Exact WKB Formulation: **Quantization** and **Particle Production**

(Preliminary Results)

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(see also Ryo's talk slides on Tuesday, April 29th)

(arXiv:2505.XXXXX!!!)

@Benasque (2025)

My Research Journey


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High energy theory
(unified theory)

Advancements in
- Physics including gravity
- QFT
- ***Mathematics:***
  Exact WKB Analysis

Low energy theory

What is Exact WKB?

Exact WKB: treats *divergent WKB* solutions and gives non-perturvaive information *quantitatively*

"*Divergent WKB*" means:

$$\psi(x) \sim \exp\left[\frac{i}{\hbar}\int^x p(x')dx'\right] \times (\text{series in } \hbar)$$

— (Formal) expansion in $\hbar$ does not converge in general

— Non-perturbative parts include particle production information

"*Quantitatively*" means:

$$\Psi(x) \sim \int_0^\infty \exp(-\eta\zeta)[B\psi](\zeta)d\zeta$$

— Borel resummation gives a corresponding analytic function

— Non-perturbative information is encoded in the singularities

Exact WKB Analysis~
Analysis of singularities (of the Borel transformed series)

Where We’re Going vs Where We Are

“a glimpse of what might be possible”

We are aiming for

— Offering new insights into particle production
e.g. preheating, particle production from oscillaring scalar field background, gravitational particle production, etc.

— Enabling (semi-)analytic solutions to problems where numerical methods fall short

Our current situation is

— We are developing foundational building blocks

— Verifying the consistency of our analysis through comparisons with previous results

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We are laying the foundation for the future, and the possibilities ahead are limitless

Today's Goal

🎯 **We will eventually focus on a specific potential:**

$$\left[-\frac{d^2}{dx^2} - \eta^2 V(x)\right]\psi(x) = 0$$

$$V(x) = -E + \frac{x^2}{4} \quad (E > 0)$$

— This potential has been studied well in the context of particle production
— Previous analyses rely on asymptotic expansions of special functions: parabolic cylinder functions

📌 **What we do today:**

— Use resummed WKB soltuions for quantization
— Derive the Bogoliubov coefficients ``without” relying on parabolic functions

🧩 **Why this matters:**

— Clarifies how non-perturbative physics emerges
— A step toward applying exact WKB to broader cosmological settings

The Starting Point: Formal WKB Solution

✅ **1-d Schrodinger-like equation:**

$$\left(-\frac{d^2}{dx^2} - \eta^2 V(x)\right)\psi(x,\eta) = 0$$

$\eta \equiv 1/\hbar$: small hbar = large η expansion

✅ **Formal WKB solutions:**

$$\boxed{\psi_\pm(x) = \frac{1}{\sqrt{S_{\mathrm{odd}}(x)}} \exp\left(\pm \int_{x_0}^{x} S_{\mathrm{odd}}(\tilde{x}) d\tilde{x}\right)}$$

$$S_{\mathrm{odd}} \equiv \sum_{j \geq 0} S_{2j-1} \eta^{1-2j}$$

$$S_{-1}^2 = -V,\ \ 2S_{-1}S_j = -\left(\sum_{k+l=j-1 \& k,l \geq 0} S_k S_l + \frac{\partial S_{j-1}}{\partial x}\right)$$

⚡ **Formal solution is divergent in general**

From Divergence to Meaning: Borel Transform and Borel Sum

♾️ **Original formal series**: divergent series of η:

$$\psi(\eta, x) = \sum_{n=0}^{\infty} f_n(x)\eta^{-n}$$

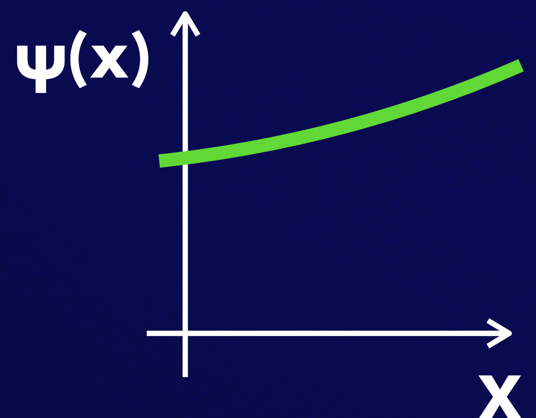
✨ **Borel transformation**: a well-behaved series of ζ:

$$\psi_B(\zeta, x) = \sum_{n=0}^{\infty} \frac{f_n(x)}{(n-1)!}\zeta^{n-1}$$

✅ **Borel sum**: Laplace integration (from ζ to η)

$$\Psi(\eta, x) = \int_0^{\infty} \exp(-\eta\zeta)\psi_B(\zeta, x)d\zeta$$

Integration in Borel plane ζ



⚡ Giving ***Analytic Function*** for divergent series

What's the Magic Behind Borel Resummation?

$$\psi(\eta, x) = \sum_{n=0}^{\infty} f_n(x)\eta^{-n} = \sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \int_0^{\infty} \exp(-\eta\zeta)\zeta^{n-1} d\zeta$$

$$\Psi(\eta, x) = \int_0^{\infty} \exp(-\eta\zeta) \left[\sum_{n=0}^{\infty} \frac{f_n(x)}{n!} \zeta^{n-1} \right] d\zeta$$

✅ **Borel resummation** formally involves an ***exchange of summation and integration*** of the original series

$$\sum \overset{⚠️}{\leftrightarrow} \int$$

This is a non-trivial mathematical step

⚡**Non-perturbative** informaiton is encoded in **singularities**

Turning Points and Stokes Geometry: Keys to Non-perturbative Data

$$\left(-\frac{d^2}{dx^2} - \eta^2 V(x)\right)\psi(x,\eta) = 0$$

✅ **Turning points:**

$$V(a) = 0$$

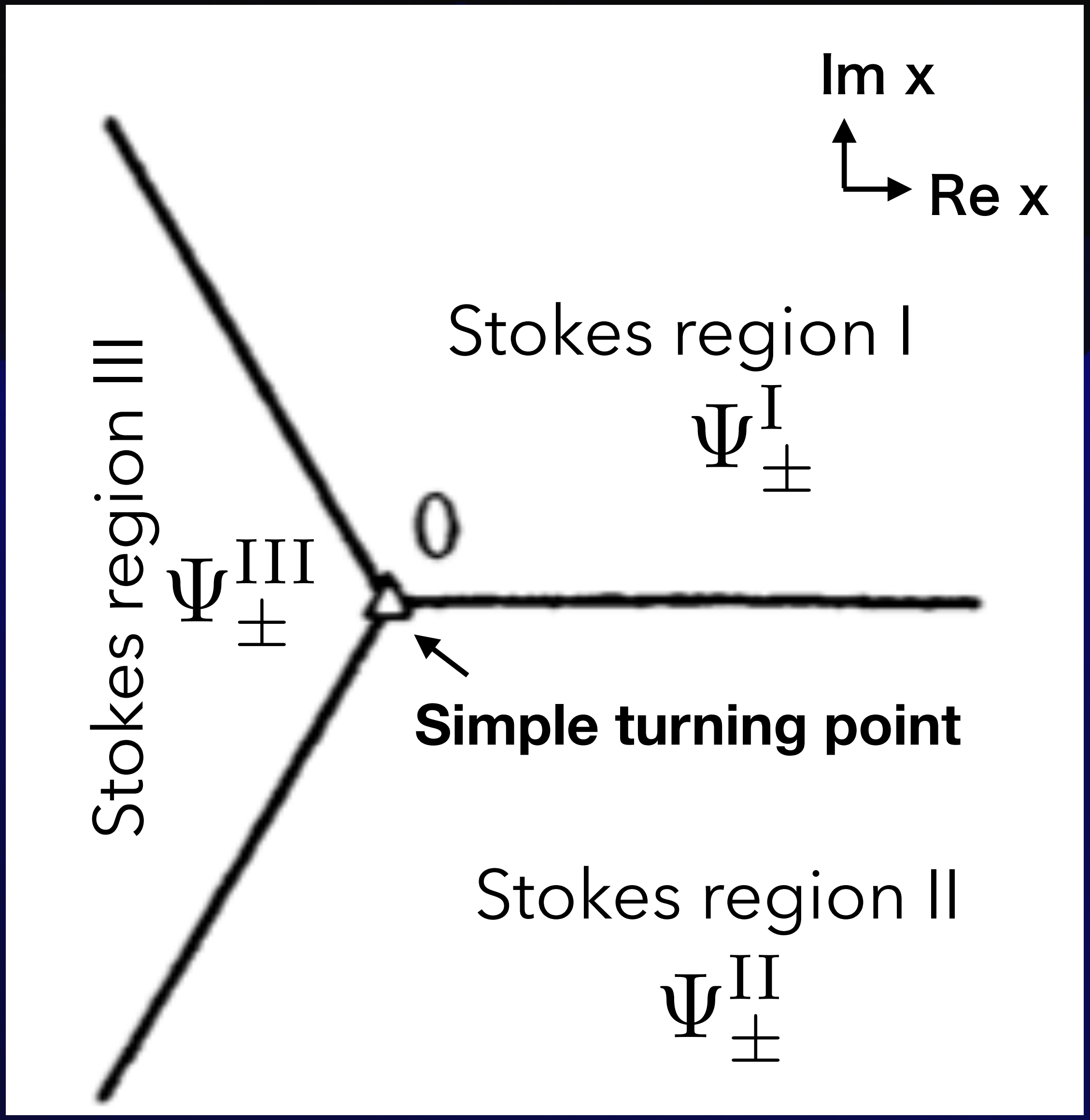
✅ **Simple Turning points:**

$$\left.\frac{dV}{dx}\right|_{x=a} \neq 0$$

✅ **Stokes lines (curves):**

$$\mathrm{Im}\int_a^x \sqrt{-V(\tilde{x})}d\tilde{x} = 0$$

e.g. Airy function: $V = -x$

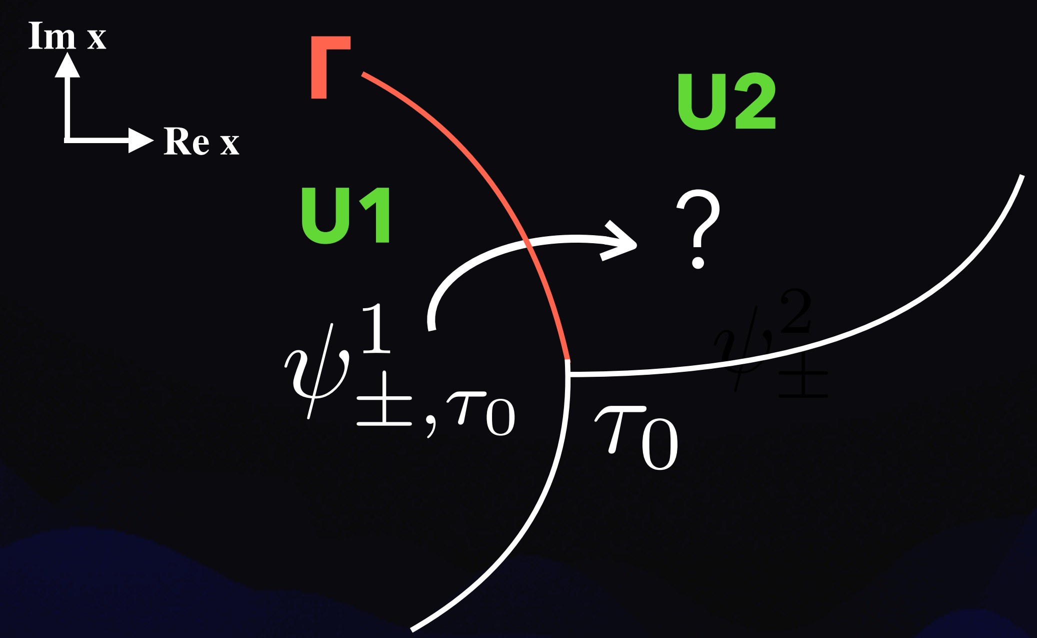


⚡ **In Stokes regions, ψ is Borel summable**

More complicated example

$$Q(x) = \frac{(x^2 - 9)(x^2 - 1/9)}{(x^3 - \exp(i\pi/8))^2}.$$

What happens if you cross the Stokes line?

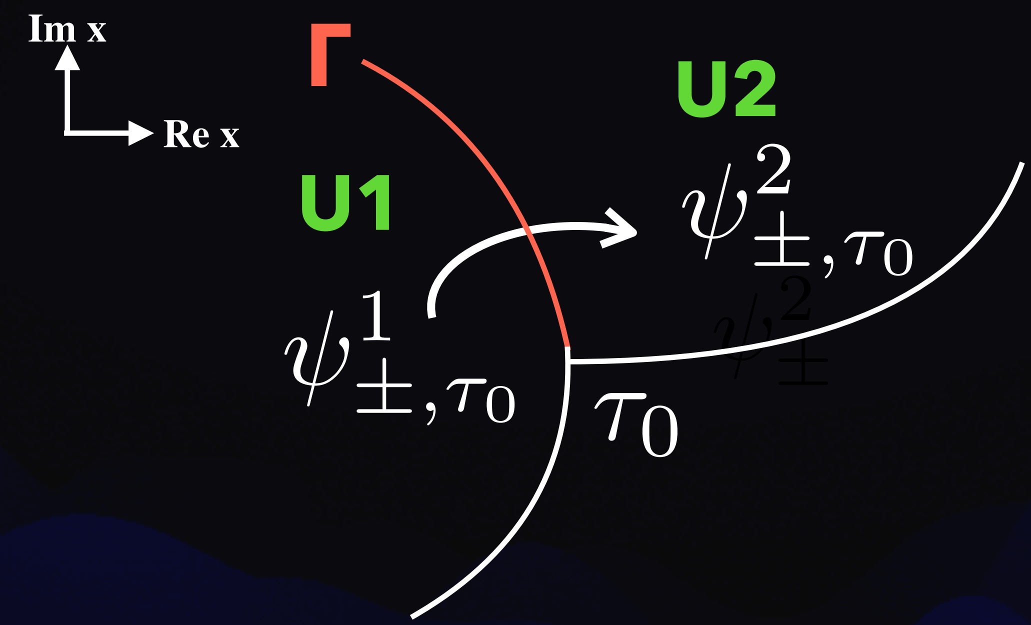


Suppose **U1** and **U2** are Stokes regions having a Stokes curve **Γ** as a common boundary

✅ **WKB solutions normalized** at a turning point

$$\psi_{\pm,\tau_0} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm \int_{\tau_0}^{x} S_{\mathrm{odd}} dx \right)$$

What happens if you cross the Stokes line?



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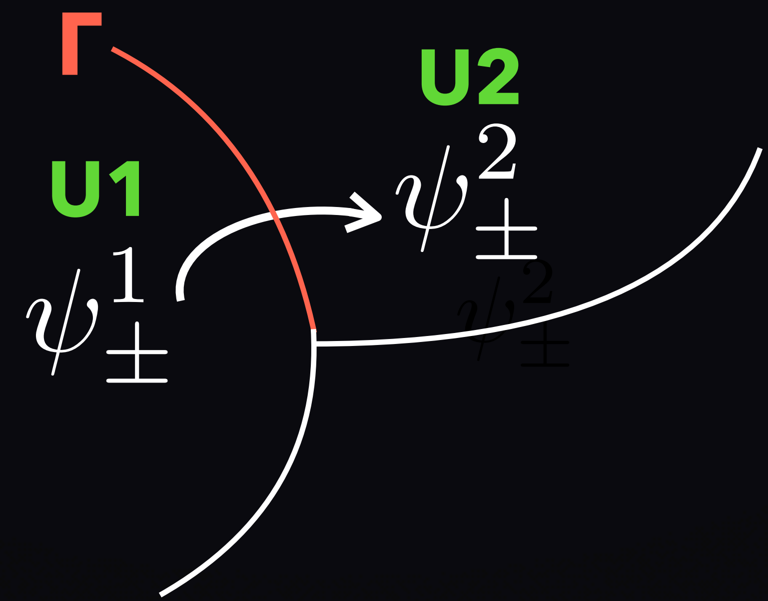
$$\psi_{\pm,\tau_0} = \frac{1}{\sqrt{S_{\text{odd}}}} \exp\left(\pm \int_{\tau_0}^{x} S_{\text{odd}}\, dx \right)$$

✅ **Connection formula** for Borel resummmed solutions**: analytical continuation**

$$\begin{aligned} \psi^1_{+,\tau_0} &= \psi^2_{+,\tau_0} + i\psi^2_{-,\tau_0} \\ \psi^1_{-,\tau_0} &= \psi^2_{-,\tau_0} \end{aligned}$$



Connection Formula *Voros (1983)*

$\psi^1_\pm$ are **analytically continued** to **U2** by

The **sign** of $\mathrm{Re}\int_a^x \sqrt{-V(\tilde{x})}d\tilde{x}$

$$\begin{cases} \Psi^1_+ = \Psi^2_+ \pm i\Psi^2_- \\ \Psi^1_- = \Psi^2_- \end{cases} \quad > 0$$

Or

$$\begin{cases} \Psi^1_+ = \Psi^2_+ \\ \Psi^1_- = \Psi^2_- \pm i\Psi^2_\pm \end{cases} \quad < 0$$

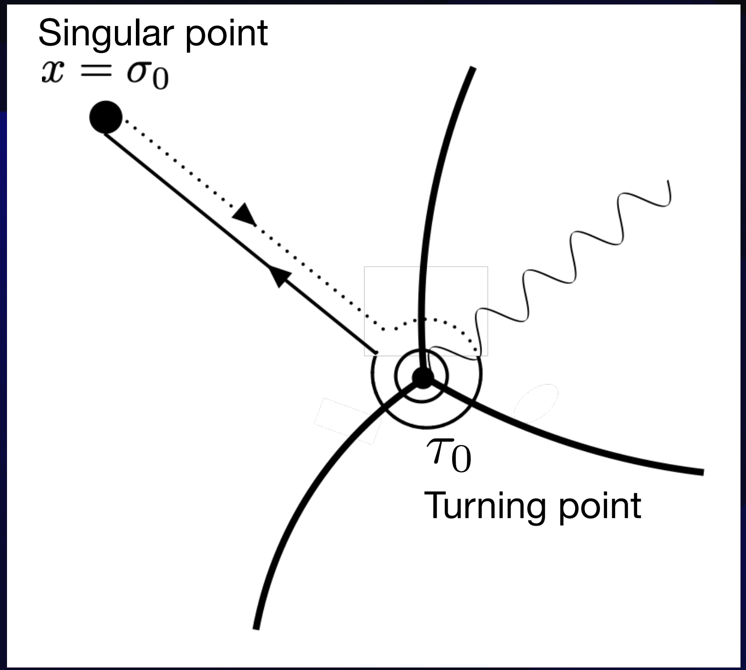
±: **Counter-clockwise** or **clockwise crossing Γ**

⚡**Singularities** in Borel plane (ζ plane) gives **connection formula** ~**non-perturbative information**

Another Source of Non-perturbative Information: Voros Coefficients

✅ **Voros coefficient** is defined with **"regularized" Sodd**

$$V_{\text{voros}} \equiv \frac{1}{2}\int_{\gamma_{\sigma_0,\tau_0}} S_{\text{odd}}^{(\text{reg})}$$



γ_{σ_0,τ_0} : Integration path is non-closed contour in general — starting from a ***singular point σ0*** (second Riemann sheet), turning around ***turning point τ0*** clockwise, to a ***singular point*** (first Riemann sheet)

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e.g. Parabolic cylinder: $V=-E+x^2/4$

$$V_{\text{voros}} \equiv \frac{1}{2}\int_{\gamma_{2\sqrt{E},\infty}} (S_{\text{odd}} - \eta S_{-1}) \sim \int_{2\sqrt{E}}^{\infty} (S_{\text{odd}} - \eta S_{-1})$$

👀 Irregular singular point: x=∞, S_{-1} is deverging at x=∞

What Does the Voros Coefficient do?

✅ The ***Voros coefficient connects*** exact ***WKB*** solutions normalized at a ***turning point*** and a ***singular point***

e.g. Parabolic cylinder: $V=-E+x^2/4$ Shen & Silverstone '08, Takei '08

— The ***WKB solution*** normalized at ***turning point*** $\tau_\pm$:

$$\psi_{\pm,\tau_\pm}(x) = \frac{1}{\sqrt{S_{\rm odd}(x)}} \exp\left[\pm \int_{\tau_\pm}^{x} S_{\rm odd}(x')dx'\right]$$

— The ***WKB solution*** normalized at ***singular points*** $x=\pm\infty$:

$$\psi_\pm^{(\pm\infty)}(x) = \exp\left[\pm\int_{\tau_\pm}^{x} \eta S_{-1}(x')dx'\right] \frac{1}{\sqrt{S_{\rm odd}(x)}} \exp\left[\pm\int_{\pm\infty}^{x} \left(S_{\rm odd}(x') - \eta S_{-1}(x')\right) dx'\right]$$

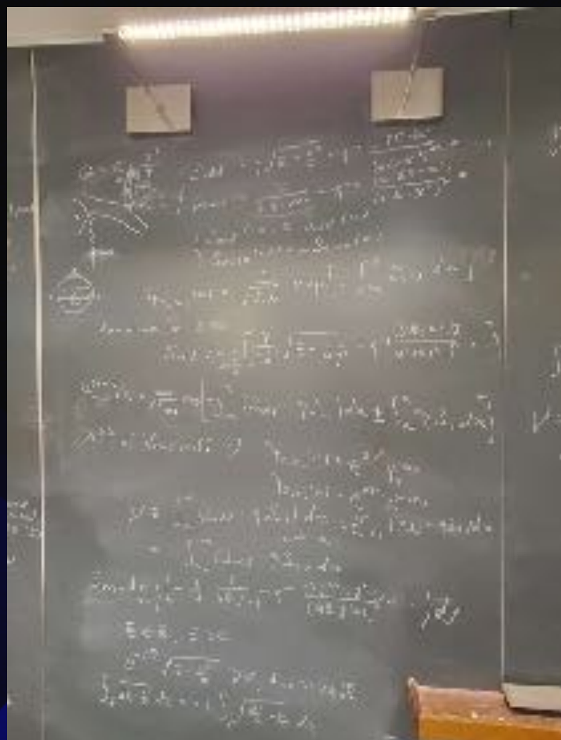
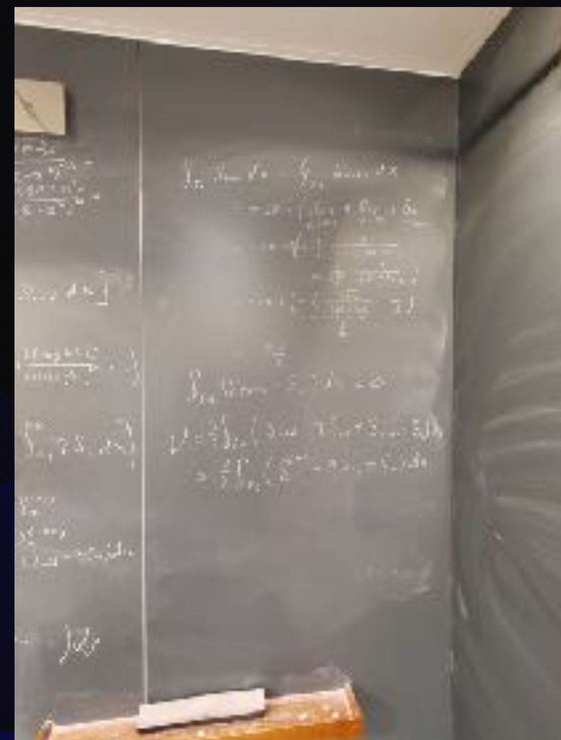
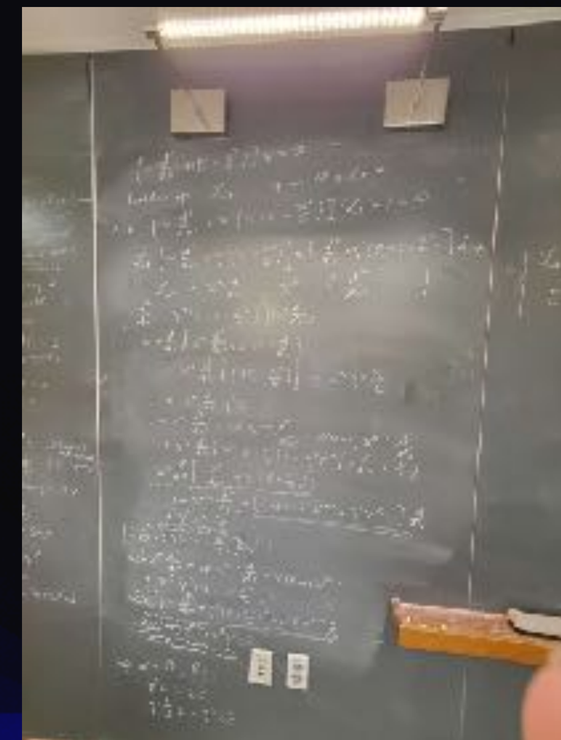
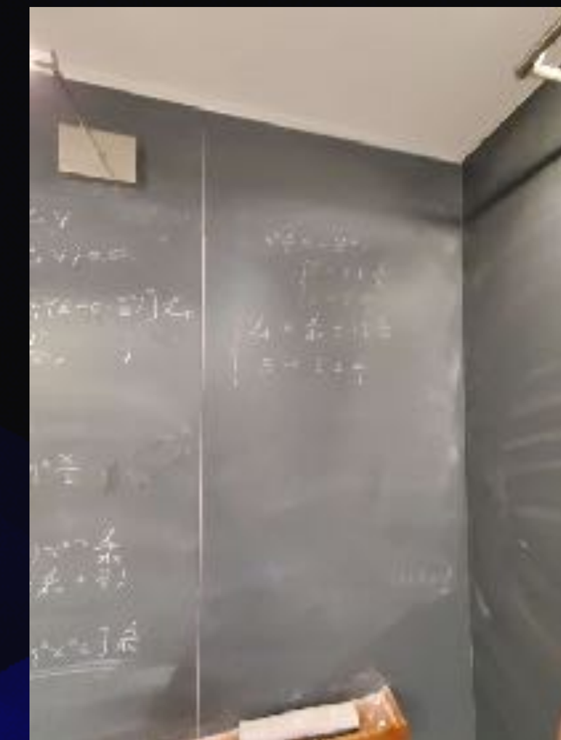
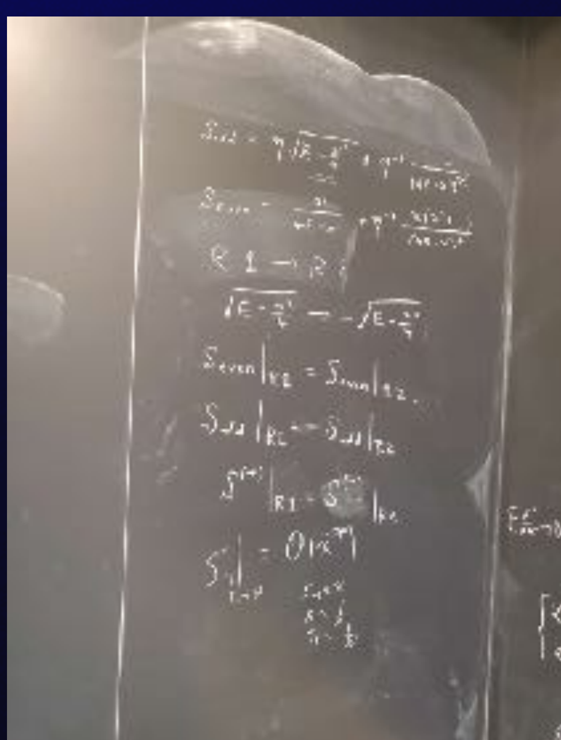
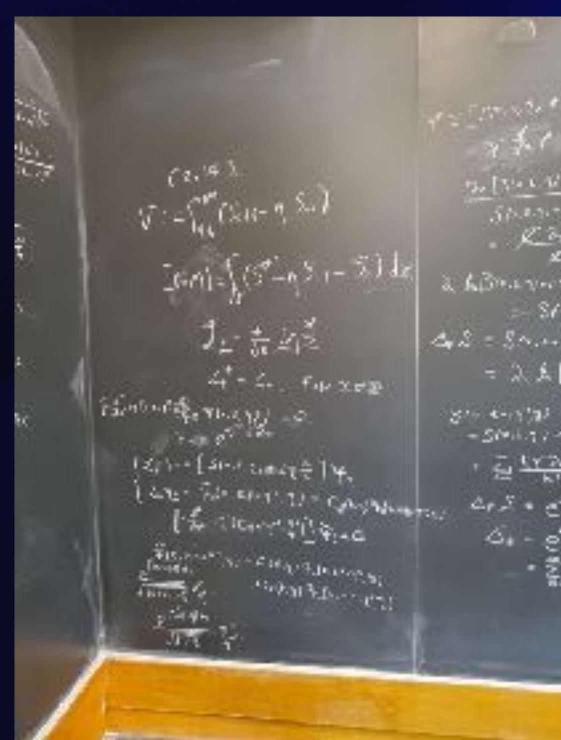
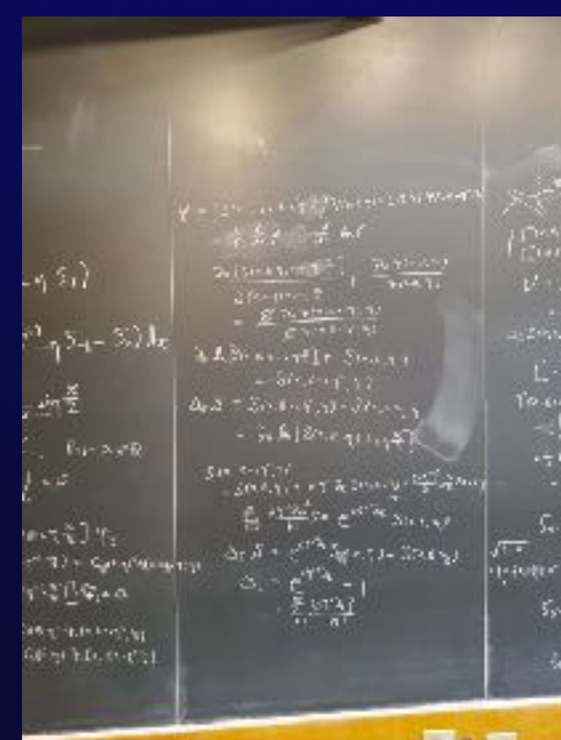
⚡ The ***Voros coefficient*** connects the two WKB solutions

$$\psi_{\pm,\tau_+}(x) = e^{\pm\mathcal{V}_{\rm voros}^{(+\infty)}} \psi_\pm^{(+\infty)}(x) \ , \ \psi_{\pm,\tau_-}(x) = e^{\pm\mathcal{V}_{\rm voros}^{(-\infty)}} \psi_\pm^{(-\infty)}(x)$$

$$\mathcal{V}_{\rm voros}^{(+\infty)} = \int_{\tau_+}^{\infty} [S_{\rm odd}(x) - \eta S_{-1}(x)]dx \ , \ \mathcal{V}_{\rm voros}^{(-\infty)} = \int_{\tau_-}^{-\infty} [S_{\rm odd}(x) - \eta S_{-1}(x)]dx$$

Remark: Like the connection formula, the Voros coefficient captures data from singularities in the Borel plane

Behind the elegance 💪

To be continued

Quantization via Exact WKB

Ryo& M.S. (arXiv:2503.XXXXX)

✅ *We now use* ***exact WKB solutions*** *to perform* ***quantization***

Hamiltonian: $\hat{H} \equiv \frac{1}{2}\left[\hat{\pi}\hat{\pi} + V(x)\,\hat{\psi}\hat{\psi}\right]$

Complete set of solutions: $\left(-\frac{d^2}{dx^2} - V\right) u_\pm = 0$

$$(u_+, u_+) = 1\ , \qquad (u_-, u_-) = -1\ , \qquad (u_+, u_-) = (u_-, u_+) = 0$$

$(f, g) \equiv -i\,(f\,\partial_x \bar{g} - \bar{g}\partial_x f)$

Mode decomposition:

$$\hat{\psi}(x) = u_+(x)\,\hat{a} + u_-(x)\,\hat{a}^\dagger\ , \qquad \hat{\pi}(x) = v_+(x)\,\hat{a} + v_-(x)\,\hat{a}^\dagger$$

$v_\pm(x) \equiv \frac{du_\pm}{dx}$

Quantization conditions:

$$[\hat{a}, \hat{a}^\dagger] = 1\ , \qquad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$$

$$u_+ v_- - u_- v_+ = i$$

Vacuum: $\hat{a}|0\rangle = 0$

We want to define **mode functions** with **exact WKB** solution $\Psi_\pm$

⚡ $$u_+(x) = \frac{1}{\sqrt{2}}\left[\alpha\,\Psi_-(x) + \beta\,\Psi_+(x)\right]\ , u_-(x) = \overline{u_+(x)}$$

Quantization via Exact WKB: A Practical Example

e.g. Parabolic cylinder: $V=-E+x^2/4$

⚡ ***We take mode functions by the exact WKB solutions normalized at asymptotic point x=∞***

$$u_+(x) = \frac{1}{\sqrt{2}}\left[\alpha\,\psi_-^{(\infty)}(x) + \beta\,\psi_+^{(\infty)}(x)\right]$$

$$u_-(x) = \overline{u_+(x)}$$

$$\psi_{\pm,\tau_+} = \exp(\pm V_{\text{voros}})\,\psi_\pm^{(\infty)}$$

$$\psi_\pm^{(\infty)} = \frac{1}{\sqrt{S_{\text{odd}}}}\exp(\pm\eta\int_{\tau_+}^{x} S_{-1}dx)\,\exp(\pm\int_{\infty}^{x}(S_{\text{odd}}-\eta\,S_{-1})dx)$$

⚡ **The asymptotic state is described by the standard WKB**

$$\psi_\pm^{(\infty)} \sim \frac{1}{\sqrt{S_{-1}}}\exp(\pm\eta\int_{\tau_+}^{x} S_{-1}dx)$$

$$x \to \infty$$

✅ Satisfying quantization conditions

✅ We can generalized this procedure to other potentials

Computing Particle Production with Exact WKB

✅ With the mode functions defined via exact WKB, we now compute the particle production.

- **Define vacua:**

$$|0\rangle \text{ such that } a|0\rangle = 0$$
$$|\tilde{0}\rangle \text{ such that } \tilde{a}|\tilde{0}\rangle = 0$$

$$\hat{\psi} = \tilde{u}_+(x)\hat{\tilde{a}} + \tilde{u}_-(x)\hat{\tilde{a}}^\dagger$$

- **Relate mode functions via Bogoliubov transformation:**

$$\begin{pmatrix} \tilde{u}_+ \\ \tilde{u}_- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

- **Bogoliubov coefficient β encodes particle production:**

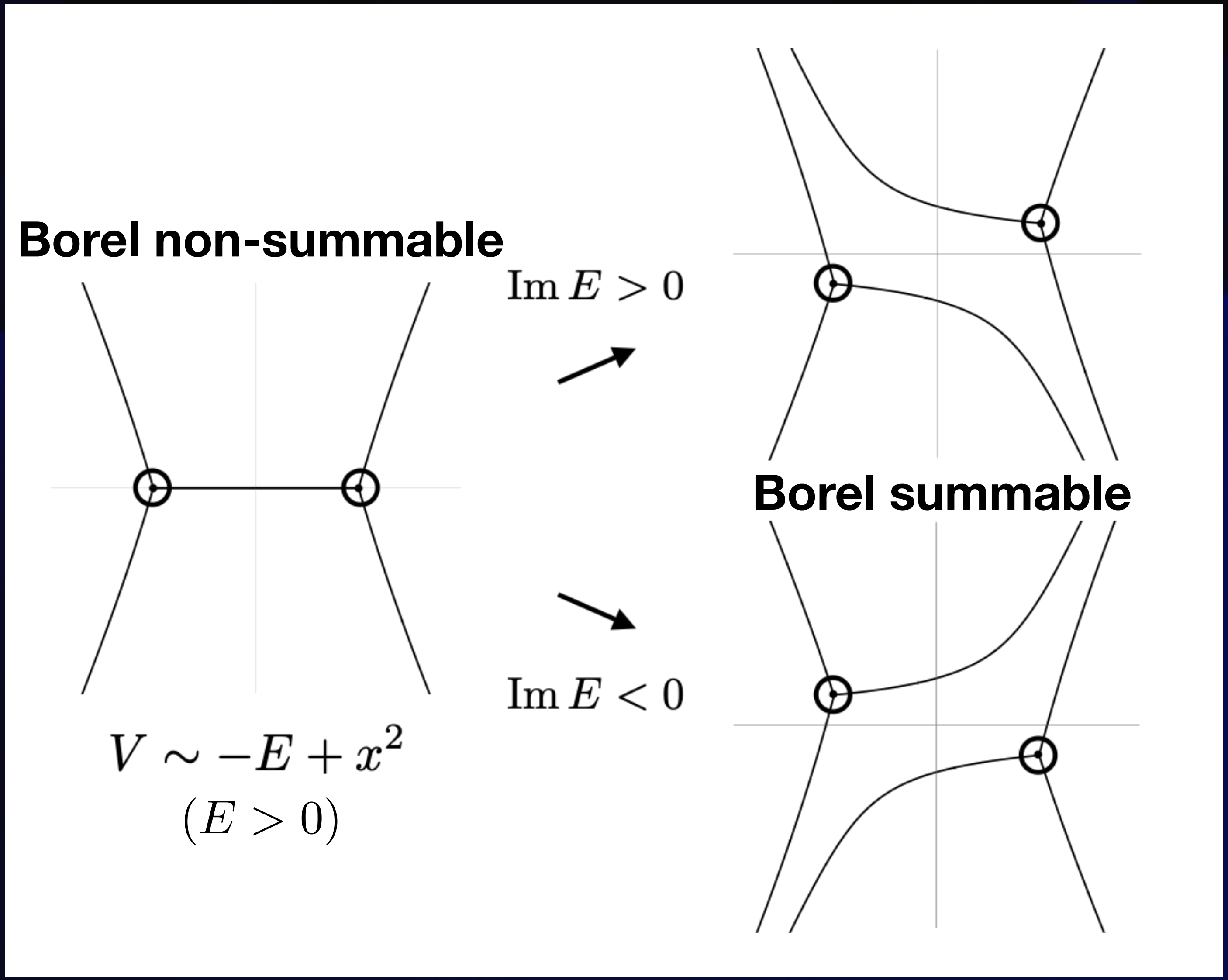
$$|\beta|^2 = \text{number density of produced particles}$$

⚡ ***In exact WKB, α and β are computed via the connection formula and Voros coefficients***

A Worked Example: V=-E+x²/4

✅ **Simple turning points:** $\tau = \pm 2\sqrt{E}$

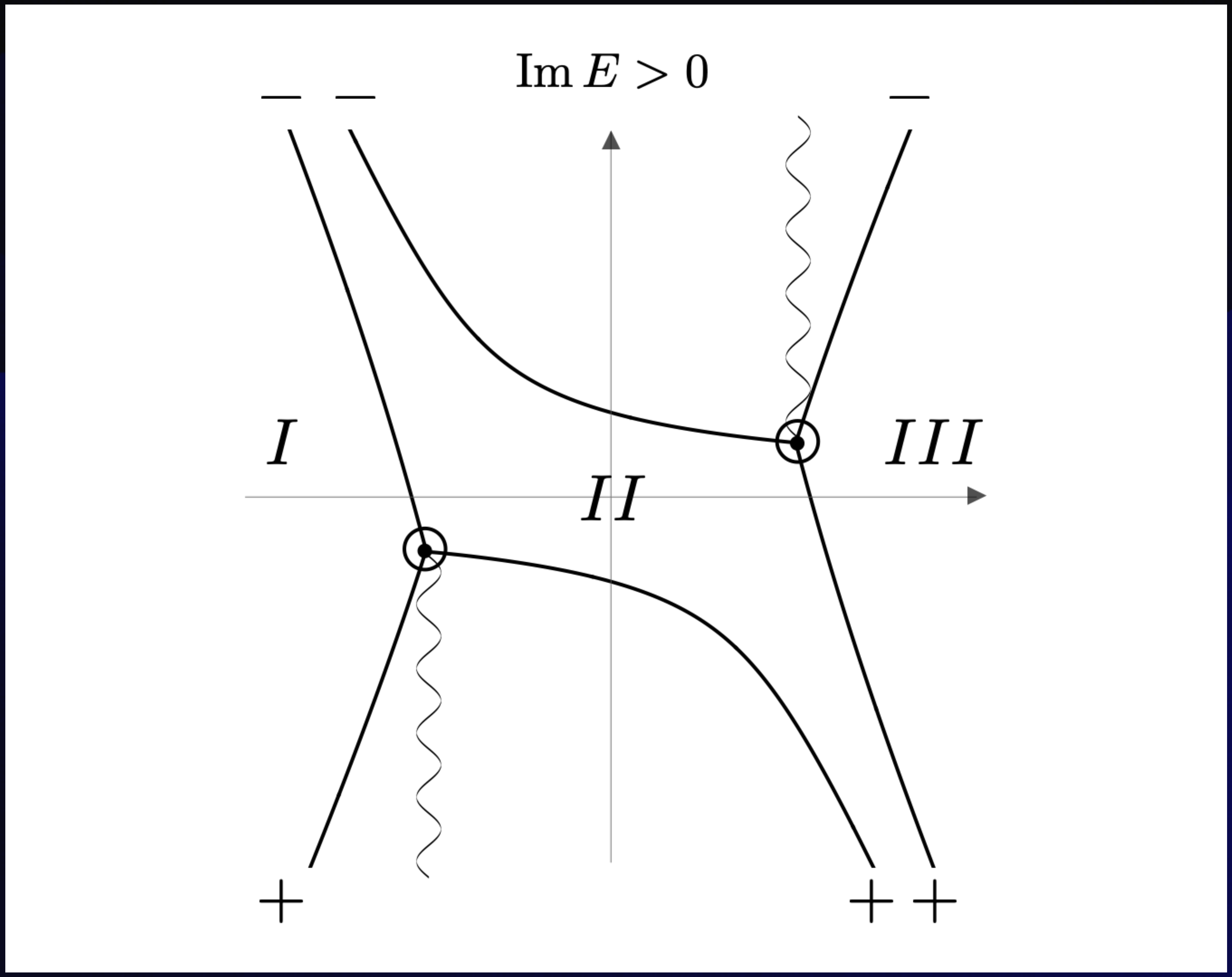
✅ **Stokes curves:**



⚡ **Both eventually give the same physics**

A Worked Example: $V=-E+x^2/4$

- ✅ **Simple turning points:** $\tau = \pm 2\sqrt{E}$
- ✅ **Stokes curves:**



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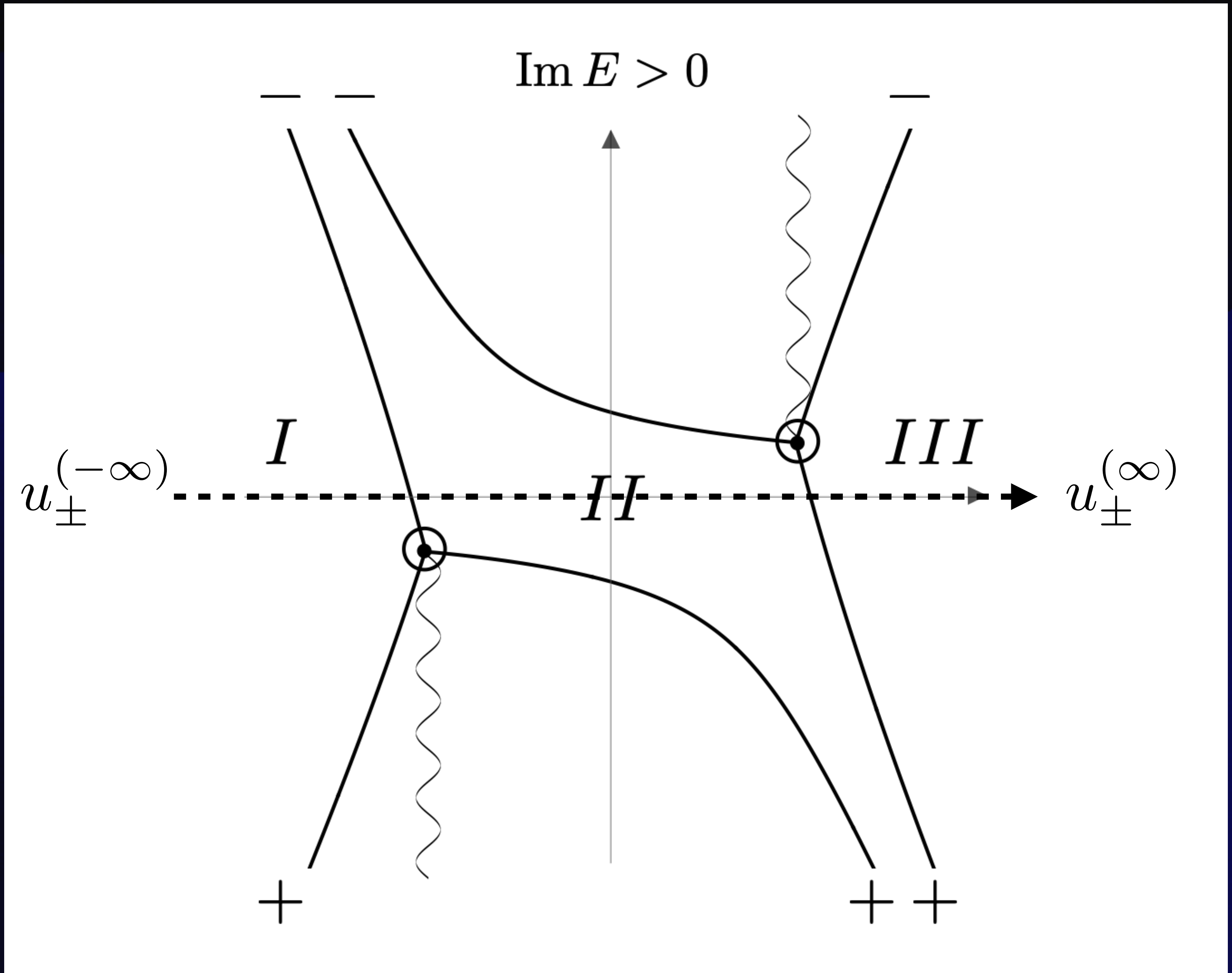

Relate two asymptotic states: $u_{\pm}^{(-\infty)} \to u_{\pm}^{(\infty)}$

A Worked Example: V=-E+x²/4

$$\begin{pmatrix} u_+^{(-\infty)} \\ u_-^{(-\infty)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} \exp(-V_{\text{voros}}^{(-\infty)}) & 0 \\ 0 & \exp(V_{\text{voros}}^{(-\infty)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} e^{\pi E \eta} & 0 \\ 0 & e^{-\pi E \eta} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

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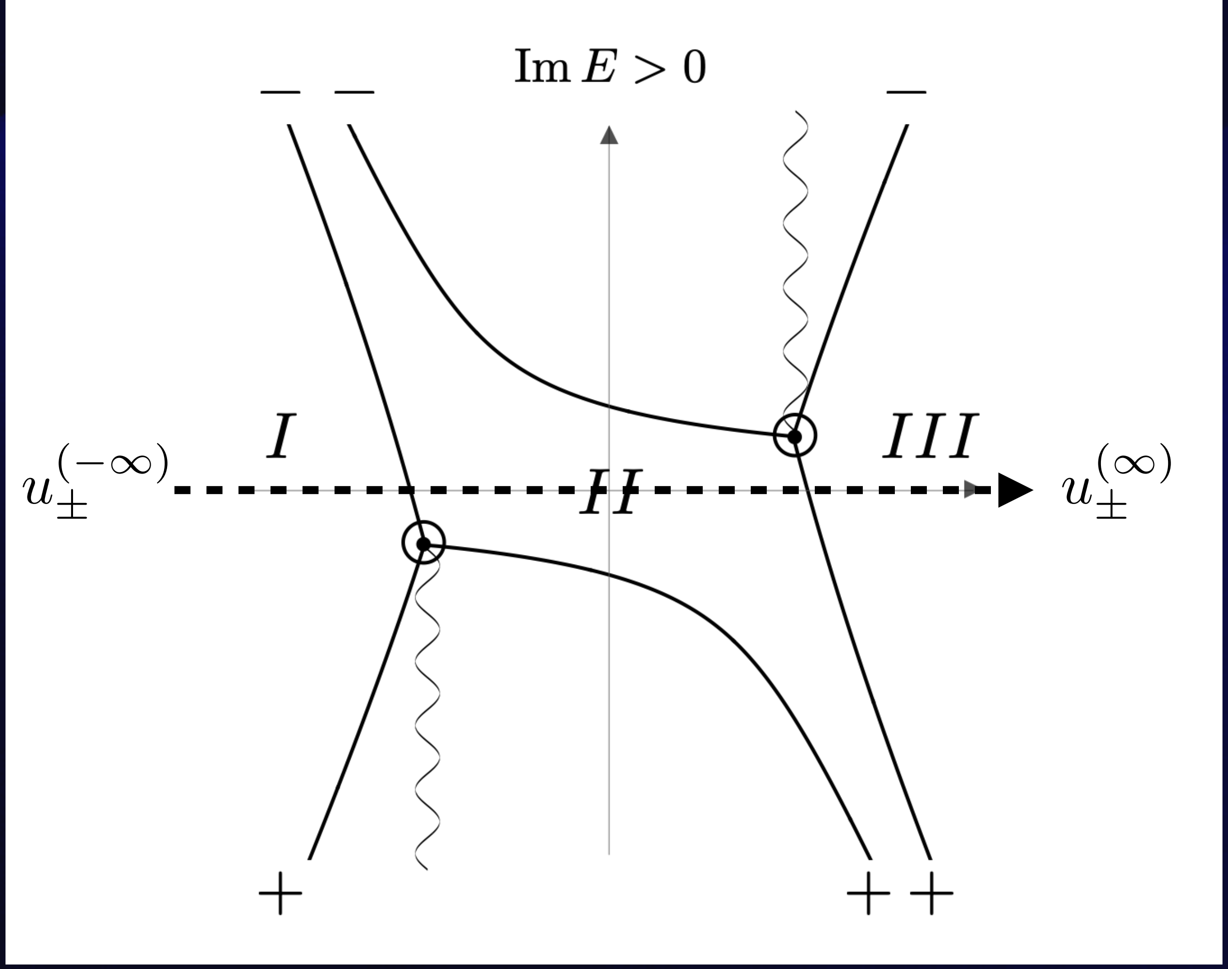

$$u_+^{(-\infty)} = \frac{1}{\sqrt{2}}\psi_-^{(-\infty)} \;,\; u_-^{(-\infty)} = \frac{i}{\sqrt{2}}\psi_+^{(-\infty)}$$

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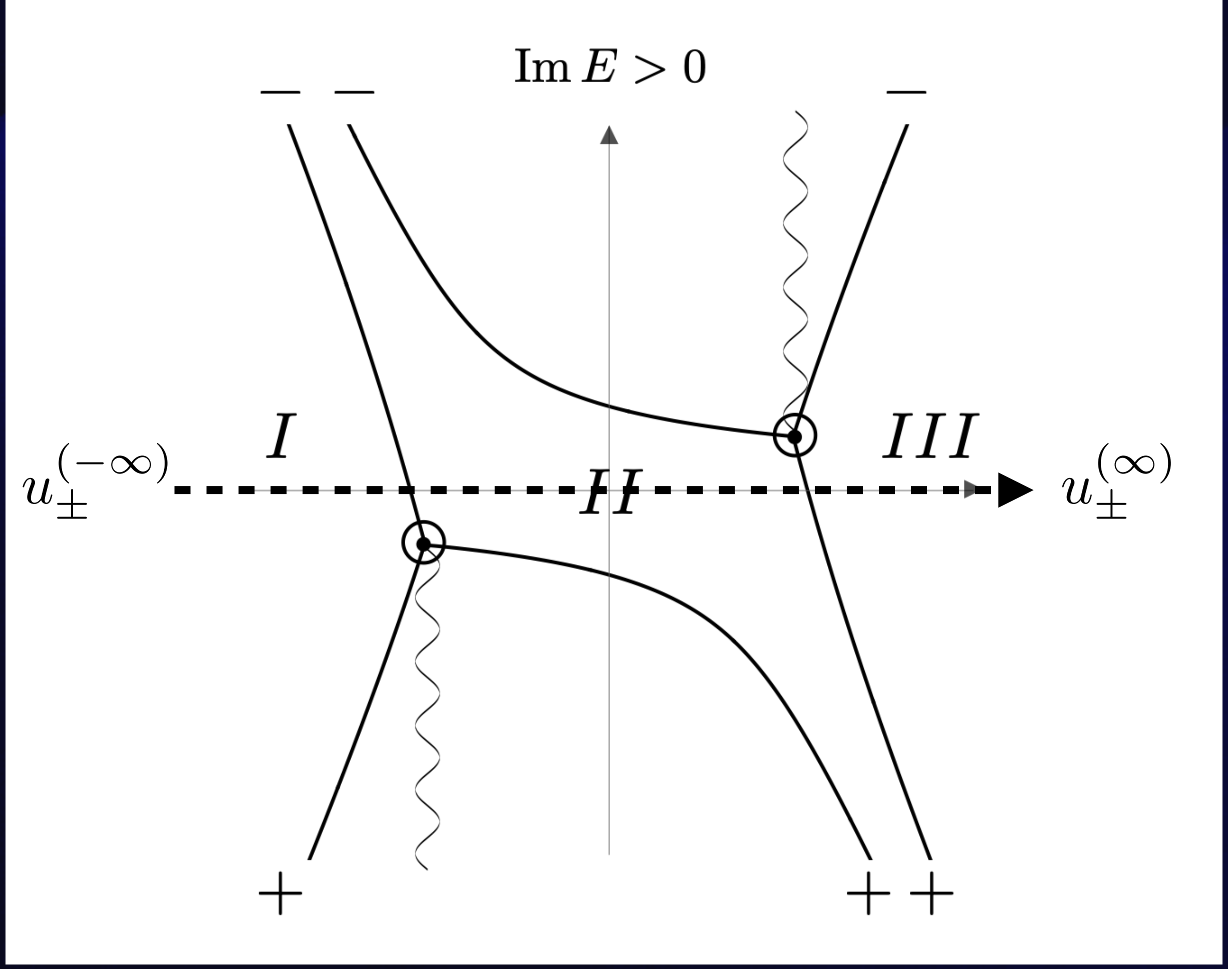

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$$\psi^I_{\pm,\tau_-} = \exp(\pm V_{\text{voros}}^{(-\infty)})\, \psi_\pm^{(-\infty)}$$

A Worked Example: V=-E+x²/4

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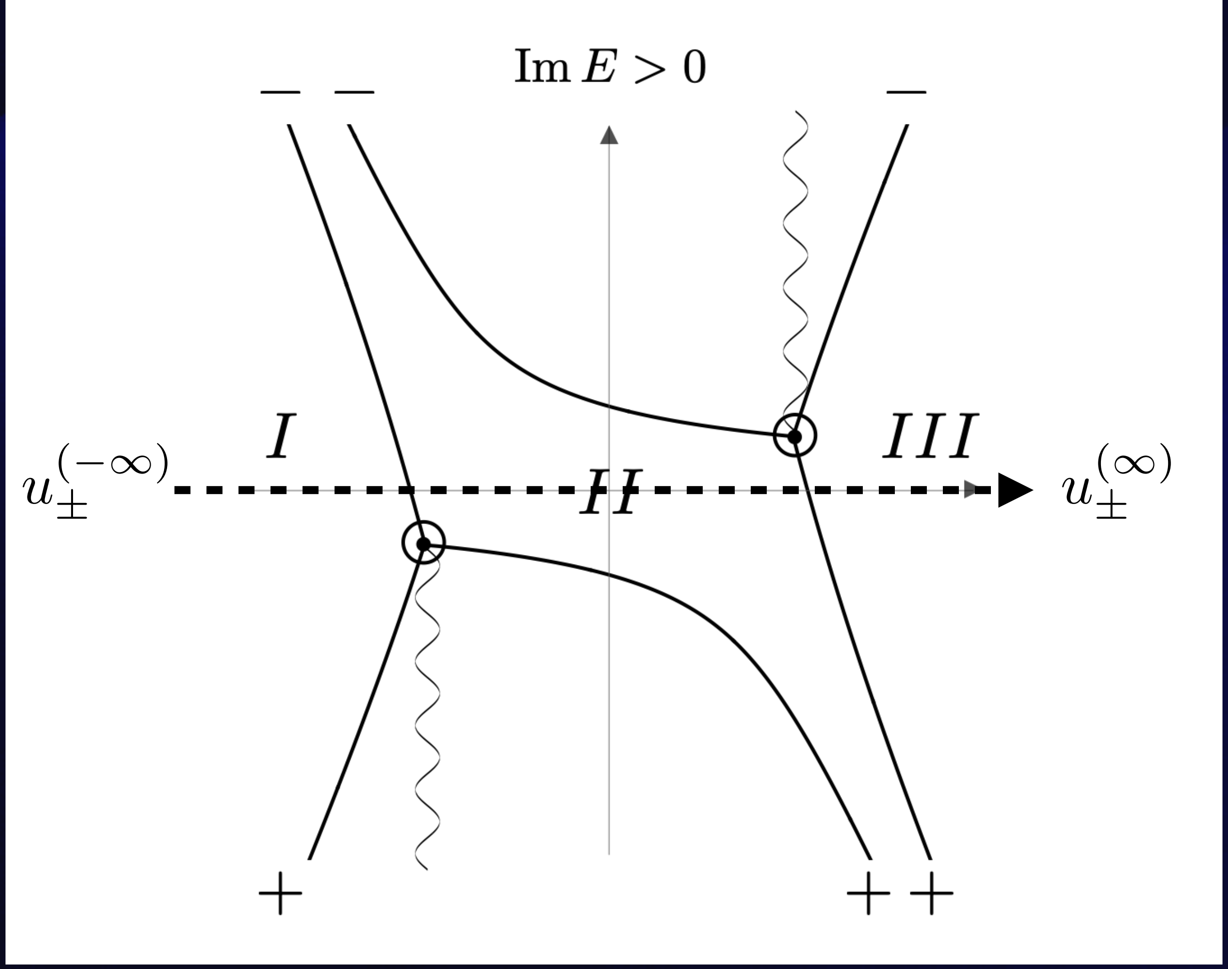

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$$\begin{pmatrix} \psi^I_{+,\tau_-} \\ \psi^I_{-,\tau_-} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \psi^{II}_{+,\tau_-} \\ \psi^{II}_{-,\tau_-} \end{pmatrix}$$

A Worked Example: V=-E+x²/4

$$\begin{pmatrix} u_+^{(-\infty)} \\ u_-^{(-\infty)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \begin{pmatrix} \exp(-V_{\text{voros}}^{(-\infty)}) & 0 \\ 0 & \exp(V_{\text{voros}}^{(-\infty)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} e^{\pi E \eta} & 0 \\ 0 & e^{-\pi E\eta} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$
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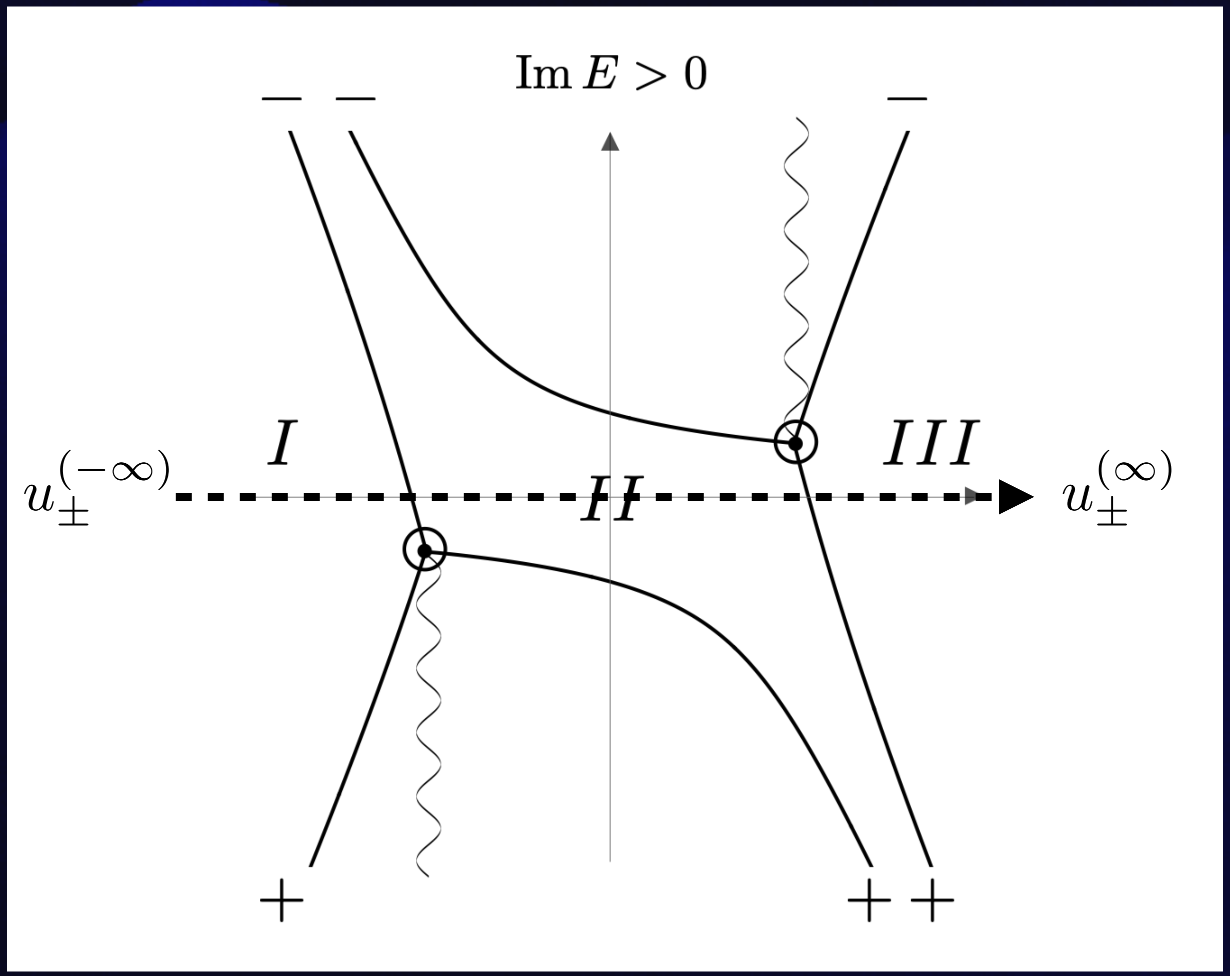

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$$\begin{pmatrix} \psi^{II}_{+,\tau_-} \\ \psi^{II}_{-,\tau_-} \end{pmatrix} = \begin{pmatrix} e^{+\int_{\tau_-}^{\tau_+} S_{\text{odd}}dx} & 0 \\ 0 & e^{-\int_{\tau_-}^{\tau_+} S_{\text{odd}}dx} \end{pmatrix} \begin{pmatrix} \psi^{II}_{+,\tau_+} \\ \psi^{II}_{-,\tau_+} \end{pmatrix}$$

⋮

Exact WKB to Particle Production: Mission Accomplished

Ryo& M.S. (arXiv:2503.XXXXX)

$$\begin{pmatrix} u_+^{(-\infty)} \\ u_-^{(-\infty)} \end{pmatrix} = \begin{pmatrix} \sqrt{1+e^{2\pi E\eta}}e^{i\theta} & -e^{\pi E\eta} \\ -e^{\pi E\eta} & \sqrt{1+e^{2\pi E\eta}}e^{-i\theta} \end{pmatrix} \begin{pmatrix} u_+^{(\infty)} \\ u_-^{(\infty)} \end{pmatrix}$$

✅ **Exact WKB Reproduces known results**

Kofman, Linde & Starobinsky '97
Salehian, Gorji, Mukohyama & Firouzjahi '20

✅ **Im E<0 gives the same results**

⚡ The non-perturbative information captured by exact WKB translates into particle production

Through singularities in the exact WKB analysis, we fully determine the particle number density — connecting formal structure to real physics

Summary and Outlook

- ✅ Starting from a divergent WKB series, we applied Borel transformation and found that non-perturbative information is encoded in singularities on the Borel plane

- ✅ Key structures like turning points, Stokes lines, and Voros coefficients allowed us to capture these non-perturbative effects precisely

- ✅ Using exact WKB solutions as mode functions, we reproduced the known particle production results by extracting Stokes data

- ✅ Exact WKB analysis can provide a systematic approach to study particle production

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Challenges: deeper mathematics and smarter physical approximations

Tackle more complicated potentials *(e.g. Mathieu equation) with exact WKB techniques*

Explore exact WKB-based approximations *to go beyond current limits*