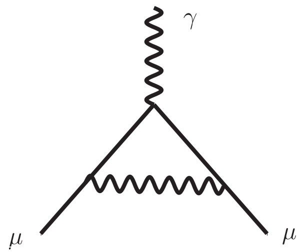
- Magnetic moment of charged leptons :

$$\vec{\mu} = g_\ell \left(\frac{Qe}{2m_\ell} \right) \vec{S}$$

- Dirac equation : $g_\ell = 2$

- In the Standard Model, quantum corrections slightly shift this value



$$a_\ell = \frac{g_\ell - 2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2)$$

- What is special with the muon ?

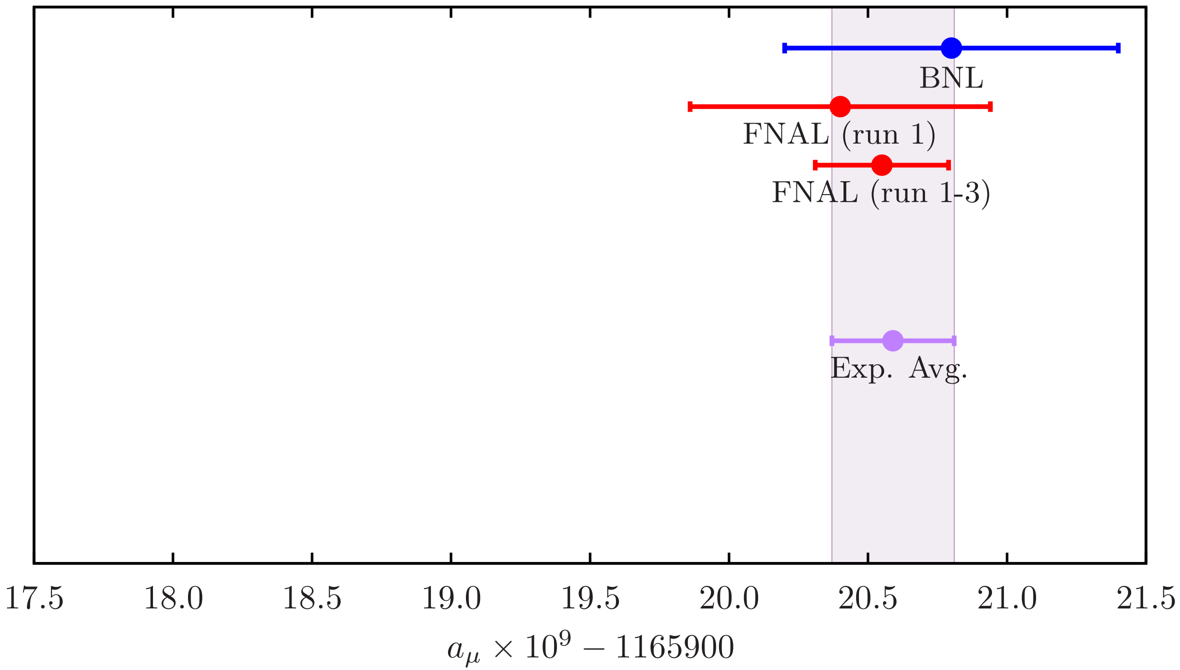
  $\rightarrow$ a_μ can be measured very precisely (0.2 ppm) ...

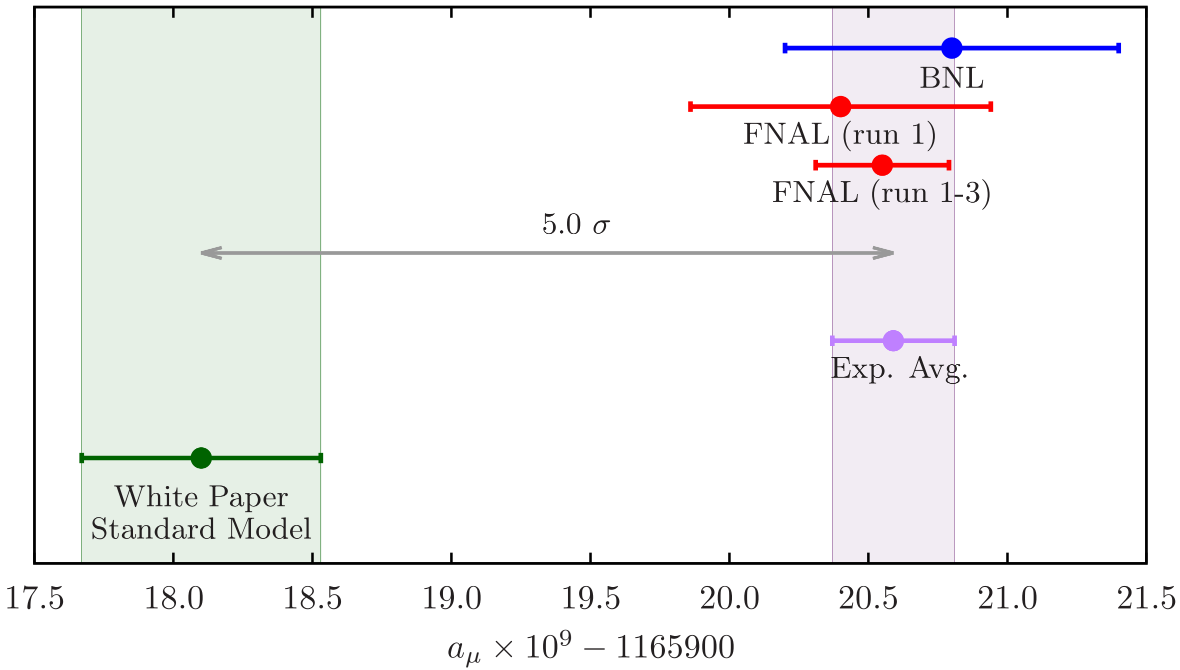
  $\rightarrow$ ... and can be computed with (comparable) precision in the SM

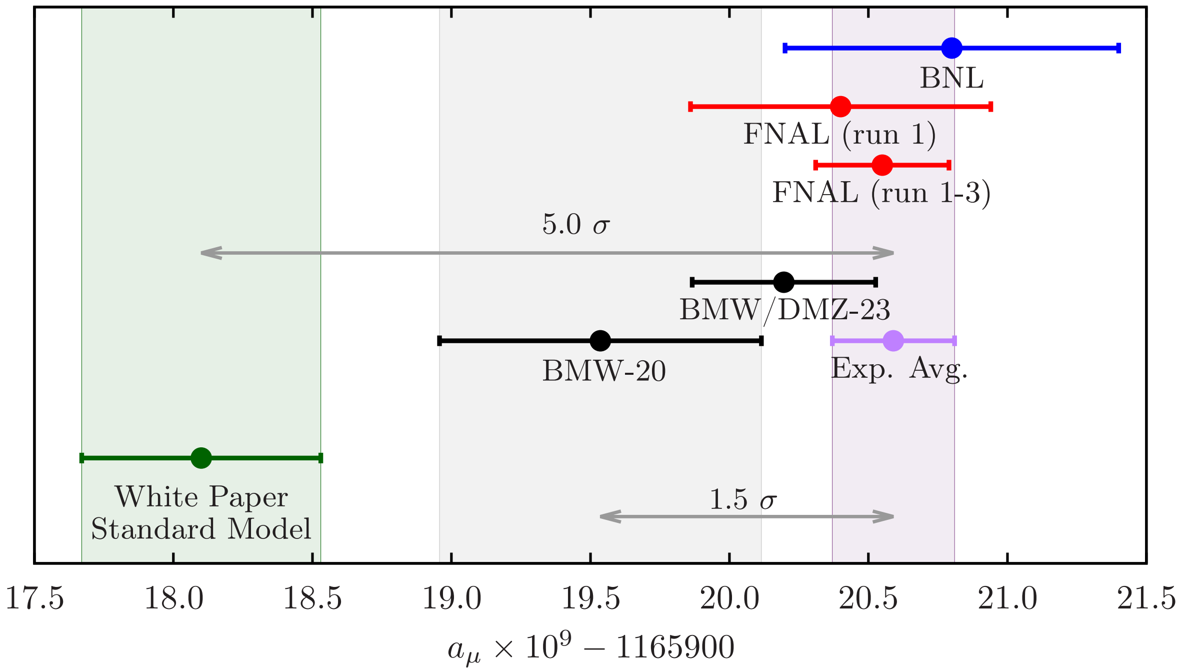
  $\rightarrow$ muons are 200 heavier than electrons (and $\tau_\mu = 2.2\ \mu s \gg \tau_\tau$)

$$\delta a_\ell^{\rm NP} = \mathcal{C}\, \frac{m_\ell^2}{\Lambda_{\rm NP}^2}$$

Status after white paper 1 / run 1-3 at Fermilab (August 2023)




BNL
FNAL (run 1)
FNAL (run 1-3)
5.0 σ
Exp. Avg.
White Paper
Standard Model
17.5
18.0
18.5
19.0
19.5
20.0
20.5
21.0
21.5
$a_\mu \times 10^9 - 1165900$


BNL
FNAL (run 1)
FNAL (run 1-3)
5.0 σ
BMW/DMZ-23
BMW-20
Exp. Avg.
White Paper
Standard Model
1.5 σ
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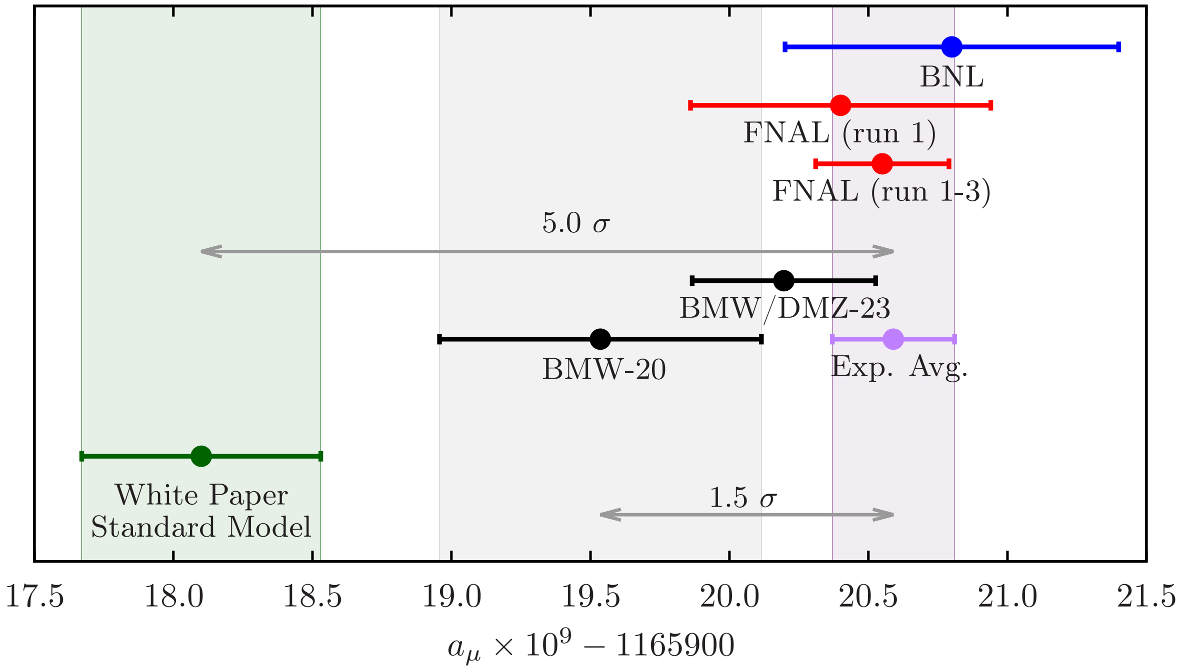
$a_\mu \times 10^9 - 1165900$

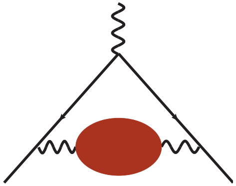


Comment : this plot can be mis-leading since only one hadronic contribution was computed in BMW-20

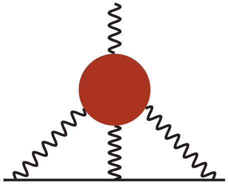
Final result from Fermilab expected in less that two months !

"*The anomalous magnetic moment of the muon in the Standard Model*" [Phys.Rept. 887 (2020) 1-166]

Contribution	$a_\mu \times 10^{11}$
- **QED** (10^{th} order)	$116\ 584\ 718.931 \pm 0.104$
- **Electroweak**	153.6 ± 1.0
- **Strong interaction**	
HVP (LO)	$6\ 931 \pm 40$
HVP (NLO + NNLO)	-85.9 ± 0.7
HLbL	92 ± 18
Standard Model	$116\ 591\ 810 \pm 43$
Experiment	$116\ 592\ 059 \pm 22$



Hadronic Vacuum Polarisation



Hadronic Light-by-Light scattering

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HLbL	92 ± 18
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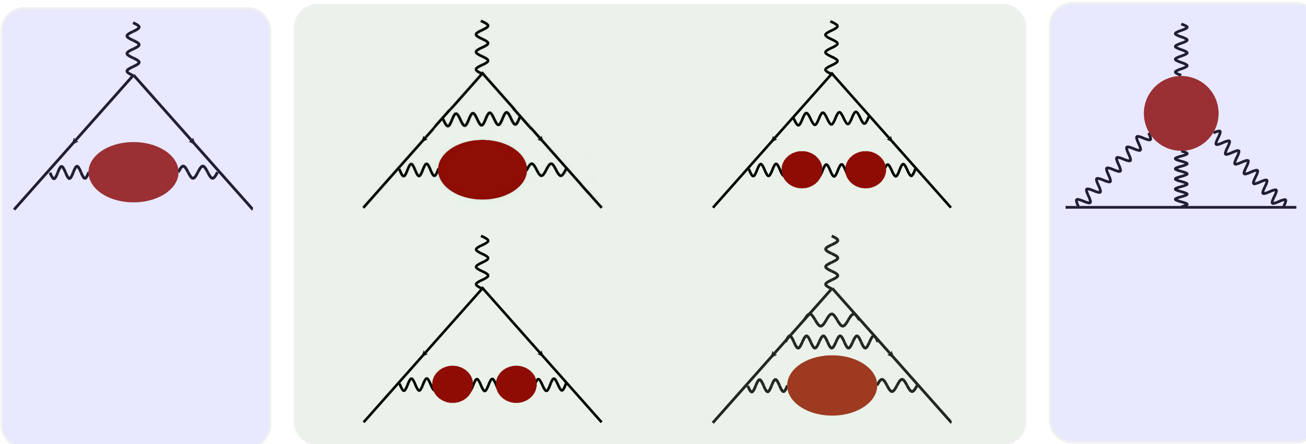
- For each hadronic contribution (HVP and HLbL)

  $\rightarrow$ dispersive (data-driven) estimate - subject to experimental errors / data availability

  $\rightarrow$ Lattice QCD calculations

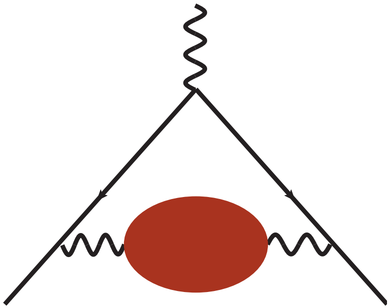
LO HVP (α^2) | **NLO HVP** (α^3) | **NNLO HVP** (α^4) | **HLbL** (α^3)



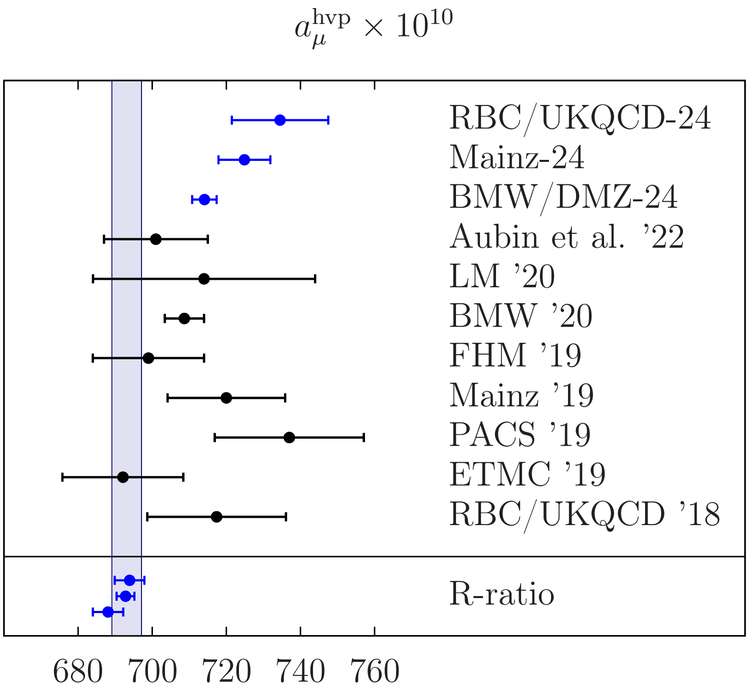
$\sim 7000 \times 10^{-11}$ | $\sim -100 \times 10^{-11}$ | $\sim 12 \times 10^{-11}$ | $\sim 120 \times 10^{-11}$

- LO HVP : includes photons in the QCD blob
- NLO HVP and NNLO HVP differ by the QED weight functions
  - $\longrightarrow$ NLO HVP : same order as HLbL
  - $\longrightarrow$ not negligible, but error under control

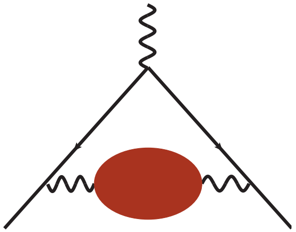
Hadronic vacuum polarization



- White paper in 2020 : final value based on the data-driven estimate
- 3 collaborations have presented complete results with $\sim 1\%$ precision
  $\rightarrow$ all flavors, including disconnected
  $\rightarrow$ QED and strong isospin-breaking corrections



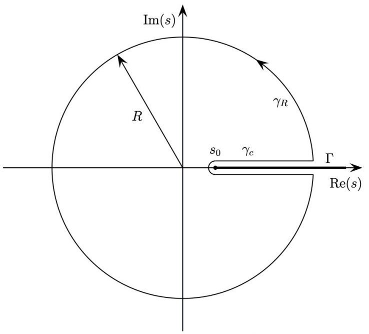
$$\boxed{a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2 \; f(Q^2) \; \left(\Pi(Q^2) - \Pi(0)\right)}$$



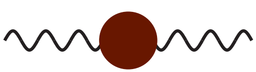
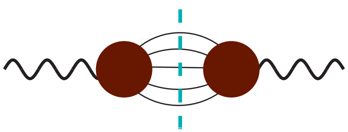
$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x \, e^{iQ\cdot x} \, \langle J_\mu(x) J_\nu(0) \rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

- Use analyticity

$$\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{s_{\mathrm{th}}}^\infty \frac{\mathrm{Im}\Pi(s')}{s'(s'-s-i\epsilon)} \mathrm{d}s'$$



- Optical theorem (unitarity)

Im  $\propto \sum_n$ 

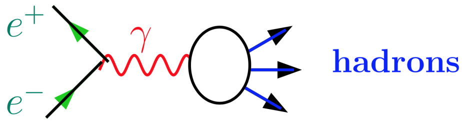
$$\mathrm{Im}\,\Pi(s) \propto \sigma(e^+e^- \to \gamma^* \to \mathrm{hadrons})$$

- Insert the VP in the definition of a_μ to get

$$a_\mu^{\mathrm{LO-HVP}} = \frac{m_\mu^3}{12\pi^2} \int_{s_{\mathrm{th}}}^\infty \mathrm{d}s \frac{K(s)}{s} \sigma(e^+e^- \to \mathrm{hadrons})$$

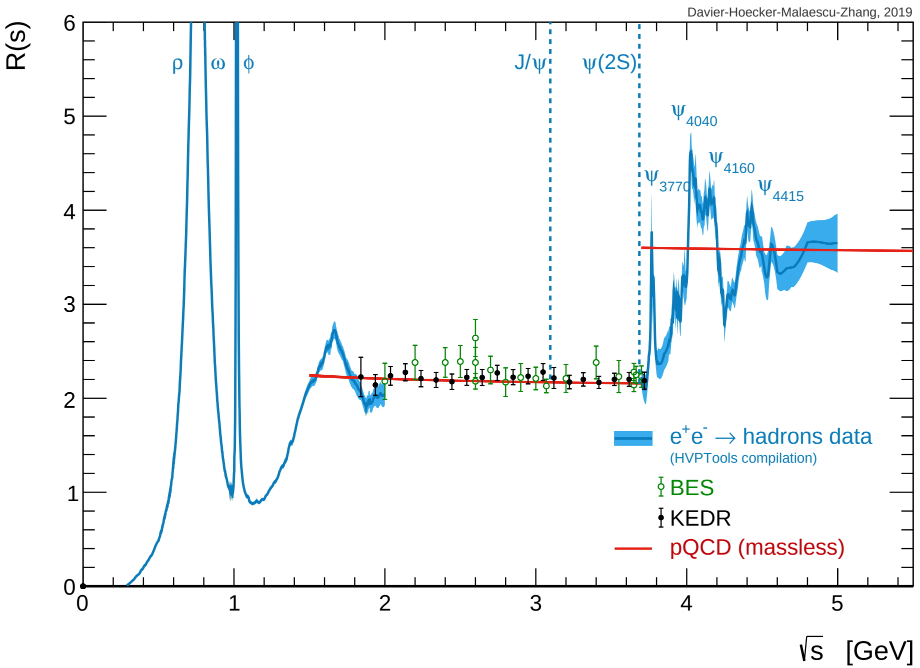
- R-ratio

$$R_{\rm had}(s) = \frac{\sigma^0(e^+e^- \to \gamma^* \to \text{hadrons})}{(4\pi\alpha^2/3s)}$$



- Compilation of experimental data from many experiments

[Davier, Hoecker, Malaescu, Zhang, 2019]



- 2020 White paper average for the dispersive approach (CMD3 data not included)

$$a_\mu^{\rm hvp} = 693.1(2.8)_{\rm stat}(0.7)_{\rm DV+QCD}(2.8)_{\rm KLOE/BABAR} \times 10^{-10} \quad [0.58\%]$$

[Davier et al. '19] [Keshavarzi et al. '20]

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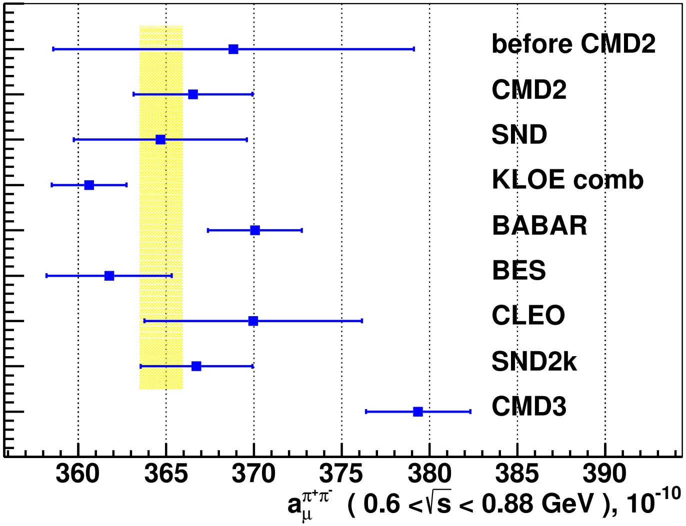
[Davier et al. '19] [Keshavarzi et al. '20]

- But large tensions between different experimental data sets

→ mostly problematic for the dominant $\pi\pi$ channel, region $\sqrt{s} \in [0.6 : 0.9]$ GeV

[CMD3 '23 [2302.08834]]



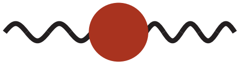
Difference pheno / exp for the $g-2$:

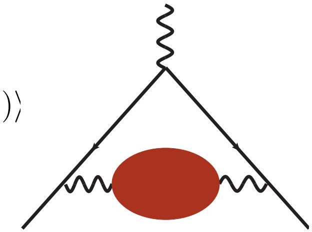
$$a_\mu^{\rm SM} - a_\mu^{\rm exp.} = 28(8) \times 10^{-10}$$

→ $\pi^+\pi^-$: 73% of the total contribution

→ CMD3 ('23) results remove the tension

→ The 5σ tension should be taken with extreme caution

$$\Pi_{\mu\nu}(Q) =$$  $$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2) = \int \mathrm{d}^4x \, e^{iQ\cdot x} \, \langle V_\mu(x) V_\nu(0) \rangle$$

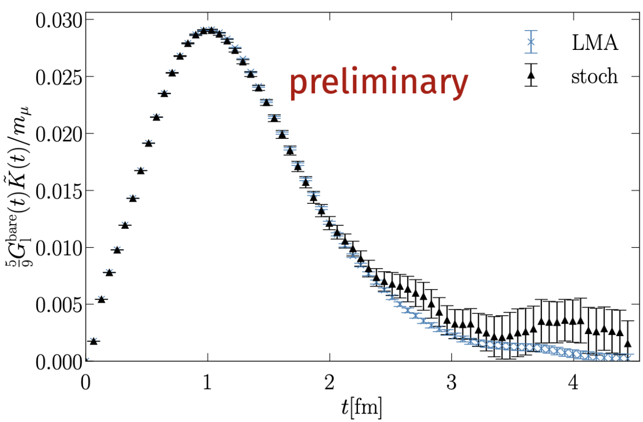


- All collaborations are now using the Time Momentum Representation (TMR)

[Blum '02] [Bernecker, Meyer '11]

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t \; K(t) \; G(t) \; , \qquad G(t) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k(x) V_k(0) \rangle$$

- In principle, straightforward. Compute VV correlator and sum over t with known weight factors



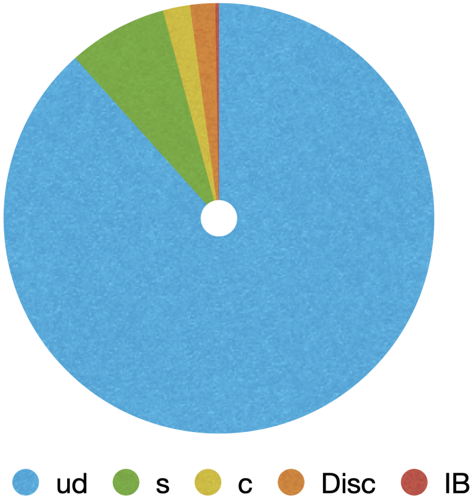
[Mainz, Kuberski et al.]

+ correct for finite-volume effects + extrapolation to the continuum limit. But need 2 permil precision.

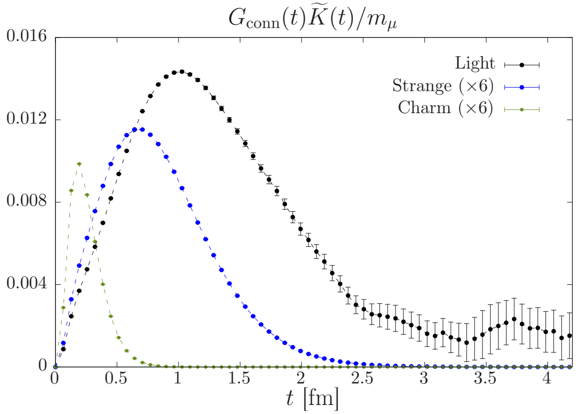
The standard decomposition :

$$a_\mu^{\text{LO-HVP}} = \left(\sum_{f=l,s,c} a_\mu^{\text{conn,f}} \right) + a_\mu^{\text{disc}} + a_\mu^{\text{IB}}$$

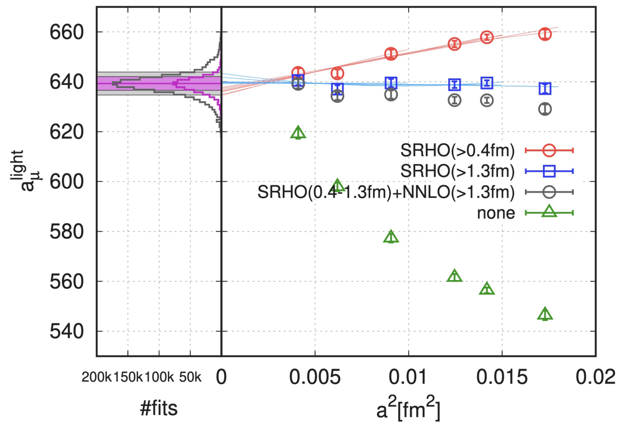
- Connected and disconnected contributions in iso-symmetric QCD
- Isospin breaking (IB) corrections treated separately $\rightarrow$ fewer calculations
- Left hand side : physical observable computed in QCD + QED
- Right hand side : each contribution is scheme dependent (def. of iso-symmetric QCD)
  $\rightarrow$ need to agree on common scheme first.
  $\rightarrow$ only relevant when comparing intermediate results



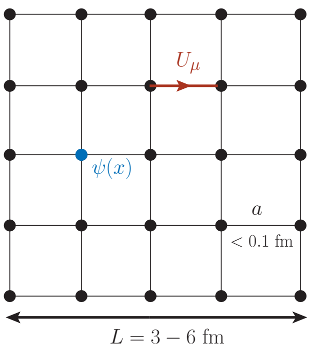
- Noise problem (light-quark contribution)



- Continuum extrapolation [BMW '20]



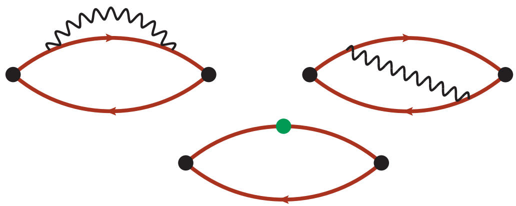
- Finite-volume effects $\mathcal{O}(3\%)$ @ $L = 6$ fm



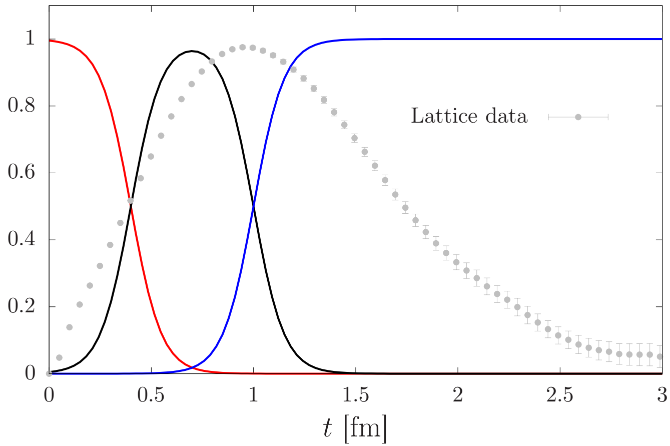
- QED / strong isospin breaking corrections

$m_u \neq m_d$: $\mathrm{O}(\frac{m_u - m_d}{\Lambda_{\rm QCD}}) \approx 1/100$

$Q_u \neq Q_d$: $\mathrm{O}(\alpha_{\rm em}) \approx 1/100$



- Also need to know the scale precisely (to convert the muon mass in lattice units).



[RBC/UKQCD 2018]

$$a_\mu^{\mathrm{win}} = \left(\frac{\alpha}{\pi}\right)^2 \sum_t G(t)\, K(t)\, W(t; t_0, t_1)$$

→ Short distances (SD)

→ Intermediate distances (ID)

→ Long distances (LD)

- By construction, the sum over the 3 windows gives the full contribution

$$a_\mu^{\mathrm{LO-HVP}} = a_\mu^{\mathrm{win,SD}} + a_\mu^{\mathrm{win,ID}} + a_\mu^{\mathrm{win,LD}}$$

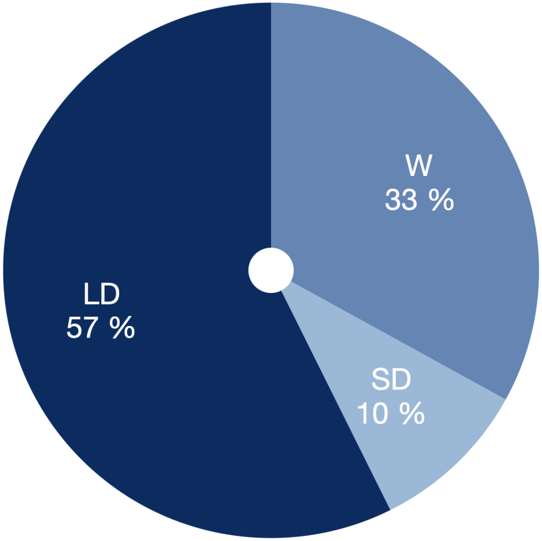
→ In principle, need to include correlations. Numerically not so relevant since $\delta a_\mu^{\mathrm{win,LD}}$ is large.

- Each window is **well defined**, and subject to very **different systematic errors**

Short-distance	Intermediate-distance	Long-distance
stat. precise	stat. precise	noise problem
discretization effects	small finite volume effect	finite volume corrections
		large taste breaking (staggered)

- Uses the same raw lattice data

$$a_\mu^{\mathrm{LO-HVP}} = a_\mu^{\mathrm{win,SD}} + a_\mu^{\mathrm{win,ID}} + a_\mu^{\mathrm{win,LD}}$$

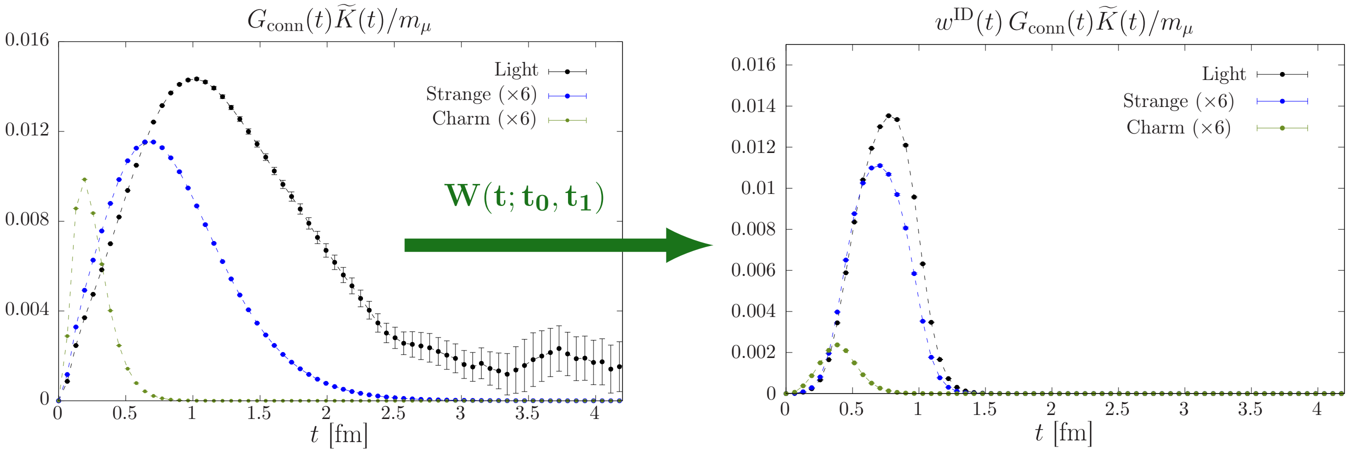


- **Assuming a target precision of 0.2 ppm** (to match expected Fermilab precision) :

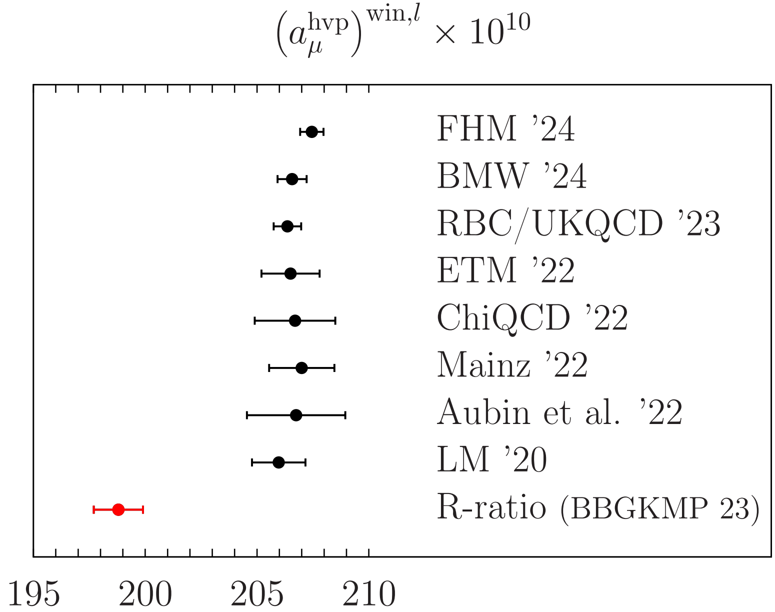
$$a_\mu^{\mathrm{LO-HVP}} = (714.1 \pm 3.3) \times 10^{-10} \quad \longrightarrow \quad \pm 1.4$$

- Next slides : results for each windows

$$a_\mu^{\mathrm{win}} = \left(\frac{\alpha}{\pi}\right)^2 \sum_t G(t)\, \widetilde{K}(t)\ \mathbf{W(t; t_0, t_1)}$$



- **Intermediate window** : $\sim 33\%$ of the total contribution. $a_\mu^{\mathrm{W,tot}} = 236.3(0.4) \times 10^{-11}$
- Easier to compute on the lattice (and accessible from R-ratio data !) :
  - $\rightarrow$ 1-2 permille statistical precision can be reached on the integrand
  - $\rightarrow$ small finite-volume effects ($\delta a_\mu = 0.49(5) \times 10^{-11}$)
  - $\rightarrow$ small isospin-breaking correction ($\delta a_\mu = 0.43(8) \times 10^{-11}$)
- Many groups have presented results (at least for the light quark contribution)



- Significant tension between (all !) lattice calculations vs data-driven approach (no CMD-3) (here shown for the light-quark connected contribution in the isospin limit)

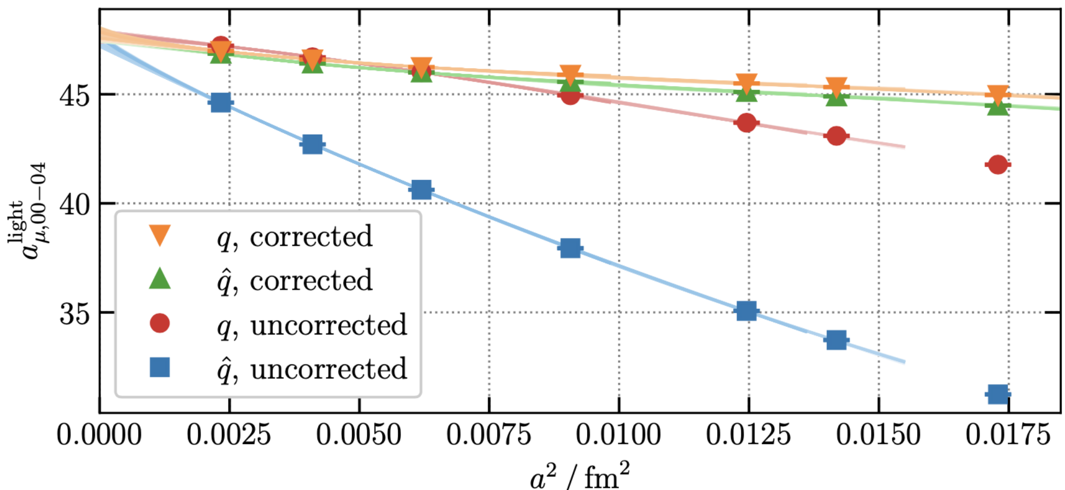
- Data-driven : 2π contribution in the region $600~\mathrm{MeV} \leq \sqrt{s} \leq 900~\mathrm{MeV}$ (around the rho peak) :

  $\rightarrow$ relative contribution of 55%-60% to both $a_\mu^{\mathrm{LO-HVP}}$ and a_μ^{win}

  $\rightarrow$ $\sqrt{s} \leq 600~\mathrm{MeV}$ slightly suppressed, $\sqrt{s} \geq 900~\mathrm{MeV}$ slightly enhanced.

Dominated by light quark contribution (70%). Contains 80% of the total charm quark contribution.

- Statistics is not a problem here
- Finite-volume effects can be neglected (sub-permil contribution)
- Disconnected and QED/SIB contributions can also be neglected here ($\Rightarrow$ no visible scheme dependence)
- Mild dependence on the quark mass
- Continuum extrapolation : enhanced $a^2 \log(a)$ terms [Cè et al. - 2106.15293], [Sommer et al., 2211.15750] :
  $\rightarrow$ needs several (small) lattice spacings. Use several fit functions $\in a^2, a^4, a^2 \log(a), \cdots$
  $\rightarrow$ lattice artifacts : tree-level improvement (BMW, ETM, FHM, RBC/UKQCD) / subtracted kernel + pQCD (Mainz)



[BMW/DMZ-24]

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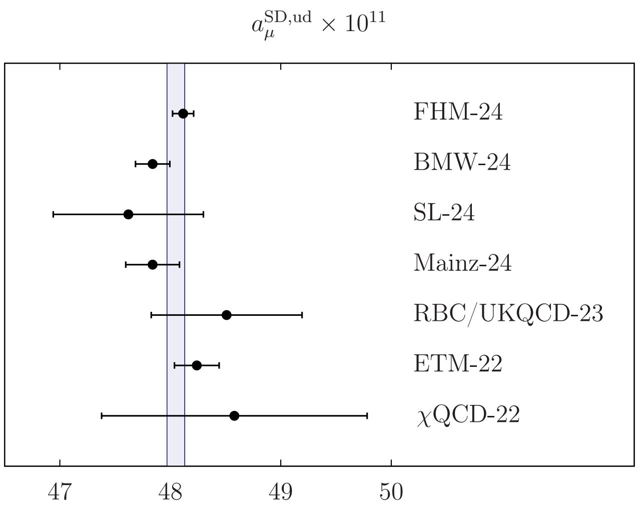


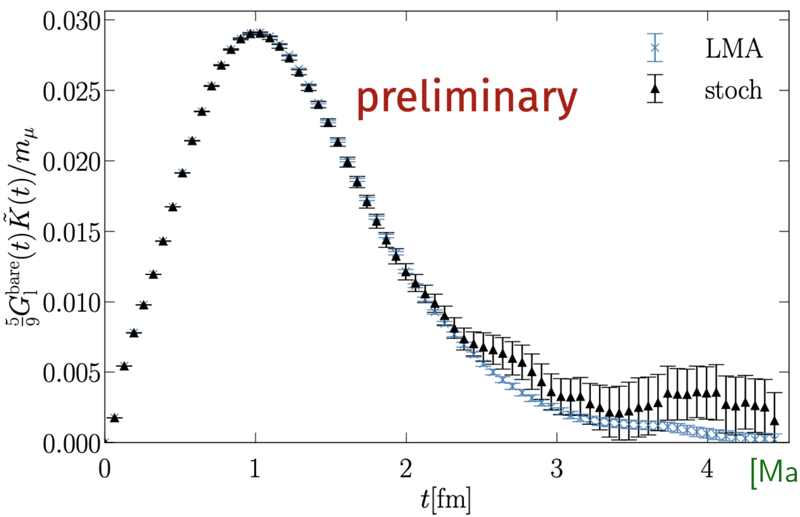
$\rightarrow$ results with different gauge/fermion discretizations

Strange $\approx 9 \times 10^{-11}$
Charm $\approx 12 \times 10^{-11})$

$$a_\mu^{\mathrm{SD,tot}} = 69.0(0.3) \times 10^{-11}$$

Reminder : target error on $a_\mu^{\mathrm{LO-HVP}}$ is 1.4×10^{-11}
$\rightarrow$ So, seems to be under control

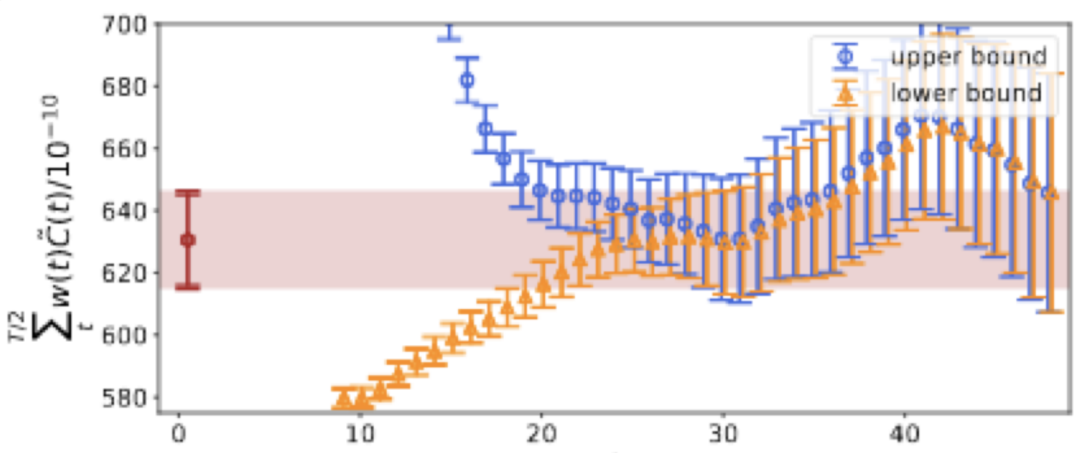
- Limiting factor for precision. About 57% of the total contribution. :
  - $\longrightarrow$ signal-to-noise problem
  - $\longrightarrow$ large finite-volume correction ($\approx 3\%$ at $L = 6$ fm)
  - $\longrightarrow$ contains most the quark-disconnected contribution
  - $\longrightarrow$ sensitive to the scale setting uncertainty
  - $\longrightarrow$ large isospin-breaking corrections
- Signal-to-noise problem : can use low-mode averaging / All-mode averaging to improve statistics
  - $\rightarrow$ now standard. Used by all groups.



[Mainz, Kuberski et al.]

$\rightarrow$ Combined with bounding methods to cut the integration range

$$0 \leq G(t_c)e^{-E_{\text{eff}}(t-t_c)} \leq \mathbf{G(t)} \leq G(t_c)e^{-E_{2\pi}(t-t_c)}$$



[Plot by A. Meyer, RBC/UKCD]

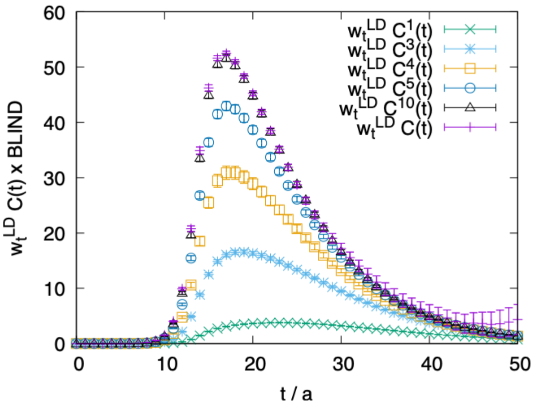
$\rightarrow$ Might not be enough to reach 2 permil precision ...

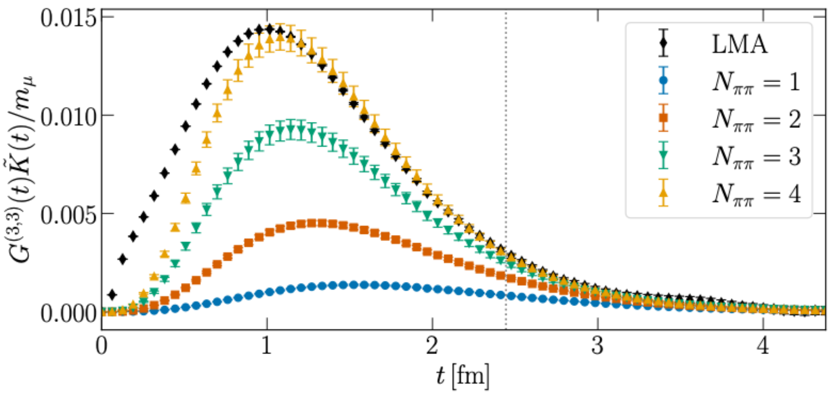
$\rightarrow$ Can we do better ?

The vector correlators admits a spectral decomposition :

$$G(t) = \sum_{\vec{x}} \langle V_k(t,\vec{x}) V_k(0) \rangle = \sum_n \langle 0|V_k|n\rangle \, \frac{1}{2E_n} \, \langle n|V_k(0)|0\rangle \, e^{-E_n t}$$

- $|n\rangle$ are the eigenstates in finite volume
- E_n and $\langle 0|V_k|n\rangle$ can be computed on the lattice using distillation methods.





- Noise grows only linearly : always best some method at large t (typically 2.4fm for Mainz)

  → Method used by Mainz ('19, '24) and RBC/UKQCD ('24).

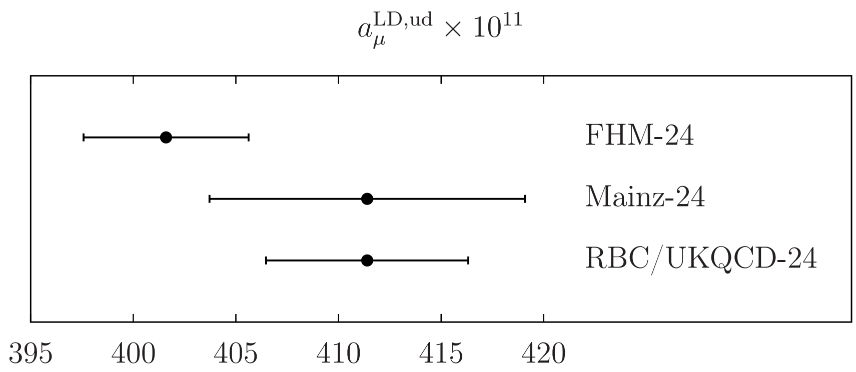
- More challenging with staggered quarks

  → first results presented by Fermilab/HPQCD/MILC [2409.00756] with staggered quarks at $a = 0.15$ fm.

  → no tension observed compared to fit strategy used at finer lattice spacing.

- Three groups have results so far

  $\rightarrow$ Light-quark only in isosymetric - QCD.
  $\rightarrow$ Scheme dependance not negligible



$\rightarrow$ Fermilab/HQPCD/MILC : small tension with RBC/UKQCD
$\rightarrow$ Fit strategy might leads to an underestimate of the tail contribution ?

$$a_\mu^{\mathrm{LD,ud}} = 411.4(4.3)(2.4) \times 10^{-10} \qquad \mathrm{RBC/UKQCD}$$
$$a_\mu^{\mathrm{LD,ud}} = 423.2(4.2)(3.4) \times 10^{-10} \qquad \mathrm{MAINZ}$$
$$a_\mu^{\mathrm{LD,ud}} = 401.2(2.3)(3.6) \times 10^{-10} \qquad \mathrm{FHM}$$

Reminder : target error on $a_\mu^{\mathrm{LO-HVP}}$ is 1.4×10^{-11}

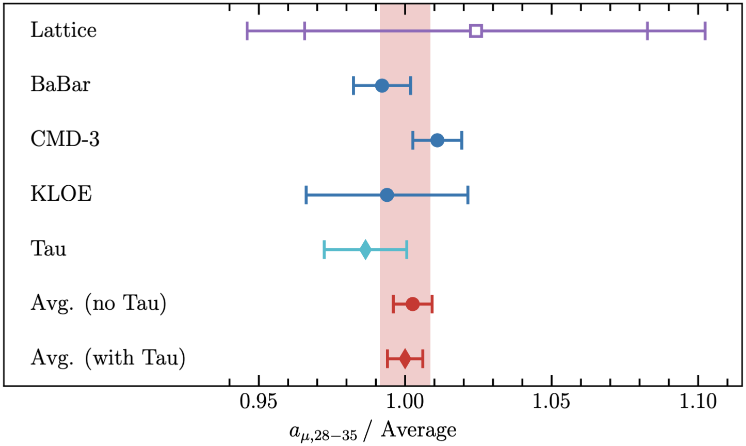
- Other strategy (BMW/DMZ-24) :
  - use lattice data for $t < 2.8$ fm
  - use data-driven estimate for $t > 2.8$ fm ($< 5\%$ of the total contribution)

$\rightarrow$ Improve the statistical precision

$\rightarrow$ Reduce finite-volume effects

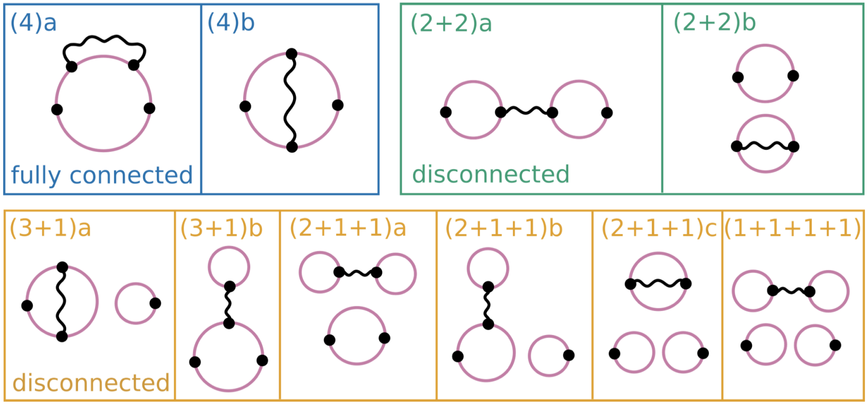
$\rightarrow$ Reduce the (largely unknown) QED corrections at large t

- Good agreement among experiments (tail dominated by cross section below ρ peak)

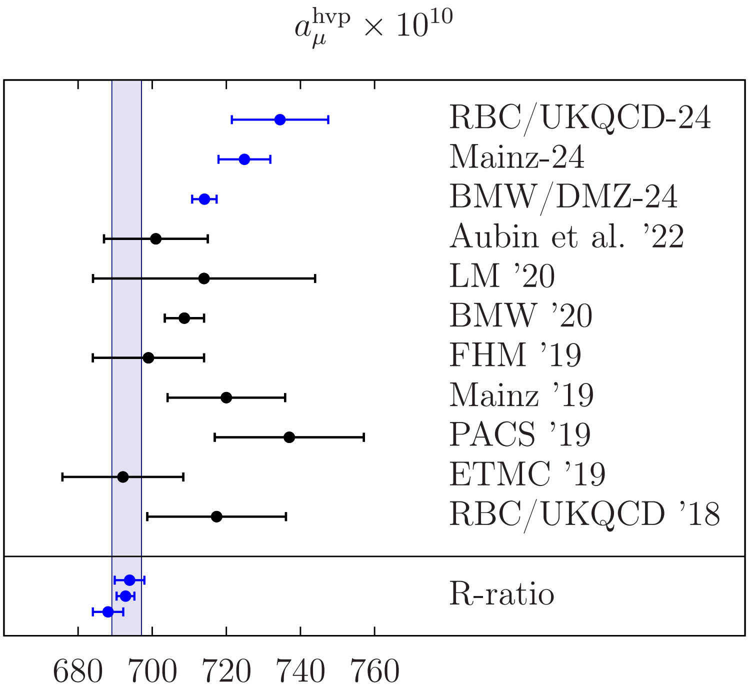


- Reduction of on square of error by 65%

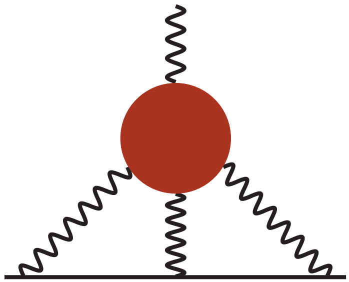
- Small effect (below 1%), but contribute significantly to the error budget.
  → mostly contribute to the long-distance window
  → fewer results



- First line : dominant contributions
  → signal usually lost at small times ($< 2\mathrm{fm}$)
  → similar diagrams in the HLbL calculation : long distance contrib. and large cancellations
  → tensions between groups
- Results in isospin symmetric QCD are scheme dependent.
  → common scheme suggested by FLAG



Lattice HVP update in the white paper very soon ...



- Challenging

  $\rightarrow$ hadronic light-by-light tensor $\Pi_{\mu\nu\lambda\sigma}(p_1, p_2, p_3) = \int_{x,y,z} \Pi_{\mu\nu\lambda\sigma}(x, y, z) e^{-i(q_1 x + q_2 y + q_3 z)}$

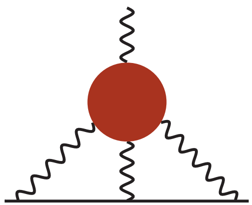
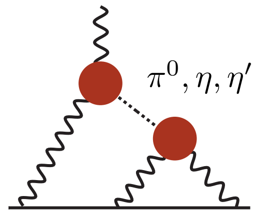
  $\rightarrow$ multi-scale system

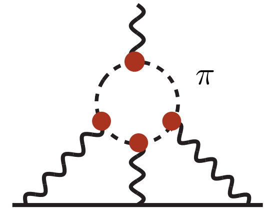
- Until 2016 : mostly based on model estimates

$$a_\mu^{\mathrm{HLbL}} = 105(26) \times 10^{-11} \qquad \text{[Prades, de Rafael, Vainshtein '09]}$$
$$a_\mu^{\mathrm{HLbL}} = 116(39) \times 10^{-11} \qquad \text{[Jegerlehner, Nyffeler '09]}$$

- Precision goal : below 10% (with controlled uncertainties)

  $\rightarrow$ requires first principle approach : data-driven dispersive framework / lattice QCD

$a_\mu^{\mathrm{HLbL}} =$  $\approx$ 

$+$ 

$+ \quad \cdots$

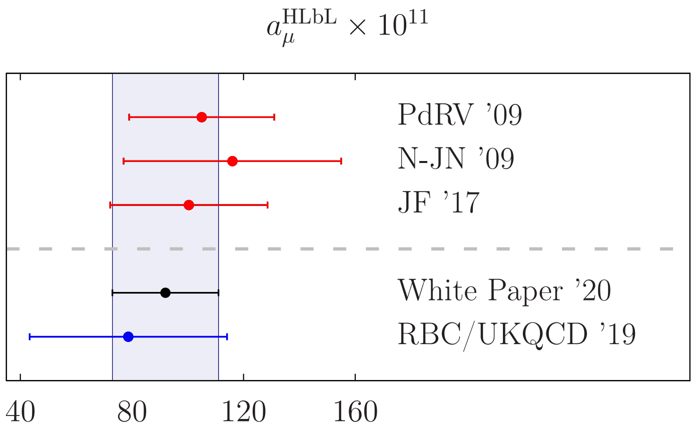
Dispersive framework ('21)	$a_\mu \times 10^{11}$
π^0, η, η'	93.8 ± 4
pion/kaon loops	-16.4 ± 0.2
S-wave $\pi\pi$	-8 ± 1
axial vector	6 ± 6
scalar + tensor	-1 ± 3
q-loops / short. dist. cstr	15 ± 10
charm + heavy q	3 ± 1
sum	92 ± 19
Mainz-22	109.6 ± 15.9
RBC/UKQCD-23	124.7 ± 15.2
BMW-24	125.5 ± 11.6

Two approaches on the lattice :

π^0, η, η' : accessible on the lattice

direct lattice calculations

- **Single lattice calculation of the HLbL diagram** at physical point & cont. limit by RBC/UKQCD



→ Lattice precision : 40%

→ Mainz results at $m_\pi = 400$ MeV [2006.16224]

- **Single calculation of the pion-pole contribution**

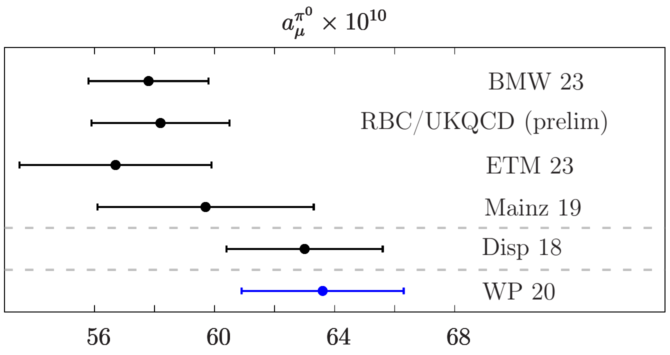
→ Mainz group ('16 + '19 update)

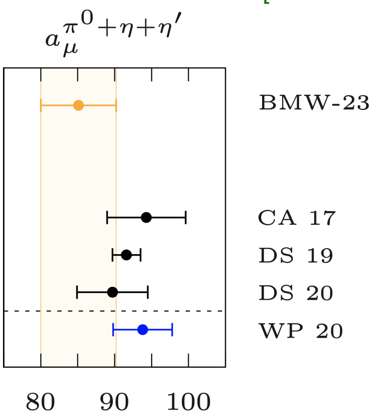
→ $a_\mu^{\pi^0\text{-pole}} = 59.7(3.6) \times 10^{-11}$. Precision $\sim 6\%$

Dispersive framework ('21)	$a_\mu \times 10^{11}$
π^0, η, η'	93.8 ± 4
pion/kaon loops	-16.4 ± 0.2
S-wave $\pi\pi$	-8 ± 1
axial vector	6 ± 6
scalar + tensor	-1 ± 3
q-loops / short. dist. cstr	15 ± 10
charm + heavy q	3 ± 1
sum (WP '20)	92 ± 19

- **Pion-pole contribution** : 3 new calculations (BMW, ETM, RBC/UKQCD).

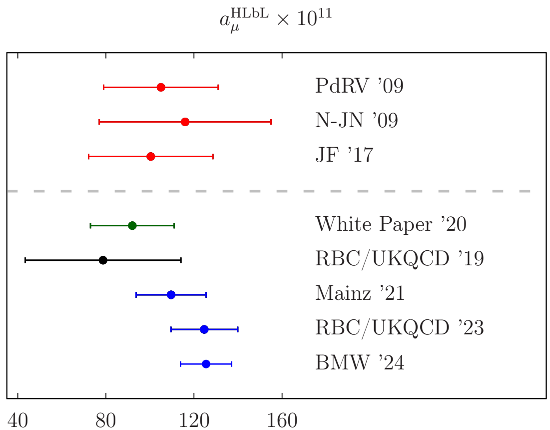
[BMW-23 , 2305.04570]





→ Extension by BMW to include η and η' : total pseudoscalar-pole contribution $a_\mu^{\mathrm{HLbL};\pi^0} = (85.1 \pm 5.2) \times 10^{-11}$

- **HLbL diagram** :
  - 3 complete calculations
  - preliminary results by ETM @Lattice2024
  - good agreement among all results

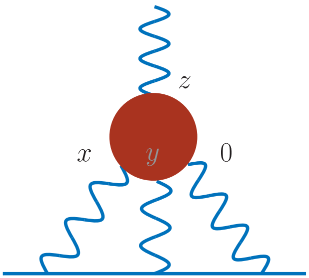


- Coordinate-space approach developed by the Mainz group (similar for RBC/UKQCD)

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \int \mathrm{d}^4 y \int \mathrm{d}^4 x \, \mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y) \, i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \, ,$$

with

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int \mathrm{d}^4 z \, z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$



and $\mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)$ computed by Mainz group [JHEP 04 (2023) 040]

- sums over x and z are done explicitly over the lattice : $a_\mu(|y|) = \int_0^{|y|} \mathcal{I}(|y'|) \, \mathrm{d}|y'|$

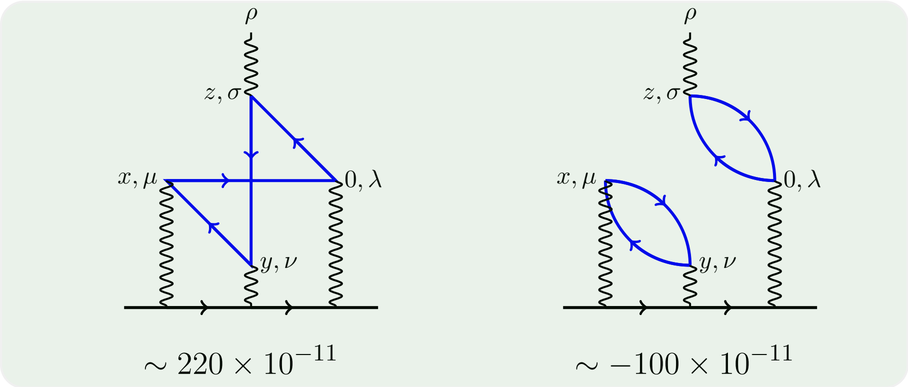
- Lattice calculations differ by the choice of the kernel function

  $\rightarrow$ direct comparison at the level of the integrand not possible among groups

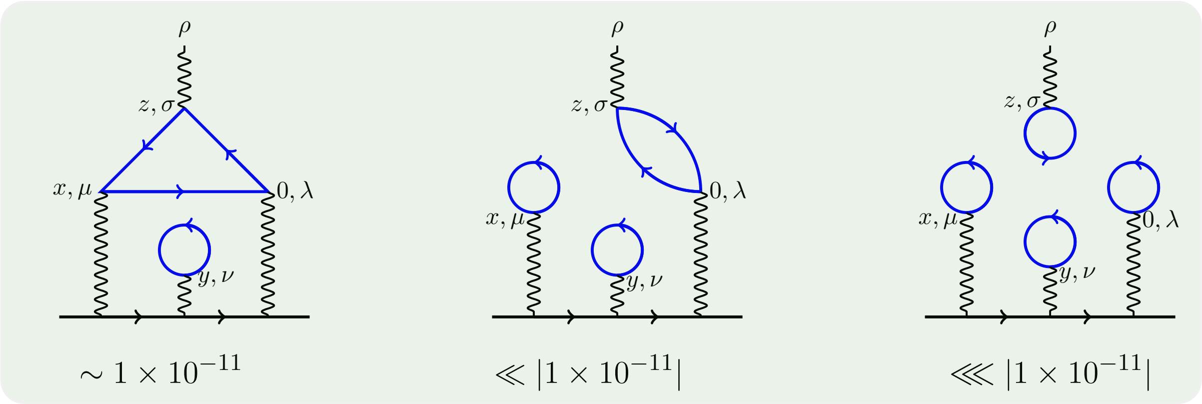
  $\rightarrow$ but should agree in the continuum and infinite volume limits

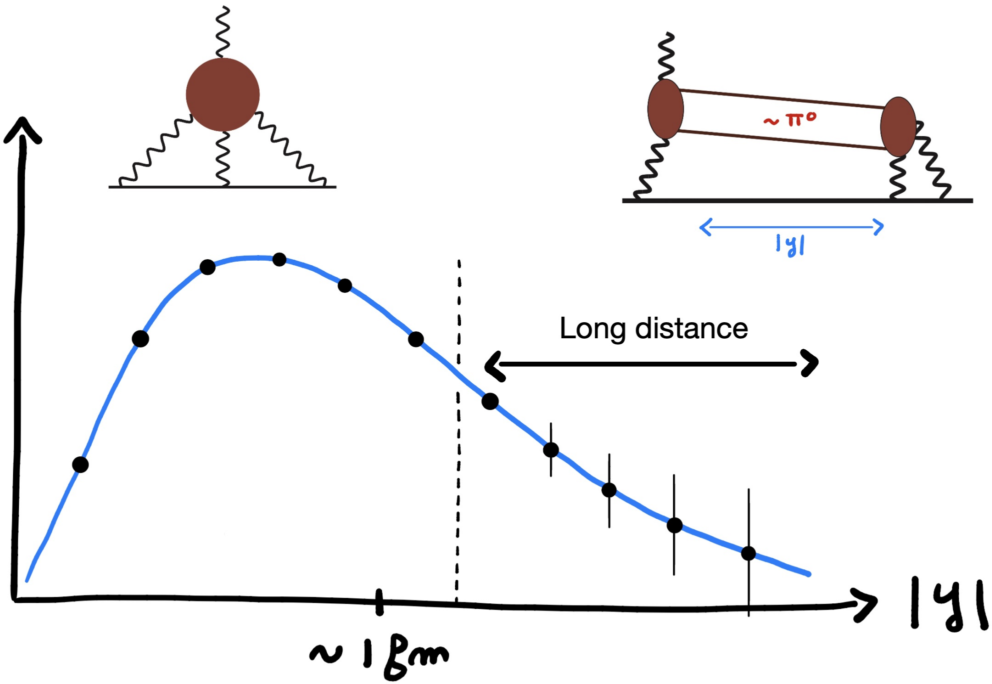
- Challenges : statistics, continuum extrapolation (+ finite-volume correction)

- Connected and leading disconnected diagrams : large cancellation (Conn ≈ -2 Disc)



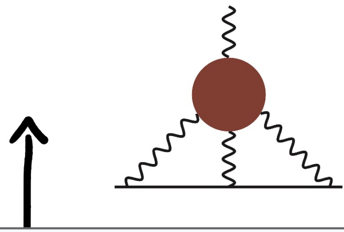
- Sub-leading diagrams : computed by Mainz-21 [2104.02632] and BMW-24 [2411.11719]

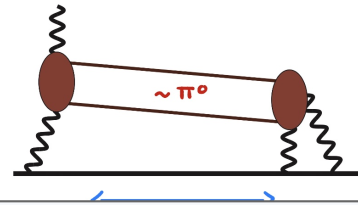


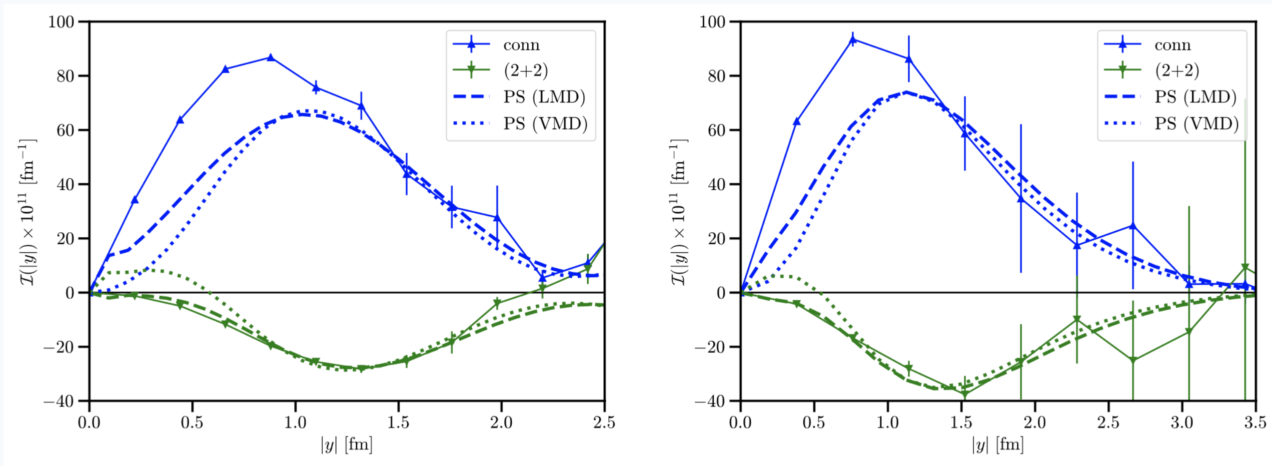


- Low-mode averaging hard to implement
- Tail : dominated by the pion-pole contribution
- Replace data by pion-pole contribution, evaluated using position-space approach

  → include lattice data as much as possible to avoid systematic bias





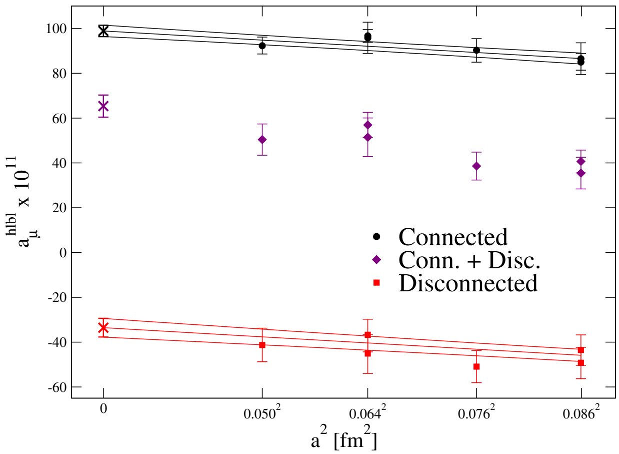


[BMW-24]

- Replace data by pion-pole contribution, evaluated using position-space approach

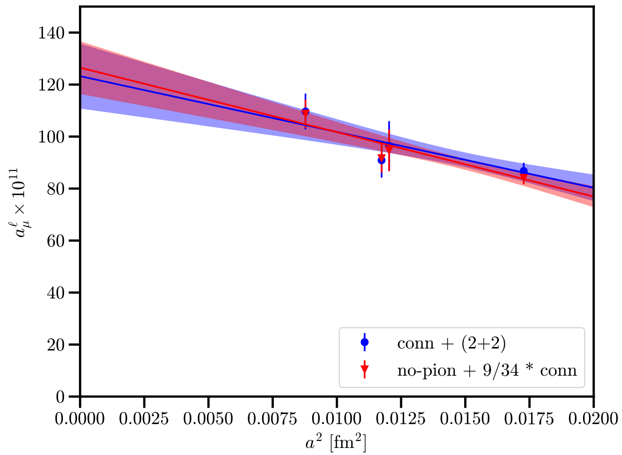
  $\rightarrow$ include lattice data as much as possible to avoid systematic bias

- **Mainz '20**



- 4 lattice spacings
- non-improved currents, $m_\pi = m_K \approx 420$ MeV

- **BMW '24**



- 3 lattice spacings (4th lattice spacing soon !)
- physical pion-mass

- **RBC/UKQCD '24** : single lattice spacing for the light quark at physical point

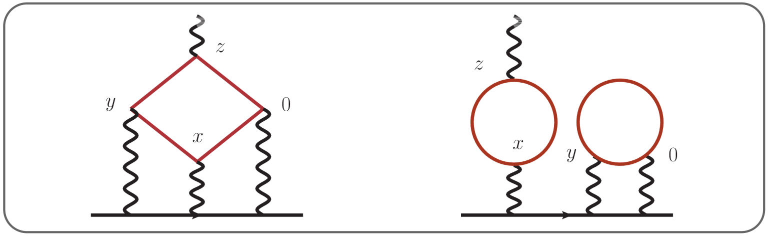
  $\rightarrow$ Systematic error estimated from lepton-loop / strange quark contribution

- **There is no "windows" for the HLbL**

  $\longrightarrow$ No simple way to separate short and long distances (there are mixed regions !)

  $\longrightarrow$ Groups use $\neq$ QED weight functions $\Rightarrow$ comparison only in the continuum & infinite volume limits !

  $\longrightarrow$ Comparison of lattice at unphysical muon mass ? But might require new simulations ...

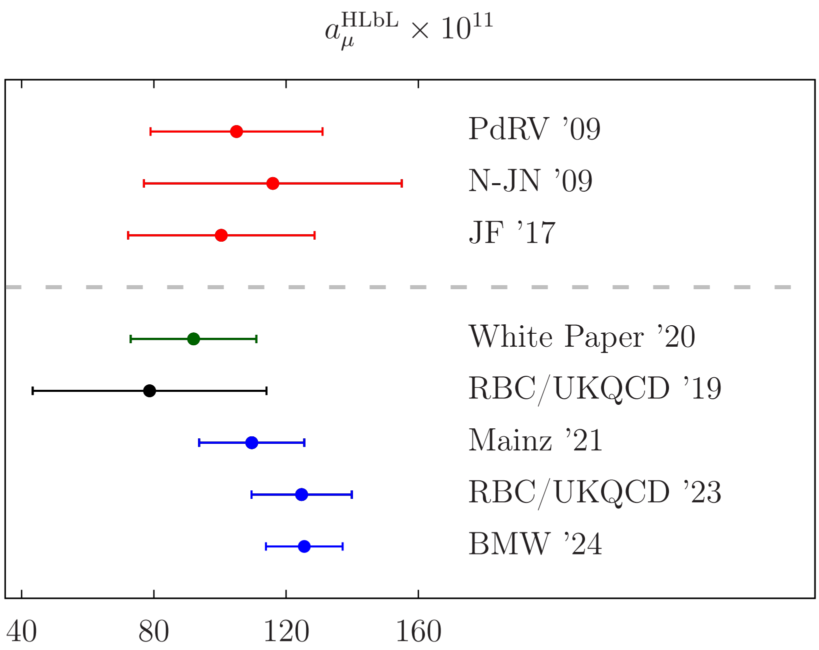


- Estimated by computing the pion-pole contribuion in finite-volume
- Finite volume corrections are enhanced on individual diagrams :

$$a_\mu^{\text{hlbl,conn}} \longleftarrow +\frac{34}{9} \times a_\mu^{\pi^0-\text{pole}}$$
$$a_\mu^{\text{hlbl,2+2}} \longleftarrow -\frac{25}{9} \times a_\mu^{\pi^0-\text{pole}}$$

- Total contribution : numerically small compared to statistical precision (with $L = 6$ fm).

Comparison between groups



- Final result from Fermilab expected in less than 2 months

  $\rightarrow$ White paper update by the $g-2$ Theory Initiative in preparation

- Several lattice calculations of the HVP have reached sub-percent precision

  $\rightarrow$ new strategy advocated by the TI is the decomposition in windows

  Advantage : comparison between groups is easy

  Disadvantage : systematic errors are likely very correlated

  $\rightarrow$ Reasonable agreement among lattice collaborations

  $\rightarrow$ But more work is needed, especially for the long distance part (including isospin-breaking)

- HLbL : three lattice calculations with precision close or below 10%

  $\rightarrow$ + preliminary results presented by ETM

  $\rightarrow$ challenges : statistics and continuum extrapolation

  $\rightarrow$ comparison between groups more difficult

Thank you !