Semiclassical description of confinement and the global structure of η′

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Based on the following works: 2201.06166 (with Mithat Ünsal),
and 2402.04320, 2405.12402 (with Yui Hayashi)

Study 4d gauge theories on $\mathbb{R}^2 \times \underbrace{T^2}_{\text{'t Hooft twist}}$

- Part 1 : Pure $SU(N)$ YM
  - Reliable semiclassics on small T^2 ($NL \lesssim \Lambda^{-1}$).
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    = KvBLLY Monopole instantons
  - Area law of $W(C)$, N-branch structure of θ-vacua
- Part 2 : QCD w/ fundamental quarks.
  - Derivation of chiral Lagrangian including η'
  - Witten - Veneziano - like η'-mass formula.
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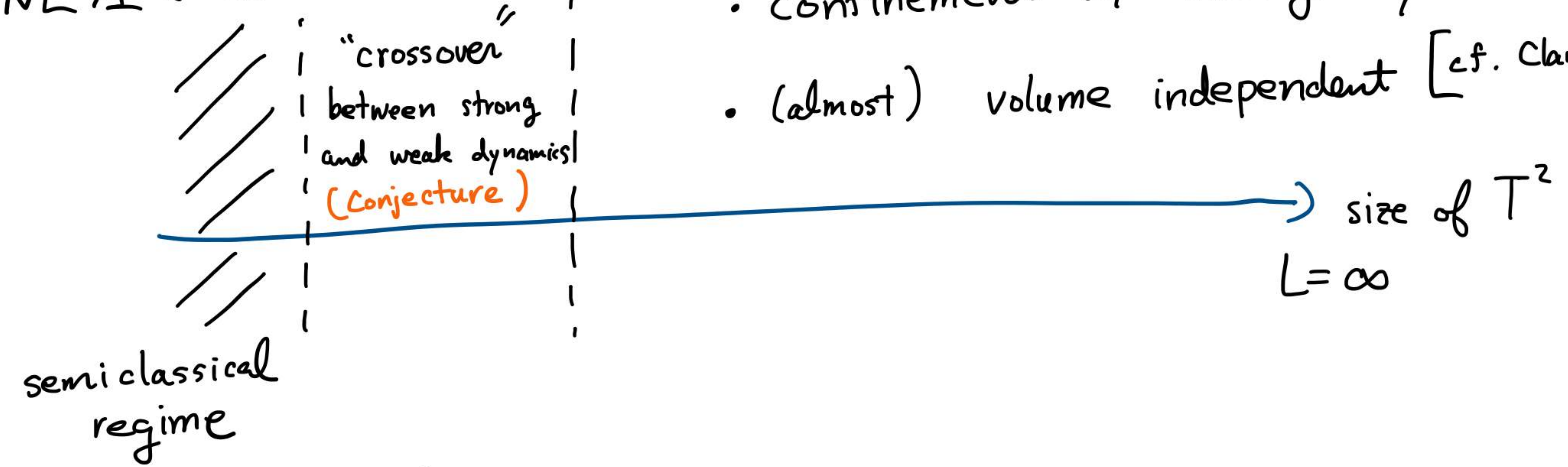
Semiclassical regime for 4d YM on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux

$L\Lambda \sim O(1)$

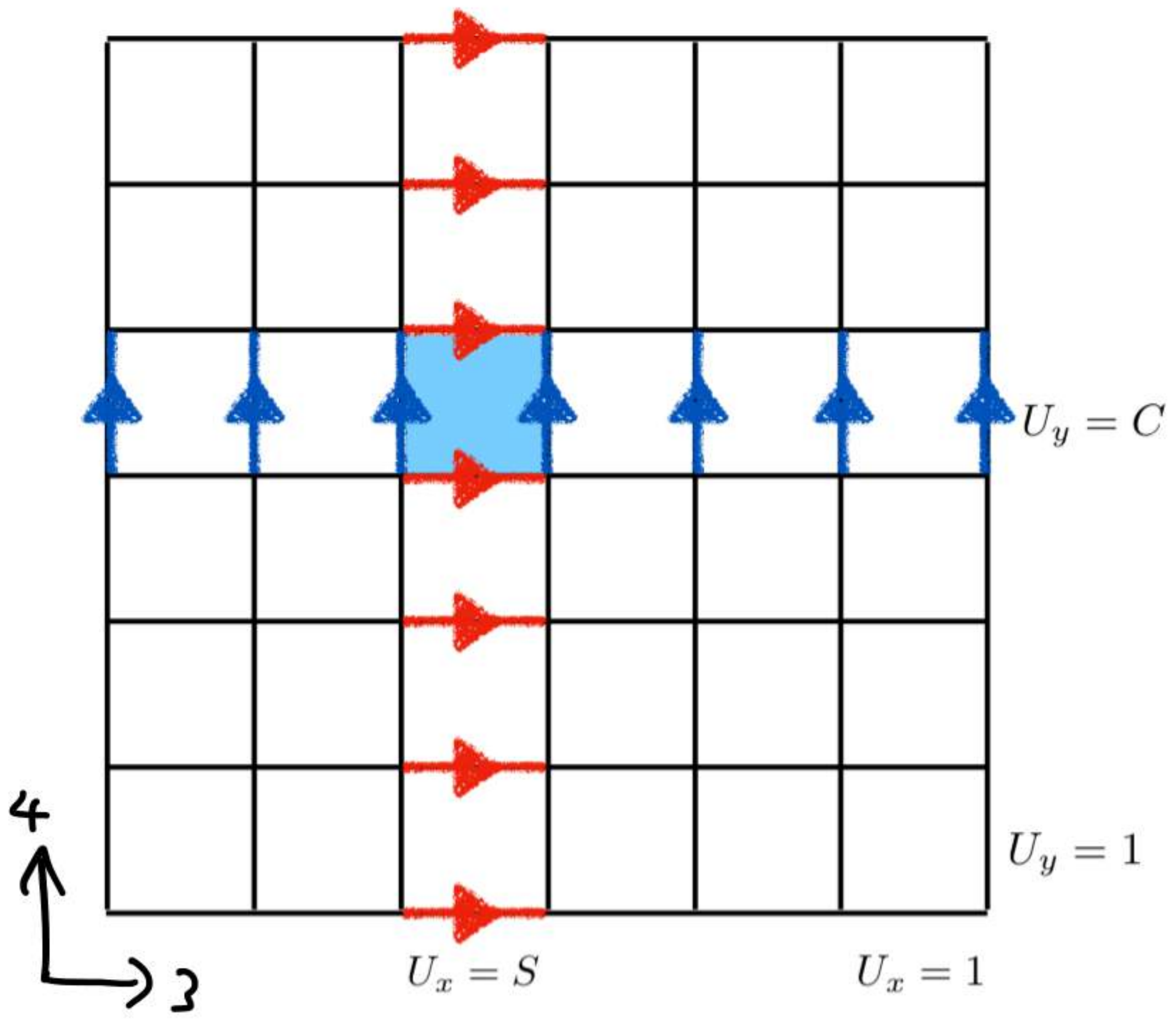
- confinement w/ strong dynamics
- (almost) volume independent [cf. Claudio's talk]



Claim (YT, Ünsal, '22)

$$\boxed{\text{We can prove confinement for } NL\Lambda \ll 1 \text{ using the semiclassical method.}}$$

't Hooft flux & Classical vacuum



Lattice action

$$S_W[U_\ell, B] = -\frac{1}{g^2}\sum_p \left(e^{-iB_p}\,\mathrm{tr}[U_p] + e^{iB_p}\,\mathrm{tr}[U_p^\dagger]\right).$$

$$B_p = \begin{cases} \frac{2\pi}{N} & \text{(for the plaquette indicated with light blue)} \\ 0 & \text{(otherwise)} \end{cases}$$

We can minimize this action by setting

$$U_\ell = \begin{cases} S = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \\ C = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix} \\ \mathbb{1} \end{cases}$$

[* Any classical minimum is gauge equivalent to this one]

$$\Rightarrow \quad P_3 = S \quad , \quad P_4 = C .$$

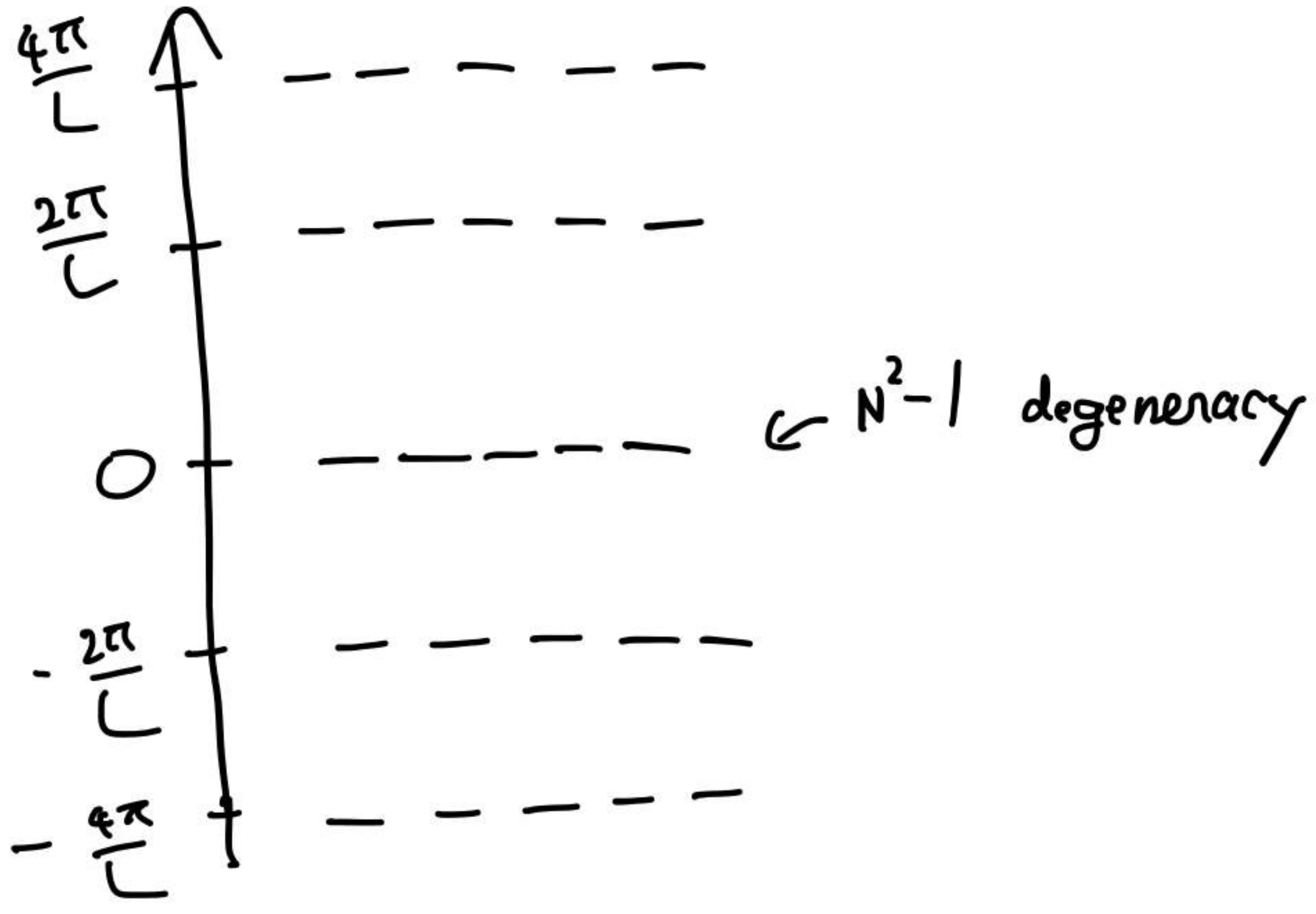
This configuration completely preserves $\mathbb{Z}_N^{[0]} \times \mathbb{Z}_N^{[0]}$.

Perturbative spectrum

Put 4d YM on $\mathbb{R}^2 \times T^2$.

Perturbative spectrum of 2d gauge fields on $\mathbb{R}^2$ part.

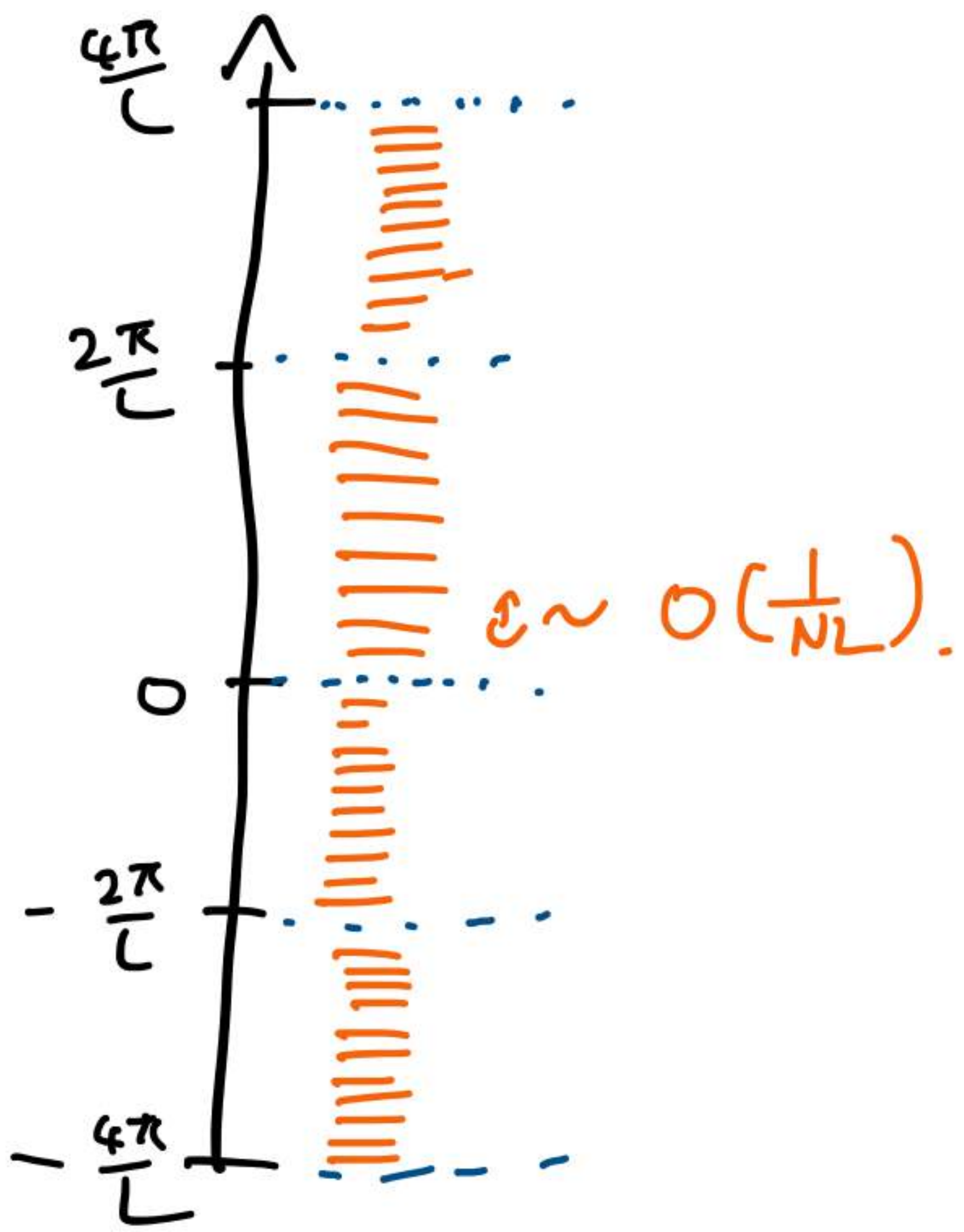
Without twist



$$M_{KK}^2 = \left(\frac{2\pi}{L}\right)^2 (l_3^2 + l_4^2)$$

- There exist zero modes.

With twist



$$M_{KK} = \left(\frac{2\pi}{L}\right)^2 \left(\left(l_3 + \frac{p_3}{N}\right)^2 + \left(l_4 + \frac{p_4}{N}\right)^2 \right)$$

for the color basis $e^{-\frac{2\pi i}{N}\frac{p_3 p_4}{2}} C^{-p_3} S^{p_4}$.

- No zero modes.
- Gap $\sim \frac{2\pi}{NL}$.

2d $\mathbb{Z}_N^{(1)}$ symmetry & center vortex

- For 2d effective theory,

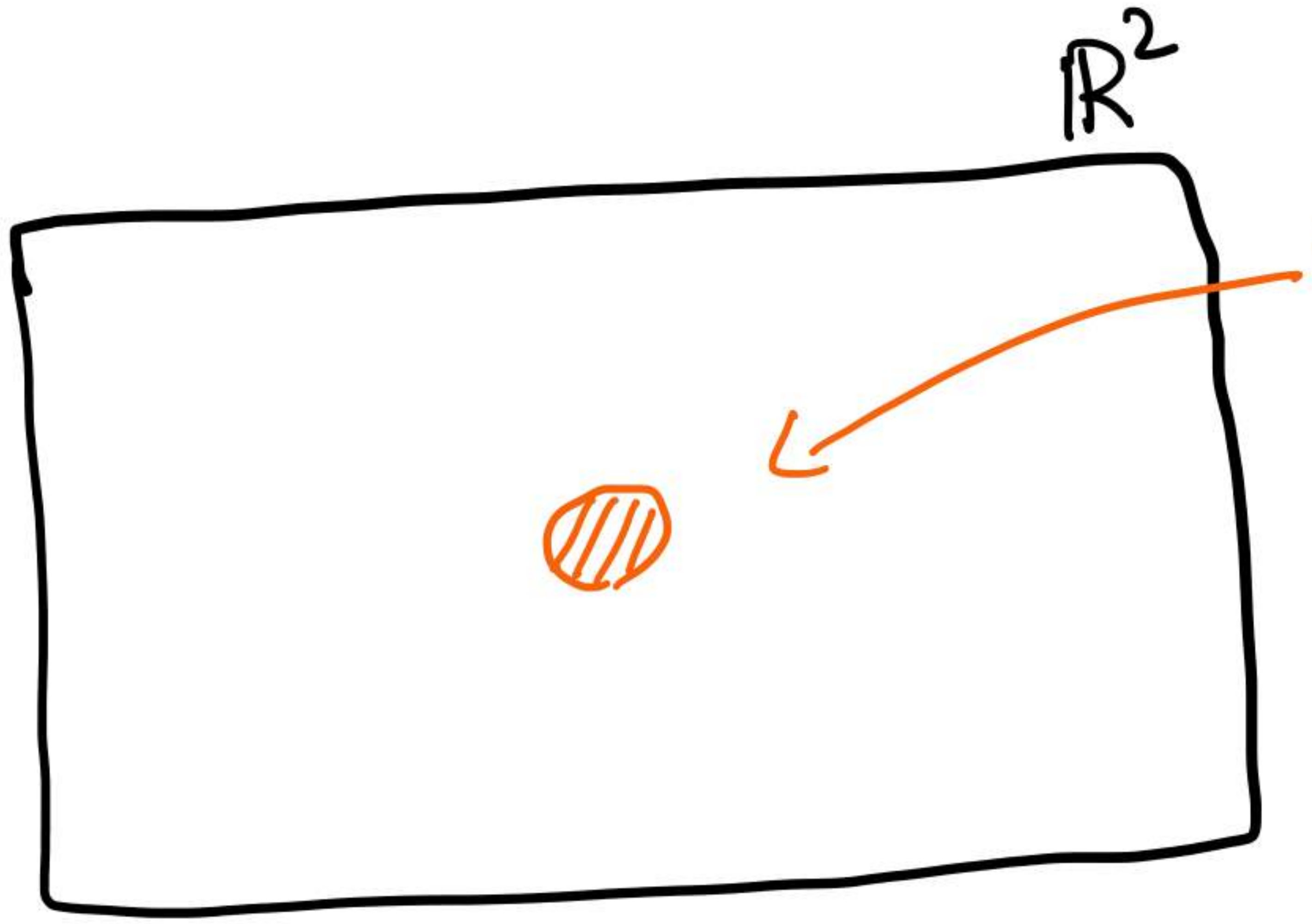
$P_3 = S$ and $P_4 = C$ behave as Adjoint Higgs with orthogonal VEV :

$$SU(N) \xrightarrow{\text{Higgs}} \mathbb{Z}_N .$$

$\Rightarrow$ 2d 1-form symmetry is spontaneously broken at classical vacua.

- Broken discrete symmetry $\Rightarrow$ Degenerate ground state

$\Rightarrow$ Topological solitons connecting them
(Domain walls for 0-form symmetry)



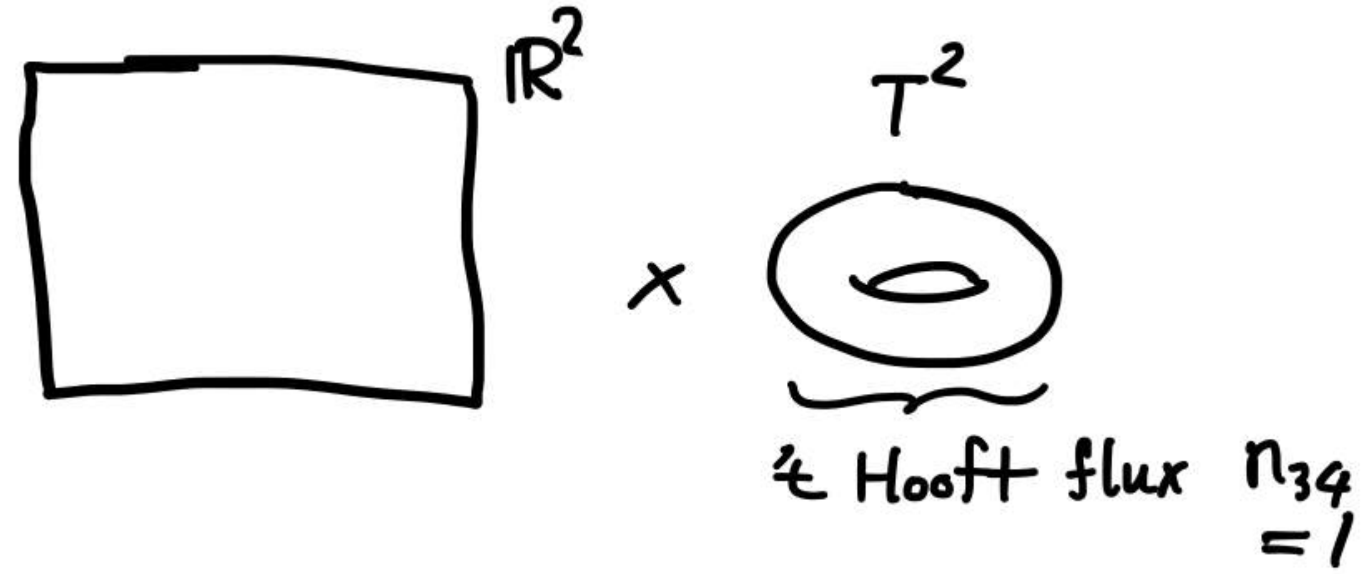
Vortex Soliton

= Center vortex

[cf. Tin's talk]

Center vortex = fractional instanton on $\mathbb{R}^2 \times T^2$

In this setup, the minimal topological charge is given by



$$\boxed{Q_{top} = \frac{1}{8\pi^2}\int \mathrm{tr}(F \wedge F) = \frac{1}{N}}$$

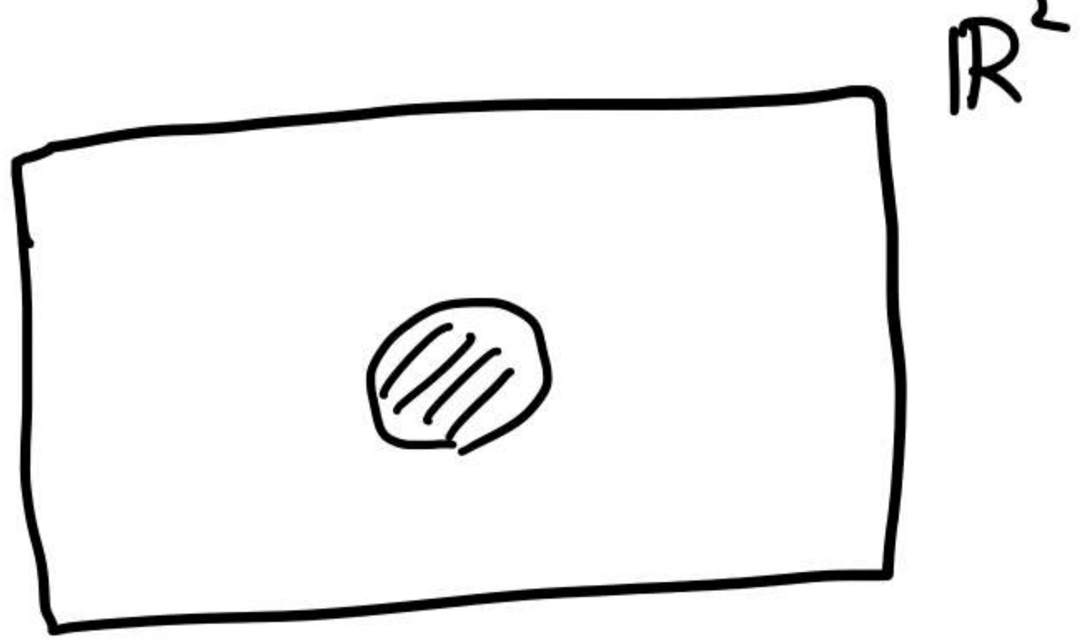
(More precisely, $Q_{top} \in \frac{1}{N}\left(\frac{-\varepsilon_{\mu\nu\rho\sigma} n_{\mu\nu} n_{\rho\sigma}}{8}\right) + \mathbb{Z}$ (van Baal '82))

If there exists a self-dual configuration, its Yang-Mills action becomes

$$\boxed{S_{YM} = \frac{8\pi^2}{g^2}|Q_{top}| = \frac{8\pi^2}{g^2 \cdot N}}$$

Gonzalez-Arroyo, Montero '98, Montero '99 numerically confirmed such a classical solution exists:

center vortex
or fractional instanton.



$$\begin{pmatrix} Q_{top} = \frac{1}{N} \\ S_{YM} = \frac{8\pi^2}{g^2 N} \end{pmatrix}$$

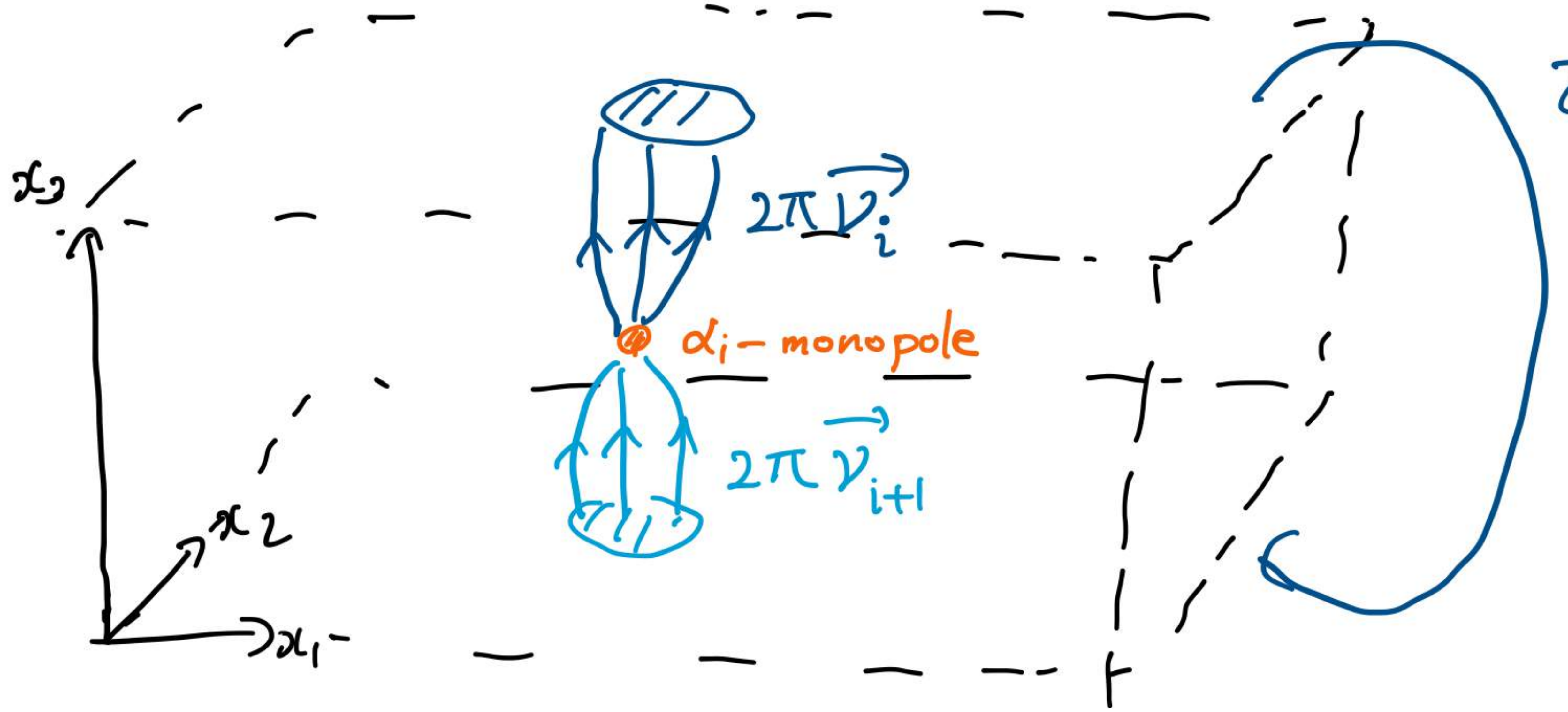
(cf. Garcia Perez, Gonzalez-Arroyo, '92, ⋯, Itou '18, Wandler '24)

Center vortex on $\mathbb{R}^2 \times \underbrace{T^2}_{\text{flux}}$ = KvBLLY monopole instanton

SU(N) gauge field on $\mathbb{R}^3 \times S^1$ w/ nontrivial holonomy : N fundamental monopoles (Lee, Yi '97; Lee, Lu '98; Kraan, van Baal '98)

[⇒ 3d semiclassics by Ünsal, ... since 2007]

α_i - monopole emits the magnetic flux $2\pi\alpha_i = 2\pi(\nu_i - \nu_{i+1})$.



$\mathbb{Z}_N$-twisted b.c. (= 't Hooft flux on T^2)

$2\pi\nu_i$

$\mapsto 2\pi\nu_{i+1}$

[Hayashi, YT 2405.12402]

$\mathbb{Z}_N$-twisted b.c. gives the pertubative gap $\frac{2\pi}{NL_3}$ ⇒ Magnetic flux localizes.

$$\boxed{\text{Monopole} = \text{Junction of the center vortex}}$$

(cf. Ambjorn, Giedt, Greensite '99, de Forcrand, Pepe '00)

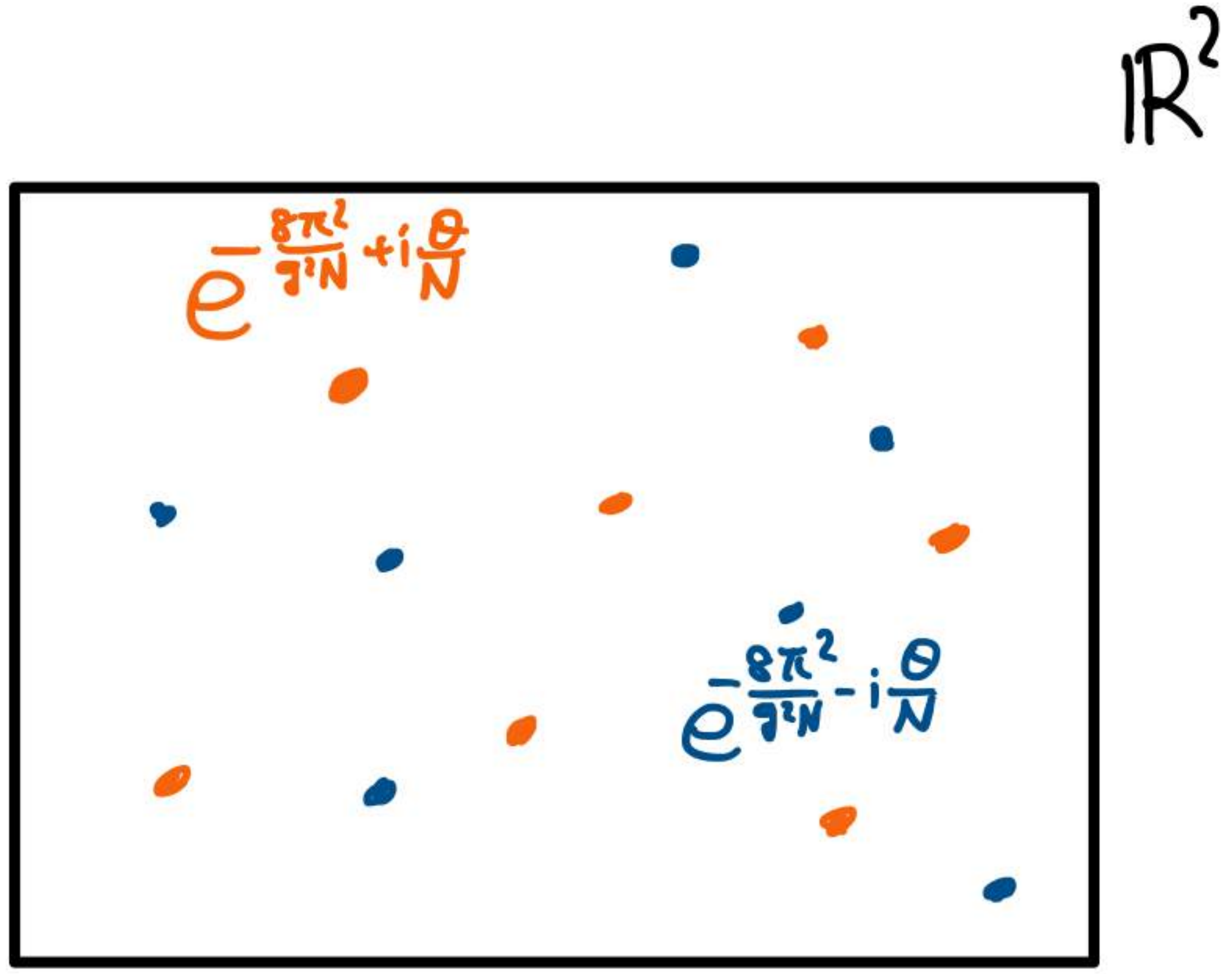
Dilute gas approximation

2d gluon fields are perturbatively gapped by 't Hooft twist.

$\Longrightarrow$ Center vortex, or fractional instanton, does NOT have the size moduli.

$\Longrightarrow$ Dilute gas approximation is available.

(* In 4d pure YM, DIGA is invalidated because of IR divergences.)



n : # of vortices

$\bar{n}$: # of anti-vortices

$$Q_{top} = \frac{n-\bar{n}}{N}.$$

Partition function on $\underset{\to \mathbb{R}^2}{M_2} \times T^2$ & θ-dependence

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

Using the 1-loop vertex of the center vortex

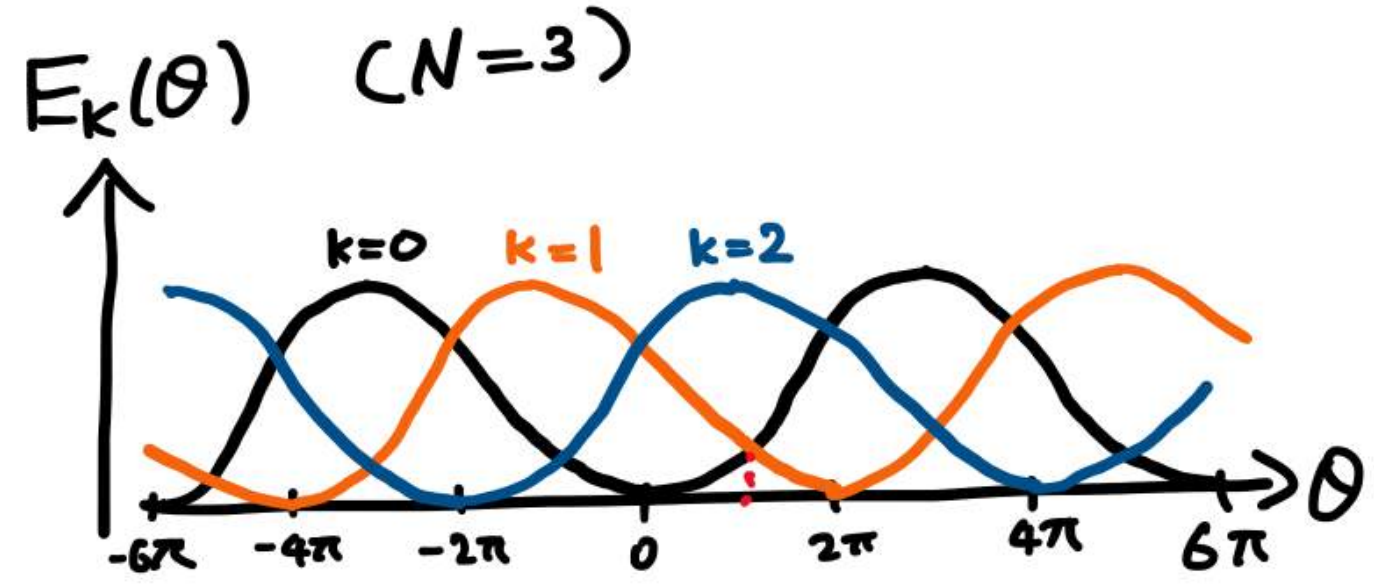
$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}_{\text{vortex}}\Big)^{n} \Big(\underbrace{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}_{\text{anti-vortex}}\Big)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\Big[-V\,\Big(\underbrace{-2K\, e^{-\frac{8\pi^2}{g^2N}} \cos\Big(\frac{\theta-2\pi k}{N}\Big)}_{E_k(\theta)\ :\ \text{Ground-state energy densities}}\Big)\Big]$$

$\Rightarrow$
- N-branch structure of ground states.
- Each branch has a fractional θ-dependence.



Partition function on $M_2 \times T^2$ & θ-dependence

$\to \mathbb{R}^2$

To make the computation well-defined, we compactify $\mathbb{R}^2$ to some closed 2-manifold M_2.

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$$K \cdot e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}},$$

we have

vortex, anti-vortex

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}k(n-\bar{n})} = \delta_{n-\bar{n}\in N\mathbb{Z}}$$

$$Z(\theta) = \sum_{n,\bar{n}\geq 0} \frac{\delta_{n-\bar{n}\in N\mathbb{Z}}}{n!\,\bar{n}!} \left(\underline{V\cdot K e^{-\frac{8\pi^2}{g^2N}+i\frac{\theta}{N}}}\right)^n \left(\underline{V\cdot K e^{-\frac{8\pi^2}{g^2N}-i\frac{\theta}{N}}}\right)^{\bar{n}}$$

$$= \sum_{k=0}^{N-1} \exp\left[-V\left(\underbrace{-2K e^{-\frac{8\pi^2}{g^2N}} \cos\left(\frac{\theta-2\pi k}{N}\right)}_{E_k(\theta)}\right)\right]$$

$E_k(\theta)$: Ground-state energy densities

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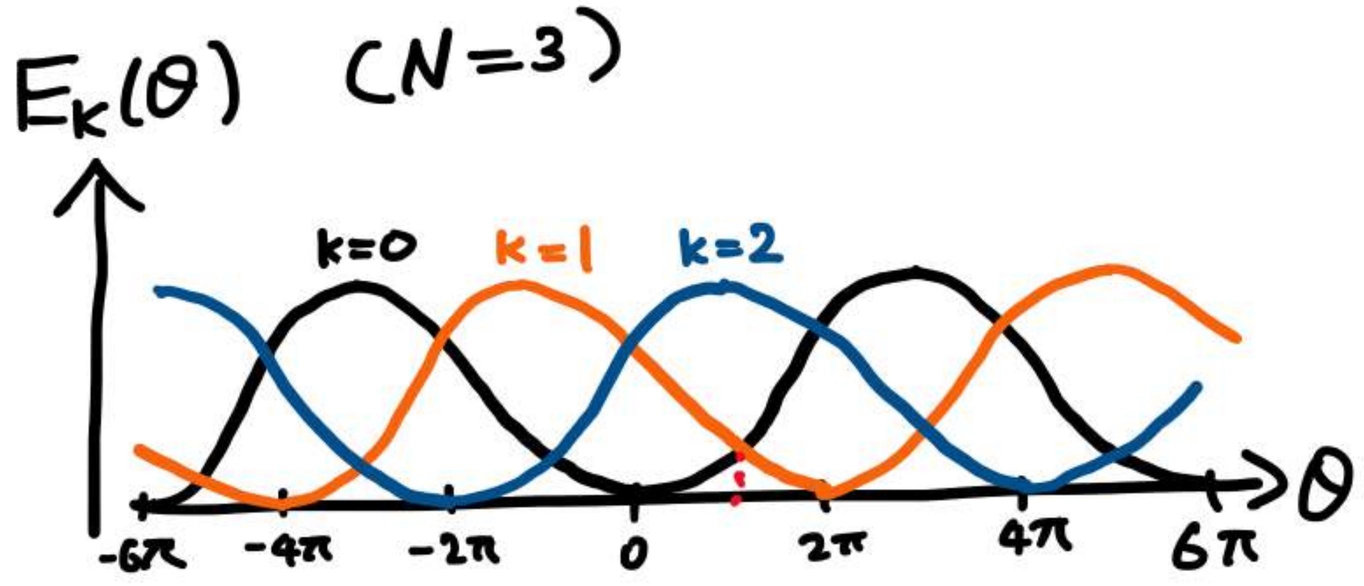

Physical Meaning of the branch label $k \in \mathbb{Z}_N$

① Distinction as Symmetry-Protected Topological (SPT) states

When 4d $\mathbb{Z}_N^{(1)}$ is unbroken,

$$Z[B] \overset{\text{low energy}}{\sim} \exp\left(i\int \frac{Nk}{4\pi} B \wedge B\right).$$

[cf. Gaiotto, Kapustin, Komargodski, Seiberg '17]

② Distinction based on Wilson-'t Hooft classification

't Hooft ('79) : Area law for $W(C)$ is not enough to classify confinement states.
Use also $H(C,\Sigma)$.

Screened line operators are generated by $H(C,\Sigma)\cdot W^k(C)$.

(ex. SU(2)

	$W(C)$	$H(C,\Sigma)$	$W(C)H(C,\Sigma)$
Higgs	$\boxed{\text{Perimeter}}$	Area	Area
Confinement $\theta \sim 0$	Area	$\boxed{\text{Perimeter}}$	Area
Confinement $\theta \sim 2\pi$	Area	Area	$\boxed{\text{Perimeter}}$

)

[Equivalence ① ⟺ ② : Nguyen, YT, Ünsal 2306.02485]

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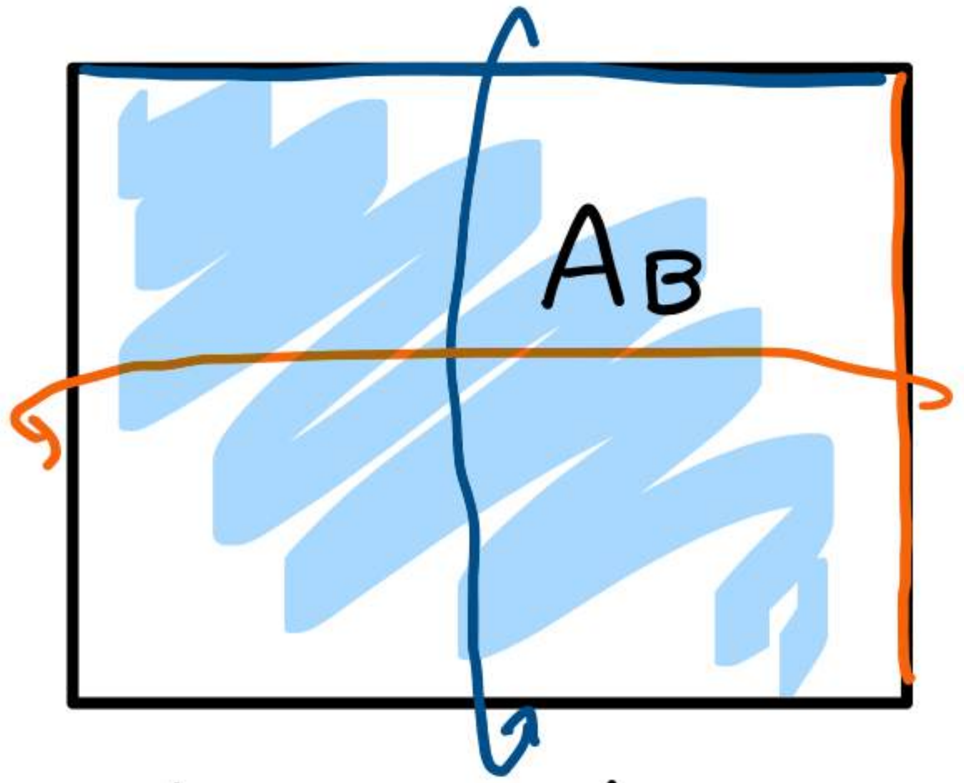
$U(1)_B$ monopole flux & 't Hooft flux on T^2

Fundamental quarks explicitly violates $\mathbb{Z}_N^{(1)}$ $\Rightarrow$ No 't Hooft B.C., naively.

Use $U(1)_B = \frac{U(1)_q}{\mathbb{Z}_N}$ monopole flux.



$$\begin{cases} \psi(L, x_4) = g_3^\dagger(x_4)\, e^{-i\frac{\phi_3(x_4)}{N}}\, \psi(0, x_4) \\ \psi(x_3, L) = \underbrace{g_4^\dagger(x_3)}_{\text{color-transition functions}}\, \underbrace{e^{-i\frac{\phi_4(x_3)}{N}}}_{U(1)_q\text{-transition functions}}\, \psi(x_3, 0) \end{cases}$$

Cocycle condition

$$g_3^\dagger(L)\, g_4^\dagger(0)\, e^{-i\frac{1}{N}(\phi_3(L)+\phi_4(0))} = g_4^\dagger(L)\, g_3^\dagger(0)\, e^{-i\frac{1}{N}(\phi_4(L)+\phi_3(0))}.$$

$U(1)_B = U(1)_q/\mathbb{Z}_N$ monopole flux

$$2\pi = \int_{T^2} dA_B = (\phi_3(L) - \phi_3(0)) - (\phi_4(L) - \phi_4(0))$$

$$\Longrightarrow \quad g_3^\dagger(L)\, g_4^\dagger(0) = g_4^\dagger(L)\, g_3^\dagger(0)\, \boxed{e^{\frac{2\pi i}{N}}}$$

't Hooft flux !!

2d Effective Lagrangian on $\mathbb{R}^2 \times T^2$ w/ baryon-'t Hooft flux

$$
\text{4d QCD} \xrightarrow[\text{Perturbative}]{\text{classical \& }} \begin{cases} \text{Gauge : Gapped gluons \& Center vortices} \\ \text{Quark : 2d Dirac fermion as 4d Dirac zero modes w/ } \int_{T^2} dA_B = 2\pi . \end{cases}
$$

$$\Big\Downarrow \ \text{Bosonization \& Semiclassical Analysis}$$

U : 2d $U(N_f)$-valued field (π, K, η & η') [YT, Ünsal 2201.06166; Hayashi, YT 2402.02430]

$$
\begin{aligned}
\mathcal{L}_{2d\ \text{effective}} = & \frac{1}{8\pi} \mathrm{tr}(\partial_\mu U^\dagger \partial_\mu U) - \mathrm{tr}[M U + \text{c.c.}] \\
& + i \frac{1}{12\pi} \mathrm{tr}\left[(U^\dagger d U)^3\right] \\
& - e^{-\frac{8\pi^2}{g^2 N}} \left(e^{-\frac{i\theta}{N}} (\det U)^{1/N} + \text{c.c.} \right)
\end{aligned}
$$

[$\Leftarrow$ If we neglect massive η' part,
this is consistent w/ 4d chiral Lagrangian on $\mathbb{R}^2 \times T^2$ w/ $\int_{T^2} dA_B = 2\pi$.]

Revisiting $U(1)_A$ problem

$U(1)_A$: η' (i.e. $e^{i\eta'} = \det U$) is too massive according to SSB of the chiral symmetry $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$.

$\Rightarrow$ $U(1)_A$ is not a symmetry of quantum theory !

η' gets the mass even if $M_{\text{quark}} = 0$.

<u>Previous proposals</u>

- Kobayashi - Maskawa - 't Hooft : $-\cos(\eta' - \theta)$
- Witten - Veneziano's large $-N$: $\eta'^2 + O\left(\frac{\eta'^4}{N^2}\right)$

$$\boxed{\text{2d center-vortex theory} \quad : \quad -\cos\left(\frac{\eta' - \theta}{N}\right)}$$

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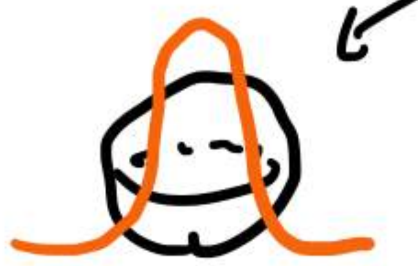
$\frac{1}{N}$ fractionalize

large-N

$$\boxed{\text{2d center-vortex theory} \quad : \quad -\cos\left(\frac{\eta' - \theta}{N}\right)}$$

How does $\frac{1}{N}$ appear? What is its fermionic picture?

't Hooft [cf. Thamas's talk @ finite-T]



4d instanton

localization of chiral zero mode

$\mathbb{R}^2 \times T^2$ w/ 't Hooft flux $\Longrightarrow$

$\frac{1}{N}$ fraction of the chiral zero mode

4d instanton = N independant center vortices

$$e^{-\frac{8\pi^2}{g^2}} \cdot e^{-i\theta} \det U(x)$$

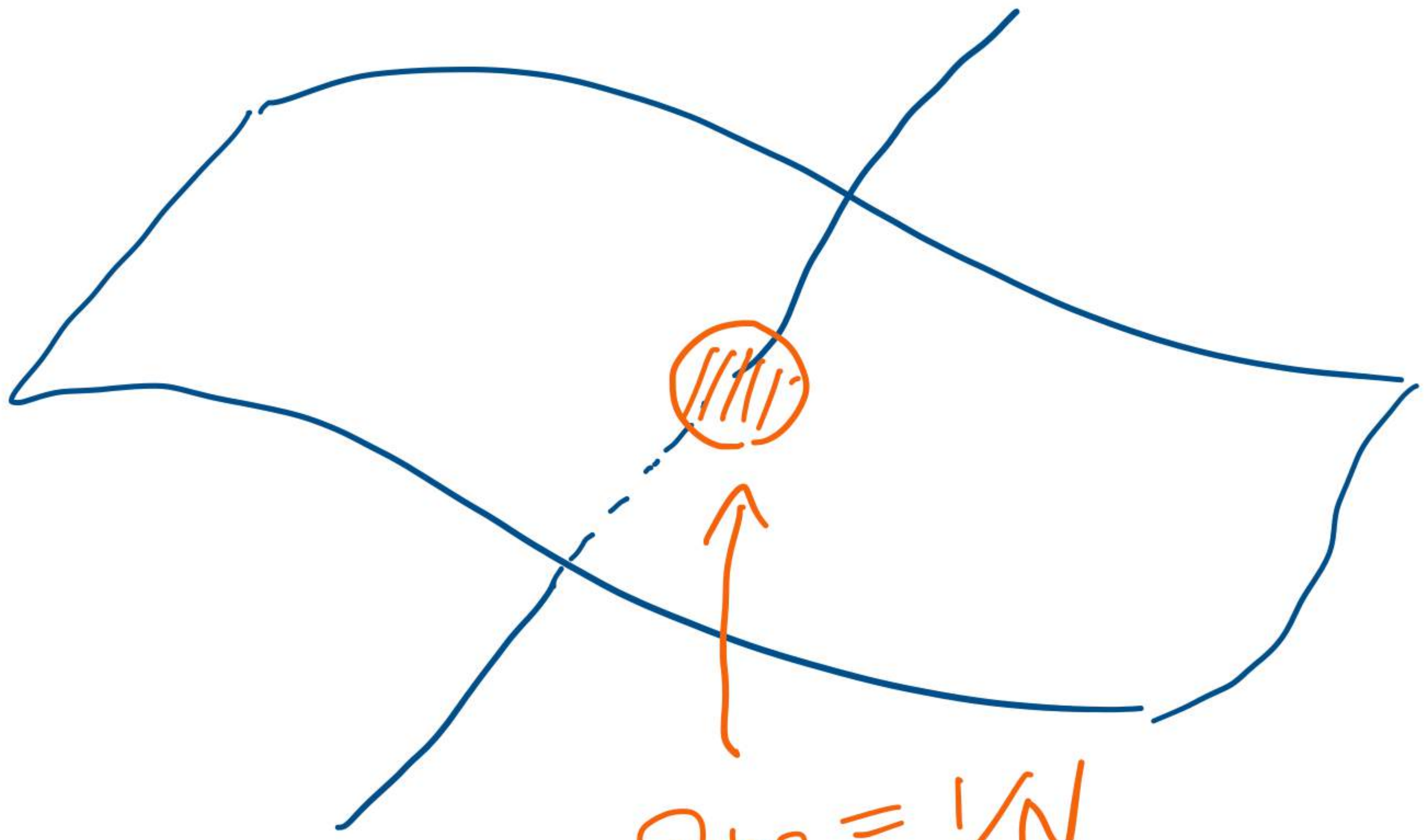
$$\prod_{i=1}^{N} \underbrace{e^{-\frac{8\pi^2}{g^2 N}} \quad e^{-i\frac{\theta}{N}} \left(\det U(x_i)\right)^{1/N}}_{\text{each term behaves almost independently}}$$

How does the fractionalization happen in 4d? [* speculative!]

4d confinement vacua have percolated center-vortex worldsheets.

[cf. Jackson, Chris, Lorenz's talk]

$\Rightarrow$ Those vortex sheets should have point-like intersections.



$Q_{top} = 1/N$

Such intersections have $1/N$ fractional Q_{top}! [Engelhardt, Reinhardt 2000, Cornwall 2000]

Periodicity of η'

Recall that we introduced η' by

$$e^{i\eta'} = \det U.$$

$\Rightarrow$ $\eta' \sim \eta' + 2\pi$, which does not fit $-\cos\left(\frac{\eta'-\theta}{N}\right)$.

[Same problem occurs for the Witten-Veneziano formula]

<u>Solution</u> Reinstate the YM confinement vacua labelled by $k \sim k+N$.

$$-e^{-\frac{8\pi^2}{g^2N}}\cos\left(\frac{\eta'-\theta}{N}\right) \Longrightarrow -e^{-\frac{8\pi^2}{g^2N}}\cos\left(\frac{\eta'-\theta+2\pi k}{N}\right).$$

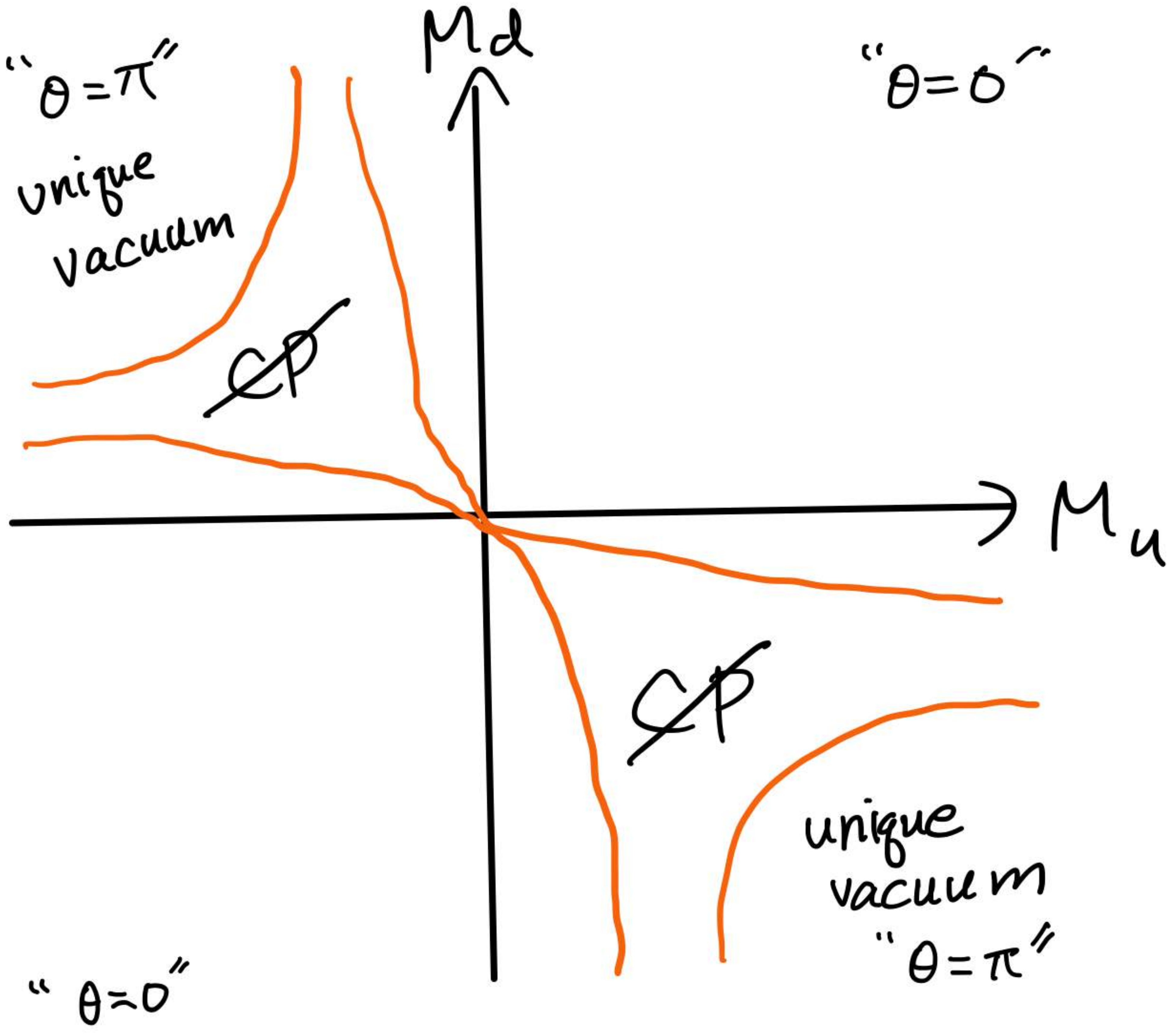
The correct periodicity for η' involves the shift of k: (Hayashi, YT 2402.04320)

$$(\eta'+2\pi, k) \sim (\eta', k+1).$$

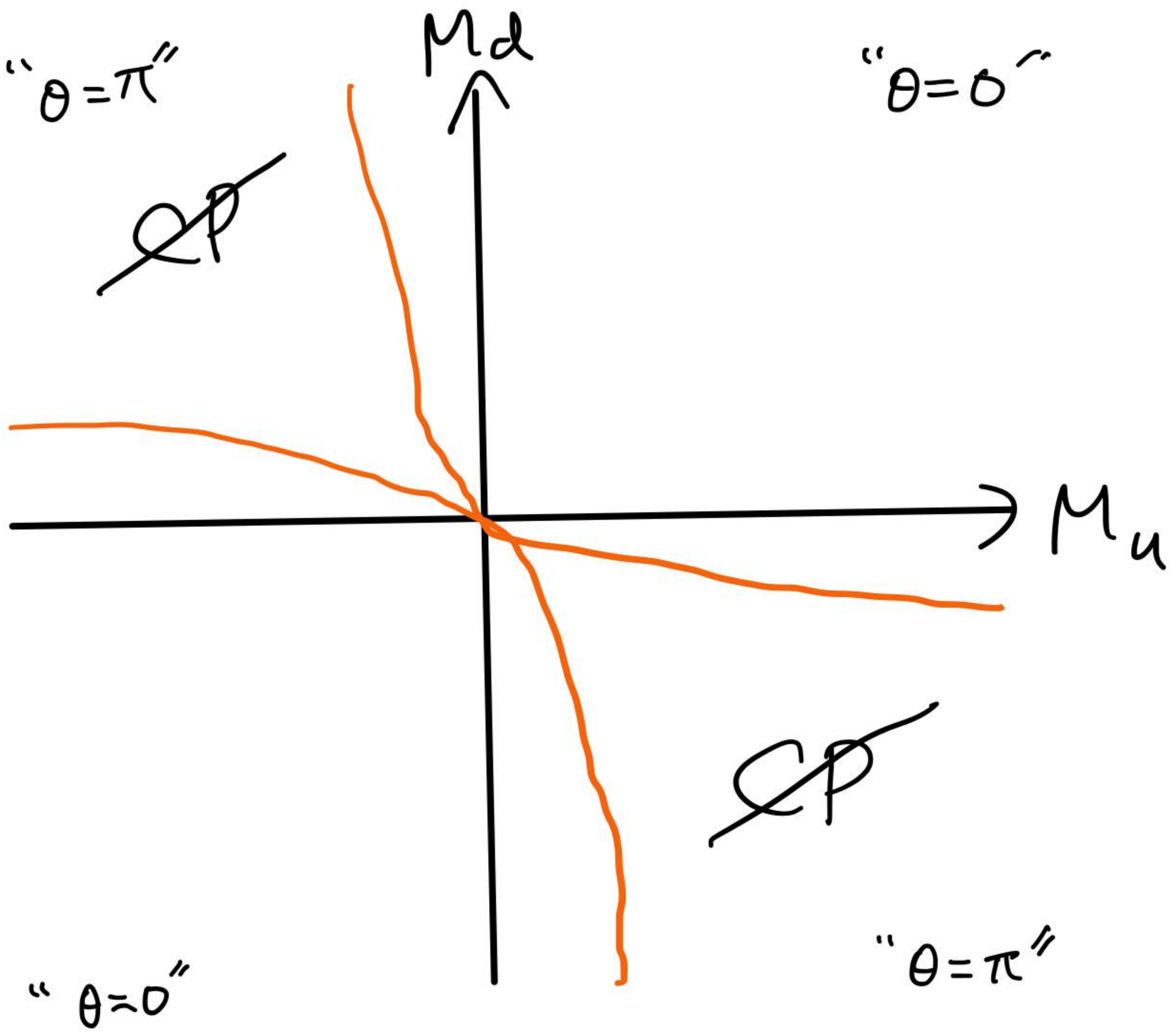
$\Rightarrow$ $\eta' \sim \eta' + 2\pi N$ by "eating" the k index.

Phase diagram in (M_u, M_d) space

KMT vertex & $\frac{1}{N}$-fractionalized vertex can give different predictions.



w/ KMT vertex
[S. Aoki, Creutz '14]
(inconsistent w/ anomaly matching)



w/ $\frac{1}{N}$-fractionalized vertex
[Hayashi, YT '24]

Summary

Studying 4d gauge theories on $\mathbb{R}^2 \times T^2$ w/ 't Hooft flux is useful to investigate some aspects of confinement.

- Center vortex can be used for analytically well-controlled semiclassics.
- Center vortex in this setup $= \frac{1}{N}$ fractional instanton
  $= $ KvBLL monopole instanton
- Fresh perspective on $U(1)_A$ & η'.
  - $\frac{1}{N}$ fractionalization of Kobayashi–Maskawa–'t Hooft vertex
  - η' is correlated w/ N-branch YM vacua : $(\eta' + 2\pi, k) \sim (\eta', k+1)$.
- Other fermions (QCD(Sym/ASym/BF)) can also be studied. (YT, Ünsal '22; Hayashi, YT, Watanabe '23, '24)
  $\Rightarrow$ Large-N orbifold equivalence.