Instanton Density Operator in Lattice QCD from Higher Category Theory

Jing-Yuan Chen

Institute for Advanced Study, Tsinghua University

Confinement and Symmetry from Vacuum to QCD Phase Diagram

Feb 14, 2025 @ Benasque

Why Wilson invented lattice gauge theory in the first place?

Why Wilson invented lattice gauge theory in the first place?

<u>fundamental purpose:</u>
to make sense of what a QFT is

- manifestly well-defined path integral
- manifestly reflection positive (Euclidean version of unitarity)
- manifestly local
- no bizarre (especially non-local) gauge fixing subtleties

Why Wilson invented lattice gauge theory in the first place?

fundamental purpose:
to make sense of what a QFT is

- manifestly well-defined path integral
- manifestly reflection positive (Euclidean version of unitarity)
- manifestly local
- no bizarre (especially non-local) gauge fixing subtleties

practical purpose (thanks to Creutz):
to numerically compute things

Why Wilson invented lattice gauge theory in the first place?

<u>fundamental purpose:</u>
to make sense of what a QFT is

- manifestly well-defined path integral
- manifestly reflection positive (Euclidean version of unitarity)
- manifestly local
- no bizarre (especially non-local) gauge fixing subtleties

<u>practical purpose (thanks to Creutz):</u>
to numerically compute things

This talk: A conceptual-level refinement of Wilson's definition,
natural, useful (sometimes necessary) for both purposes,
unifies with multiple other branches in physics and math.

motivation, principle & math: [2406.06673]

explicitly: with Peng Zhang [2411.07195]

The Big Picture

Refined Lattice Yang-Mills Path Integral

Plaquette d.o.f. and weight

Cube d.o.f. and weight; Chern-Simons saddle

Hypercube d.o.f. and weight; instanton

Further Remarks

Problem: continuum $\mathcal{I} := \frac{1}{2}\,\mathbf{tr}\left[\frac{F}{2\pi}\wedge\frac{F}{2\pi}\right]$, $I := \int_{\mathcal{M}} \mathcal{I} \in \mathbb{Z}$

Wilson's lattice gauge theory: *G* d.o.f. on each link

$$\prod_{\text{links } l} G_l \longrightarrow \mathbb{Z}$$

config space instanton number

Problem: continuum $\mathcal{I} := \frac{1}{2}\,\mathbf{tr}\left[\frac{F}{2\pi} \wedge \frac{F}{2\pi}\right]$, $I := \int_{\mathcal{M}} \mathcal{I} \in \mathbb{Z}$

Wilson's lattice gauge theory: *G* d.o.f. on each link

$$\prod_{\text{links } l} G_l \longrightarrow \mathbb{Z}$$

config space *connected*	instanton number *discrete*
any two configs can continuously deform to each other	either single-valued or discontinuous map

Problem: continuum $\mathcal{I} := \frac{1}{2}\mathbf{tr}\left[\frac{F}{2\pi}\wedge\frac{F}{2\pi}\right],\qquad I := \int_{\mathcal{M}} \mathcal{I} \in \mathbb{Z}$

Wilson's lattice gauge theory: *G* d.o.f. on each link

$$\prod_{\text{links } l} G_l \longrightarrow \mathbb{Z}$$

config space
connected

instanton number
discrete

any two configs can continuously deform to each other

either single-valued or discontinuous map

current methods all involve discontinuities
(restricted config space, cooling/flow, fermion index)

Problem: continuum $\mathcal{I} := \frac{1}{2}\mathbf{tr}\left[\frac{F}{2\pi}\wedge\frac{F}{2\pi}\right], \qquad I := \int_{\mathcal{M}} \mathcal{I} \in \mathbb{Z}$

Wilson's lattice gauge theory: *G* d.o.f. on each link

$$\prod_{\text{links } l} G_l \longrightarrow \mathbb{Z}$$

config space
connected

instanton number
discrete

any two configs can continuously deform to each other

either single-valued or discontinuous map

same issue in those group cohomology lattice models for TQFT, when the group becomes Lie group

Very general problem:

topological operators lost

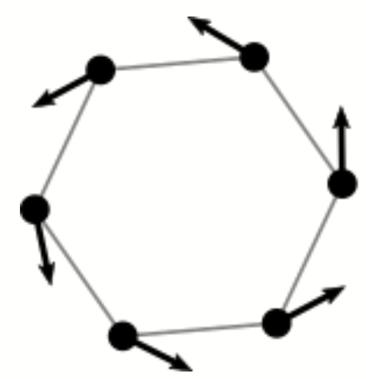
when putting continuous-valued d.o.f. on lattice

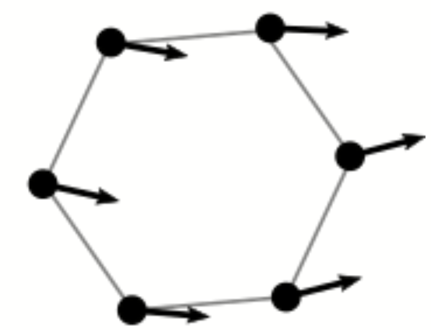
$$\prod_{\text{local}} X \longrightarrow \text{topological number}$$

connected *discrete*

either single-valued
or discontinuous

S^1 nlsm (XY), 1d winding number





we feel: *w=1* *w=0*

S^1 nlsm (XY), 1d winding number

we feel: *w=1* *w=0*

Continuous deformable to each other!
(only on lattice, not in continuum)

topological operators lost
when putting continuous-valued d.o.f. on lattice

- S^1 nlsm (XY): winding, vortex
- *U(1)* gauge: Dirac quantization, monopole, abelian CS, abelian instanton

} π_1 physics

- S^2 nlsm: Berry phase, skyrmion, hedgehog

} π_2 physics

- S^3 nlsm: WZW, skyrmion, hedgehog
- *SU(N)* gauge: CS, instanton, Yang monopole

… whatever …

} π_3 (or higher) physics

topological operators lost
when putting continuous-valued d.o.f. on lattice

- S^1 nlsm (XY): winding, vortex
- *U(1)* gauge: Dirac quantization, monopole, abelian CS, abelian instanton

} π_1 physics: solved by Villainization

- S^2 nlsm: Berry phase, skyrmion, hedgehog

} π_2 physics: solved by CP^N (spinon-decomposition)

- S^3 nlsm: WZW, skyrmion, hedgehog
- *SU(N)* gauge: CS, instanton, Yang monopole

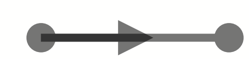
… whatever …

} π_3 (or higher) physics: groups/fibre bundles fail

higher category theory necessary

$$Z = \int D\Phi \; e^{i\int \mathcal{L}(\Phi, \partial\Phi, \cdots)}$$

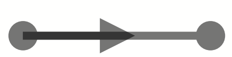
$$\partial\Phi \sim \Phi_{v'} - \Phi_v$$



becomes finite dimensional

$$Z = \int D\Phi \; e^{i\int \mathcal{L}(\Phi,\partial\Phi,\cdots)}$$

~~$\partial\Phi \sim \Phi_{v'} - \Phi_v$~~

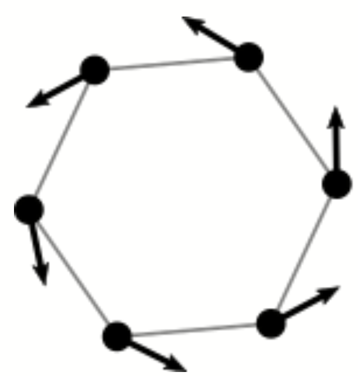


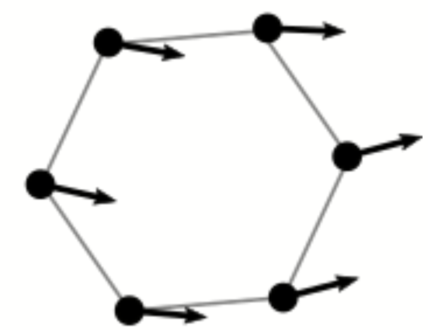
becomes finite dimensional

but not in the naive way

conceptually refine to capture
homotopy / interpolation

S^1 nlsm (XY)





we feel: *w=1* *w=0*

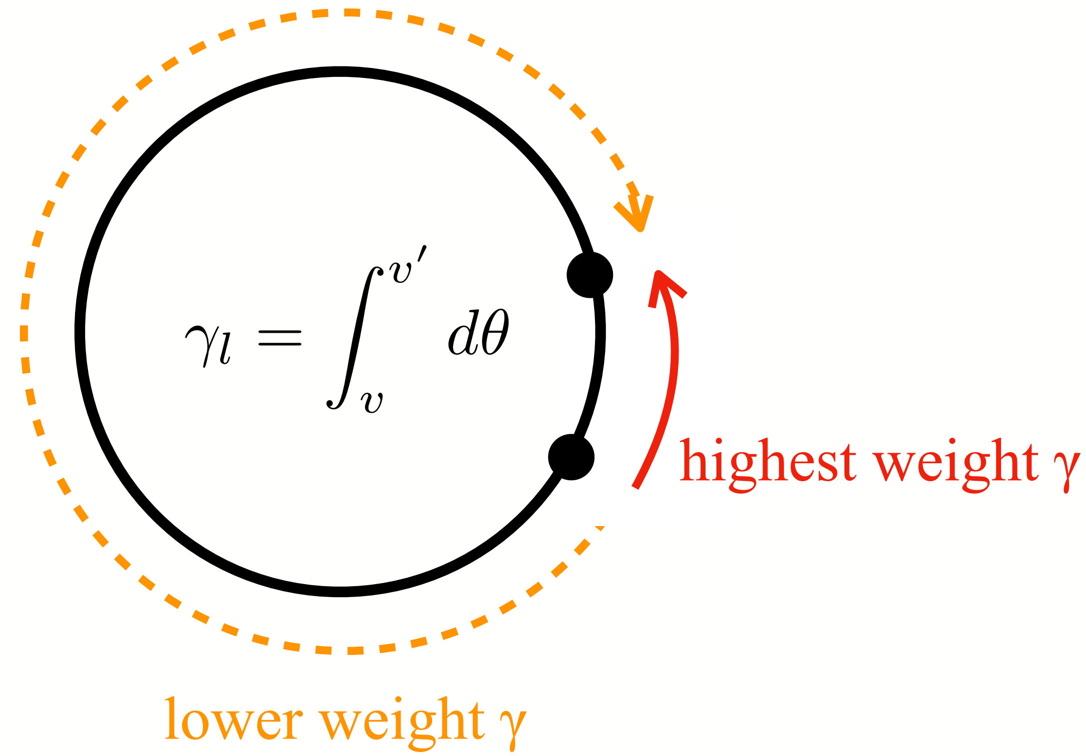
Continuous deformable to each other!
(only on lattice, not in continuum)

S^1 nlsm (XY): Villainization

If link embedded in continuum,
how does θ interpolate?

S^1 nlsm (XY): Villainization

If link embedded in continuum,
how does θ interpolate?



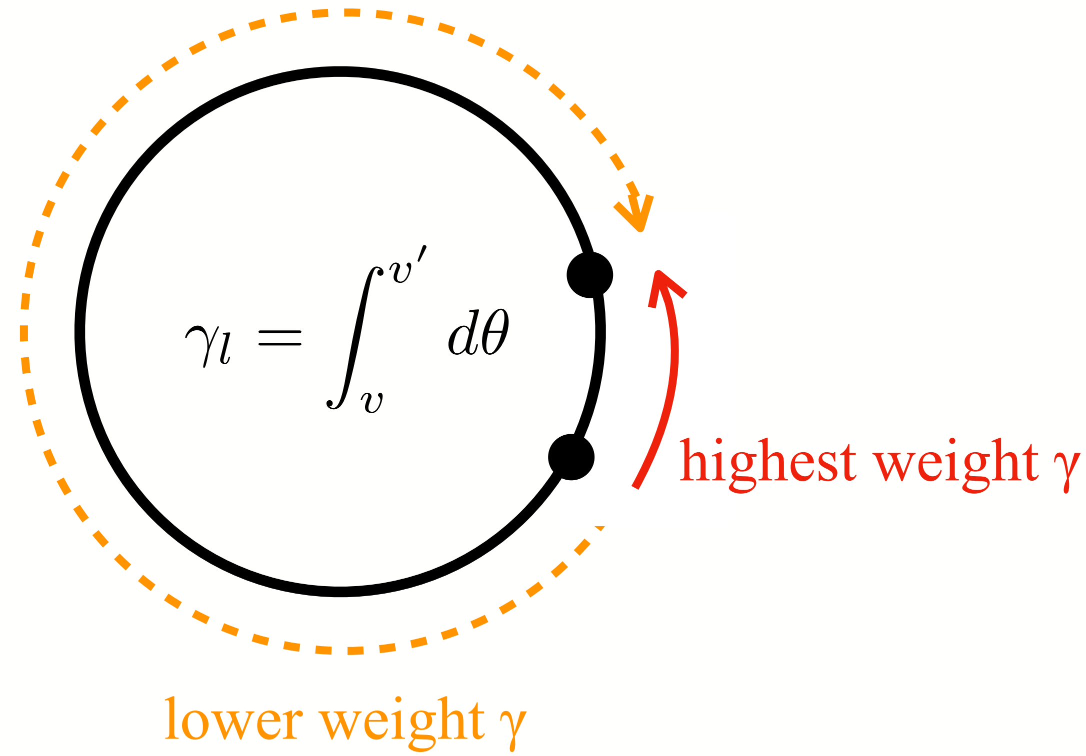
$$\gamma_l \in \mathbb{R} \qquad e^{i\gamma_l} = e^{i(\theta_{v'} - \theta_v)}$$

$2\pi\mathbb{Z}$ part unfixed $\quad \gamma_l = \theta_{v'} - \theta_v + 2\pi m_l$

sum over all possibilities with suitable weights

S^1 nlsm (XY): Villainization

If link embedded in continuum,
how does θ interpolate?

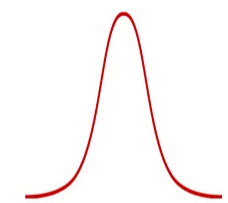


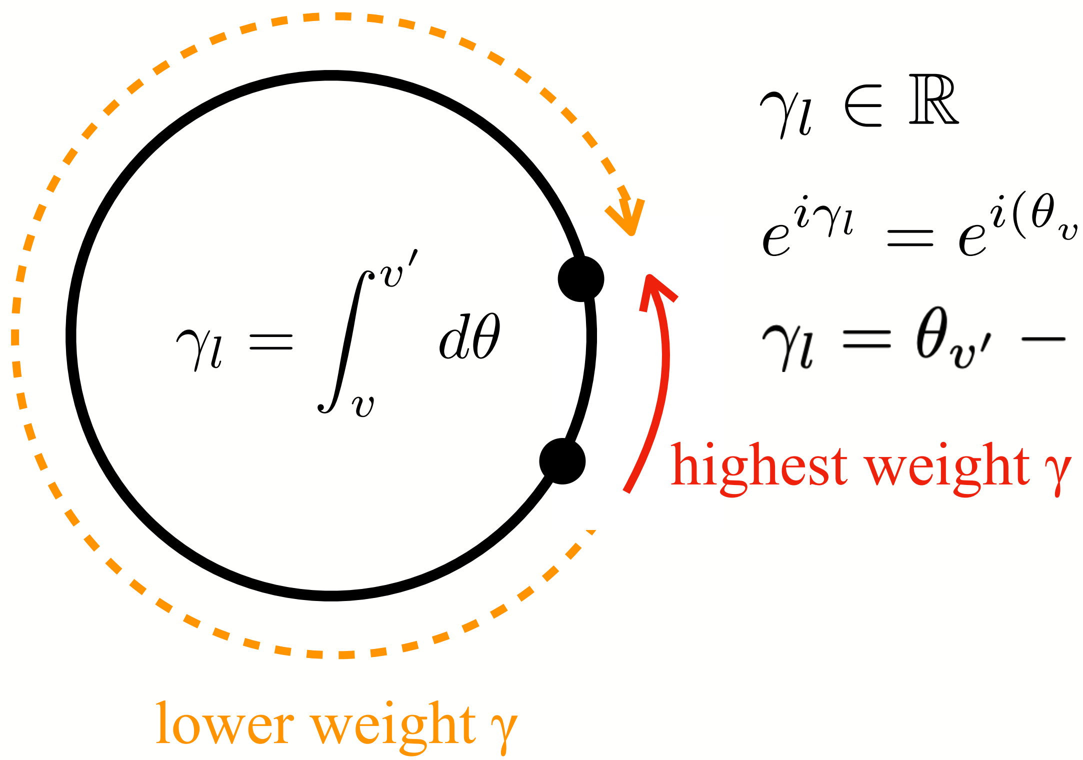
$$w := \sum_{l \text{ on loop}} \gamma_l/2\pi = \sum_l m_l$$

winding well-defined

S^1 nlsm (XY): Villainization

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi}\right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}}\right] \prod_{l} W_1(\gamma_l)$$





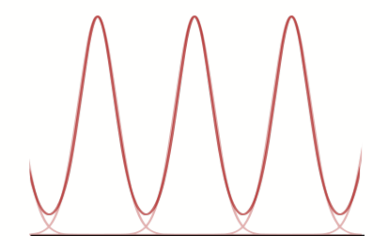
$$\gamma_l \in \mathbb{R}$$

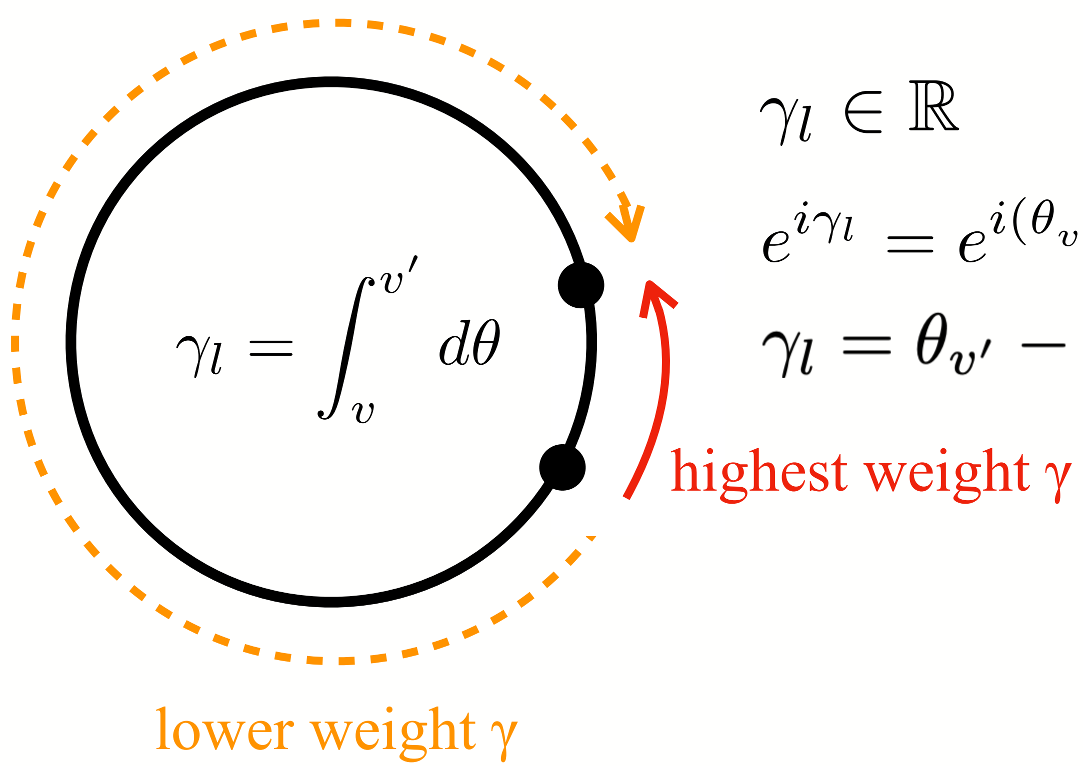
$$e^{i\gamma_l} = e^{i(\theta_{v'} - \theta_v)}$$

$$\gamma_l = \theta_{v'} - \theta_v + 2\pi m_l$$

S^1 nlsm (XY): Villainization

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi}\right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}}\right] \prod_{l} W_1(\gamma_l)$$





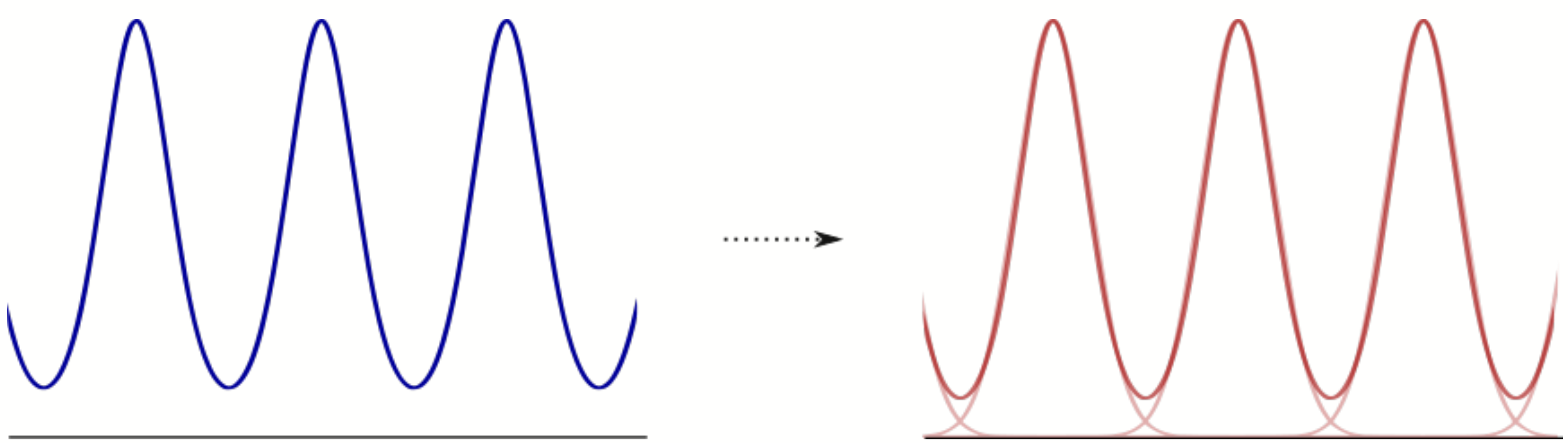
$$\gamma_l \in \mathbb{R}$$

$$e^{i\gamma_l} = e^{i(\theta_{v'} - \theta_v)}$$

$$\gamma_l = \theta_{v'} - \theta_v + 2\pi m_l$$

S^1 nlsm (XY): Villainization

$$W_{XY}(e^{id\theta_l} + c.c) \approx \sum_{m_l \in \mathbb{Z}} W_1(\gamma_l)$$



$$\gamma_l = \theta_{v'} - \theta_v + 2\pi m_l$$

S^1 nlsm (XY): Villainization

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi}\right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}}\right] \prod_{l} W_1(\gamma_l)$$

but Villain model allows us to do more:

S^1 nlsm (XY): Villainization

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi}\right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}}\right] \prod_{l} W_1(\gamma_l)$$

but Villain model allows us to do more:

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi}\right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}}\right] \prod_{l} W_1(\gamma_l) \boxed{\prod_{p} W_2(v_p)}$$

$$v_p := \frac{d\gamma_p}{2\pi} = dm_p \in \mathbb{Z}$$

vortex fugacity

or even a delta function
(manifests dual symm)

S^1 nlsm (XY): Villainization

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi} \right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}} \right] \prod_{l} W_1(\gamma_l)$$

but Villain model allows us to do more:

$$Z = \left[\prod_{v'} \int_{-\pi}^{\pi} \frac{d\theta_{v'}}{2\pi} \right] \left[\prod_{l'} \sum_{m_{l'} \in \mathbb{Z}} \right] \prod_{l} W_1(\gamma_l) \boxed{\prod_{p} W_2(v_p)}$$

$$v_p := \frac{d\gamma_p}{2\pi} = dm_p \in \mathbb{Z}$$

vortex fugacity

or even a delta function
(manifests dual symm)

invented by Berezinsky to discover BKT
helps with renormalization flow

U(1) lattice gauge theory: Villainization

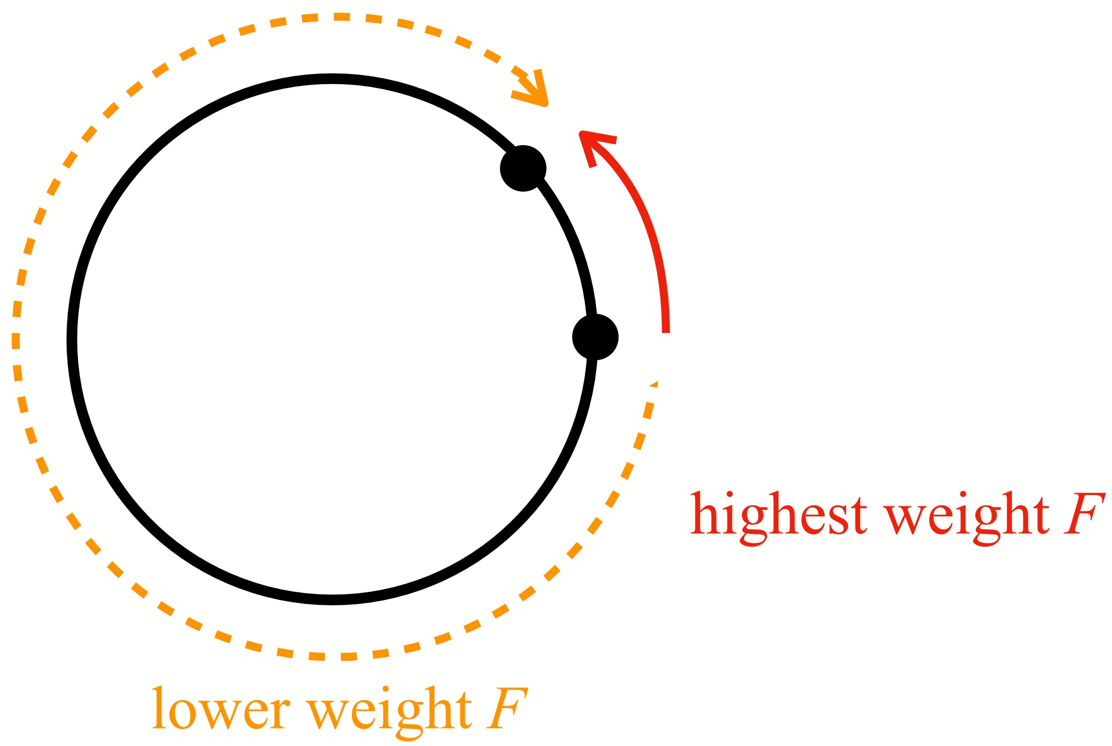
S^1 field on vertex — now U(1) gauge field A on link

real field on link — now real gauge flux F on plaquette

U(1) lattice gauge theory: Villainization

S^1 field on vertex — now U(1) gauge field A on link

real field on link — now real gauge flux F on plaquette



U(1) lattice gauge theory: Villainization

S^1 field on vertex — now U(1) gauge field A on link

real field on link — now real gauge flux F on plaquette

winding in 1d — now Dirac quantized flux in 2d

vortex in 2d — now monopole in 3d

→ *"delooping of cat"*

U(1) lattice gauge theory: Villainization

S^1 field on vertex — now U(1) gauge field A on link

real field on link — now real gauge flux F on plaquette

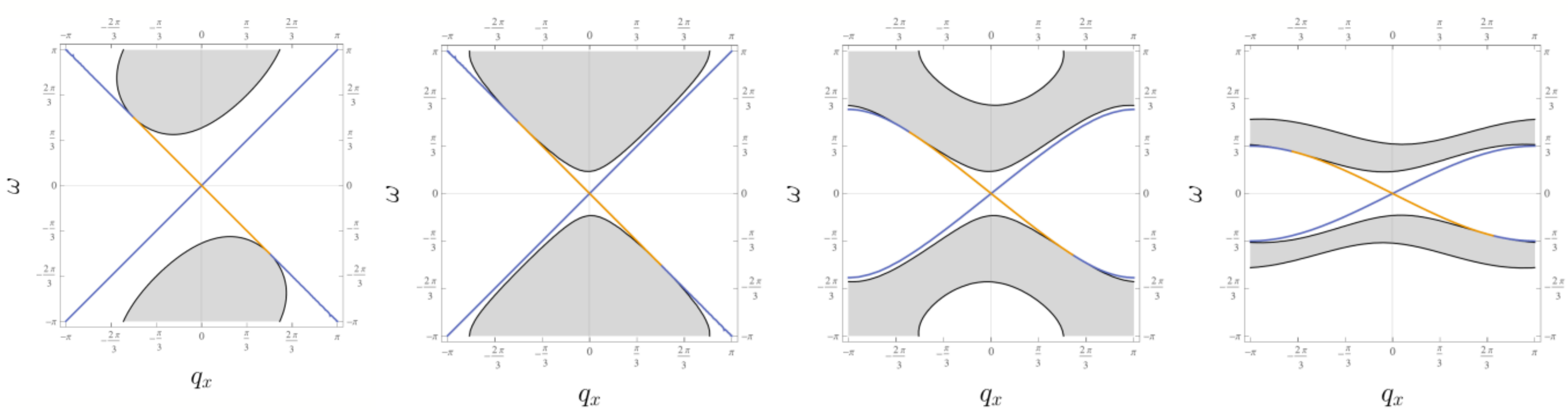
winding in 1d — now Dirac quantized flux in 2d

vortex in 2d — now monopole in 3d

lattice abelian instanton Tin Sulejmanpasic, Christof Gattringer [1901.02637]

lattice fractional Hall conductivity JYChen [1902.06756]

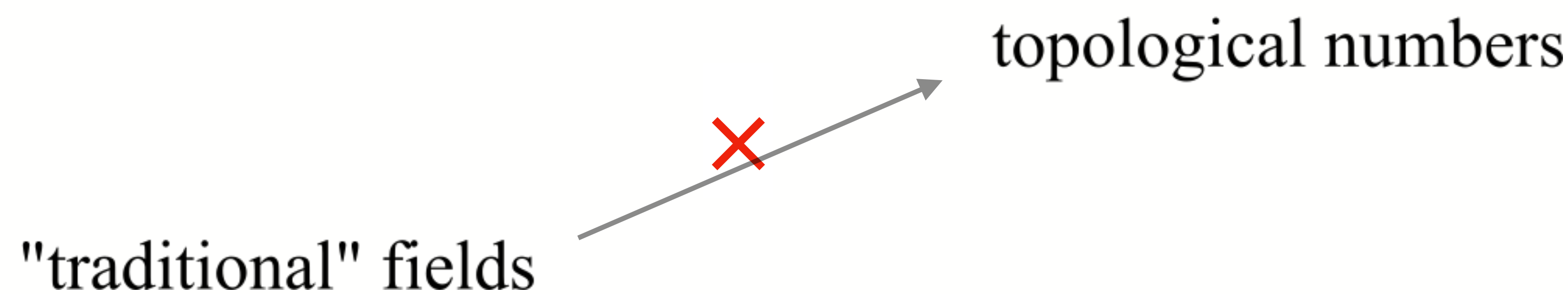
lattice U(1) chiral Chern-Simons-Maxwell Ze-An Xu, JYChen [2410.11034]

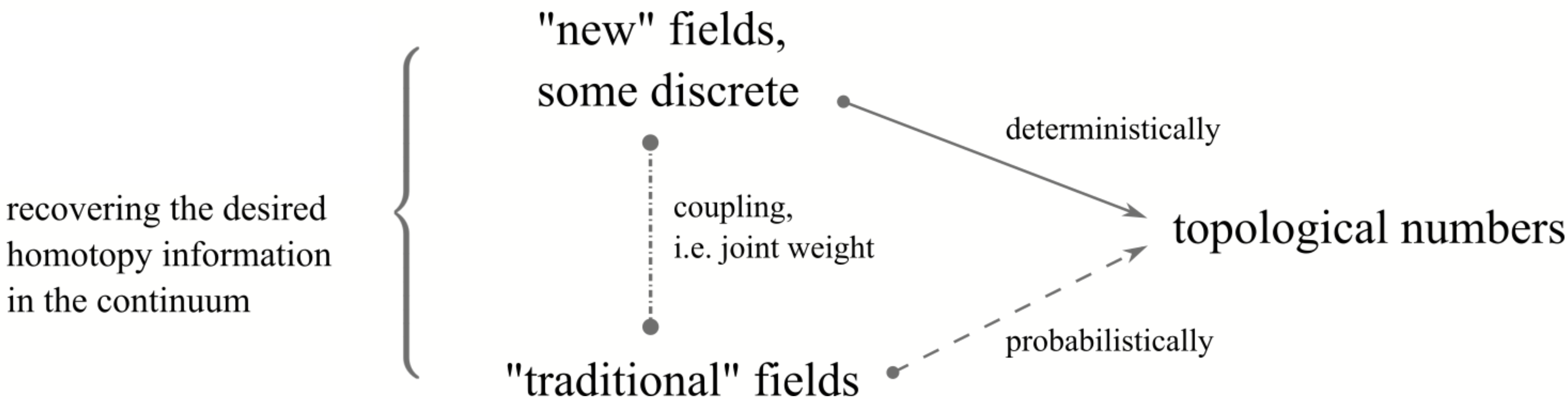


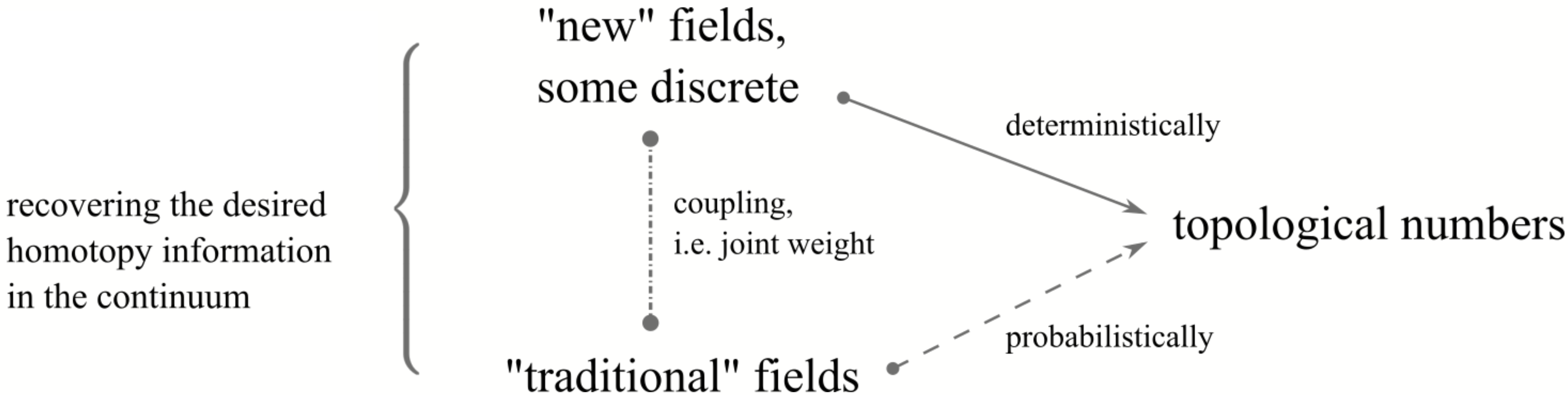
Related to:

Theo Jacobson,
Tin Sulejmanpasic,
[2303.06160]

Peng et al,
[2407.20225]

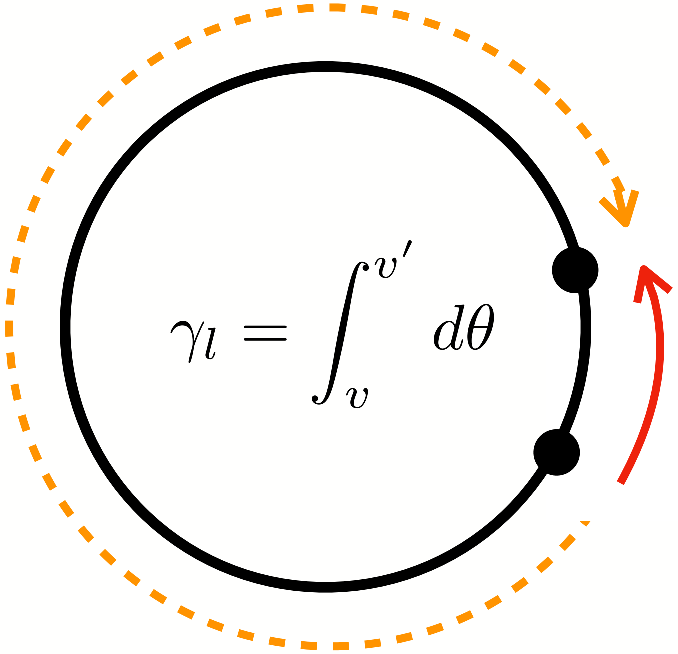

topological numbers
"traditional" fields

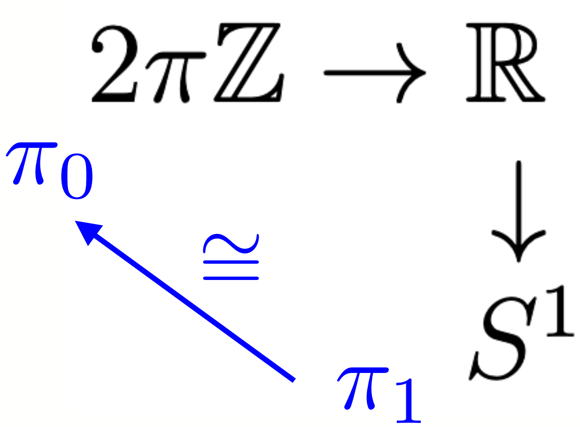

"new" fields,
some discrete
deterministically
recovering the desired
homotopy information
in the continuum
coupling,
i.e. joint weight
topological numbers
probabilistically
"traditional" fields



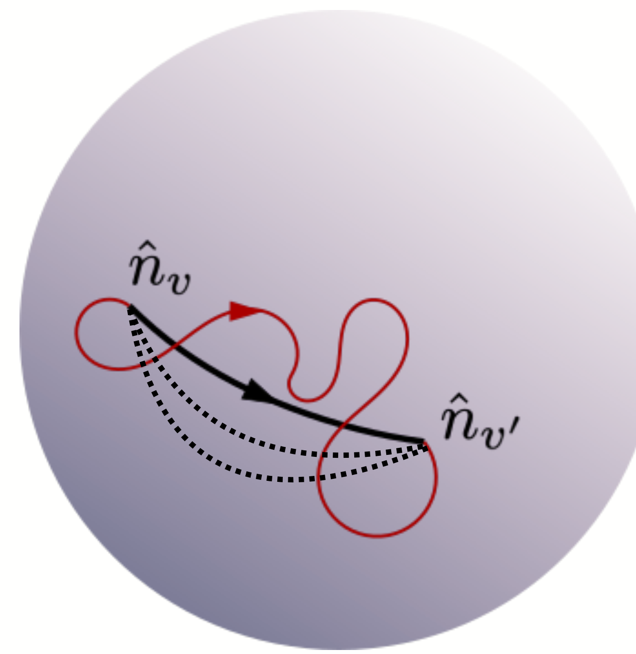
π_1 and π_2 physics on lattice (known):
form principal bundles

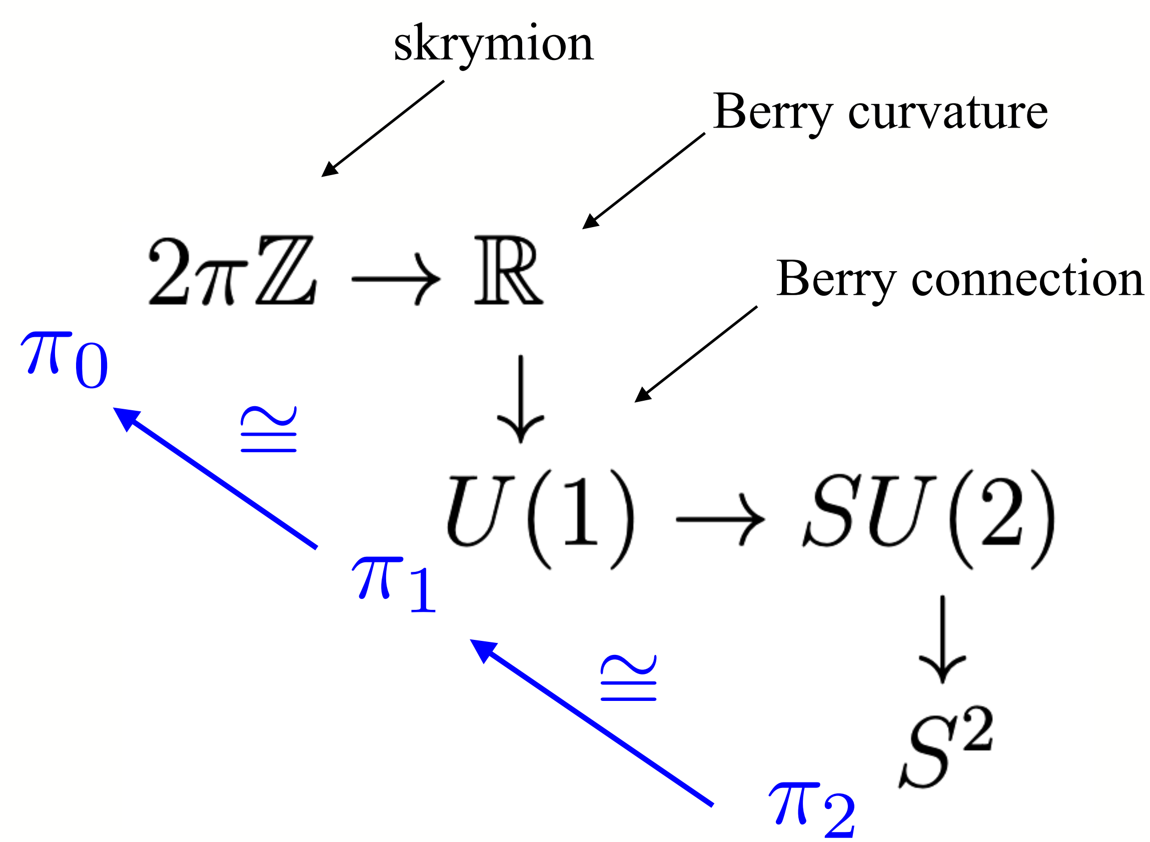
S^1 nlsm (XY): Villainization



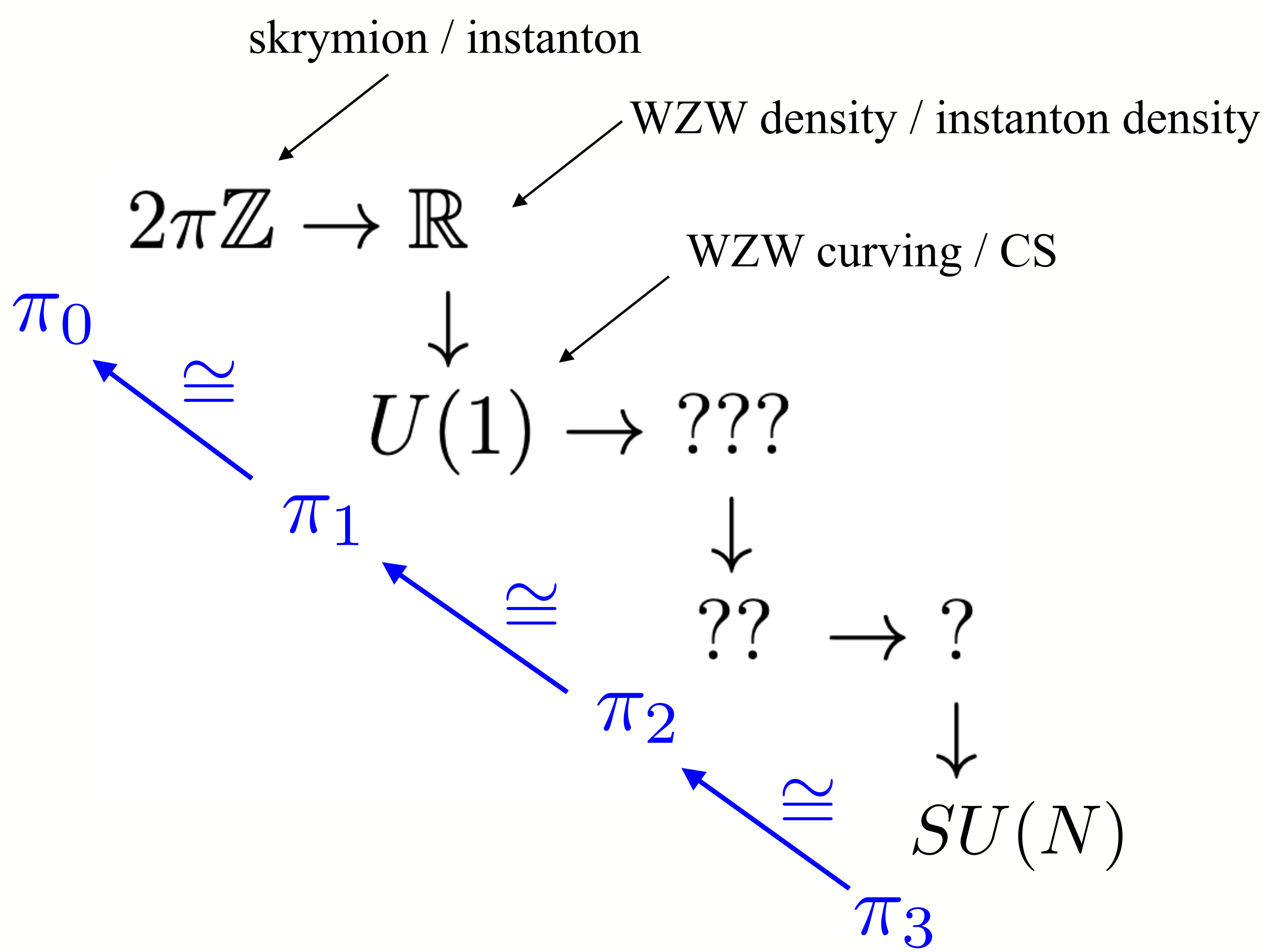


S^2 nlsm: spinon-decomposition





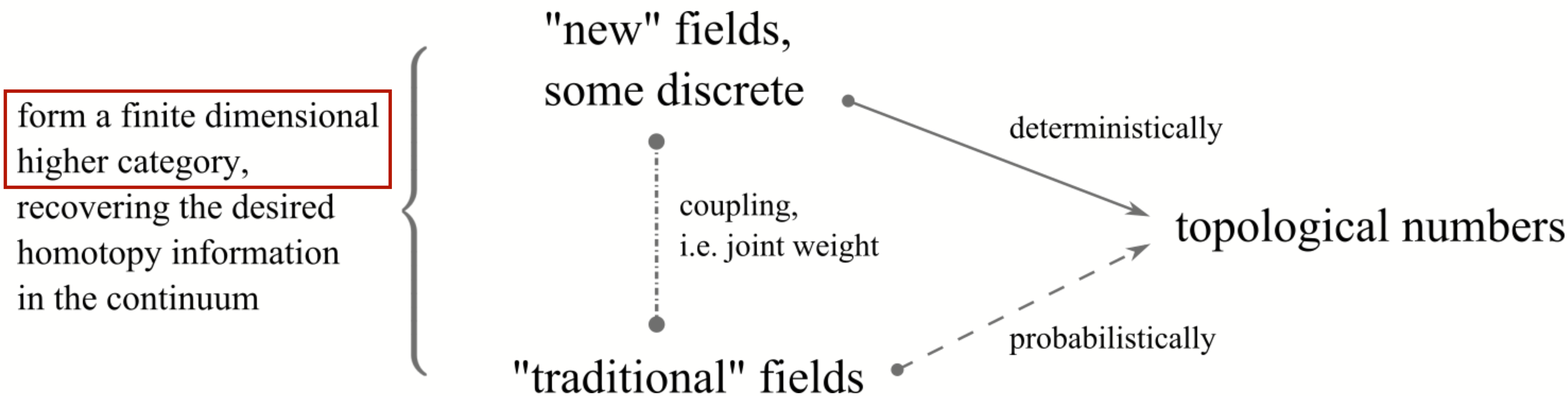
seems now we want for NLsM / YM would be



The "??" is on link (NLsM) or plaquette (YM), should compose.

But finite dimensional Lie group always has trivial π_2 !

anafunctor in cat theory

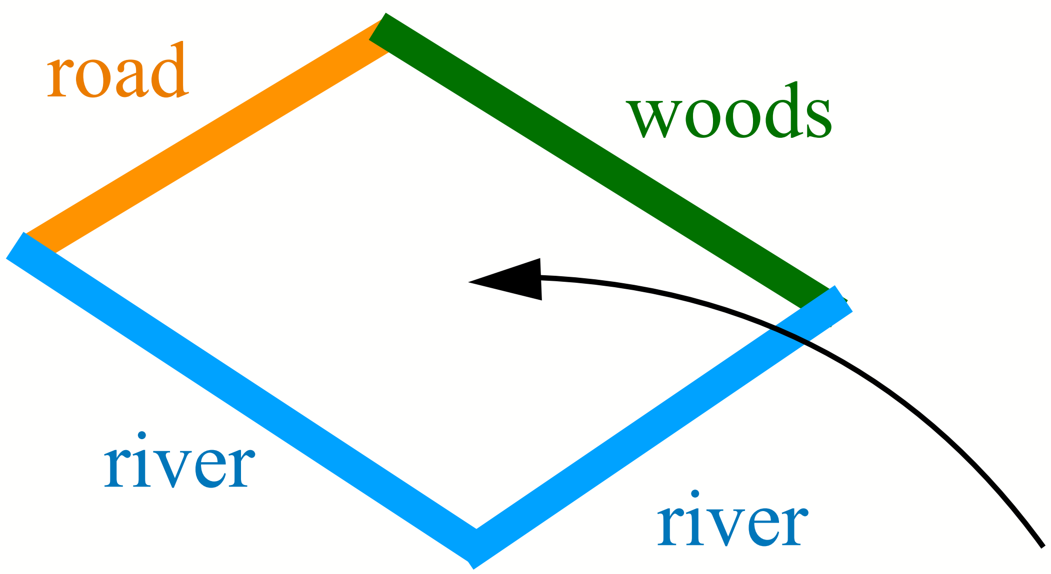


π_1 and π_2 physics on lattice (known):
form principal bundles

More general cases:
Mathematically impossible for
group theory / fibre bundles to fulfill goal

need more flexible “rules of the game”

much like some kind of board game:



Which types of castle
are allowed to play here?

continuum gauge field:
looks simple,
but infinite dim. path int.

lattice gauge field:
looks complicated,
but finite dim. path int.

$$
\begin{array}{ccc}
\mathcal{PM} & & G \\
\downdownarrows & \to & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad\Longrightarrow\quad
\begin{array}{ccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d|BG| & & \mathbf{BEG}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \xrightarrow{\sim} & \downdownarrows & \to & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2|BG| & & \mathbf{BEG}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{PM} & & \mathcal{P}|BG| & & G \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

continuum gauge field:
looks simple,
but infinite dim. path int.

lattice gauge field:
looks complicated,
but finite dim. path int.

$$
\begin{array}{ccc}
\mathcal{P}\mathcal{M} & & G \\
\downdownarrows & \to & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d|BG| & & \mathbf{BEG}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \xrightarrow{\sim} & \downdownarrows & \to & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2|BG| & & \mathbf{BEG}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{P}\mathcal{M} & & \mathcal{P}|BG| & & G \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

Wilson's theory

continuum gauge field:
looks simple,
but infinite dim. path int.

lattice gauge field:
looks complicated,
but finite dim. path int.

$$
\begin{array}{ccc}
\mathcal{PM} & & G \\
\downdownarrows & \to & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad\Longrightarrow\quad
\begin{array}{c}
\mathcal{L}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{L}_2 \\ \downdownarrows \\ \mathcal{L}_1 \\ \downdownarrows \\ \mathcal{L}_0
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\mathcal{P}^d\mathcal{M} \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{P}^2\mathcal{M} \\ \downdownarrows \\ \mathcal{PM} \\ \downdownarrows \\ \mathcal{M}
\end{array}
\to
\begin{array}{c}
\mathcal{P}^d|BG| \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{P}^2|BG| \\ \downdownarrows \\ \mathcal{P}|BG| \\ \downdownarrows \\ |BG|
\end{array}
\xrightarrow[\text{what we care}]{\text{equiv up to}}
\begin{array}{c}
\mathbf{BEG}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathbf{BEG}_2 \\ \downdownarrows \\ G \\ \downdownarrows \\ *
\end{array}
$$

The math reduces to Dijkgraaf-Witten model when G is discrete,
encompasses those TQFT stuff
—— *a unifying language for TQFT & more realistic QFT.*

continuum gauge field:
looks simple,
but infinite dim. path int.

lattice gauge field:
looks complicated,
but finite dim. path int.

$$
\begin{array}{ccc}
\mathcal{PM} & & G \\
\downdownarrows & \to & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d|BG| & & \mathbf{BEG}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \stackrel{\sim}{\longrightarrow} & \downdownarrows & \to & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2|BG| & & \mathbf{BEG}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{PM} & & \mathcal{P}|BG| & & G \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

A grand picture where

Wilson's dream — use lattice QFT to make sense of QFT

and

Grothendieck's dream — use weak higher categories as foundation of homotopy theory

come into splice.

The Big Picture

Refined Lattice Yang-Mills Path Integral

Plaquette d.o.f. and weight

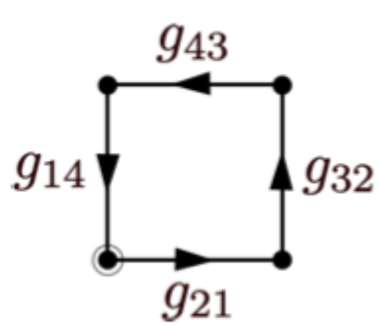
Cube d.o.f. and weight; Chern-Simons saddle

Hypercube d.o.f. and weight; instanton

Further Remarks

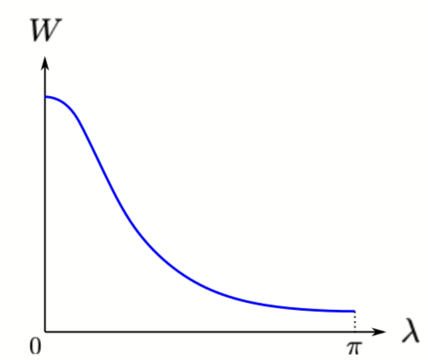
Wilson's traditional lattice gauge theory (we focus on *SU(2)* now)

$$Z = \left[\prod_{l'} \int_{g_{l'} \in SU(2)} \right] \prod_p W(\lambda_p)$$



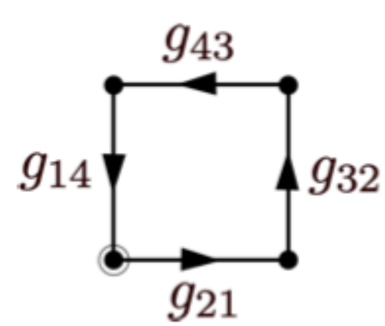
$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1}$$

W decreases with λ



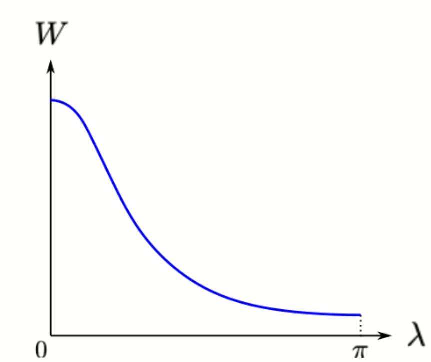
Wilson's traditional lattice gauge theory (we focus on *SU(2)* now)

$$Z = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \prod_p W(\lambda_p)$$



$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1}$$

W decreases with λ



W only depends on eigenvalue — gauge invariance

refined version — categorical generalization of Villainization

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

with Peng Zhang
[2411.07195]

refined version — categorical generalization of Villainization

traditional link d.o.f.

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

refined version — categorical generalization of Villainization

plaquette d.o.f. — NOT group or fibre bundle

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

plaquette weight

refined version — categorical generalization of Villainization

cube d.o.f. — dynamical CS phase

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

cube weight — CS saddle & CS sensitivity (technical part)

refined version — categorical generalization of Villainization

hypercube d.o.f. — Villainization

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

topological theta term

hypercube weight
over instanton density

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h$$

The Big Picture

Refined Lattice Yang-Mills Path Integral

Plaquette d.o.f. and weight

Cube d.o.f. and weight; Chern-Simons saddle

Hypercube d.o.f. and weight; instanton

Further Remarks

refined version — categorical generalization of Villainization

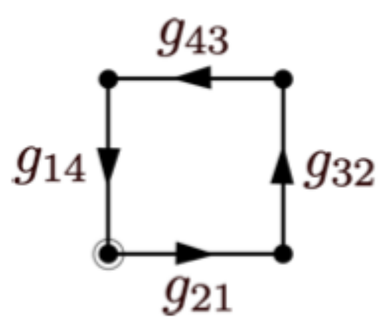
plaquette d.o.f. — NOT group or fibre bundle

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

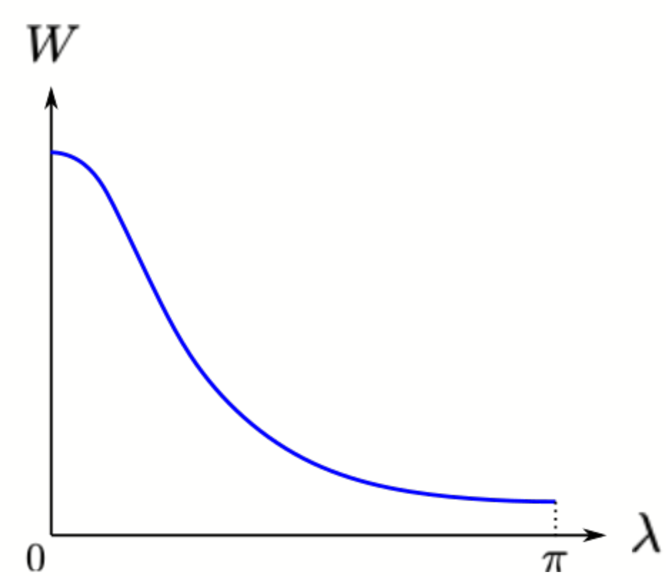
plaquette weight

traditionally:



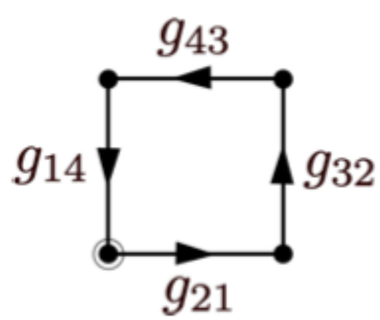
$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

traditional link weight:

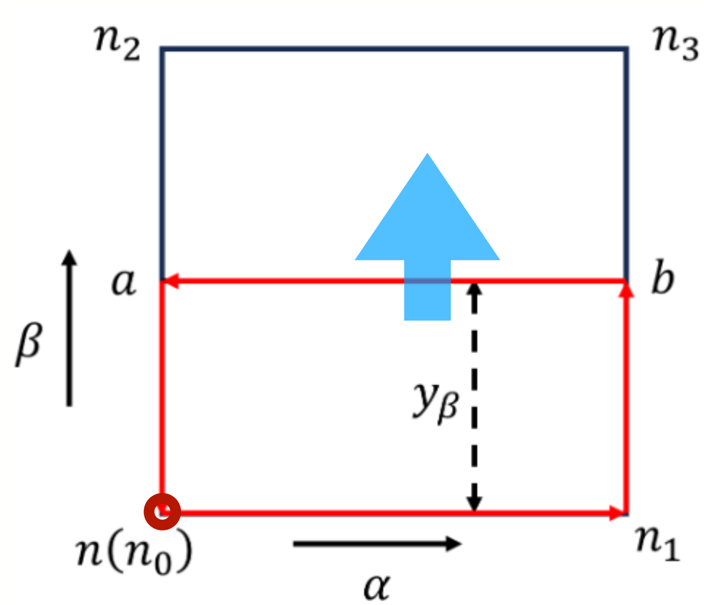


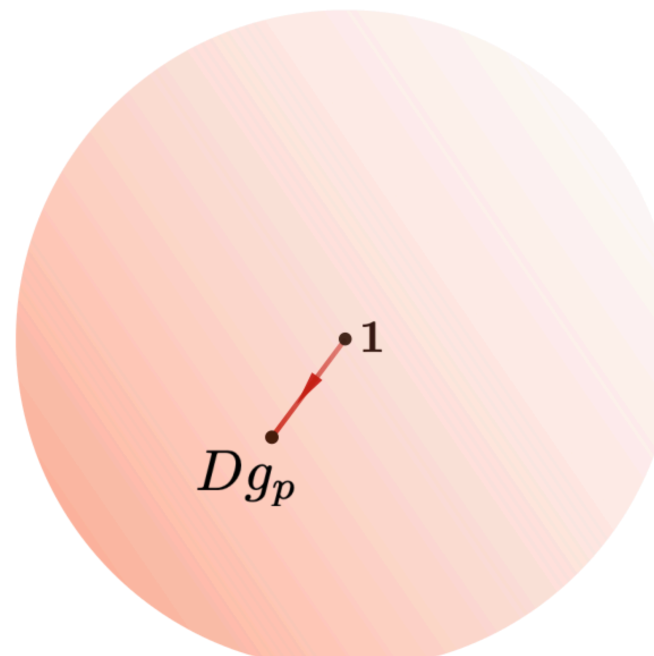
W only depends on eigenvalue — gauge invariance

traditionally:



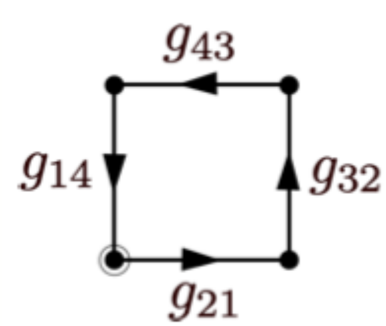
$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$



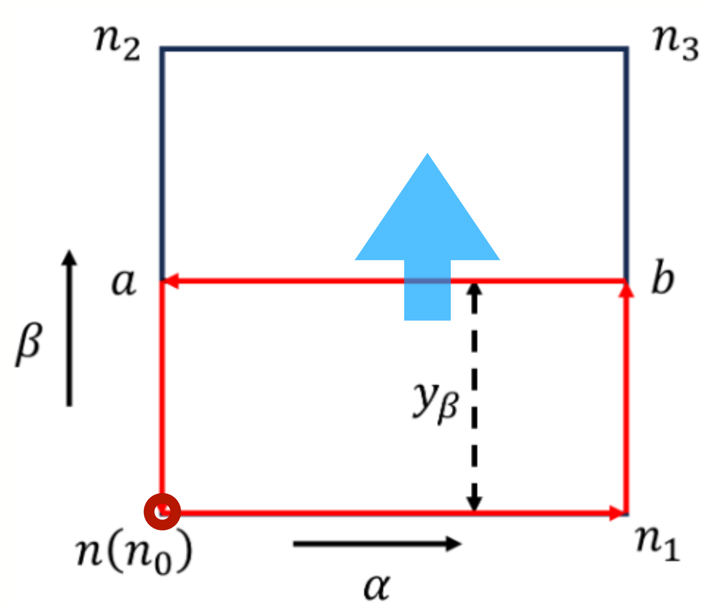


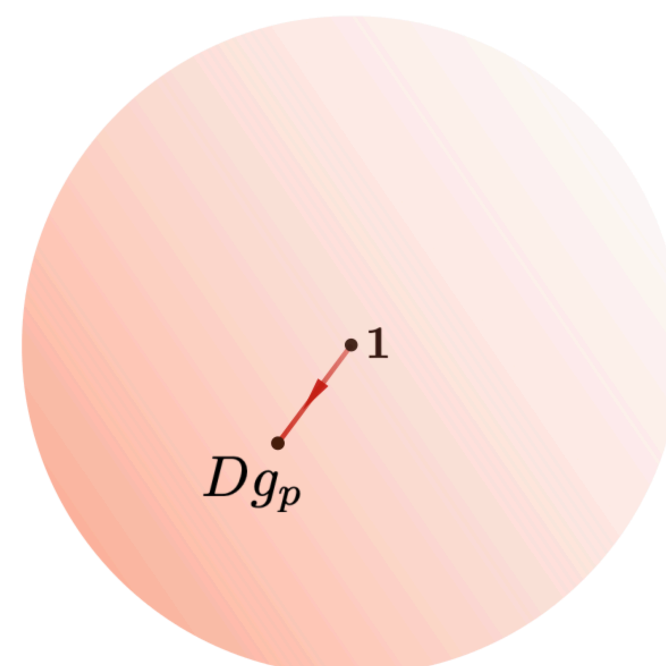
think of the interpolation in *SU(2)*
pictured as a 3-ball with center=+1, surface=-1

traditionally:



$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

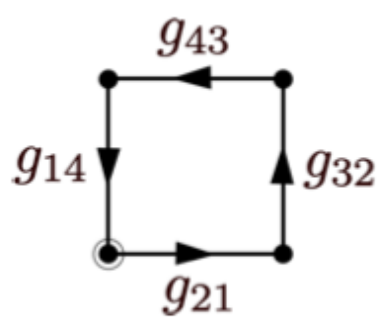




think of the interpolation in *SU(2)*
pictured as a 3-ball with center=+1, surface=-1

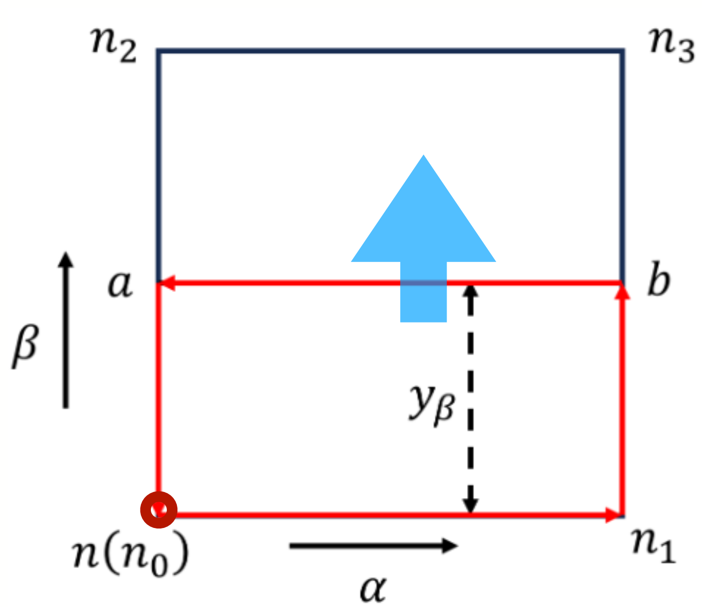
ambiguous when: $Dg_p \to -1$

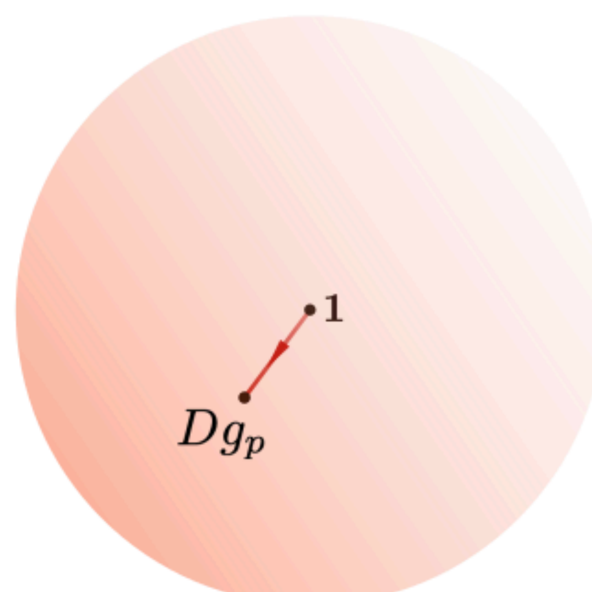
refined — different ways of interpolation



$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

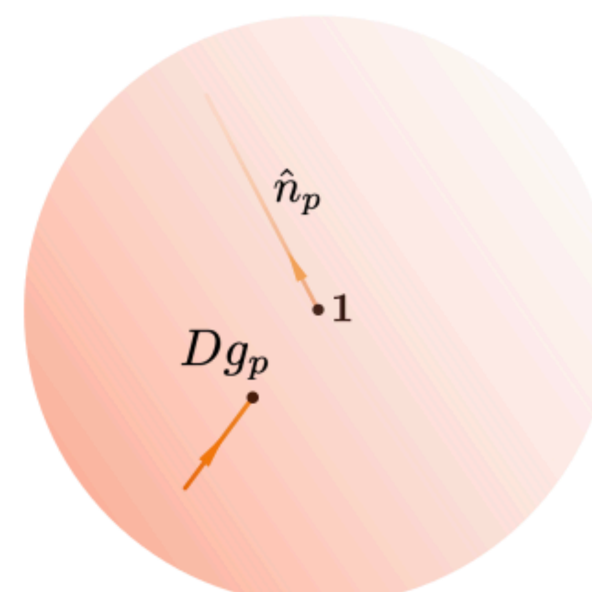
$$y_p = (Dg_p, m_p, \hat{n}_p) \in Y$$





$m_p = +$

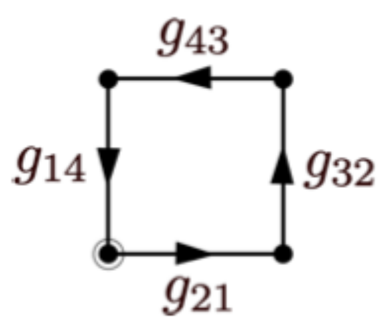
$Dg_p \neq -1$



$m_p = -$

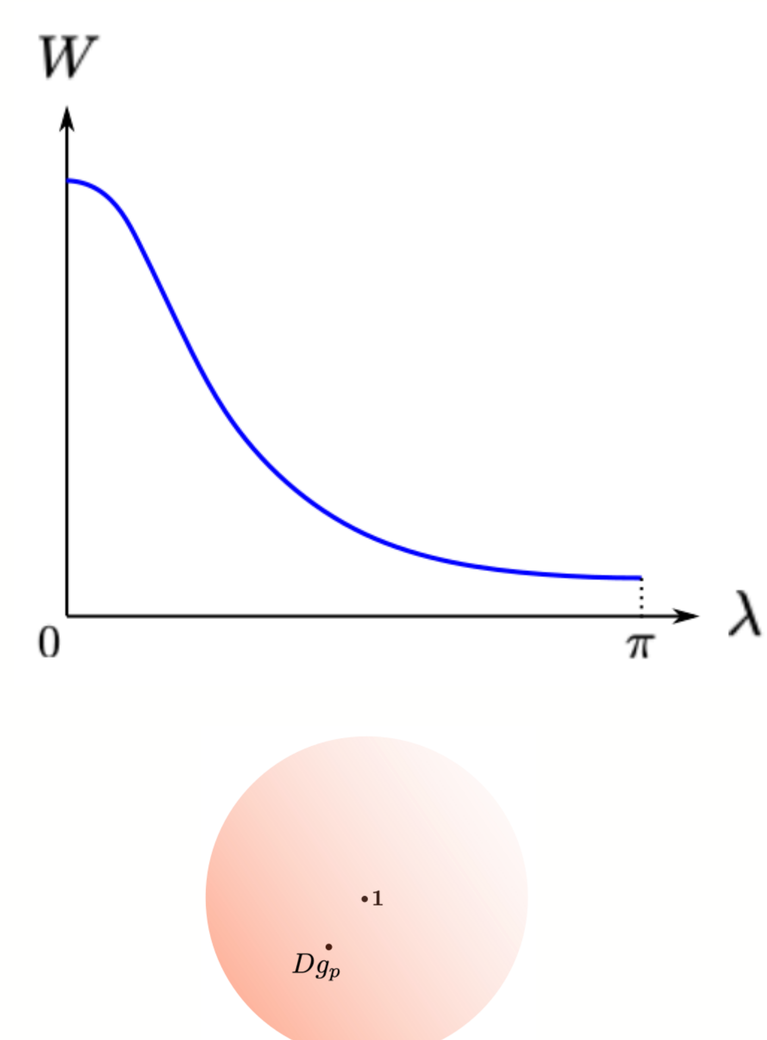
$Dg_p \neq +1$

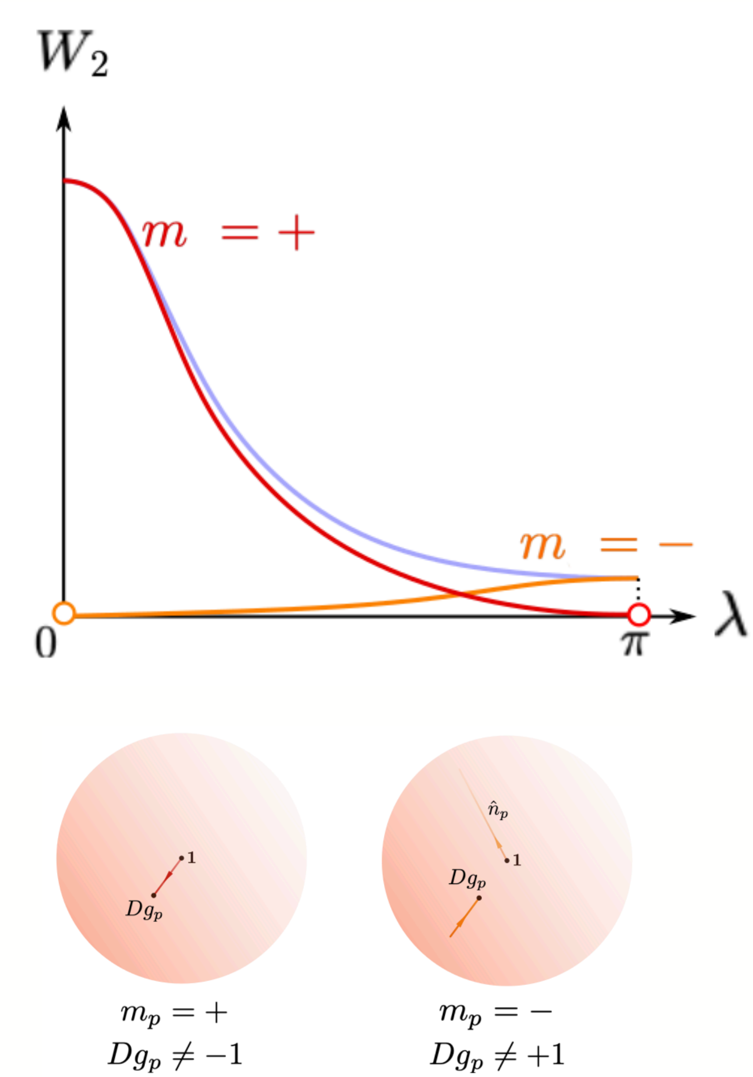
refined — different ways of interpolation



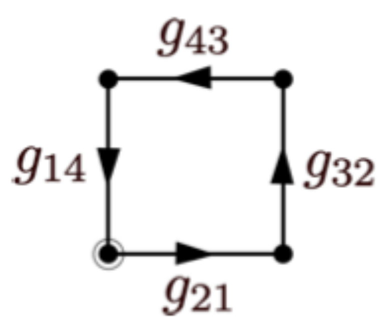
$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

$$y_p = (Dg_p, m_p, \hat{n}_p) \in Y$$



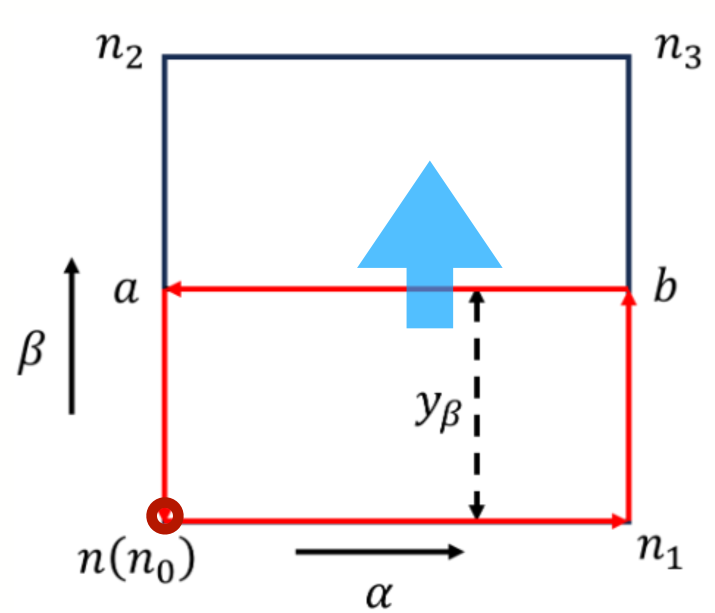


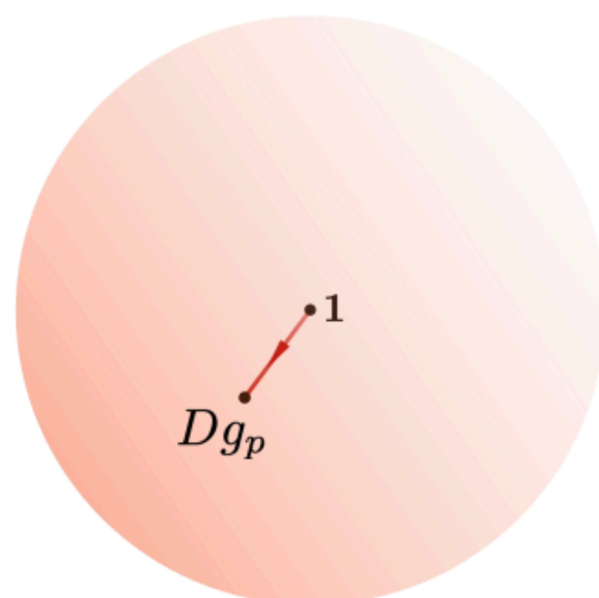
refined — different ways of interpolation



$$Dg_p = U_p e^{i\lambda_p \sigma^3} U_p^{-1} \in SU(2)$$

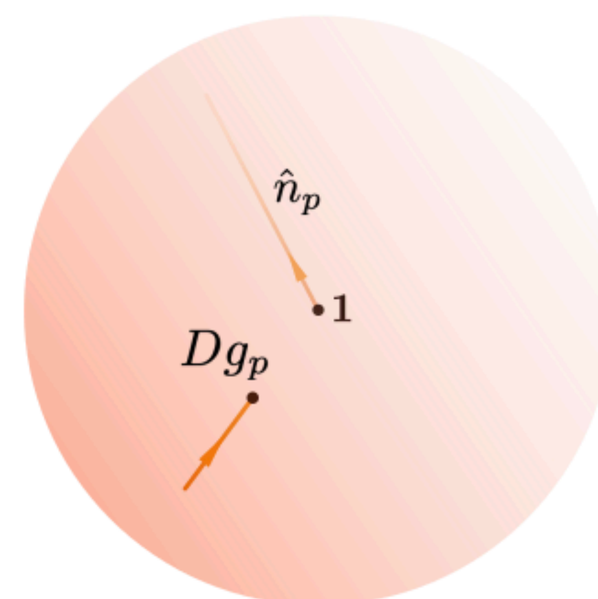
$$y_p = (Dg_p, m_p, \hat{n}_p) \in Y$$





$m_p = +$

$Dg_p \neq -1$

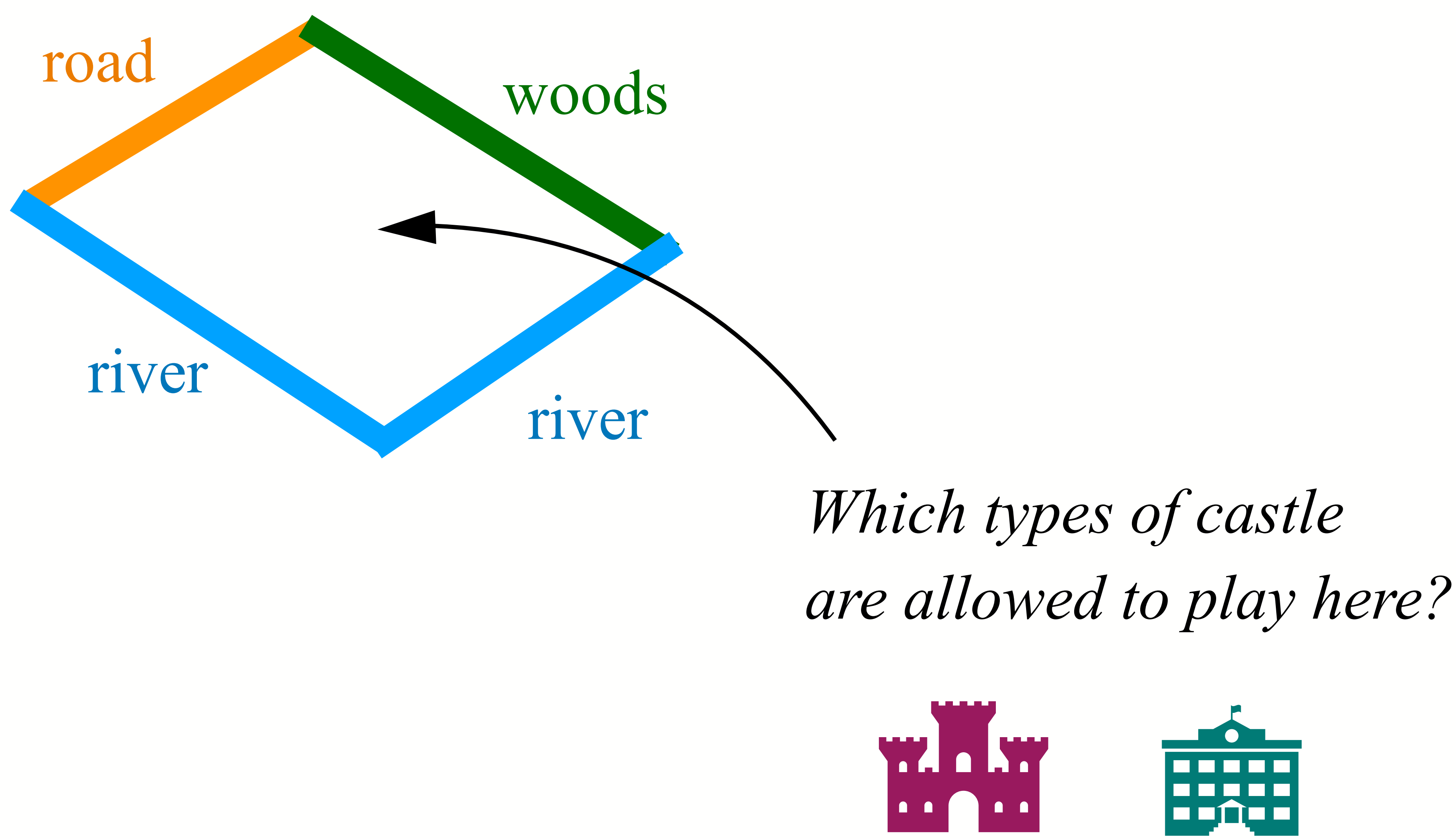


$m_p = -$

$Dg_p \neq +1$

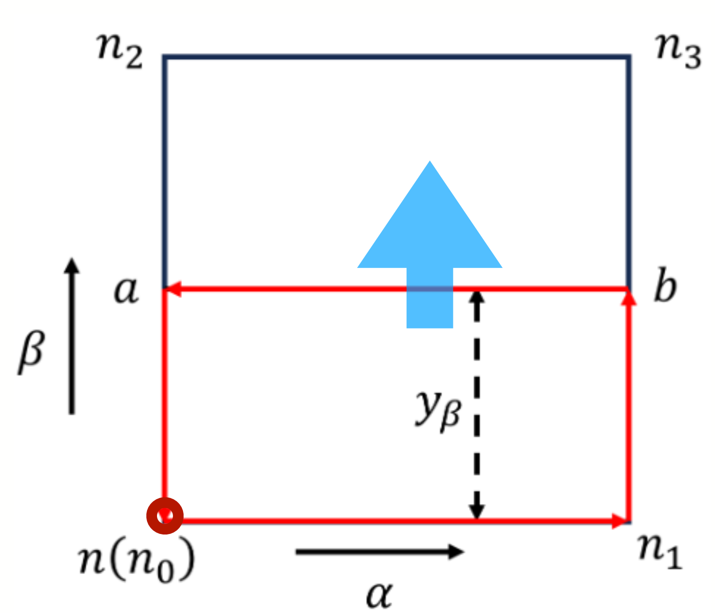
The *m label does not form a group! And Y not fibre bundle!*

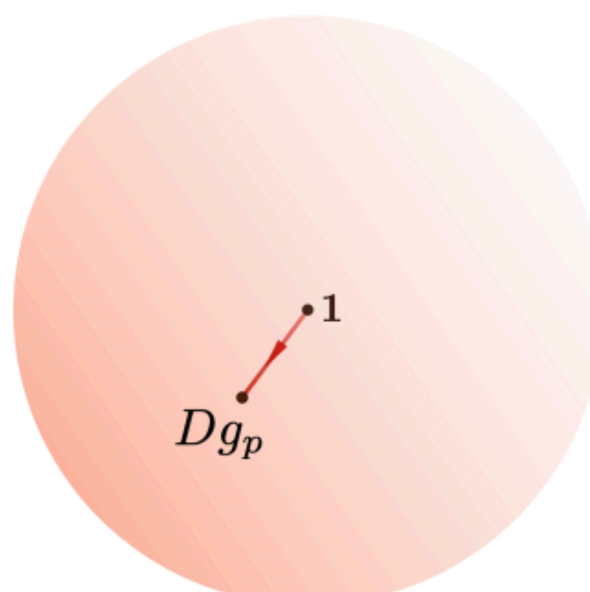
much like some kind of board game:



Tin Sulejmanpasic suggests this might be related to the center vortex.

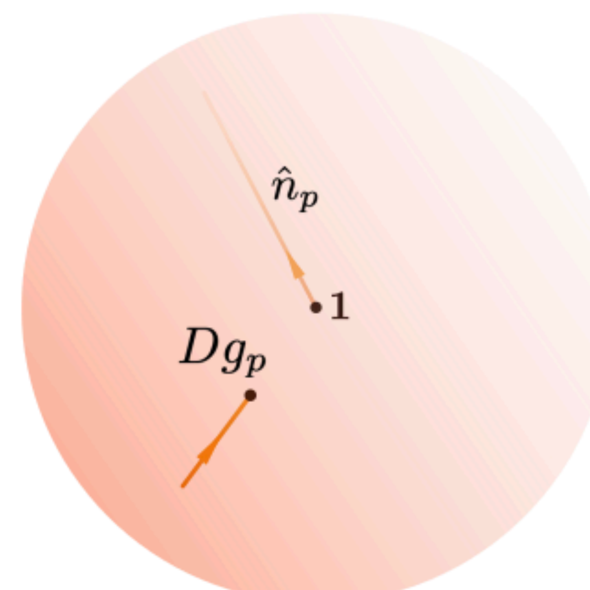
I don't know the answer yet, but seems encouraging and exciting!





$m_p = +$

$Dg_p \neq -1$



$m_p = -$

$Dg_p \neq +1$

The *m label does not form a group! And Y not fibre bundle!*

The Big Picture

Refined Lattice Yang-Mills Path Integral

Plaquette d.o.f. and weight

Cube d.o.f. and weight; Chern-Simons saddle

Hypercube d.o.f. and weight; instanton

Further Remarks

refined version — categorical generalization of Villainization

cube d.o.f. — dynamical CS phase

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

cube weight — CS saddle & CS sensitivity (technical part)

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c},m_{p\in\partial c},\hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

$e^{i\mathcal{C}_c}$ *dynamical CS phase d.o.f. over the cube*

saddle point when $e^{i\mathcal{C}_c} = \nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c},m_{p\in\partial c},\hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

$e^{i\mathcal{C}_c}$ *dynamical CS phase d.o.f. over the cube*

saddle point when $e^{i\mathcal{C}_c} = \nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$

CS saddle $\nu/|\nu|$ constructed by standard interpolation of gauge holonomy into cube (~ Lüscher 1982)

so deviation of $e^{i\mathcal{C}_c}$ from CS saddle $e^{i\mathcal{C}_c^{(0)}}$
~ deviation from standard interpolation

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c},m_{p\in\partial c},\hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

$e^{i\mathcal{C}_c}$ *dynamical CS phase d.o.f. over the cube*

saddle point when $e^{i\mathcal{C}_c} = \nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$

CS saddle $\nu/|\nu|$ constructed by standard interpolation of gauge holonomy into cube (~ Lüscher 1982)

so deviation of $e^{i\mathcal{C}_c}$ from CS saddle $e^{i\mathcal{C}_c^{(0)}}$
~ deviation of interpolation from standard interpolation

CS sensitivity $|\nu|$ approaches 0 when standard interpolation into cube becomes ambiguous

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

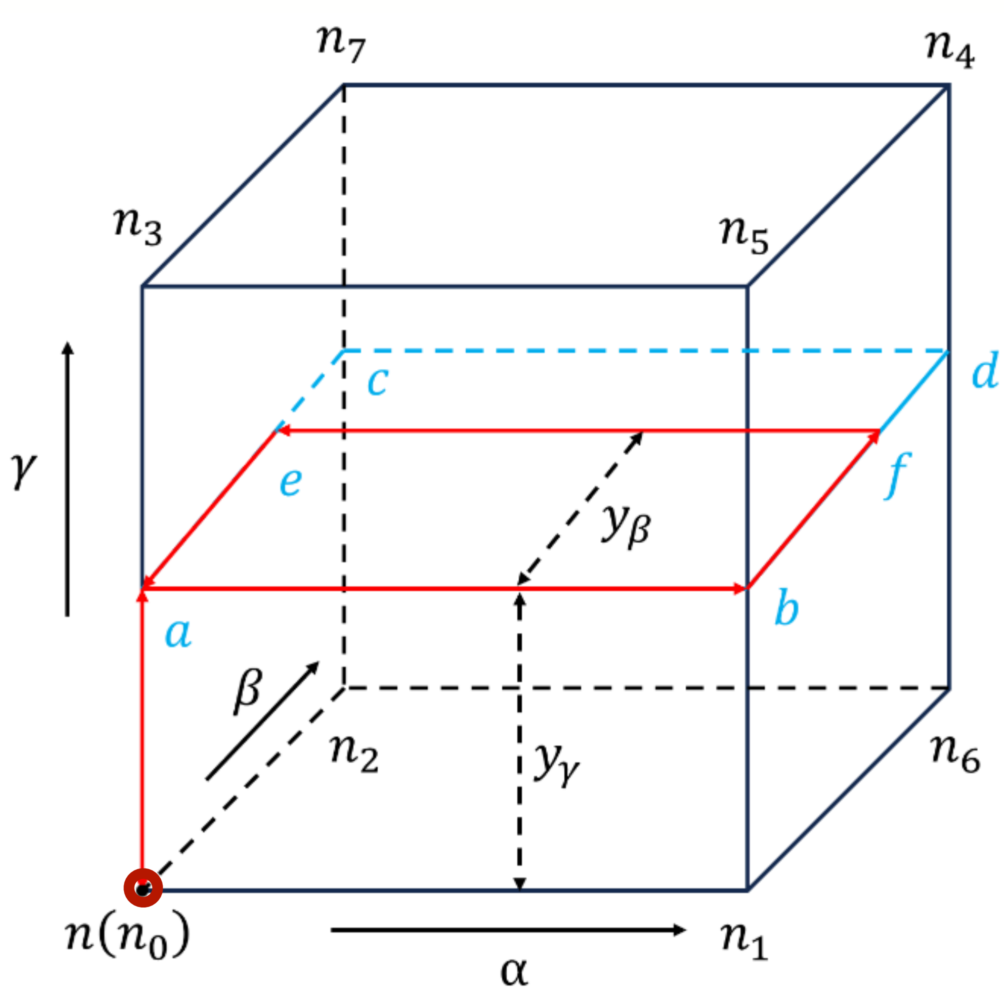
— for a given interpolation, an evaluation of the cube's CS phase

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

crucial: we only consider interpolation of Wilson loops starting at vertices
—to avoid worrying about unnecessary gauge choices in the interior of cube



given d.o.f. and interpolations
on the plaquettes around

construct a standard interpolation
into the interior of cube

CS sensitivity $|\nu|$ approaches 0
when standard interpolation ambiguous

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

in fact we only need the CS phase over all cubes on the boundary of hypercube

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

in fact we only need the CS phase over all cubes on the boundary of hypercube

in cube weight $$e^{i\mathcal{C}_c}\nu^* = e^{i(\mathcal{C}-\mathcal{C}^{(0)})_c}|\nu| = e^{i\tilde{\mathcal{C}}_c}|\nu|$$

instanton density over hypercube $$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h = \frac{d\tilde{\mathcal{C}}_h + d\mathcal{C}_h^{(0)}}{2\pi} + \iota_h$$

only need $d\mathcal{C}_h^{(0)} \mod 2\pi$

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

Lüscher has an expression for instanton density over a hypercube given the interpolation in the cubes around

$$\frac{\varepsilon_{\mu\nu\rho\sigma}}{24\pi^2}\Big\{\int_{c(n+\hat{\mu},\mu)} \mathrm{d}^3x \mathrm{Tr}[(S_{n+\hat{\mu},\mu})^{-1}\partial_\nu S_{n+\hat{\mu},\mu}(S_{n+\hat{\mu},\mu})^{-1}\partial_\rho S_{n+\hat{\mu},\mu}(S_{n+\hat{\mu},\mu})^{-1}\partial_\sigma S_{n+\hat{\mu},\mu}]$$
$$+\int_{p(n+\hat{\mu}+\hat{\nu},\mu,\nu)} \mathrm{d}^2x \mathrm{Tr}[P_{n+\hat{\mu}+\hat{\nu},\mu\nu}\partial_\rho (P_{n+\hat{\mu}+\hat{\nu},\mu\nu})^{-1}(R_{n+\hat{\mu},\mu;\nu})^{-1}\partial_\sigma R_{n+\hat{\mu},\mu;\nu}]$$
$$-\int_{c(n,\mu)} \mathrm{d}^3x \mathrm{Tr}[(S_{n,\mu})^{-1}\partial_\nu S_{n,\mu}(S_{n,\mu})^{-1}\partial_\rho S_{n,\mu}(S_{n,\mu})^{-1}\partial_\sigma S_{n,\mu}]$$
$$-\int_{p(n+\hat{\nu},\mu,\nu)} \mathrm{d}^2x \mathrm{Tr}[P_{n+\hat{\nu},\mu\nu}\partial_\rho (P_{n+\hat{\nu},\mu\nu})^{-1}(R_{n,\mu;\nu})^{-1}\partial_\sigma R_{n,\mu;\nu}]\Big\},$$

but now we are using it differently:

—we allow general gauge holonomies, not just those close to 1

—only the phase, not a real number, since integer part depends on gauge

—CS phase saddle, not "the CS phase"; the latter fluctuates

—re-express the phase saddle in a manifestly gauge invariant form
to avoid unnecessary gauge choices ambiguities in the interior of cube

To construct CS saddle $\nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$, we need:

— a standard interpolation of gauge holonomy into cube

— for a given interpolation, an evaluation of the cube's CS phase

$$
\begin{aligned}
&-\frac{\epsilon^{1\mu\nu\rho}}{12\pi}\int_{c(n+\hat{1},1)} \mathrm{d}^3x\, \mathbf{tr}\left[T(x)^{-1}\partial_\mu T(x)T(x)^{-1}\partial_\nu T(x)T(x)^{-1}\partial_\rho T(x)\right] \\
&-\frac{\epsilon^{1\mu\nu\rho}}{4\pi}\int_{p(n+\hat{1}+\hat{\mu},1,\mu)} \mathrm{d}^2x\, \mathbf{tr}\left[T(x)^{-1}\left(R_{n,1;\mu}(x-\hat{1})\partial_\nu R_{n+\hat{\mu},\mu;1}(x)\partial_\rho R_{n+\hat{1},1;\mu}(x)^{-1}\right.\right. \\
&\qquad\qquad +\partial_\nu R_{n,1;\mu}(x-\hat{1})R_{n+\hat{\mu},\mu;1}(x)\partial_\rho R_{n+\hat{1},1;\mu}(x)^{-1} \\
&\qquad\qquad \left.\left.+\partial_\nu R_{n,1;\mu}(x-\hat{1})\partial_\rho R_{n+\hat{\mu},\mu;1}(x)R_{n+\hat{1},1;\mu}(x)^{-1}\right)\right] \\
&\mathrm{mod}\ 2\pi\ .
\end{aligned}
$$

CS phase saddle around hypercube *manifestly* gauge inv

beautiful relation between CS in the space of gauge fields
and WZW in the space of Wilson lines

$$W_3(e^{i\mathcal{C}_c}\nu^*_{g_{l\in\partial c},m_{p\in\partial c},\hat{n}_{p\in\partial c}} + c.c.)$$

positive, increasing function

$e^{i\mathcal{C}_c}$ *dynamical CS phase d.o.f. over the cube*

saddle point when $e^{i\mathcal{C}_c} = \nu/|\nu| = e^{i\mathcal{C}_c^{(0)}}$

CS sensitivity $|\nu|$ approaches 0 when standard interpolation into cube becomes ambiguous

The Big Picture

Refined Lattice Yang-Mills Path Integral

Plaquette d.o.f. and weight

Cube d.o.f. and weight; Chern-Simons saddle

Hypercube d.o.f. and weight; instanton

Further Remarks

refined version — categorical generalization of Villainization

hypercube d.o.f. — Villainization

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

topological theta term

hypercube weight
over instanton density

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h$$

Villainize the U(1) dynamical CS field on cube

instanton density over hypercube

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h = \frac{d\tilde{\mathcal{C}}_h + d\mathcal{C}_h^{(0)}}{2\pi} + \iota_h \ \in \mathbb{R}$$

instanton number

$$I := \oint_{4d} \mathcal{I} = \sum_h \mathcal{I}_h = \sum_h \iota_h \in \mathbb{Z}$$

hypercube weight decreases with instanton density

$$W_4(\mathcal{I}_h)$$

(5d cell Yang monopole $d\mathcal{I}$)

“delooped CS bundle 2-gerbe (as a cubicial weak 4-group)” construction

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction

hypercube d.o.f. — Villainization

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

topological theta term

hypercube weight
over instanton density

$$\mathcal{I}_h = \frac{d\mathcal{C}_h}{2\pi} + \iota_h$$

"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction

cube d.o.f. — dynamical CS phase

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2 \hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

cube weight — CS saddle & CS sensitivity (technical part)

"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction

plaquette d.o.f. — NOT group or fibre bundle

$$Z_\Theta = \left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right] \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

plaquette weight

"delooped CS bundle 2-gerbe (as a cubicial weak 4-group)" construction

traditional link d.o.f.

$$Z_\Theta = \boxed{\left[\prod_{l'} \int_{g_{l'} \in SU(2)}\right]} \left[\prod_{p'} \sum_{m_{p'}=\pm} \int \frac{d^2\hat{n}_p}{4\pi}\right] \left[\prod_{c'} \int_{-\pi}^{\pi} \frac{d\mathcal{C}_{c'}}{2\pi}\right] \left[\prod_{h'} \sum_{\iota_{h'} \in \mathbb{Z}}\right]$$

$$e^{i\Theta I} \prod_p W_2(\lambda_p, m_p) \prod_c W_3(e^{i\mathcal{C}_c} \nu^*_{g_{l\in\partial c}, m_{p\in\partial c}, \hat{n}_{p\in\partial c}} + c.c.) \prod_h W_4(\mathcal{I}_h)$$

The Big Picture

Refined Lattice Yang-Mills Path Integral

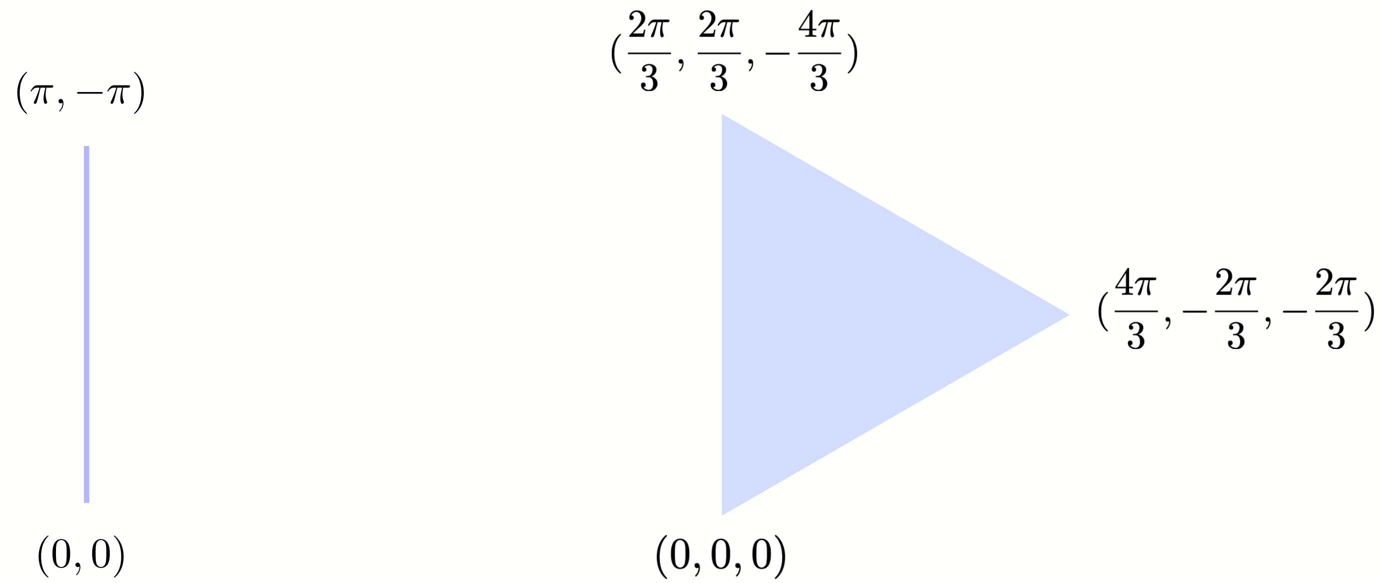
Plaquette d.o.f. and weight

Cube d.o.f. and weight; Chern-Simons saddle

Hypercube d.o.f. and weight; instanton

Further Remarks

— Generalization to SU(N)



SU(2) SU(3)

Weyl alcove in Cartan subalgebra (space of eigenvalues)

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

— Integrating out d.o.f. on higher dim. cells generates
beyond-nearest-neighbor coupling for d.o.f. on lower higher dim. cells

better control of renormalization
(recall vortex fugacity in BKT)

relation to Symanzik improvement?

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

— Integrating out d.o.f. on higher dim. cells generates
beyond-nearest-neighbor coupling for d.o.f. on lower higher dim. cells

better control of renormalization
(recall vortex fugacity in BKT)

relation to Symanzik improvement?

— Extract physics of instanton density fluctuations

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

— Integrating out d.o.f. on higher dim. cells generates
beyond-nearest-neighbor coupling for d.o.f. on lower higher dim. cells

better control of renormalization
(recall vortex fugacity in BKT)

relation to Symanzik improvement?

— Extract physics of instanton density fluctuations

— Relation to center vortex??

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

— Integrating out d.o.f. on higher dim. cells generates
beyond-nearest-neighbor coupling for d.o.f. on lower higher dim. cells

better control of renormalization
(recall vortex fugacity in BKT)

relation to Symanzik improvement?

— Extract physics of instanton density fluctuations

— Relation to center vortex??

— Similar construction for S^3 pion NLsM, get π_3 baryons
but still need 4d WZW term, due to π_5

— Generalization to SU(N)

— Optimization of the weight functions in numerics (not my expertise)

— Integrating out d.o.f. on higher dim. cells generates
beyond-nearest-neighbor coupling for d.o.f. on lower higher dim. cells

better control of renormalization
(recall vortex fugacity in BKT)

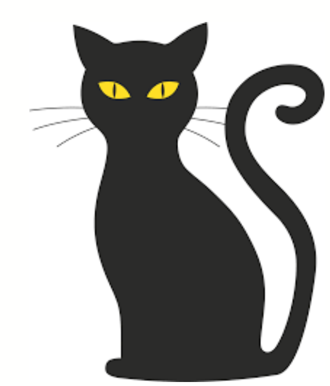
relation to Symanzik improvement?

— Extract physics of instanton density fluctuations

— Relation to center vortex??

— Similar construction for S^3 pion NLsM, get π_3 baryons
but still need 4d WZW term, due to π_5

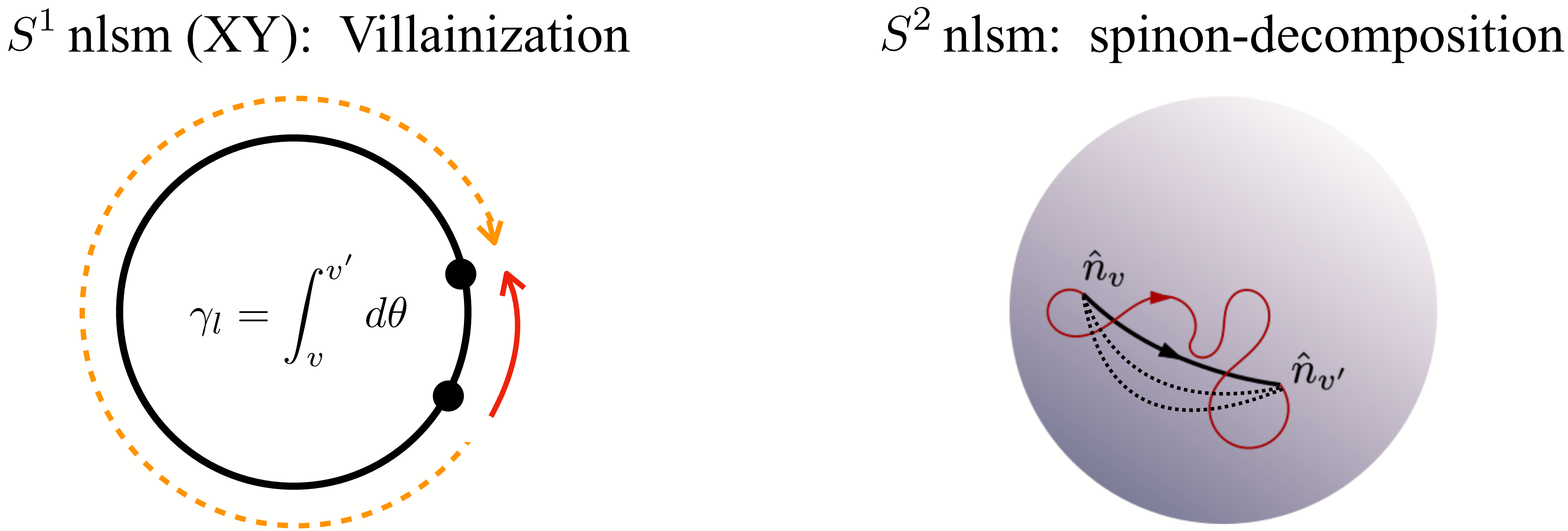
— A lot of beautiful, deep themes in cat theory



thank you

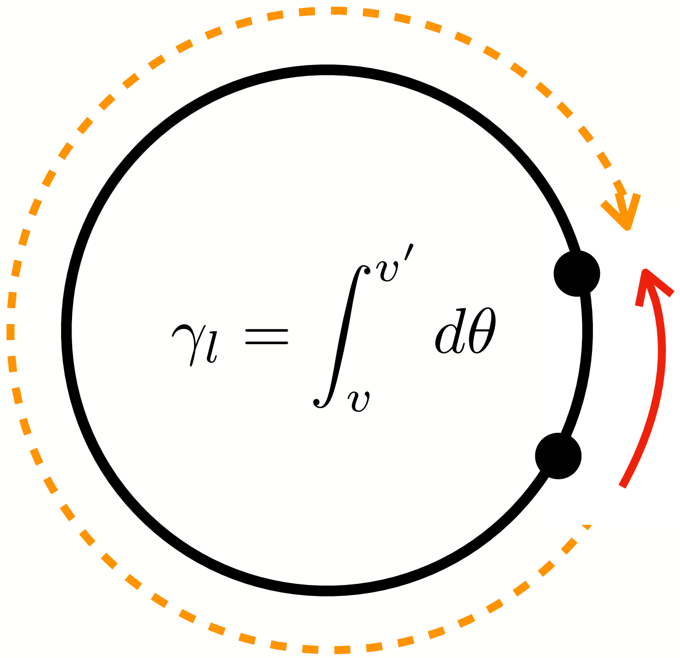
back up

why cat(egory) necessary



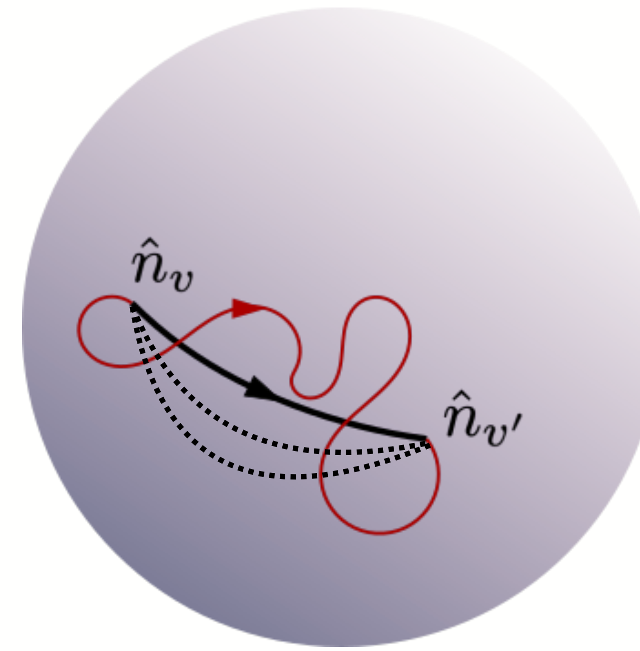
S^1 nlsm (XY): Villainization
$\gamma_l = \int_v^{v'} d\theta$
S^2 nlsm: spinon-decomposition
$\hat{n}_v$
$\hat{n}_{v'}$

S^1 nlsm (XY): Villainization



$$2\pi\mathbb{Z} \to \mathbb{R}$$
$$\downarrow$$
$$S^1$$

S^2 nlsm: spinon-decomposition



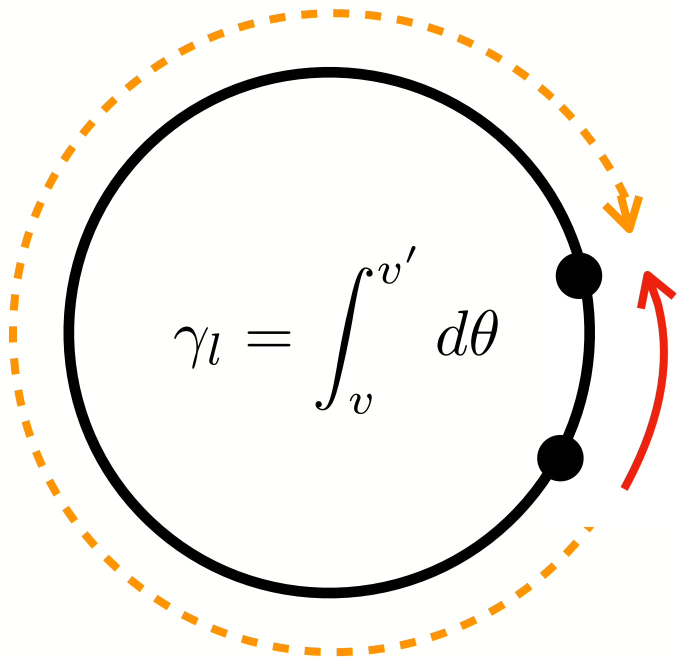
skrymion

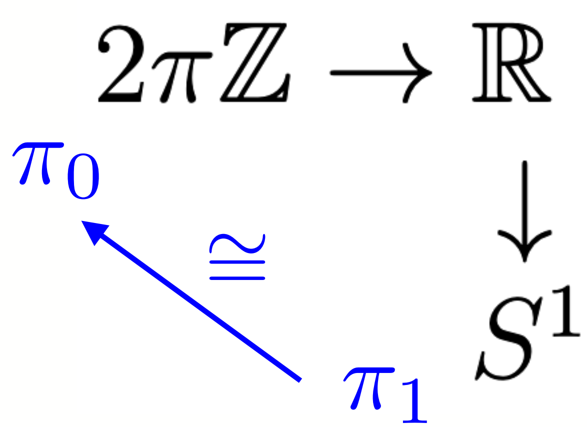
Berry curvature

Berry connection

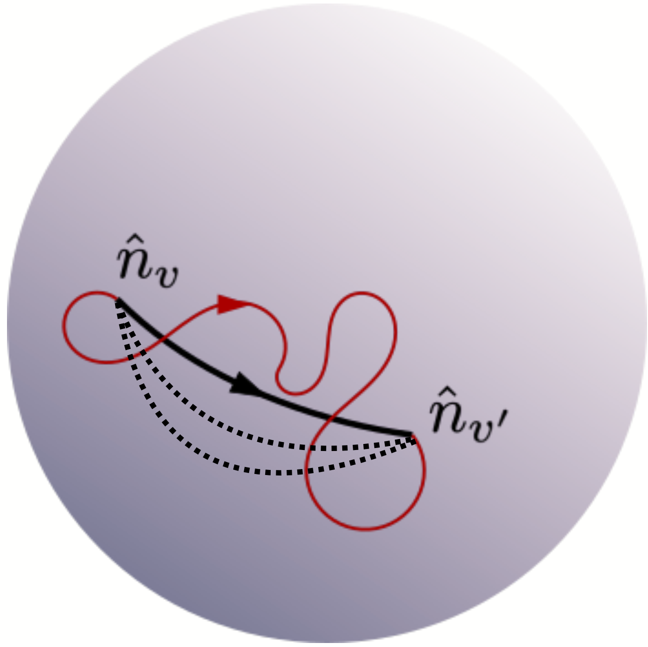
$$2\pi\mathbb{Z} \to \mathbb{R}$$
$$\downarrow$$
$$U(1) \to SU(2)$$
$$\downarrow$$
$$S^2$$

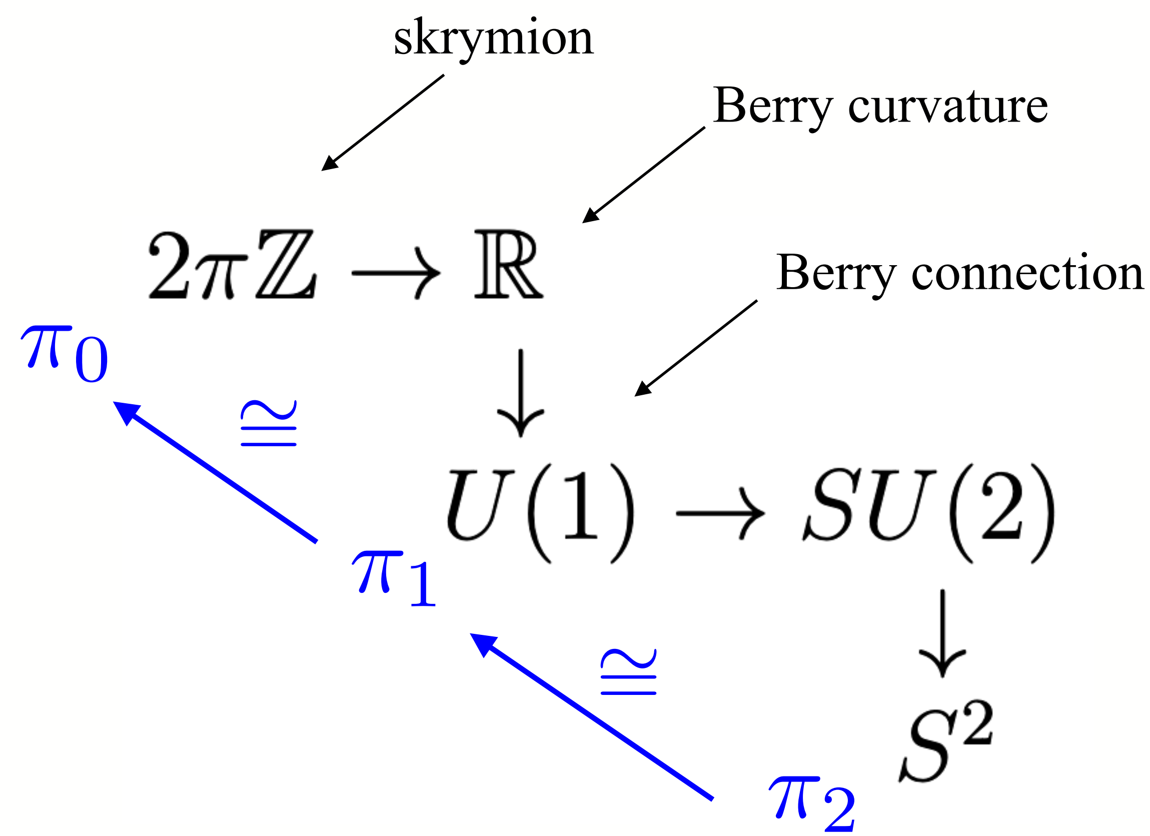
S^1 nlsm (XY): Villainization



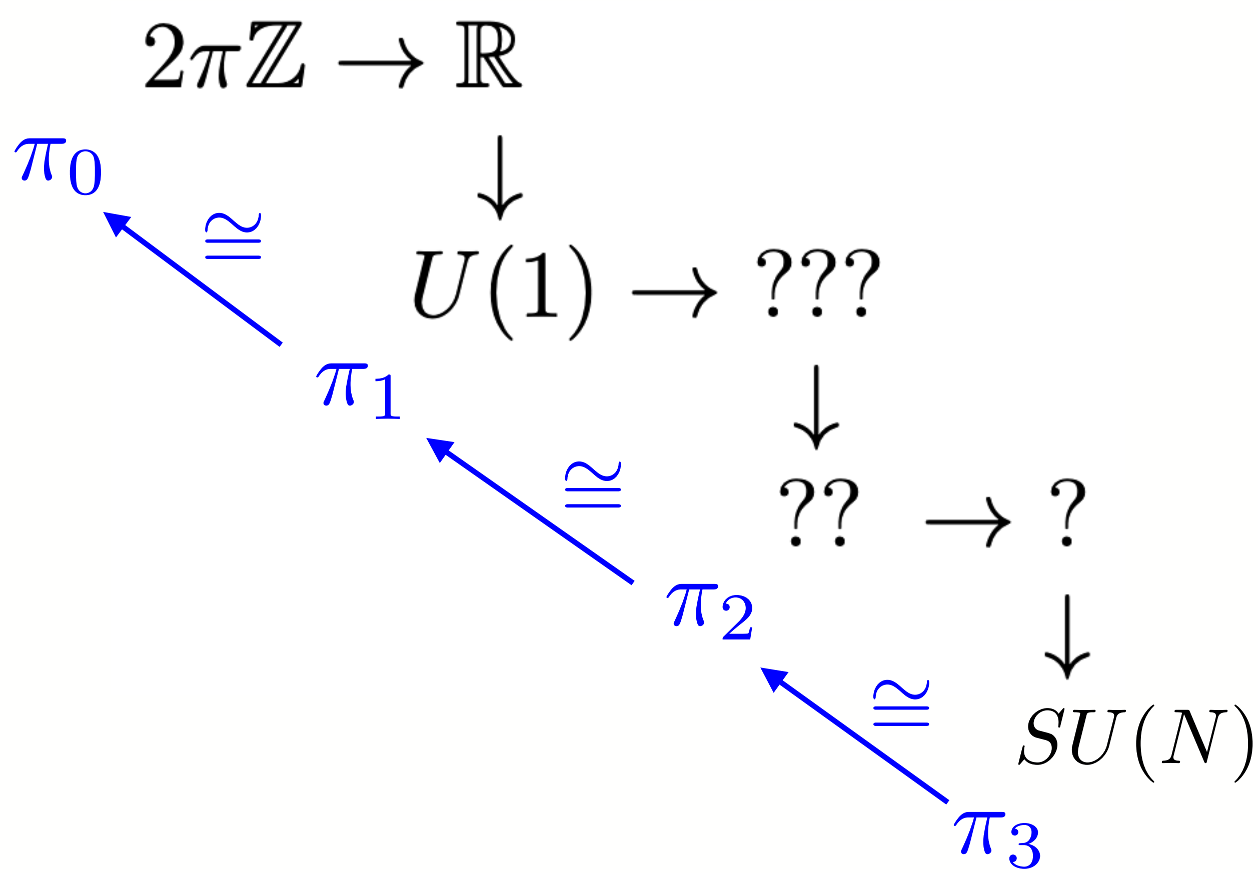


S^2 nlsm: spinon-decomposition

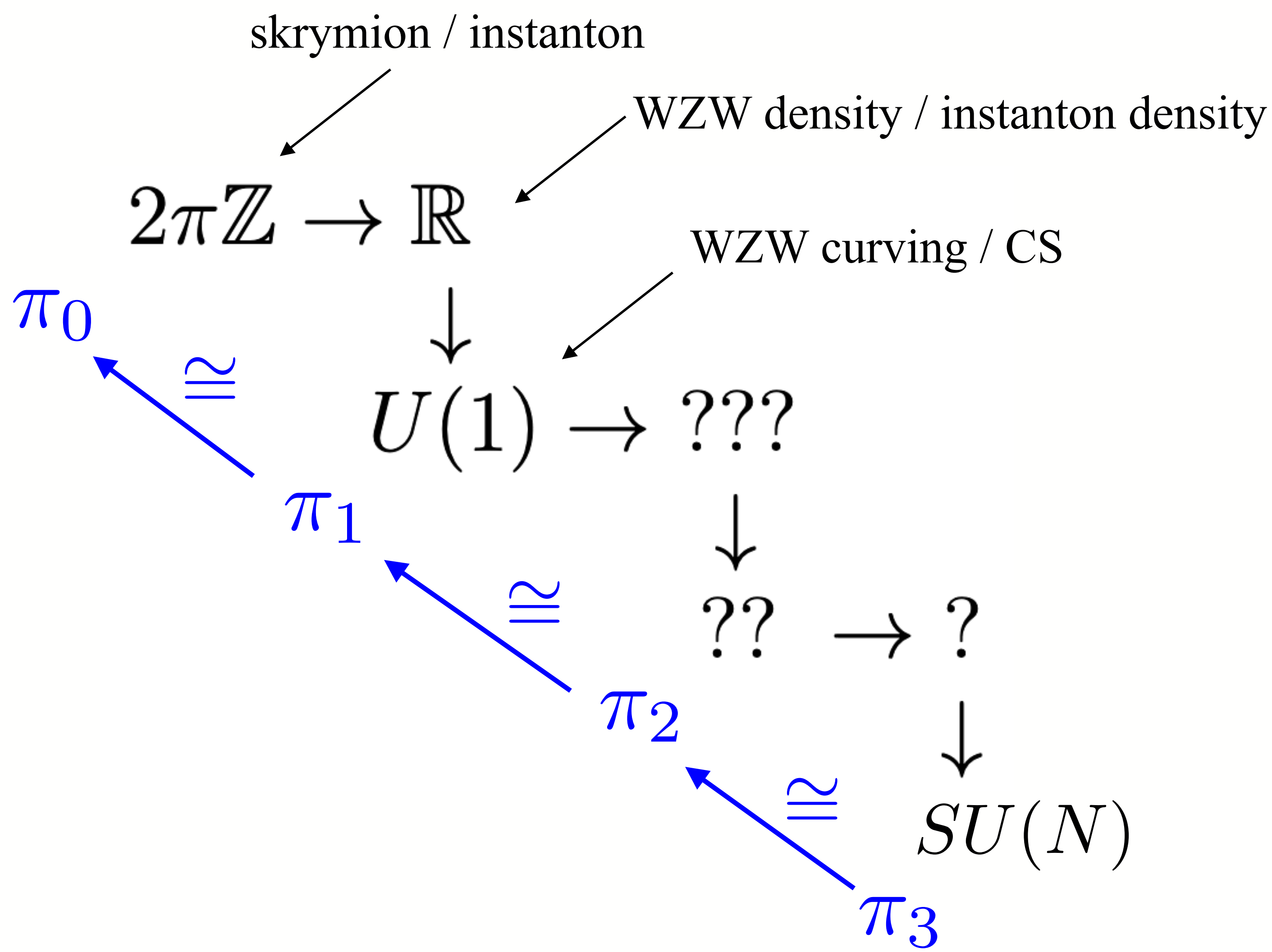




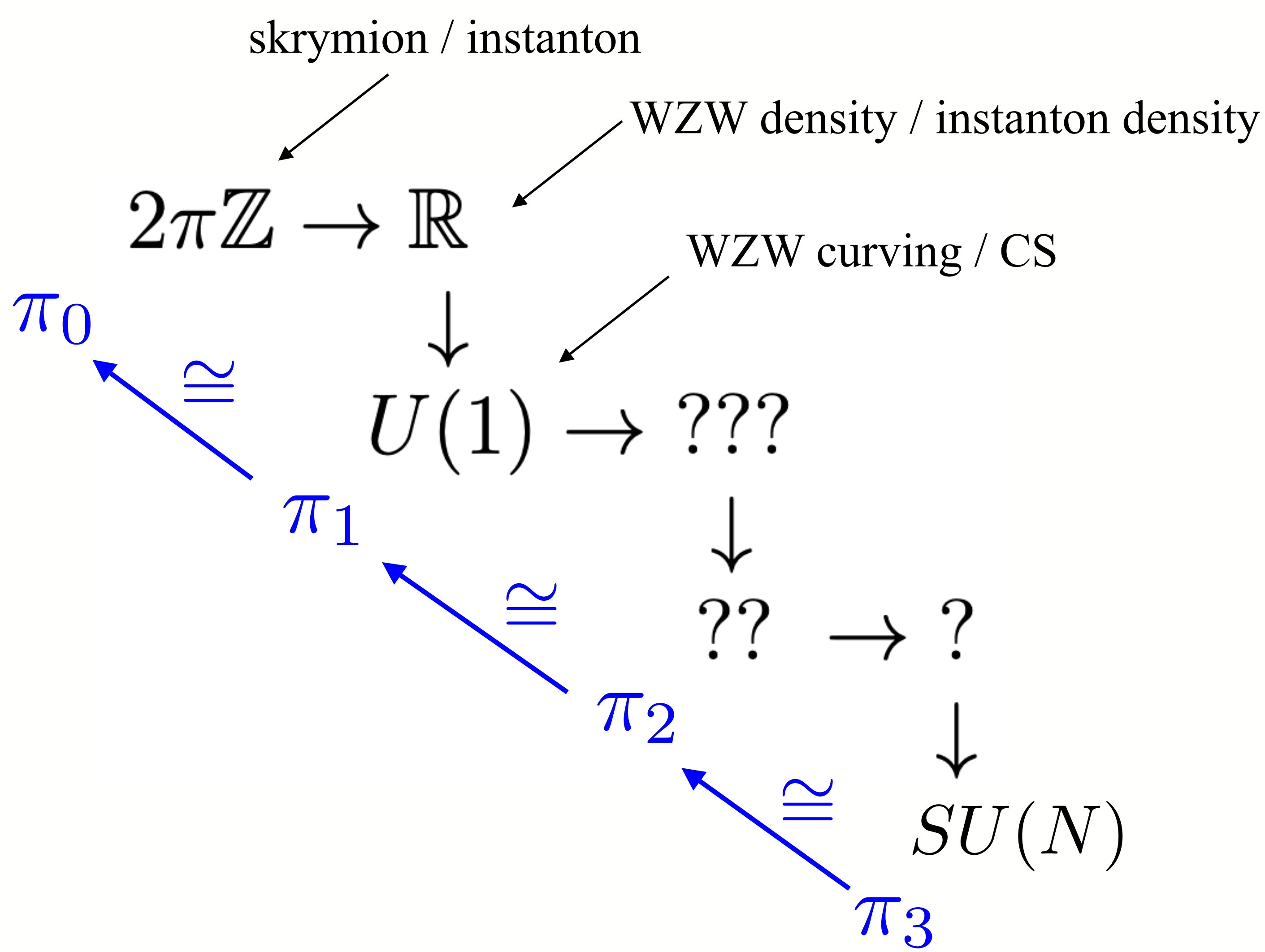
seems now we want



seems now we want



seems now we want



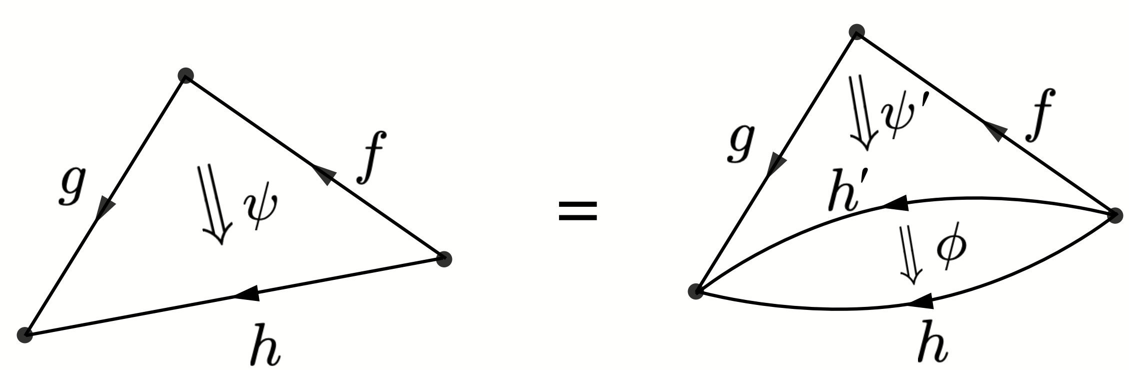
The “??” is on link (for nlsm), should be able to compose.

But finite dimensional Lie group always has trivial π_2 !

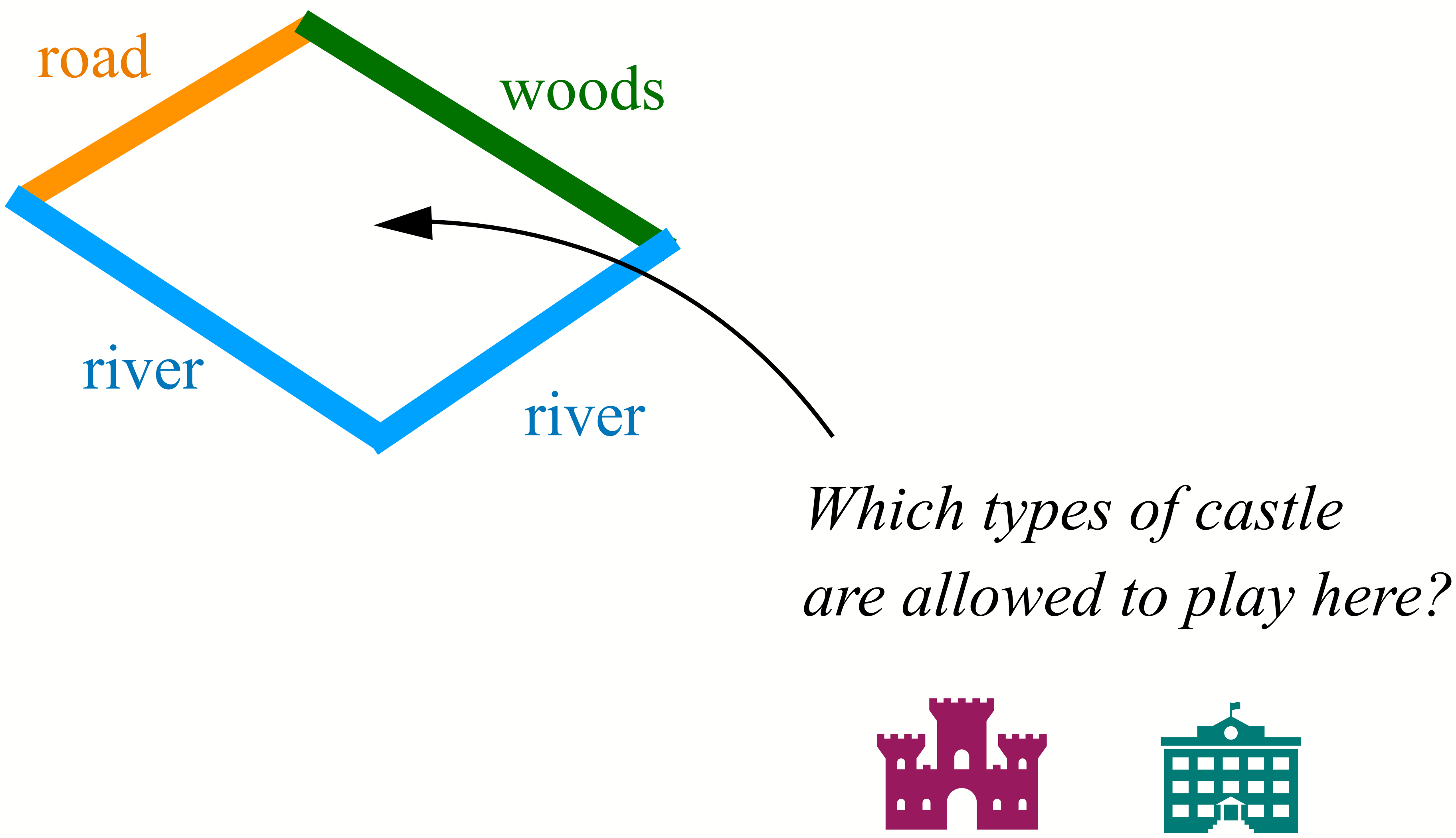
We need:

coverings that are not fibre bundles

d.o.f. that can be composed, but result is non-unique, and can talk about relations between different results — more flexible “game rules” than groups

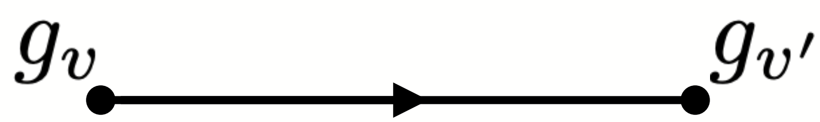


much like some kind of board game:



pion non-linear sigma model

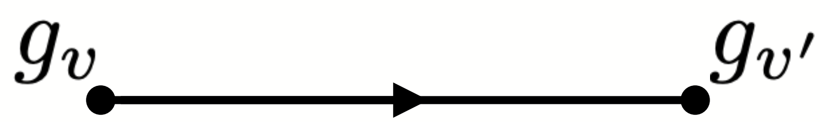
S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog



traditional link field: $g_{v'} g_v^{-1} \in SU(2)$

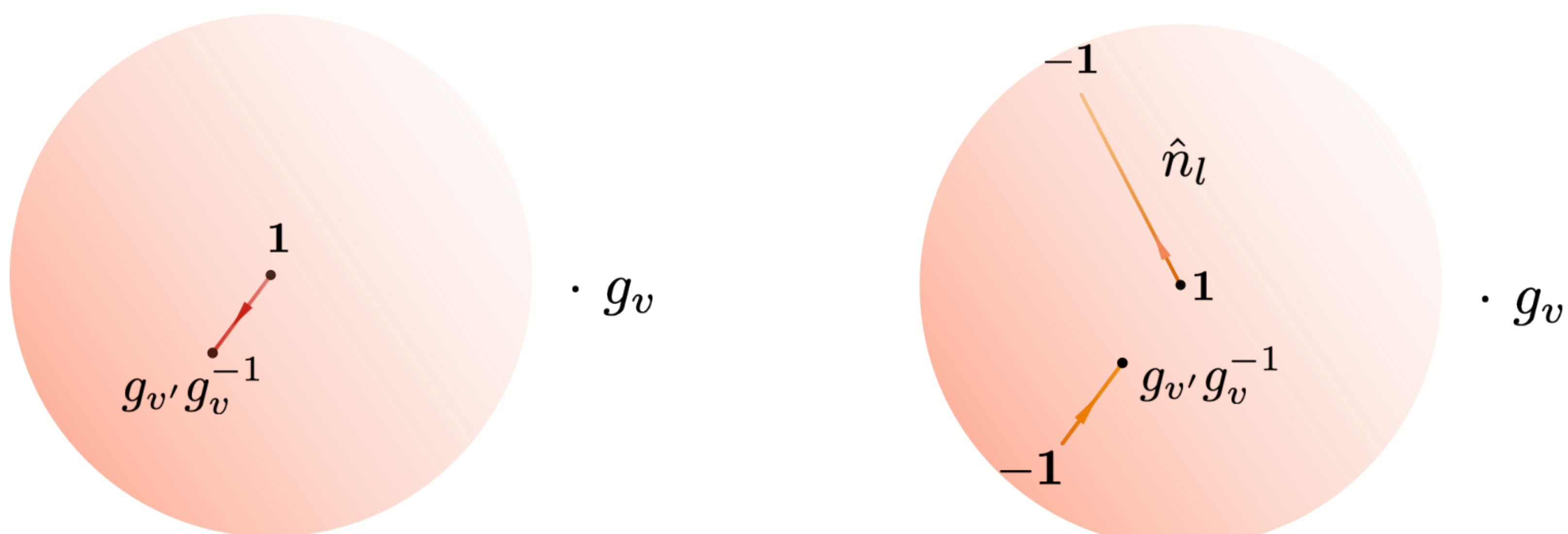
new link field: $y_l = \left(g_{v'} g_v^{-1}, m_l, \hat{n}_l\right) \in Y$ which covers $SU(2)$

S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog



traditional link field: $g_{v'} g_v^{-1} \in SU(2)$

new link field: $y_l = \left(g_{v'} g_v^{-1}, m_l, \hat{n}_l\right) \in Y$ which covers $SU(2)$

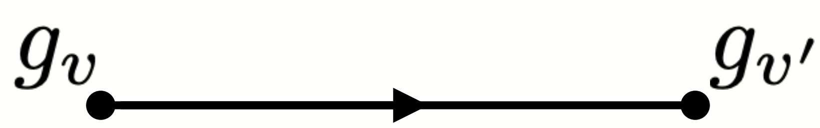


$m_l = +$ $(g_{v'} g_v^{-1} \neq -\mathbf{1})$

$m_l = -$ $(g_{v'} g_v^{-1} \neq +\mathbf{1})$

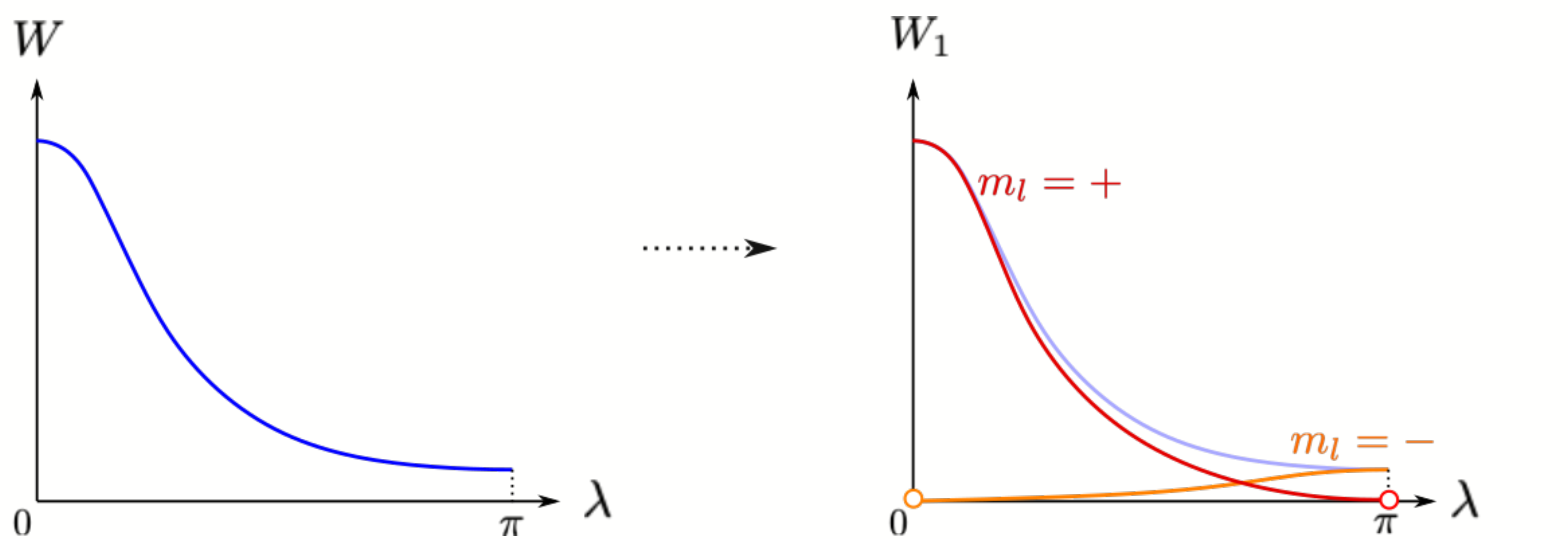
$$Y = (SU(2) \backslash \{-\mathbf{1}\}) \sqcup \left(SU(2) \backslash \{+\mathbf{1}\} \times S^2\right)$$

S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog



traditional link field: $g_{v'} g_v^{-1} \in SU(2)$

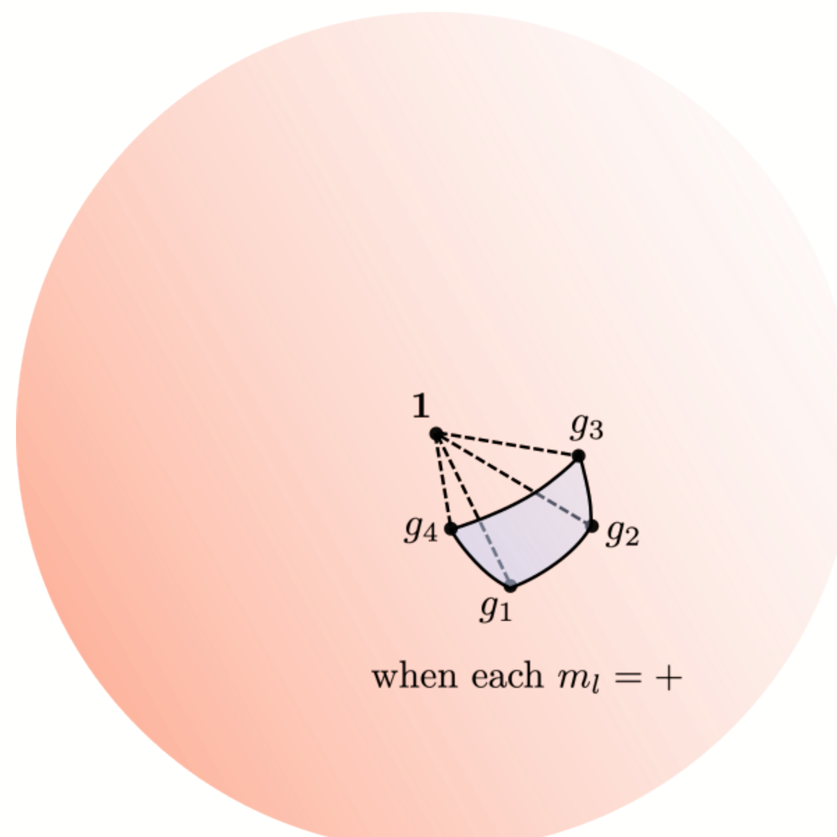
new link field: $y_l = \left(g_{v'} g_v^{-1}, m_l, \hat{n}_l\right) \in Y$ which covers $SU(2)$



S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

plaquette field: given the vertex fields and link fields,
want to sample the surface, but not too much details:
deviation from min surface captured by U(1) — WZW integral

$$W_2(e^{i\mathcal{W}_p}\mu^*_{g_{v\in\partial p},m_{l\in\partial p},\hat{n}_{l\in\partial p}} + c.c.)$$



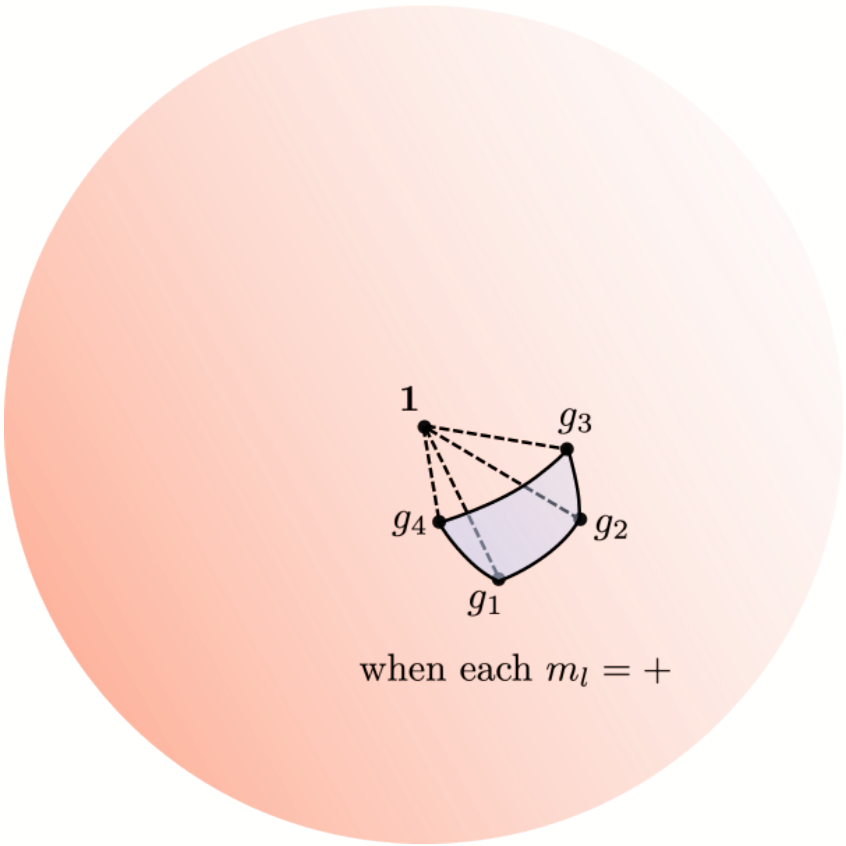
S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

plaquette field: given the vertex fields and link fields,
want to sample the surface, but not too much details:
deviation from min surface captured by U(1) — WZW integral

$$W_2(e^{i\mathcal{W}_p}\mu^*_{g_{v\in\partial p},m_{l\in\partial p},\hat{n}_{l\in\partial p}} + c.c.)$$

phase of μ: volume of pyramid

$|\mu|$ decreases as loop grows,
$|\mu|$=0 when min surf ambiguous



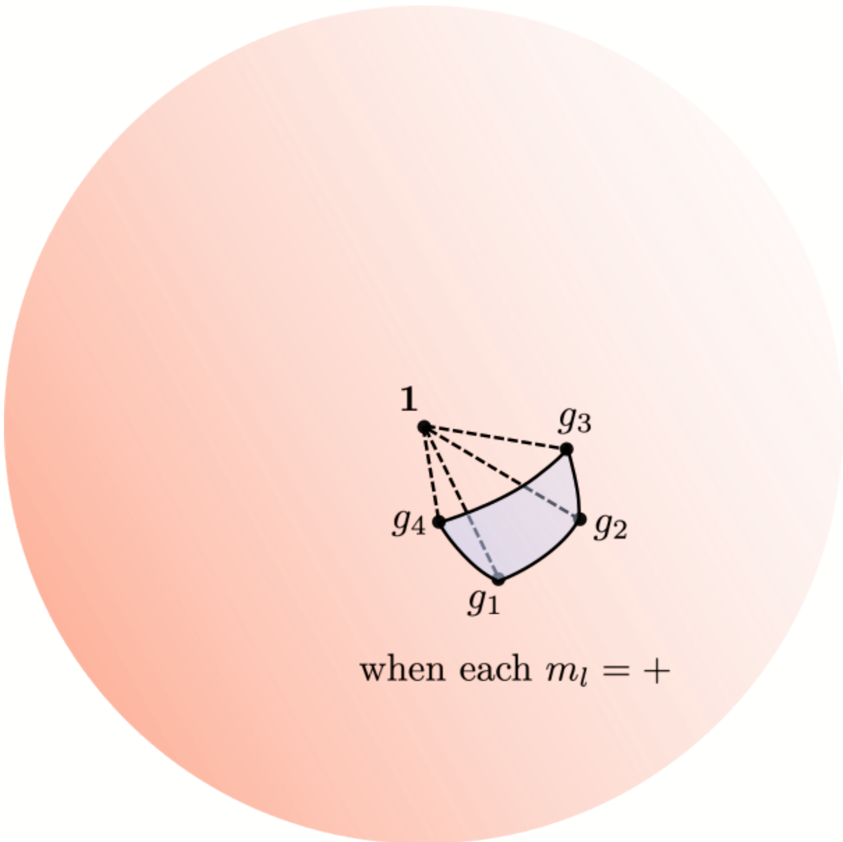
S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

plaquette field: given the vertex fields and link fields,
want to sample the surface, but not too much details:
deviation from min surface captured by U(1) — WZW integral

$$W_2(e^{i\mathcal{W}_p}\mu^*_{g_{v\in\partial p}, m_{l\in\partial p}, \hat{n}_{l\in\partial p}} + c.c.)$$

phase of μ: volume of pyramid

$|\mu|$ decreases as loop grows,
$|\mu|$=0 when min surf ambiguous



2d lattice WZW: $e^{ik\sum_p \mathcal{W}_p}$

S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

cube field: Villainize the U(1) WZW field on plaquette

skrymion density $\mathcal{S}_c := d\mathcal{W}_c/2\pi + s_c \in \mathbb{R}$

cube weight can be e.g. $e^{-V\mathcal{S}_c^2/2}$

3d theta term: $e^{i\Theta \sum_c \mathcal{S}_c}$

S^3 nlsm: 2d WZW, 3d skyrmion, 4d hedgehog

cube field: Villainize the U(1) WZW field on plaquette

skrymion density $\mathcal{S}_c := d\mathcal{W}_c/2\pi + s_c \in \mathbb{R}$

cube weight can be e.g. $e^{-V\mathcal{S}_c^2/2}$ (being Gaussian is not crucial)

3d theta term: $e^{i\Theta\sum_c \mathcal{S}_c}$

hypercube defect: $d\mathcal{S}_h = ds_h \in \mathbb{Z}$ baryon non-conservation

if forbid this by Lagrange multiplier $e^{i\phi_h d\mathcal{S}_h}$

then the U(1) baryon conservation is manifest

a general relation between

continuum QFT and lattice QFT

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow$$

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow$

$\mathcal{L}_0 \quad \hookrightarrow \quad \mathcal{M} \quad \to \quad \mathcal{T}$

traditional lattice theory

lost info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccc} \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ \downdownarrows & & \downdownarrows \\ \vdots & \cdots & \vdots \\ \downdownarrows & & \downdownarrows \\ \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ \downdownarrows & & \downdownarrows \\ \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ \downdownarrows & & \downdownarrows \\ \mathcal{M} & \to & \mathcal{T} \end{array}$$

still continuum theory

no new info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc} & & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \vdots & \cdots & \vdots \\ & & \downdownarrows & & \downdownarrows \\ & & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T} \end{array}$$

traditional lattice theory

lost info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc} & & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \vdots & \cdots & \vdots \\ & & \downdownarrows & & \downdownarrows \\ & & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T} \end{array}$$

more info

but still lost info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc} & & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\ & & \downdownarrows & & \downdownarrows \\ & & \vdots & \cdots & \vdots \\ & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_2 & \hookrightarrow & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T} \end{array}$$

more info

but still lost info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow \qquad \begin{array}{ccccc}
\mathcal{L}_d & \hookrightarrow & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_2 & \hookrightarrow & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T}
\end{array}$$

still continuum theory, no new info
but lattice perspective, no lost info

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \qquad \Longrightarrow$$

inf dim

$$\begin{array}{ccccc}
\mathcal{L}_d & \hookrightarrow & \mathcal{P}^d\mathcal{M} & \to & \mathcal{P}^d\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\vdots & \cdots & \vdots & \cdots & \vdots \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_2 & \hookrightarrow & \mathcal{P}^2\mathcal{M} & \to & \mathcal{P}^2\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & \hookrightarrow & \mathcal{P}\mathcal{M} & \to & \mathcal{P}\mathcal{T} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & \hookrightarrow & \mathcal{M} & \to & \mathcal{T}
\end{array}$$

still continuum theory
no new info
but lattice perspective

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

find finite dim equiv

$$
\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad
\begin{array}{c}
\mathcal{L}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{L}_2 \\ \downdownarrows \\ \mathcal{L}_1 \\ \downdownarrows \\ \mathcal{L}_0
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\mathcal{P}^d\mathcal{M} \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{P}^2\mathcal{M} \\ \downdownarrows \\ \mathcal{P}\mathcal{M} \\ \downdownarrows \\ \mathcal{M}
\end{array}
\longrightarrow
\begin{array}{c}
\mathcal{P}^d\mathcal{T} \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathcal{P}^2\mathcal{T} \\ \downdownarrows \\ \mathcal{P}\mathcal{T} \\ \downdownarrows \\ \mathcal{T}
\end{array}
\xrightarrow[\text{what we care}]{\text{equiv up to}}
\begin{array}{c}
\mathbf{ET}_d \\ \downdownarrows \\ \cdots \\ \downdownarrows \\ \mathbf{ET}_2 \\ \downdownarrows \\ \mathbf{ET}_1 \\ \downdownarrows \\ \mathcal{T}
\end{array}
$$

vague phys problem becomes
well-posed math problem !

systematically rethink about lattice QFT

field in continuum nlsm | what we need on lattice

$$
\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad
\begin{array}{c}
\mathcal{L}_d \\
\downdownarrows \\
\cdots \\
\downdownarrows \\
\mathcal{L}_2 \\
\downdownarrows \\
\mathcal{L}_1 \\
\downdownarrows \\
\mathcal{L}_0
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
\mathbf{ET}_d \\
\downdownarrows \\
\cdots \\
\downdownarrows \\
\mathbf{ET}_2 \\
\downdownarrows \\
\mathbf{ET}_1 \\
\downdownarrows \\
\mathcal{T}
\end{array}
$$

systematically rethink about lattice QFT

field in continuum nlsm

what we need on lattice

$$\mathcal{M} \to \mathcal{T} \quad \Longrightarrow \quad \begin{array}{ccccccc} \mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d\mathcal{T} & & \mathbf{ET}_d \\ \downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\ \cdots & & \cdots & & \cdots & & \cdots \\ \downdownarrows & \xrightarrow{\sim} & \downdownarrows & \longrightarrow & \downdownarrows & \xrightarrow[\text{what we care}]{\text{equiv up to}} & \downdownarrows \\ \mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2\mathcal{T} & & \mathbf{ET}_2 \\ \downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_1 & & \mathcal{P}\mathcal{M} & & \mathcal{P}\mathcal{T} & & \mathbf{ET}_1 \\ \downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\ \mathcal{L}_0 & & \mathcal{M} & & \mathcal{T} & & \mathcal{T} \end{array}$$

Resonates with development of homotopy theory in math!

Grothendieck's dream in his *Pursuing Stacks*

systematically rethink about lattice QFT

field in continuum Yang-Mills

what we need on lattice

$$
\begin{array}{ccc}
\mathcal{P}\mathcal{M} & & G \\
\downdownarrows & \rightarrow & \downdownarrows \\
\mathcal{M} & & *
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccc}
\mathcal{L}_d & & \mathcal{P}^d\mathcal{M} & & \mathcal{P}^d|BG| & & \mathbf{BEG}_d \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downdownarrows & \xrightarrow{\sim} & \downdownarrows & \rightarrow & \downdownarrows & \underset{\text{what we care}}{\xrightarrow{\text{equiv up to}}} & \downdownarrows \\
\mathcal{L}_2 & & \mathcal{P}^2\mathcal{M} & & \mathcal{P}^2|BG| & & \mathbf{BEG}_2 \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_1 & & \mathcal{P}\mathcal{M} & & \mathcal{P}|BG| & & G \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\mathcal{L}_0 & & \mathcal{M} & & |BG| & & *
\end{array}
$$

Take $\mathcal{T} = G$ and then “deloop” (involves Yang-Baxter equation) to get Yang-Mills

Yang-Baxter *automatically* resolved by the physically intuitive construction