Benasque Workshop

Goldstone bosons at finite temperature

(Based on: PL, O. Philipsen, 2501.17120)

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Outline

1. Goldstone's theorem in vacuum
2. Generalisation to finite temperature
3. A thermal particle? $\rightarrow$ "Thermoparticle"
4. Thermal Goldstone bosons
5. Goldstone signatures from the lattice

1. Goldstone's theorem in vacuum

- In words: Goldstone's theorem states that the spontaneous breaking of a continuous symmetry implies the existence of massless (Goldstone) bosons

- QFT language: if j^{μ} is the conserved current associated with the symmetry, and A is some local field whose transformation under the symmetry has a non-trivial vev: $<\delta \mathrm{A}> = \lim_{\mathrm{R}\to\infty}<[Q_R, A]>\neq 0$, then:

  → *The Fourier transform of* $<[j^0(\mathrm{x}), A(\mathrm{y})]>$ *contains a* $\delta(p^2)$ *singularity*

- In fact... current conservation and field locality means that $<\delta \mathrm{A}>\neq 0$ implies the Fourier transform of $<[j^0(\mathrm{x}), A(\mathrm{y})]>$ contains a $\delta(\omega)$ component as $\boldsymbol{p}\to 0$. This is ***independent*** of the properties of the background state*

- The Goldstone "quasi-particle" $\delta(\omega)$ becomes a stable massless particle state $\delta(p^2)$ for relativistic QFTs

* See: [F. Strocchi, Symmetry Breaking, Lect. Notes Phys. 732 (2008)]

2. Generalisation to finite temperature

- For $T = 1/\beta > 0$, one defines: $<\Phi(x_1)\ldots\Phi(x_n)>_\beta = Z^{-1}\,\mathrm{Tr}\; e^{-\beta H}\,\Phi(x_1)\ldots\Phi(x_n)$
- There are some immediate implications:
  - **Lorentz invariance** ✘ $\rightarrow$ but can retain rotational invariance
  - **Spectral condition ($H > 0$)** ✘ $\rightarrow$ replaced by KMS condition
  - **Field locality (**causality**)** ✓ $\rightarrow$ this is important!
- Since current conservation and field locality are unaffected by T, the Fourier transform of $<[j^0(x), A(y)]>_\beta$ ***still*** contains a $\delta(\omega)$ as $\boldsymbol{p} \rightarrow 0$
- Can we learn anything else about the properties of thermal Goldstone modes, e.g. what happens for $\boldsymbol{p} > 0$?

Yes! The key is to determine how T modifies spectral functions $\rho_{AB}(\omega,\boldsymbol{p})$, the Fourier transform of the thermal expectation values $<[\Phi_A(x), \Phi_B(y)]>_\beta$ [Bros, Buchholz, *PRD* 58 (1998)]

2. Generalisation to finite temperature

- For (complex) scalar fields, the constraints imposed for $T > 0$ imply that the spectral function has the representation*

$$\boxed{\rho(\omega,\vec{p}) = \int_0^{\infty} ds \int \frac{d^3\vec{u}}{(2\pi)^2}\, \epsilon(\omega)\, \delta\big(\omega^2 - (\vec{p}-\vec{u})^2 - s\big)\, \widetilde{D}_{\beta}(\vec{u},s)}$$

"Thermal spectral density"

This is the $T > 0$ generalisation of the textbook *Källén-Lehmann* representation!

$$\boxed{\rho(\omega,\vec{p}) = 2\pi\epsilon(\omega)\int_0^{\infty} ds\, \delta(p^2 - s)\, \varrho(s)}$$

- $T > 0$ effects amount to understanding: $\rho(s) \rightarrow \widetilde{D}_{\beta}(\boldsymbol{u},s)$, which tells us about the possible excitations that can exist in a thermal medium

- The non-trivial $\boldsymbol{u}$ dependence of $\widetilde{D}_{\beta}(\boldsymbol{u},s)$ controls the extent to which the spectral function can be off the mass-shell $p^2 = s$

- The s dependence determines whether the spectral function has energy ω thresholds, much like in the $T = 0$ case

* See: [J. Bros and D Buchholz, *Z. Phys. C* 55 (1992), *Ann. Inst. H.Poincare Phys.Theor.* 64 (1996)]

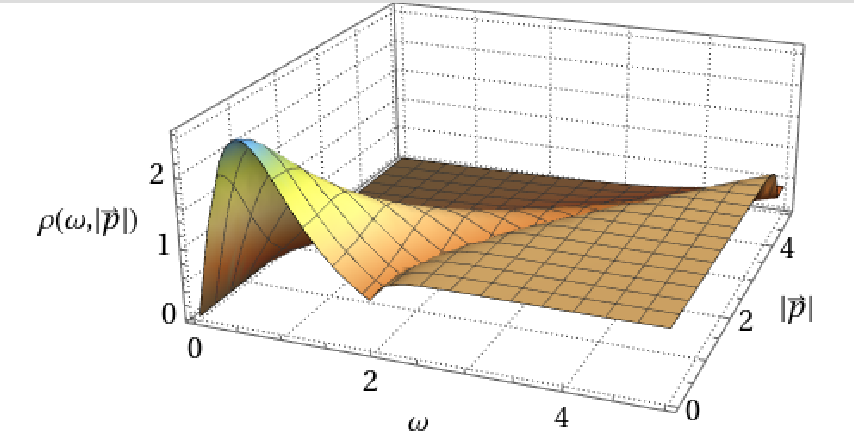
3. A thermal particle? $\rightarrow$ "Thermoparticle"

Proposition: the medium contains "Thermoparticles": particle-like excitations which differ from collective quasi-particle modes, and show up as ***discrete*** contributions to $\widetilde{D}_\beta(\boldsymbol{u},s)$ [Bros, Buchholz, *NPB* 627 (2002)]

$$\boxed{\widetilde{D}_\beta(\vec{u},s) = \widetilde{D}_{m,\beta}(\vec{u})\,\delta(s-m^2) + \widetilde{D}_{c,\beta}(\vec{u},s)}$$

$\rightarrow$ Thermoparticle components $\widetilde{D}_\beta(\boldsymbol{u})\delta(s\text{-}m^2)$ reduce to those of a vacuum particle with mass m in the limit $T \rightarrow 0$

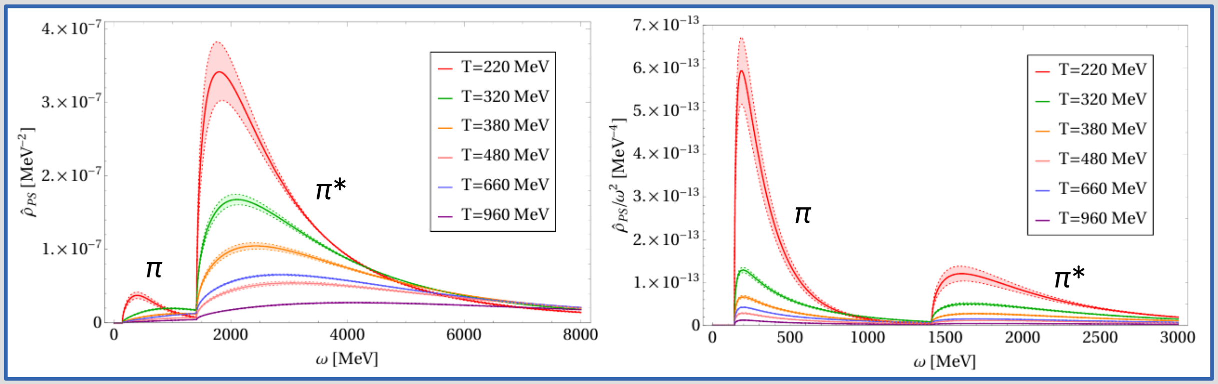
$\rightarrow$ Non-trivial "Damping factor" $\widetilde{D}_\beta(\boldsymbol{u})$ results in thermally-broadened peaks in the spectral function: this parametrises the effects of collisional broadening



$\rightarrow$ Component $\widetilde{D}_{c,\beta}(\boldsymbol{u},s)$ contains all other types of excitations, including those that are *continuous* in s

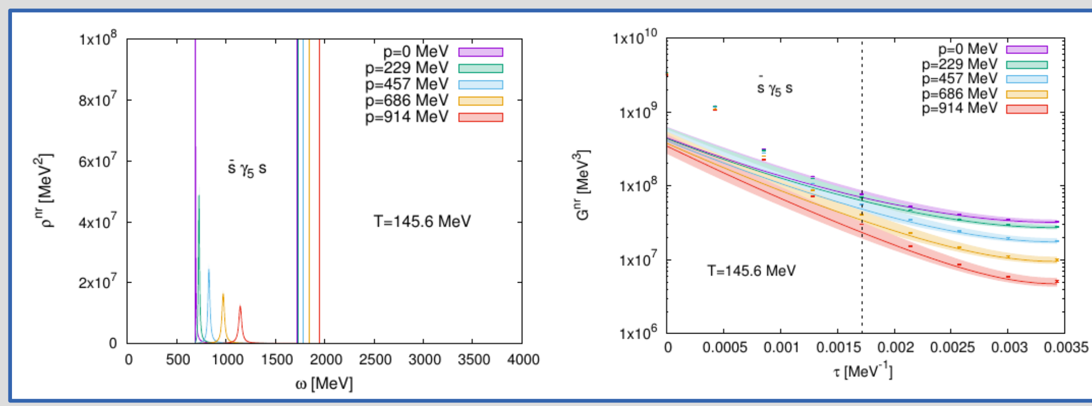
3. A thermal particle? → “Thermoparticle”

- There is mounting evidence for low-energy thermoparticle excitations, e.g. spatial correlator $C_{\text{PS}}(z)$ of the pseudo-scalar meson operator $\boxed{\mathcal{O}^a_{\text{PS}} = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi}$ in lattice QCD
- Studies extracting pseudo-scalar spectral function in various channels:



Light-light pseudo-scalar meson (pion) channel [P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

Light-strange (kaon) and strange-strange (eta) pseudo-scalar meson channels [D. Bala, O. Kaczmarek, P. L., O. Philipsen, and T. Ueding, *JHEP* 05, 332 (2024)]



Data in *all* channels consistent with a thermoparticle-type ground state: suggests light pseudo-scalar mesons (pions, kaons,..) still have a bound-state-like structure, even at high T

4. Thermal Goldstone bosons

- Using the extra information given by the $T > 0$ spectral representation, in [Bros, Buchholz, 1998] the authors were able to prove that the SSB condition $<\delta A> \neq 0$ implies that $<[j^0(x), A(0)]>_\beta$ contains a *massless* thermoparticle component, which in position space has the form:

$$\boxed{D_\beta^G(\vec{x}, s) = D_\beta^G(\vec{x})\delta(s)}$$

- This a **thermal Goldstone boson**: in the $T \to 0$ limit $D^G(\boldsymbol{x}) \to 1$, hence the current-field spectral function contains a vacuum Goldstone component $\delta(p^2)$, as expected!

- When the damping factor $D^G(\boldsymbol{x})$ is non-trivial, this causes the stable massless Goldstone peak at $p^2 = 0$ to become broadened

  $\rightarrow$ Describes the dissipative effects of the Goldstone boson moving through the thermal medium

4. Thermal Goldstone bosons

- This analysis reveals some very important characteristics:

  $\rightarrow$ The thermal Goldstone mode is never on-shell for $T > 0$

  $\rightarrow$ This component can persist at ***any*** temperature, even if the symmetry is restored when $T > T_{\mathrm{c}}$

Why? $\rightarrow$ Whether the vev v vanishes or not depends on the functional form of the damping factor $D^{G}(\boldsymbol{x})$

$D^{G}(\boldsymbol{x})$ has a rapid decay $\rightarrow$ $v = 0$, ***symmetry-restored phase***

$T = T_{\mathrm{c}}$

$D^{G}(\boldsymbol{x})$ does not decay too rapidly $\rightarrow$ $v > 0$, ***broken phase***

$T = 0$ $\quad$ $D^{G}(\boldsymbol{x}) = 1$, i.e. no dissipative effects, hence $p^2 = 0$ mode

- This captures the physics! Sufficiently strong dissipative effects destroy the long-range order and lead to symmetry restoration

4. Thermal Goldstone bosons

- If these thermal Goldstone modes are present one can look for their signatures in (Euclidean) correlation functions
- For simplicity, consider the QFT of a single complex scalar field at finite temperature, with two-point function $C(\tau,\vec{x}) = \langle \phi(\tau,\vec{x})\phi^{\dagger}(0)\rangle_{\beta}$
- If a thermal Goldstone mode is present, it follows from the thermoparticle structure and the spectral function representation that:

$$\boxed{C^{G}(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|} D^{G}_{\beta}(\vec{x})}$$

- The mode dissipation is determined by the (model-dependent) factor $D^{G}(\boldsymbol{x})$

  $\rightarrow$ For $T \rightarrow 0$ the vacuum behaviour is recovered: $\boxed{C^{G}(0,\vec{x}) \xrightarrow{T\rightarrow 0} \frac{1}{4\pi^{2}|\vec{x}|^{2}}}$

- For the spatial correlator

$$\begin{aligned} C(z) &= \int dx\, dy\, d\tau\, C(\tau,\vec{x}) \\ &= \frac{1}{2}\int_{0}^{\infty} ds \int_{|z|}^{\infty} dR\, e^{-R\sqrt{s}} D_{\beta}(R,s) \end{aligned} \quad \Longrightarrow \quad \boxed{C^{G}(z) = \frac{1}{2}\int_{|z|}^{\infty} dR\, D^{G}_{\beta}(R)}$$

See: [PL,O. Philipsen, 2022]

5. Goldstone signatures from the lattice

- Now we know what the signatures of thermal Goldstone modes are, one can look for them in lattice data!

- Consider a simple model with SSB: U(1) complex scalar field theory

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi^\dagger \partial^\mu \phi - \frac{1}{2}m^2 \phi^\dagger \phi - \frac{\lambda}{4!}(\phi^\dagger \phi)^2}$$

- In the broken phase at $T = 0$ the model contains a massless Goldstone boson and a resonance-like mode

  $\rightarrow$ Model expected to undergo a second-order phase transition: for $T > T_c$ the U(1) symmetry is restored, and $|v|^2 = <\Phi><\Phi^\dagger> = 0$

- Standard perturbative intuition comes from neglecting the p-dependence of the Goldstone self-energy* $\rightarrow$ i.e. Goldstone has purely real poles $\omega = E(\boldsymbol{p})$

- But this approximation is flawed at finite temperature...

"*Narnhofer-Requardt-Thirring Theorem*" [Commun. *Math. Phys.* 92, 247 (1983)]

* See eg. [J. I. Kapusta and C. Gale, *Finite-temperature Field Theory*]

5. Goldstone signatures from the lattice

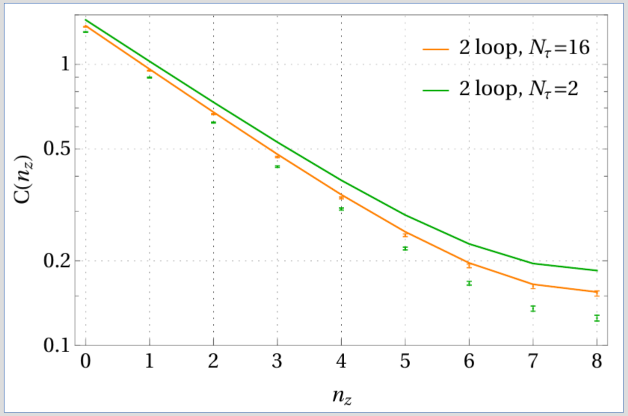
- The NRT theorem implies that states with purely real dispersion relations $\omega = E(\boldsymbol{p})$ ***cannot exist*** in interacting theories when $T > 0$

  QFT reason: *thermal states satisfy the KMS condition, and this gives rise to very different spectral constraints than in the vacuum case*

  Physics reason: *Dissipative effects of the thermal medium are everywhere-present* $\rightarrow$ *always need to take these into account* (*i.e. always a width!*)

- This has significant implications for perturbation theory: *neither free field, nor quasi-particle propagators with real poles can form the basis of finite-temperature perturbative expansions* [Landsman, *Ann. Phys.* 186, 141 (1988)]

- There is both analytic [Weldon, *PRD* 65 (2002)] and now numerical [PL, O. Philipsen, JHEP 08, (2024)] evidence for this perturbative breakdown



5. Goldstone signatures from the lattice

- Investigate U(1) theory on a $L_\tau \times L^3$ lattice ($L_\tau = aN_\tau$, $L = aN_s$) with action

$$S = a^4 \sum_{x \in \Lambda_a} \left[\sum_\mu \left(\frac{1}{2} \Delta^f_\mu \phi^*(x) \Delta^f_\mu \phi(x) \right) + \frac{m_0^2}{2} \phi^*(x)\phi(x) + \frac{g_0}{4!} \left(\phi^*(x)\phi(x) \right)^2 \right]$$

- Avoid potential triviality of the model by keeping lattice spacing a fixed throughout, hence $T = (aN_\tau)^{-1}$ is varied in discrete steps

- Require a sufficiently fine and large lattice to ensure that the lattice temperature covers both the symmetry-broken and restored phases for large and small values of N_τ, respectively

- In the broken symmetry phase at $T=0$ the vev $|v|$ sets the scale of the system, and separates long $|\boldsymbol{x}||v| > 1$ and short $|\boldsymbol{x}||v| < 1$ distances

- To reduce cutoff effects one requires that $\Lambda/|v|$ is large

$\rightarrow$ Choose lattice parameter set (am_0, g_0) which satisfies these conditions!

5. Goldstone signatures from the lattice

- SSB does not occur in a finite spatial volume $V = L^3$, i.e. there is no notion of a "vev" on the lattice

  $\rightarrow$ One needs to perform an $L \rightarrow \infty$ extrapolation of lattice results!

- For $T = 0$ one has the general property: $\boxed{\lim_{L\to\infty} C_L(\tau, \vec{x}) \xrightarrow{|\vec{x}|\to\infty} |v|^2}$
- Based on this property there are different approaches* for extracting $|v|^2$
- Given periodic spatial boundary conditions one can use: $\boxed{|v|^2 = \lim_{L\to\infty} C_L(0, |\vec{x}| = L/2)}$

  $\rightarrow$ But need a choice of parametrisation for $C_L(0, z = |\boldsymbol{x}|)$

- For $T = 0$ the correlator will be dominated by the massless Goldstone, so this should also be true if the lattice is sufficiently cold, i.e. large N_τ

  $\rightarrow$ $C_L(0,z)$ fits and extrapolation ansatz need to take this into account!

* See eg. [H. Neuberger, *PRL* 60,(1988).]

5. Goldstone signatures from the lattice

Results from the coldest lattice (N_τ=32):

- To test the hypothesis that the system is in the vacuum-like broken phase we therefore fit the functional form:

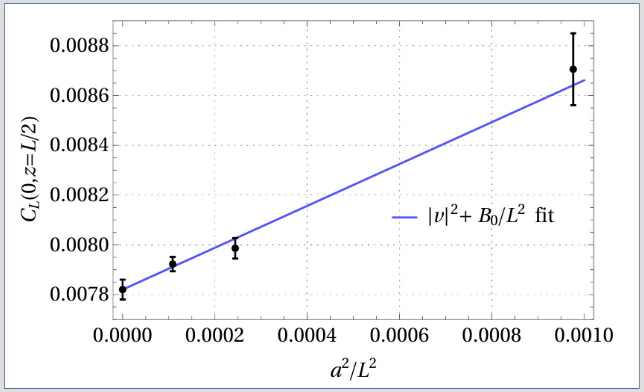
$$\boxed{C_L(0,z) = c_0 + b_0\left[\tfrac{1}{z^2} + \{z \to (L-z)\}\right]}$$

Non-zero if in broken phase

Massless mode for finite *L*

- Functional form provides very good description of the data for each of the volumes considered ($N_s = 32,\ 64,\ 96$)

$\rightarrow$ To extrapolate vev perform fits of $C_L(0,z=L/2)$ using $|v|^2 + B_0/L^2$



$$\boxed{a^2|v|^2 = 0.00782(4)}$$

$T/|\mathrm{v}| \sim 0.35$, **hence cold!**

5. Goldstone signatures from the lattice

Results from the hottest lattice (N_τ=2):

- In this case: $T/|\mathrm{v}| \sim 6$, which indicates the system is hot and potentially in the symmetry-restored phase (for $L\to\infty$)
- As outlined previously, the Goldstone mode can still persist in this regime, and would have the structure of a massless thermoparticle
- What is the structure of the damping factor $D^G(\boldsymbol{x})$? Use *spatial* correlator:

$$\boxed{C_L(z) = d_L\left[e^{-m_L z} + \{z \to (L-z)\}\right]} \longrightarrow \boxed{D_\beta^G(\vec{x}) = \alpha\, e^{-\gamma|\vec{x}|}}$$

- Spatial correlator fits provide excellent description of data over full range $[0, L/2]$ for each of the (large) volumes considered ($N_s = 64, 96, 128$)
- This strongly indicates: (i) In symmetry-restored phase for $L\to\infty$

  (ii) Goldstone damping factor is pure exponential

5. Goldstone signatures from the lattice

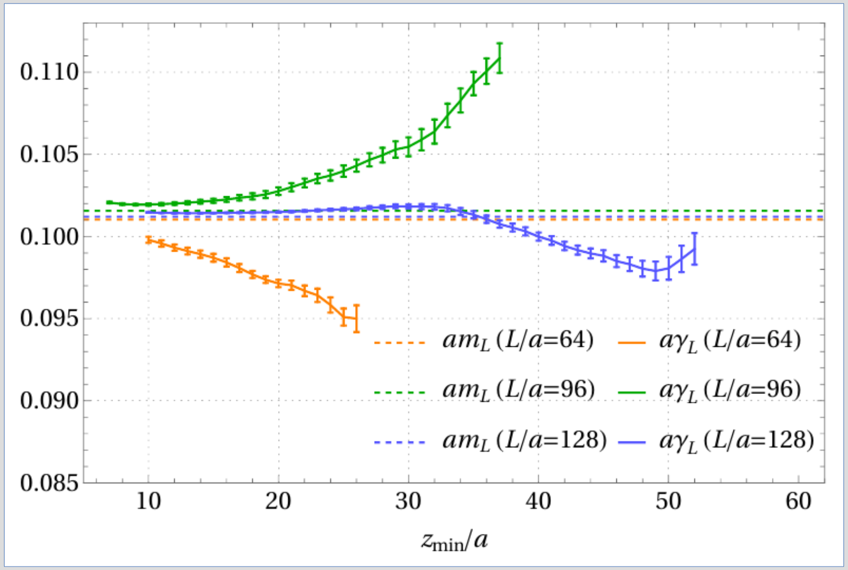
Results from the hottest lattice (N_τ=2):

- In this case, Goldstone two-point function has the form: $$C^G(0,\vec{x}) = \frac{\coth\left(\frac{\pi|\vec{x}|}{\beta}\right)}{4\pi\beta|\vec{x}|} \alpha\, e^{-\gamma|\vec{x}|}$$

- To test the consistency of this conclusion one fits:

$$C_L(0,z) = b_L\left[\frac{\coth\left(\frac{\pi z}{\beta}\right)}{z} e^{-\gamma_L z} + \{z \to (L-z)\}\right]$$

- Find that the functional form describes data increasingly better for larger volumes, and changes to the parametrisation lead to significant fit instabilities



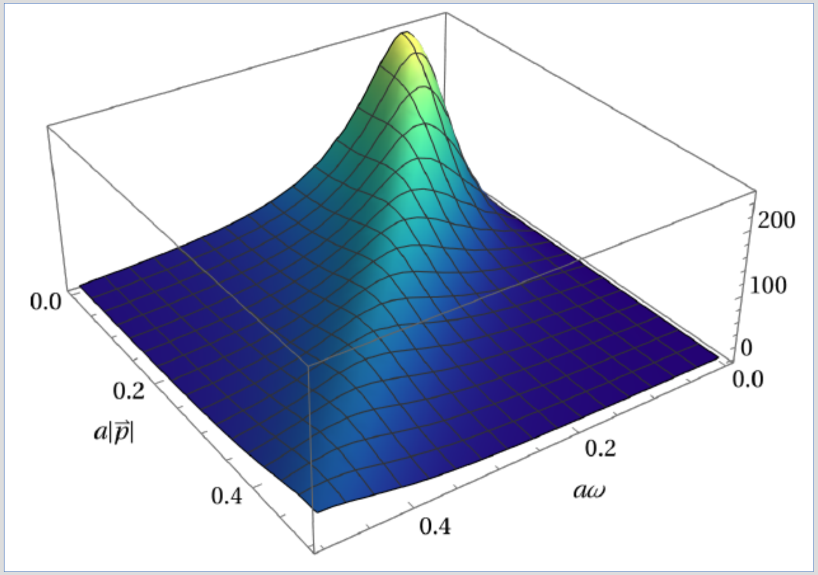
- Non-trivial test: $$\lim_{L\to\infty} \gamma_L = \lim_{L\to\infty} m_L = \gamma$$

5. Goldstone signatures from the lattice

Results from the hottest lattice (N_τ=2):

- Data strongly indicates that the correlator is dominated by massless thermoparticle component $\longrightarrow$ a thermal Goldstone boson!
- One can use the extracted damping factor $D^G(\boldsymbol{x}) = \alpha\, e^{-\gamma|\boldsymbol{x}|}$ to compute the corresponding spectral function $\rho_G(\omega,\boldsymbol{p})$ of the Goldstone mode

$$\boxed{\rho_G(\omega,\vec{p}) = \frac{4\alpha\,\omega\gamma}{(\omega^2-|\vec{p}|^2-\gamma^2)^2+4\omega^2\gamma^2}}$$



- Spectral properties are *very* different to the vacuum case: $\rho_G(\omega,\boldsymbol{p}) \sim \delta(p^2)$

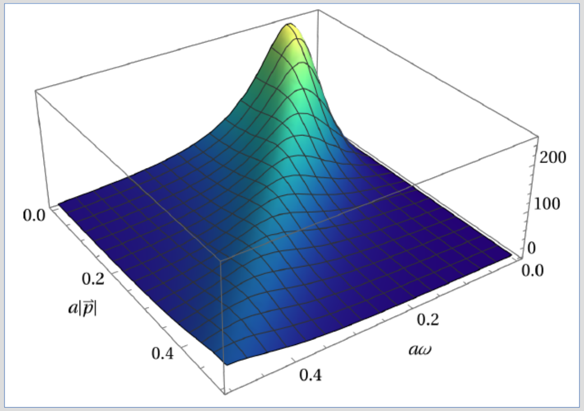
  $\longrightarrow$ Broadened peak structure around the $T = 0$ singularity $p^2 = 0$

- $\rho_G(\omega,\boldsymbol{p})$ very similar to vacuum resonance state, except the width $\gamma = \gamma(T)$ arises from collisional processes with the medium, ***not*** mixing effects

Summary & outlook

- Goldstone's theorem in vacuum has well-known consequences, but at finite temperature there still remain open questions
- One can use the non-perturbative constraints imposed by *causality* to gain new insights $\longrightarrow$ $\rho(\omega, \boldsymbol{p})$ have spectral representations
- This narrows down the potential excitations that can exist in a thermal medium $\longrightarrow$ particle-like excitations: "Thermoparticles"
- SSB for $T > 0$ implies existence of Goldstone modes which have the structure of *massless* thermoparticles $\longrightarrow$ these modes can persist at ***any*** temperature, even if the symmetry is restored
- We find evidence for the existence of such modes in the U(1) complex scalar field theory on the lattice

  $\longrightarrow$ Suggests pions in QCD could remain important degrees of freedom in chiral limit for $T > T_c$



Backup: Rigorous definition of SSB

- In order to define SSB rigorously one needs to define a regularised charged operator Q_R, and this only converges for $R \to \infty$ within commutator $[Q_R, A]$

$$\boxed{Q_R \doteq \int d^4x\, f(x_0) g(\boldsymbol{x}/R)\, j_0(x)}$$

(i) $g(\boldsymbol{x})=1$ for $|\boldsymbol{x}| \leq 1$, $g(\boldsymbol{x})=0$ for $|\boldsymbol{x}| \geq 1+\varepsilon$

(ii) $f_d(x_0)$ has compact support, and $f_d(x_0) \to \delta(x_0)$ for $d \to 0$

- The condition: $\lim_{R\to\infty} <[Q_R, A]>=q$ is then ***always*** well-defined, and $q=0$ is a necessary and sufficient condition for the existence of a unitary charge operator Q, defined by: $<u, Qv> := \lim_{R\to\infty} <u, Q_R v>$

$\rightarrow$ If this is non-vanishing for *any* A then no such charge exists, i.e. SSB!

- The condition for SSB reduces to: $\boxed{\int_0^\infty ds \int d^3\vec{u}\, \widetilde{D}_\beta^{(+)}(\vec{u}, s) \tilde{f}(\sqrt{|\vec{u}|^2 + s}) = iq(2\pi)^{3/2} \tilde{f}(0)}$

- In the vacuum case this implies: $\widetilde{D}^{(+)}(\boldsymbol{u},s) \to iq\, (2\pi)^{3/2}\, \delta(\boldsymbol{u})\delta(s)$

- Whether q vanishes or not is ***determined*** by functional form of $\widetilde{D}^{(+)}(\boldsymbol{u},s)$

Backup: Perturbation theory

- How might one resolve inconsistencies of $T > 0$ perturbation theory?
- In the vacuum theory we know that the basis of perturbation theory is the Gell-Mann/Low formula [Landsman, *Ann. Phys.* 186, 141 (1988)]:

$$\boxed{iG(x_1 \cdots x_n) = \sum_{i_1 \cdots i_n} Z_{i_1}^{1/2} \cdots Z_{i_n}^{1/2} \frac{\langle \Omega |\, T[\Phi_{i_1}(x_1) \cdots \Phi_{i_n}(x_n)\, U(\infty, -\infty)]\, | \Omega \rangle}{\langle \Omega |\, U(\infty, -\infty)\, | \Omega \rangle}.}$$

$$\boxed{\begin{aligned} U(t_1, t_2) &= e^{iH_0[\Phi] t_1} e^{-iH[\Phi](t_1 - t_2)} e^{-iH_0[\Phi] t_2} \\ &= T \exp\left(-i \int_{t_2}^{t_1} dt\, H_I(t)\right), \end{aligned}}$$

- Correlation functions of *interacting* fields can be computed from correlators of *free* fields $\rightarrow$ This is *derived* from the fact that at large times the interactions between fields diminish: the asymptotic fields/states are free!
- The standard perturbative series is defined by expanding the exponential in the evolution operator as a series in the coupling parameter $\rightarrow$ each term in the expansion is determined by the propagators of the asymptotic fields

Key point: free field propagators form the basis of perturbation theory at $T = 0$ because the large-time states experience no interactions

Backup: Thermoparticles in QCD data

Goal: Extract information about the finite T spectral function $\rho_\Gamma(\omega,\boldsymbol{p})$ from data of *Euclidean* correlator $C_\Gamma(\tau,\vec{x}) = \langle O_\Gamma(\tau,\vec{x})\, O_\Gamma(0,\vec{0})\rangle_T$ O_Γ = scalar operator

- Standard approach: extract $\rho_\Gamma(\omega,\boldsymbol{p})$ from temporal correlator $\widetilde{C}_\Gamma(\tau,\boldsymbol{p})$

$$\boxed{\widetilde{C}_\Gamma(\tau,\vec{p}) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh\left[\left(\frac{\beta}{2} - |\tau|\right)\omega\right]}{\sinh\left(\frac{\beta}{2}\omega\right)} \rho_\Gamma(\omega,\vec{p})}$$

→ *Problem is ill-conditioned, need more information!*

- Instead, one can use the ***spatial*** correlator, where one integrates $C_\Gamma(\tau,\boldsymbol{x})$ over $\{\tau,x,y\}$ and fixes a spatial direction z

$$\boxed{C_\Gamma(z) = \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} e^{ip_z z} \int_0^\infty \frac{d\omega}{\pi\omega}\, \rho_\Gamma(\omega, p_x = p_y = 0, p_z)}$$

- It turns out that *if* thermoparticles exist, then they will give a distinct contribution to $C(z)$

$$\boxed{C(z) \approx \frac{1}{2}\int_{|z|}^{\infty} dR\, e^{-mR} D_{m,\beta}(R)}$$

[P.L., *PRD* 106 (2022); P.L., O. Philipsen, *JHEP* 10, 161 (2022)]

→ This component can be extracted *directly* from data

Backup: Thermoparticle characteristics

Given a specific QFT, what form should the damping factors take?

Idea: thermal scattering states are defined by imposing an asymptotic field condition [Bros, Buchholz, (2002)]:

Asymptotic fields Φ_0 are assumed to satisfy dynamical equations, but only at large x_0

In Φ^4 theory

$$(\partial^2 + m^2)\phi_0(x) + \frac{\lambda}{3!}\phi_0^3(x) \xrightarrow{|x_0|\to\infty} 0$$

- Since thermoparticles dominate the large-time behaviour of correlators, they are natural candidates for describing such states. It turns out that their damping factors $\widetilde{D}_{m,\beta}(\boldsymbol{u})$ are **uniquely fixed** by the asymptotic condition
- In Φ^4 theory one finds (where κ is a thermal width):

For $g < 0$:

$$\widetilde{D}^{(-)}_{m,\beta}(\vec{u}) = \frac{2\pi^2}{\kappa^2}\delta(|\vec{u}| - \kappa),$$

$$\widetilde{G}^{(-)}_{\beta}(k_0,\vec{p}) = \frac{1}{4|\vec{p}|\kappa}\ln\left[\frac{-k_0^2 + m^2 + (|\vec{p}| + \kappa)^2}{-k_0^2 + m^2 + (|\vec{p}| - \kappa)^2}\right]$$

For $g > 0$:

$$\widetilde{D}^{(+)}_{m,\beta}(\vec{u}) = \frac{4\pi}{\kappa_0\left(|\vec{u}|^2 + \kappa^2\right)},$$

$$\widetilde{G}^{(+)}_{\beta}(k_0,\vec{p}) = \frac{i}{2|\vec{p}|\kappa_0}\ln\left[\frac{\sqrt{-k_0^2 + m^2} - i|\vec{p}| + \kappa}{\sqrt{-k_0^2 + m^2} + i|\vec{p}| + \kappa}\right]$$

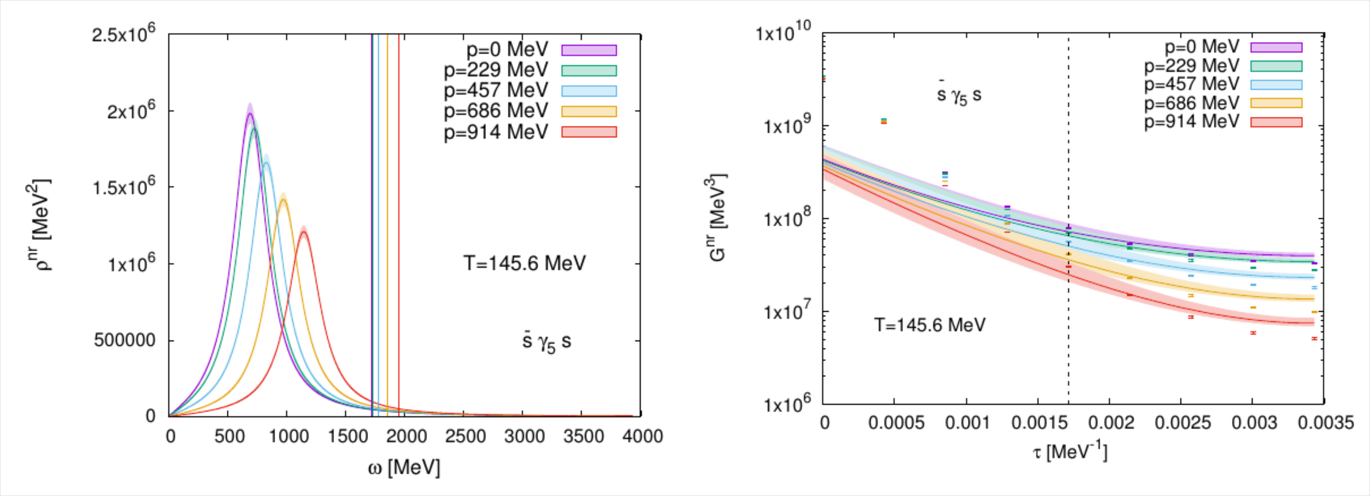
Backup: Robustness of spectral approach

- The robustness of the thermoparticle hypothesis can also be tested by comparing with different causal models, e.g. a Breit Wigner

$$\boxed{\rho_{\mathrm{BW}}(\omega,\vec{p}) = \frac{4\omega\Gamma}{(\omega^2 - |\vec{p}|^2 - m^2 - \Gamma^2)^2 + 4\omega^2\Gamma^2},}$$

$$\boxed{C_{\mathrm{BW}}(z) = \frac{e^{-\sqrt{m^2+\Gamma^2}|z|}}{2\sqrt{m^2+\Gamma^2}}.}$$

- Same procedure as with the thermoparticle case: (i) extract the width parameter Γ and coefficient from the spatial lattice data (ii) use this to predict the corresponding temporal correlator



→ Data is *not* consistent with a Breit-Wigner-type ground state!

Backup: Vacuum spectral representations

- The standard textbook derivation (e.g. Peskin & Schroeder) depends on inserting a complete set of states $\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}(\lambda)} |\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|$

$\rightarrow$ *But this assumes one knows what the spectrum is!*

Instead: use QFT constraints: (i) field locality, (ii) Lorentz invariance, (iii) spectral condition

Field locality $\Rightarrow$

$$\boxed{\rho(\omega,\vec{p}) = \int_0^\infty ds \int d^4u \; \epsilon(\omega - u_0)\, \delta\big((\omega-u_0)^2 - (\vec{p}-\vec{u})^2 - s\big)\, \Psi(u,s)}$$

$[\Phi(x),\Phi(y)]=0$ for $(x\text{-}y)^2 < 0$

"Jost-Lehmann-Dyson (JLD) representation"

[R. Jost, H. Lehmann Nuovo Cim. 5, 1957; F.J. Dyson, Phys. Rev. 110, 1958]

Lorentz invariance $\Rightarrow$

$$\boxed{\rho(\omega,\vec{p}) = 2\pi\epsilon(\omega)\int_0^\infty ds\, \delta\big(p^2 - s\big)\, \varrho(s)}$$

$\leftarrow$ *KL representation for* $\rho(\omega,\mathbf{p})$

- Note: the splitting $\rho(\omega,\vec{p}) = \widetilde{\mathcal{W}}(\omega,\vec{p}) - \widetilde{\mathcal{W}}(-\omega,-\vec{p})$ does *not* uniquely relate the (p-space) two-point function to the spectral function $\rho(\omega,\mathbf{p})$
- But... if we impose the **spectral condition** $\Rightarrow$ $\boxed{\widetilde{\mathcal{W}}(\omega,\vec{p}) = \theta(\omega)\rho(\omega,\vec{p})}$

When T>0

$$\boxed{\widetilde{\mathcal{W}}(\omega,\vec{p}) = \frac{\rho(\omega,\vec{p})}{1-e^{-\beta\omega}}}$$