Topological charge and chiral symmetry in hot QCD

Tamás G. Kovács

ELTE Eötvös Loránd University, Budapest, Hungary
and
Institute for Nuclear Research, Debrecen, Hungary

Partly based on TGK, PRL 132 (2024) 131902

Benasque, February 12, 2025

Symmetries of QCD and their realization

- partition function $Z = \int \mathscr{D}U \; \prod_f \det\left(D[U] + m_f\right) \cdot e^{-S_g[U]}$

- $m_u \approx m_d \approx 0$

- Symmetries: $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$
  - $U(1)_A$ anomalous
  - $SU(2)_A$ spontaneously broken below T_c

- Order parameter of $SU(2)_A$ (Banks-Casher formula):

$$\langle \bar{\psi}\psi \rangle \;\propto\; \frac{1}{V}\sum_i \frac{1}{\lambda_i + m} \;\propto\; \int_{-\Lambda}^{\Lambda} d\lambda \, \frac{m}{\lambda^2 + m^2} \, \rho(\lambda) \;\xrightarrow[m\to 0]{}\; \rho(0)$$

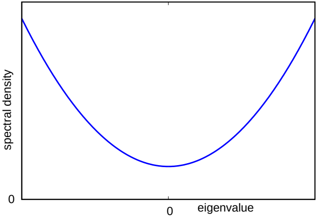
λ_i: eigenvalues of the Dirac operator, $\rho(\lambda)$: its spectral density

The finite temperature transition

Standard picture

Below T_c

- Chiral symmetry broken
- Order parameter:
  $\rho(0) \neq 0$

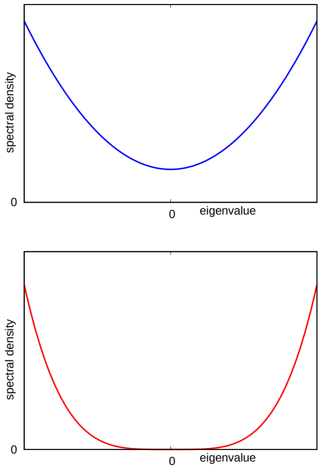


Below T_c

- Chiral symmetry broken
- Order parameter: $\rho(0) \neq 0$

Above T_c

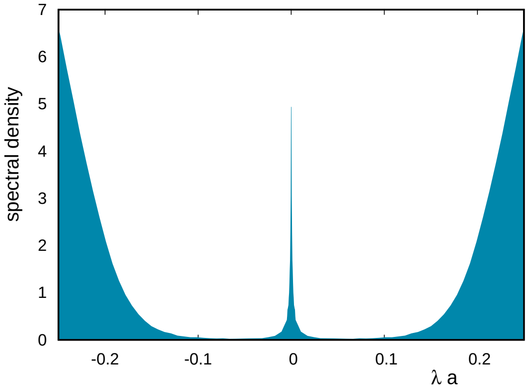
- Chiral symmetry restored
- Order parameter $\rho(0) = 0$
- (Pseudo)gap (lowest Matsubara mode)



spectral density at 0 $\Longleftrightarrow$ realization of chiral symmetry

Spectral density at $T = 1.1\,T_c$ on the lattice

quenched (quark back reaction omitted), exact zero modes not shown



$$Z = \int \mathscr{D}U \prod_f \cancel{\det\left(D[U]+m_f\right)} \cdot e^{-S_g[U]}$$

Peak at zero in the spectral density!

Edwards et al. PRD 61 (2000) 074504; Alexandru & Horvath, PRD 92 (2015); 2404.12298; Kaczmarek, Mazur, Sharma, PRD 104 (2021) 094518

Questions

- Why is there a peak at zero?
- How is it suppressed if the quark determinant is included?
- How does the peak influence chiral symmetry as $m \to 0$?

Instantons $\rightarrow$ zero eigenvalues of $D(A)$

- (Anti)instanton
  $\rightarrow$ zero eigenvalue of $D(A)$ with $(-)+$ chirality eigenmode

- High T:
  large instantons "squeezed out" in the temporal direction
  $\rightarrow$ dilute gas of instantons and anti-instantons

- Zero modes exponentially localized:

$$\psi(r) \propto e^{-\pi T r}$$

Disclaimer: when I say instanton, I mean

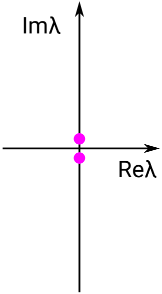
- Not instantons
- Not even calorons
- Isolated lumps of unit (± 1) topological charge
- Maybe not even that... more on that later

The Dirac operator in the subspace of zero modes

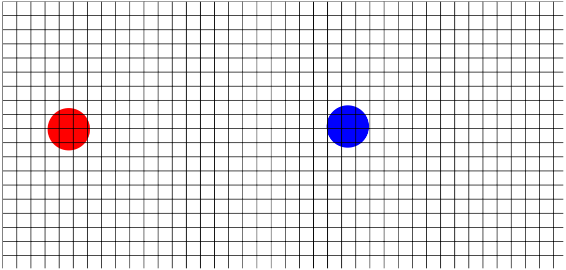
Pair of opposite charges

$$D(A) = \begin{pmatrix} 0 & iw \\ iw & 0 \end{pmatrix} \qquad w \propto e^{-\pi Tr}$$

Spectrum of $D(A)$



Positive and negative charge



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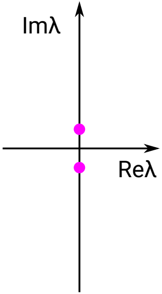

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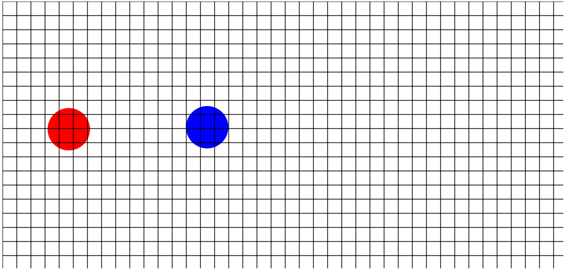

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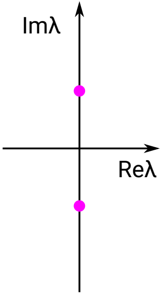

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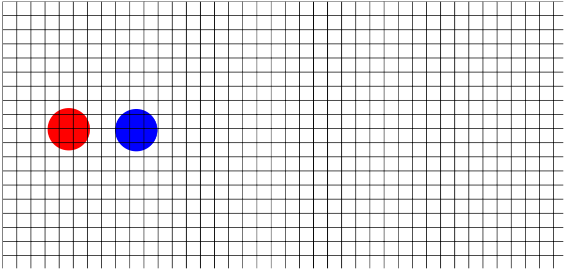

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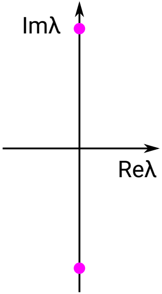

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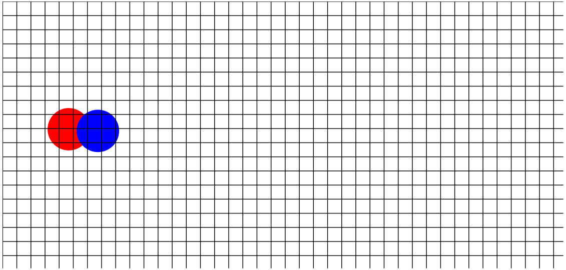
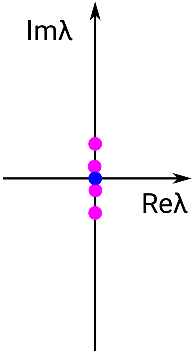


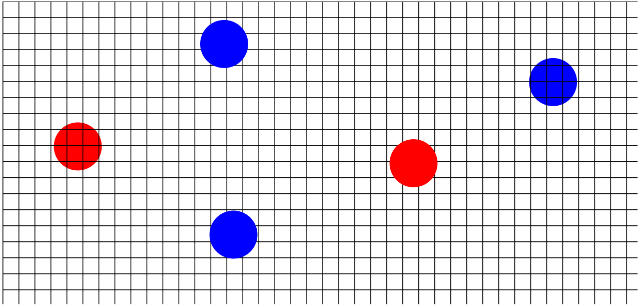
Spectrum of $D(A)$ in dilute gas of topological charges

The Dirac operator in the subspace of zero modes

Spectrum of $D(A)$



Ensemble of charges



n_i instantons n_a anti-instantons

$\rightarrow$ $|n_i - n_a|$ exact zero modes + mixing near zero modes

Dirac operator in the subspace of zero modes (ZMZ)

Work by E.V. Shuryak, J.J.M. Verbaarschot, T. Schäfer... (1990-2000)

- Given n_i instantons, n_a anti-instantons in 3d box of size L^3
- Construct $(n_i + n_a) \times (n_i + n_a)$ matrix:

$$D = \begin{pmatrix} \overbrace{0}^{n_i} & \overbrace{iW}^{n_a} \\ iW^\dagger & 0 \end{pmatrix}$$

- $w_{ij} = A \cdot \exp(-\pi T \cdot r_{ij})$ $\quad$ r_{ij} is the distance of instanton i and anti-instanton j

Random matrix model of $D(A)$ in the zero mode zone

- How to choose instanton numbers (n_i, n_a) and locations?

- Quenched lattice $T > 1.05\,T_c \quad \rightarrow \quad$ free instanton gas

  Bonati et al. PRL 110 (2013); Vig R. & TGK, PRD 103 (2021)

  - n_i and n_a independent identical Poisson-distributed

$$p(n_i, n_a) = e^{-\chi V} \cdot \frac{(\chi V/2)^{n_i}}{n_i!} \cdot \frac{(\chi V/2)^{n_a}}{n_a!}$$

    χ is the topological susceptibility

  - Locations random (uniform)

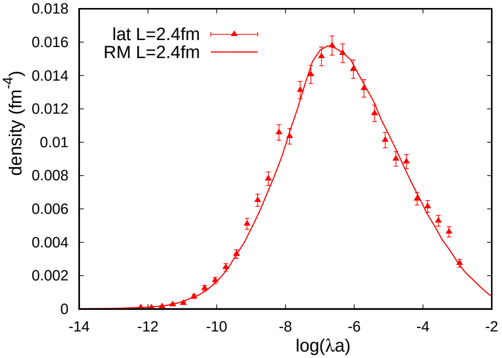
- $\rightarrow$ D(A) in quenched QCD: ensemble of random matrices

Fit parameters to quenched lattice Dirac spectrum

$T = 1.1\, T_c$ overlap Dirac spectrum

- Two parameters:
  - χ – topological susceptibility: from exact zero modes $\rightarrow\ \chi = \langle Q^2 \rangle / V$
  - A – prefactor of the exponential mixing between zero modes
- Fit A to distribution of Dirac eigenvalues (lowest eigenvalue)

$L = 2.4$fm fit



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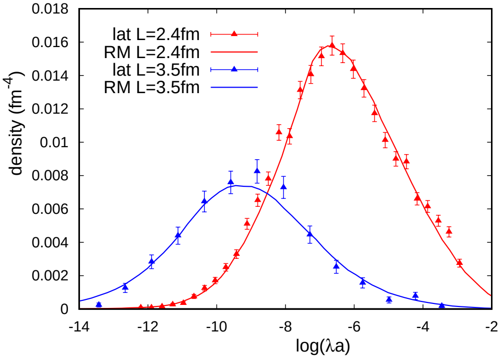
$L = 3.5$fm prediction



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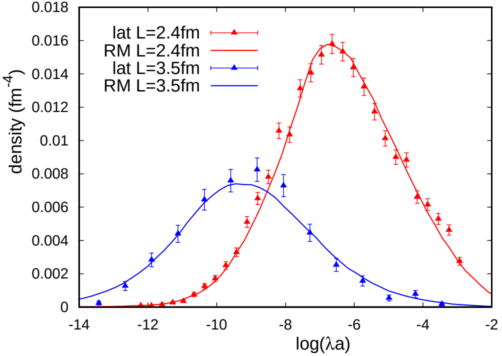
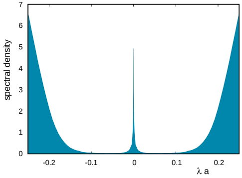


$T(T_c)$	$\langle Q^2 \rangle$	A
1.05	5.12	0.50
1.10	2.58	0.35
1.15	1.66	0.32

Random matrix model of full QCD zero mode zone

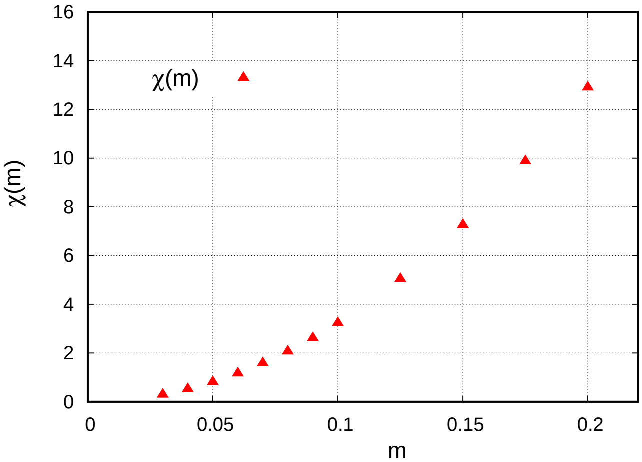
- Include $\det(D+m)^{N_\mathrm{f}}$ in Boltzmann weight
- $\det(D+m) = \prod_{\mathrm{zmz}}(\lambda_i + m) \times \prod_{\mathrm{bulk}}(\lambda_i + m)$
- Bulk weakly correlated with zero mode zone
- Approximate det with $\prod_{\mathrm{zmz}}(\lambda_i + m)$
- Consistently included in RM model:



$$P(n_\mathrm{i}, n_\mathrm{a}) = \underbrace{e^{-\chi_0 V} \frac{1}{n_\mathrm{i}!} \frac{1}{n_\mathrm{a}!} \left(\frac{\chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}}}_{\text{free instanton gas with random locations}} \times \det(D+m)^{N_\mathrm{f}}$$

Random matrix simulation: results for $N_f = 2$

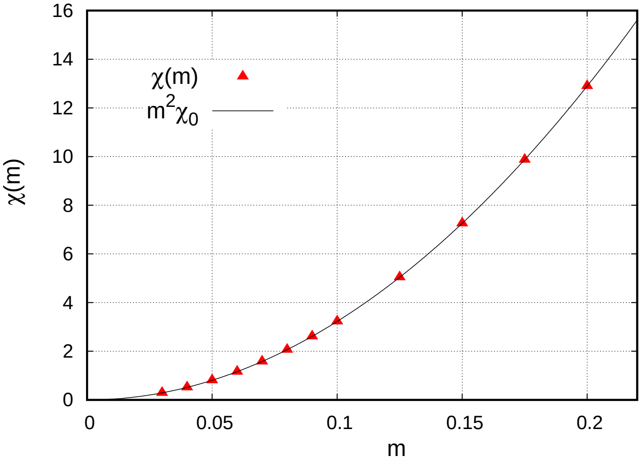
Topological susceptibility:



Random matrix simulation: results for $N_f = 2$

Topological susceptibility: $\chi(m) = m^2 \chi_0$ not a fit!

↑ quenched susceptibility



Explanation: free instanton gas

- Quark determinant for n_i instantons and n_a anti-instantons:

$$\det(D+m)^{N_f} = \prod_{n_\mathrm{i},n_\mathrm{a}} (\lambda_i + m)^{N_f} \approx m^{N_f(n_\mathrm{i}+n_\mathrm{a})}$$

$$\text{if } |\lambda_i| \ll m$$

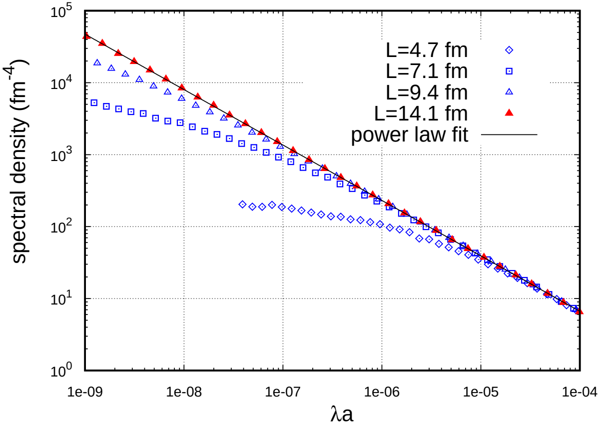
- Reweighting depends on number of topological objects, not on their type or location

$$P(n_\mathrm{i}, n_\mathrm{a}) \propto \left(\frac{\chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}} \times \det(D+m)^{N_\mathrm{f}} \approx \left(\frac{m^{N_\mathrm{f}} \chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}}$$

- Free gas, but susceptibility suppressed as $\chi_0 \to m^{N_\mathrm{f}} \chi_0$
- As $m \to 0$ instanton gas more dilute $\Rightarrow$ $|\lambda_i|$ smaller
- Even in the chiral limit $|\lambda_i| \ll m \implies$ free instanton gas

Spectral density singular at the origin for $V \to \infty$

RM model simulation, parameters from quenched $T = 1.1\,T_c$ overlap spectrum



$$\rho(\lambda) \propto \lambda^{\alpha}$$

fit: $\alpha = -0.770(5)$

Singular spectral density from similar instanton model:

Sharan and Teper, PRD 60 (1999) 054501

Banks-Casher for a singular spectral density?

“Banks-Casher” for singular spectral density

$$\langle \bar{\psi}\psi \rangle \;\propto\; \left\langle \sum_i \frac{m}{m^2+\lambda_i^2} \right\rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m} \;=\; m^{N_\mathrm{f}-1}\chi_0 V$$

$$|\lambda_i| \ll m$$

"Banks-Casher" for singular spectral density

$$\langle\bar{\psi}\psi\rangle \;\propto\; \langle\sum_i \frac{m}{m^2+\lambda_i^2}\rangle \approx \underbrace{\binom{\text{avg. number of in-}}{\text{stantons in free gas}}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m} \;=\; m^{N_\mathrm{f}-1}\chi_0 V$$

$$|\lambda_i| \ll m$$

Can $U(1)_\mathrm{A}$ breaking be seen in $\chi_\pi - \chi_\delta$?

$$\chi_\pi - \chi_\delta \;\propto\; \langle\sum_i \frac{m^2}{(m^2+\lambda_i^2)^2}\rangle \approx \underbrace{\binom{\text{avg. number of in-}}{\text{stantons in free gas}}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m^2} \;=\; m^{N_\mathrm{f}-2}\chi_0 V$$

$$\rightarrow \quad \lim_{m\to 0}(\chi_\pi - \chi_\delta) \neq 0 \qquad \text{for } N_f = 2$$

Possible connection to chiral spin symmetry?

speculations

- Chiral polarization in low Dirac eigenmodes

  Alexandru and Horvath, Nucl.Phys.B 891 (2015) 1, and other papers..

  - Spatial regions where L or R chirality dominates in low Dirac modes
  - Spectral region of polarized modes much narrower above T_c
  - With increasing temperature polarized modes rapidly disappear

- Possible connection to topological charge fluctuations

- Above T_c: polarized modes $\approx$ zero mode zone?

- Maybe L-R polarization is responsible for CSS breaking?

Related developments

- RM model @ small m $\rightarrow$ also has instanton–anti-instanton molecules

  do not contribute to $\langle\bar{\psi}\psi\rangle$ and $\chi_\pi - \chi_\delta$ in the chiral limit

- Spatial structure of topological charge $\neq$ structure of Dirac modes?
  - Caloron field power-law, zero-mode exponential
  - Nontrivial structure in topological charge above T_c Mickley, Kamleh and Leinweber, Phys.Rev.D 109 (2024) 094507

- Constraints on the Dirac spectrum from chiral symmetry restoration

  $\rightarrow$ consistent with free instanton gas M. Giordano, 2404.03546 (2024)

- Localization properties of eigenmodes in ZMZ

  M. Giordano and TGK, Universe 7 (2021); A. Alexandru and I Horvath, PRL 127 (2021), PLB 833 (2022)

Conclusions

- Above T_c, breaking of chiral symmetry controlled by ideal topologal gas

- $N_f = 2: \quad U(1)_A$ breaking remains in $\chi_\pi - \chi_\delta$ even as $m \to 0$, at any $T < \infty$

  In agreement with Cohen, PRD 54 (1996) R1867, Lee and Hatsuda, PRD 54 (1996) R1871, Evans, Hsu and Schwetz, PLB 375 (1996) 262, Birse, Cohen and McGovern, PLB 388 (1996) 137

- Dirac spectral density has singular peak at zero

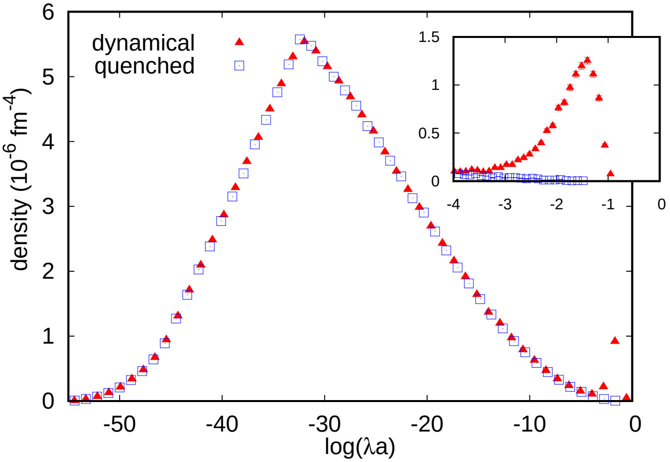
$$\rho(\lambda) \propto \lambda^{-p}$$

  - $p < 1$ (integrable!)
  - Smaller m_q or higher $T \rightarrow p$ increases (peak more singular)
  - Conjecture: if $m \to 0$ or $T \to \infty$, then $p \to 1$

BACKUP SLIDES

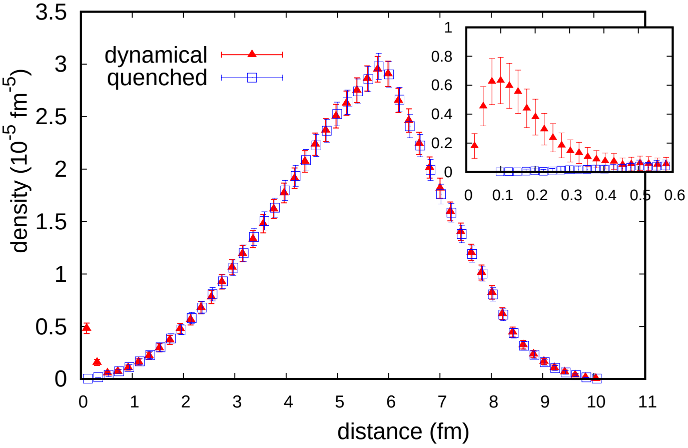
Spectral density – full QCD vs. ideal instanton gas

random matrix model, same topological susceptibility



Instanton-anti-instanton molecules

density of closest opposite charge pairs at given distance



Direct lattice simulations?

- Important to resolve small Dirac eigenvalues
  $\rightarrow$ chiral action needed JLQCD, PRD 103 (2021)

- To see spectral peak: large volume, close to T_c needed

- $\frac{\chi_\pi - \chi_\delta}{\chi_{\text{top}}} \propto m^{-2}$ instanton contribution independent of T

- Explore how far down in T free instanton gas persists
  - Compare eigenvalue statistics to prediction of free instanton gas
  - Can be done in each topological sector separately

Density of lowest eigenvalue and full spectral density

Lattice versus random matrix model, $L = 3.5$ fm