Neutrino physics (theory) II

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(IFIC Valencia)

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Lecture 1

Introduction: neutrinos and their history

Neutrinos in the Standard Model

Neutrino oscillations

Lecture 2

Values of neutrino masses

Introduction to neutrino cosmology

Cosmological bounds on some neutrino properties

Values of neutrino masses

3-neutrino oscillations: the global picture

neutrino mixing

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

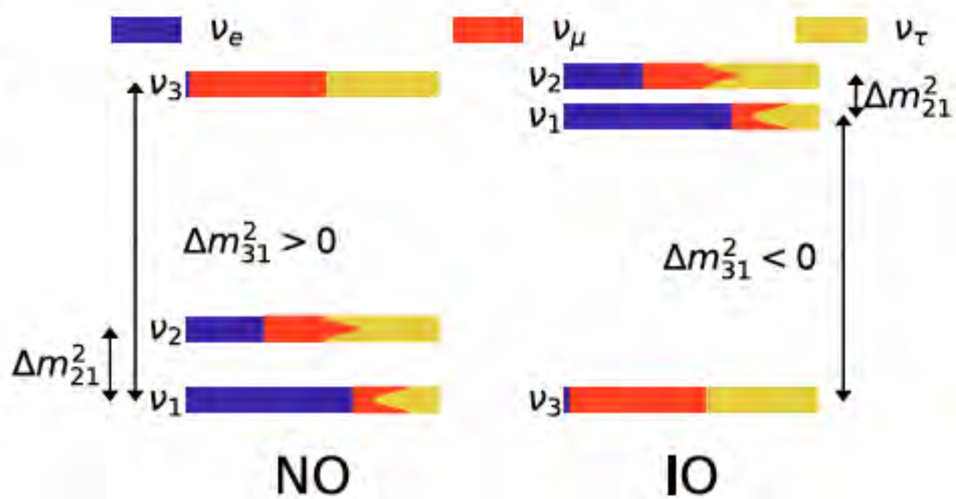
- ✓ 3 mixing angles: θ_{12}, θ_{23}, θ_{13}
- ✓ 3 CP phases: 1 Dirac + 2 Majorana
- ✓ 3 masses: m_1, m_2, m_3

⇒ absolute neutrino mass: m_0

⇒ two mass splittings:

$$\Delta m^2_{21}, \Delta m^2_{31}$$

neutrino mass spectrum



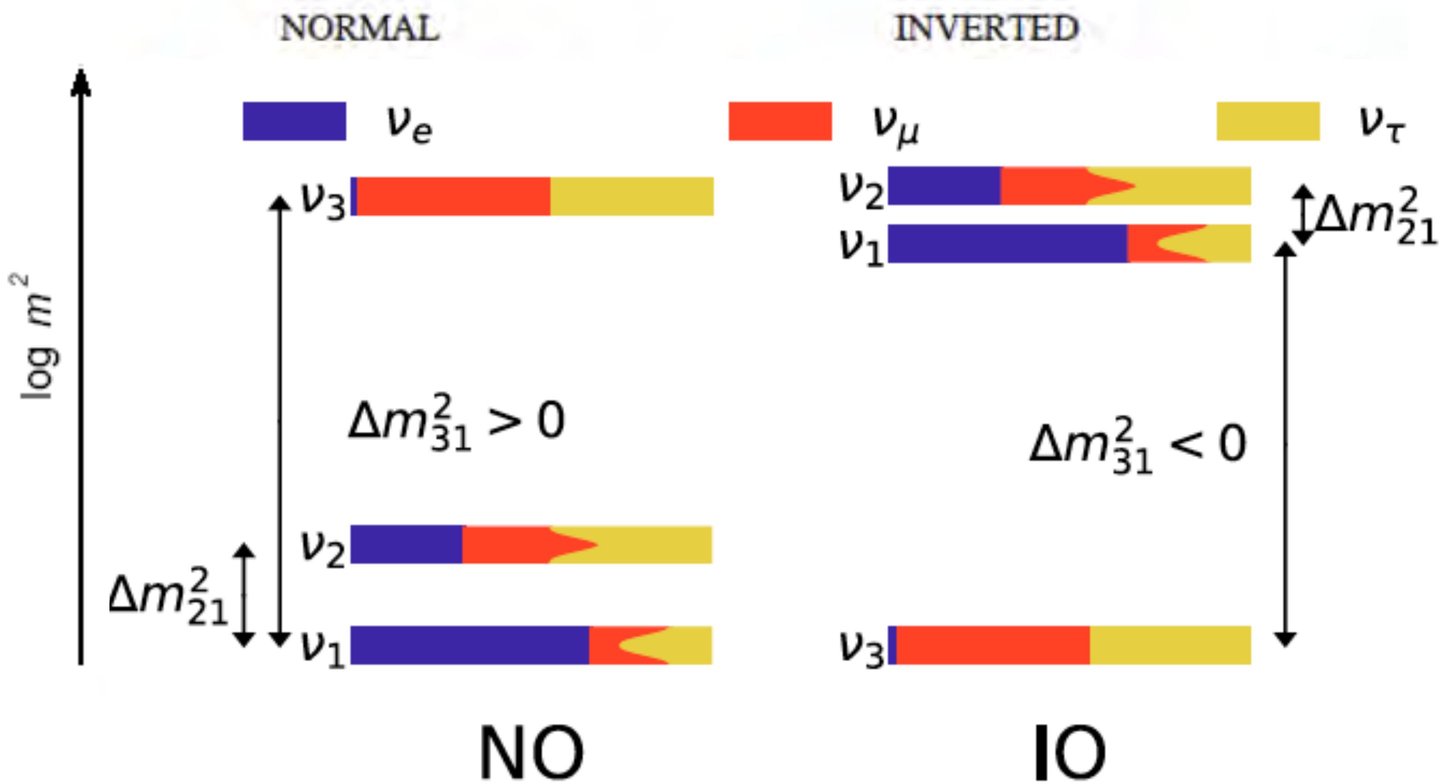
Global fit to neutrino oscillation parameters

parameter	best fit $\pm 1\sigma$	3σ range	relative 1σ uncert	
Δm^2_{21} $[10^{-5}\text{eV}^2]$	$7.55^{+0.22}_{-0.20}$	6.98–8.19	2.7 %	
$\|\Delta m^2_{31}\|$ $[10^{-3}\text{eV}^2]$ (NO)	$2.51^{+0.02}_{-0.03}$	2.43–2.58	1.0 %	mass ordering?
$\|\Delta m^2_{31}\|$ $[10^{-3}\text{eV}^2]$ (IO)	$2.41^{+0.03}_{-0.02}$	2.34-2.49		
$\sin^2\theta_{12}/10^{-1}$	$3.04^{\pm}0.16$	2.57–3.55	5.4%	
$\sin^2\theta_{23}/10^{-1}$ (NO)	$5.64^{+0.15}_{-0.21}$	4.23–6.04	3-4%	octant?
$\sin^2\theta_{23}/10^{-1}$ (IO)	$5.64^{+0.15}_{-0.18}$	4.27–6.03		
$\sin^2\theta_{13}/10^{-2}$ (NO)	$2.20^{+0.05}_{-0.06}$	2.03–2.38	2.6%	
$\sin^2\theta_{13}/10^{-2}$ (IO)	$2.20^{+0.07}_{-0.04}$	2.04–2.38		
δ/π (NO)	$1.12^{+0.16}_{-0.12}$	0.76–2.00	10-15%	maximal CP violation??
δ/π (IO)	$1.50^{+0.13}_{-0.14}$	1.11–1.87		

Valencia global fit (https://globalfit.astroparticles.es)

Two possible neutrino mass orderings

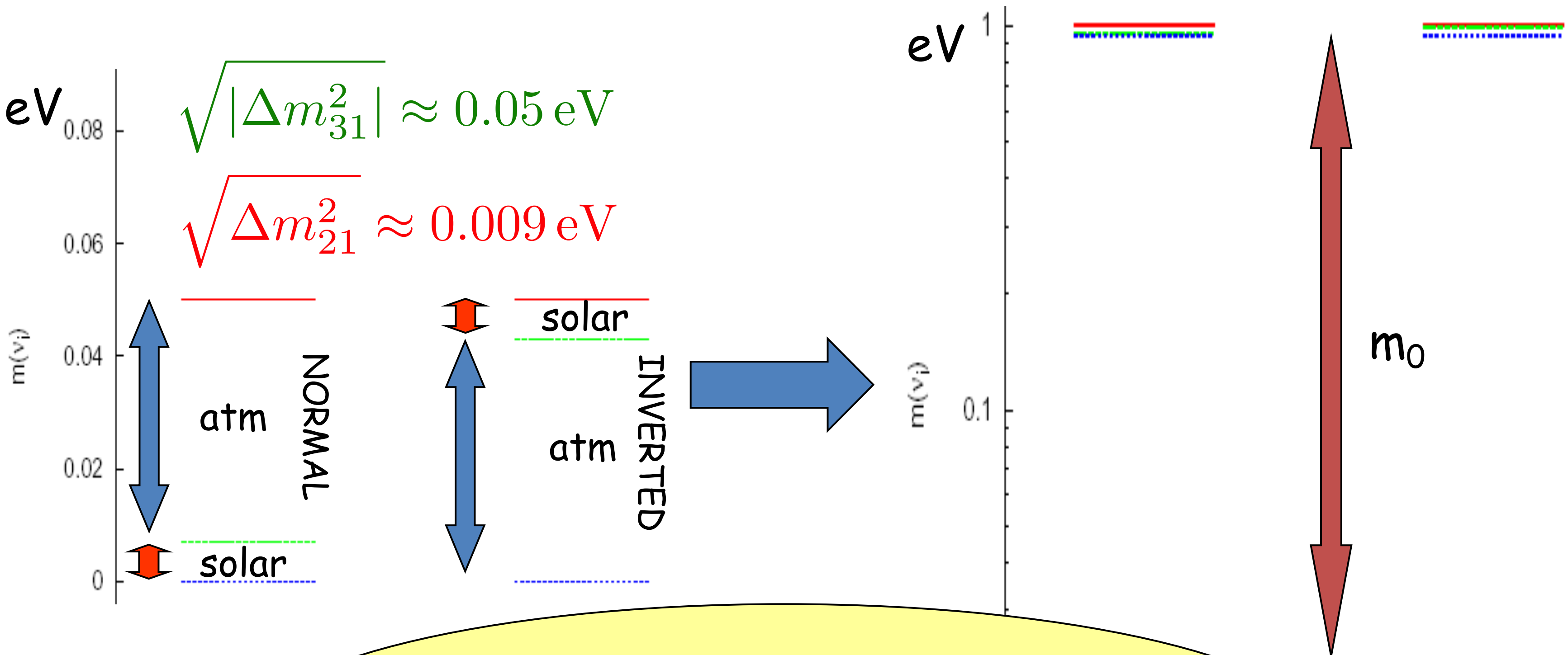
absolute neutrino mass scale?



$\Delta m^2_{31} > 0$ $\Delta m^2_{31} < 0$

Neutrino masses

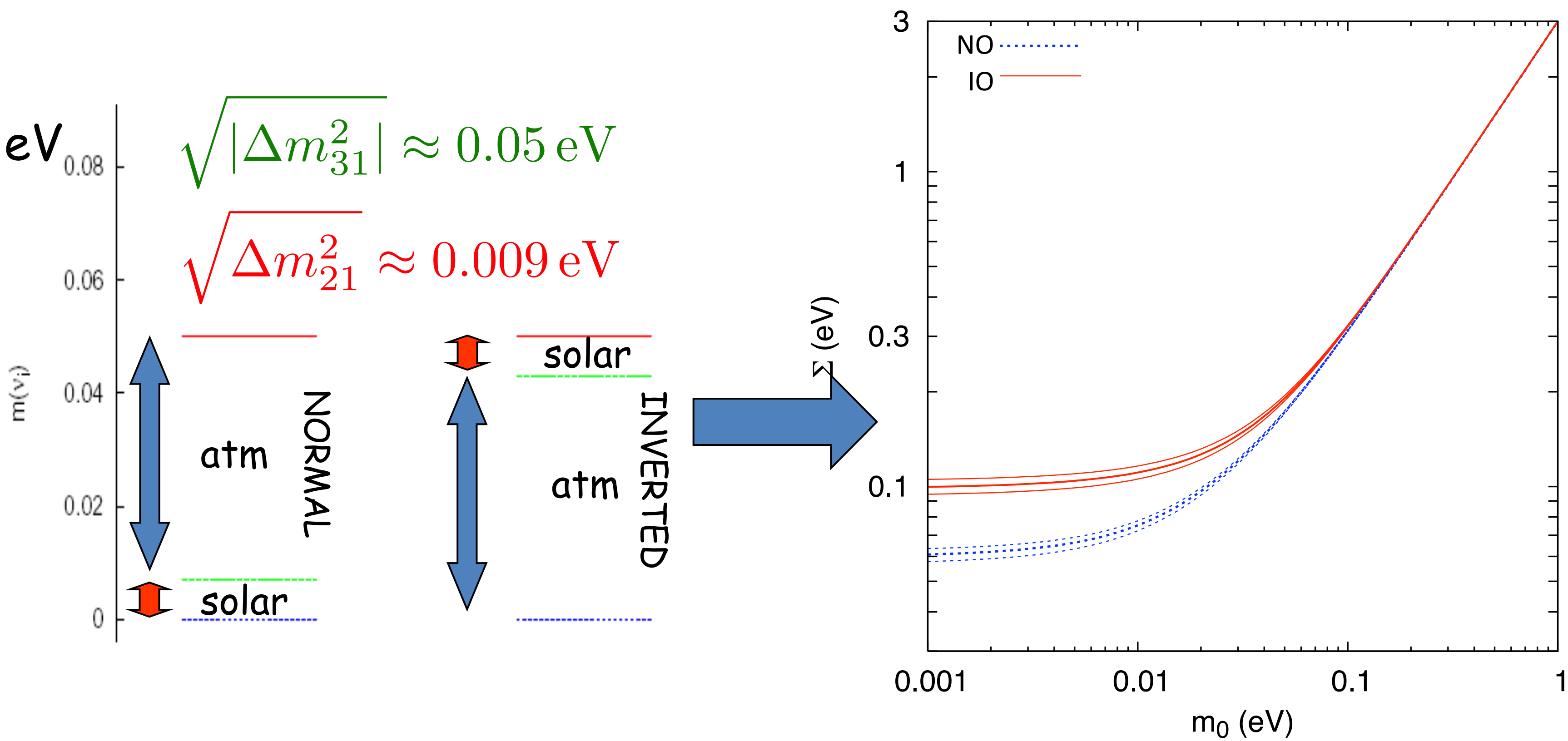
Data on flavour oscillations do not fix the absolute scale of neutrino masses



What is the value of m_0 ?

Neutrino masses

Data on flavour oscillations do not fix the absolute scale of neutrino masses



$$0.05(0.09)\ \mathrm{eV} \lesssim \sum_i m_i = m_1 + m_2 + m_3$$

oscillations

Probing the absolute neutrino mass scale

neutrino oscillations

$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

$$m_0 \geq 0.05\,\mathrm{eV}$$

cosmology

$$\sum_i m_i$$

β decay kinematics

$$m_\beta = \left(\sum_i |U_{ei}|^2 m_i^2 \right)^{1/2}$$

$$[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2]^{1/2}$$

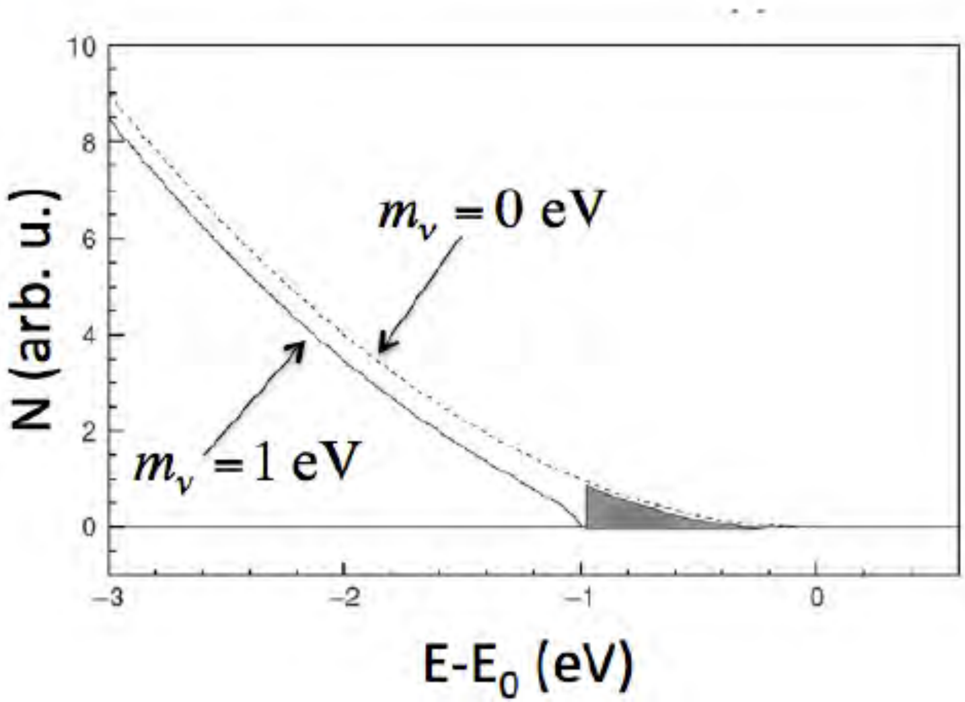
neutrinoless ββ decay

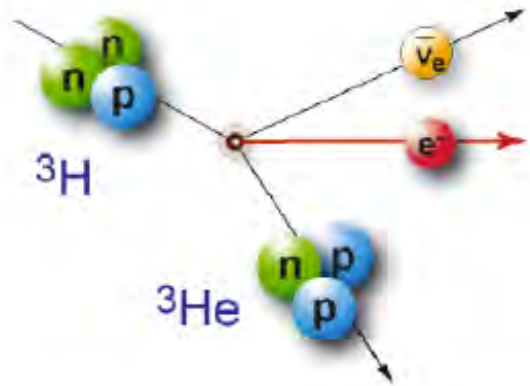
$$m_{\beta\beta} = \left| \sum_i U_{ei}^2 m_i \right|$$

$$|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}|$$

Tritium β decay experiments

- β-decay spectrum close to the endpoint is very sensitive to the neutrino mass:





$$m_\beta^2 = \sum |U_{ei}|^2 m_i^2$$

KATRIN (2024): $m_\beta < 0.45\,\mathrm{eV}\ (90\%\,\mathrm{CL})$

Neutrinoless double decay experiments

$2\nu\beta\beta$: rare process in the SM with $t_{1/2} \sim 10^{21}$ years

$0\nu\beta\beta$: possible for massive Majorana neutrinos.

$(A,Z) \rightarrow (A,Z+2) + e^- + e^-$ **test ν nature**

→ not observed yet

→ $t_{1/2} > 10^{26}$-10^{27} years

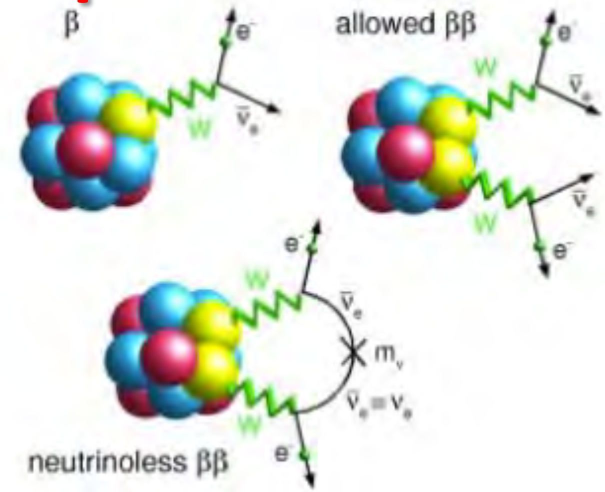
→ violates Lepton Number

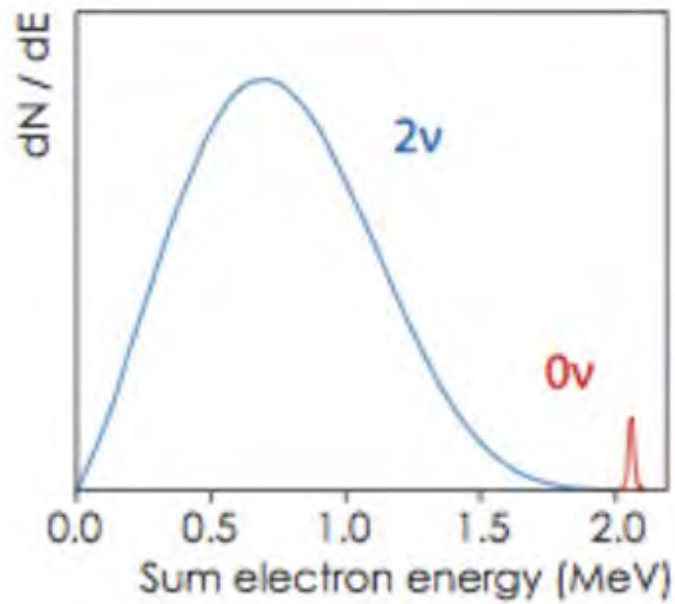
phase space — Nuclear matrix elements

$$\Gamma_{0\nu\beta\beta} = G^{0\nu} |M^{0\nu}|^2 \langle m_{\beta\beta} \rangle^2$$

Effective Majorana neutrino mass

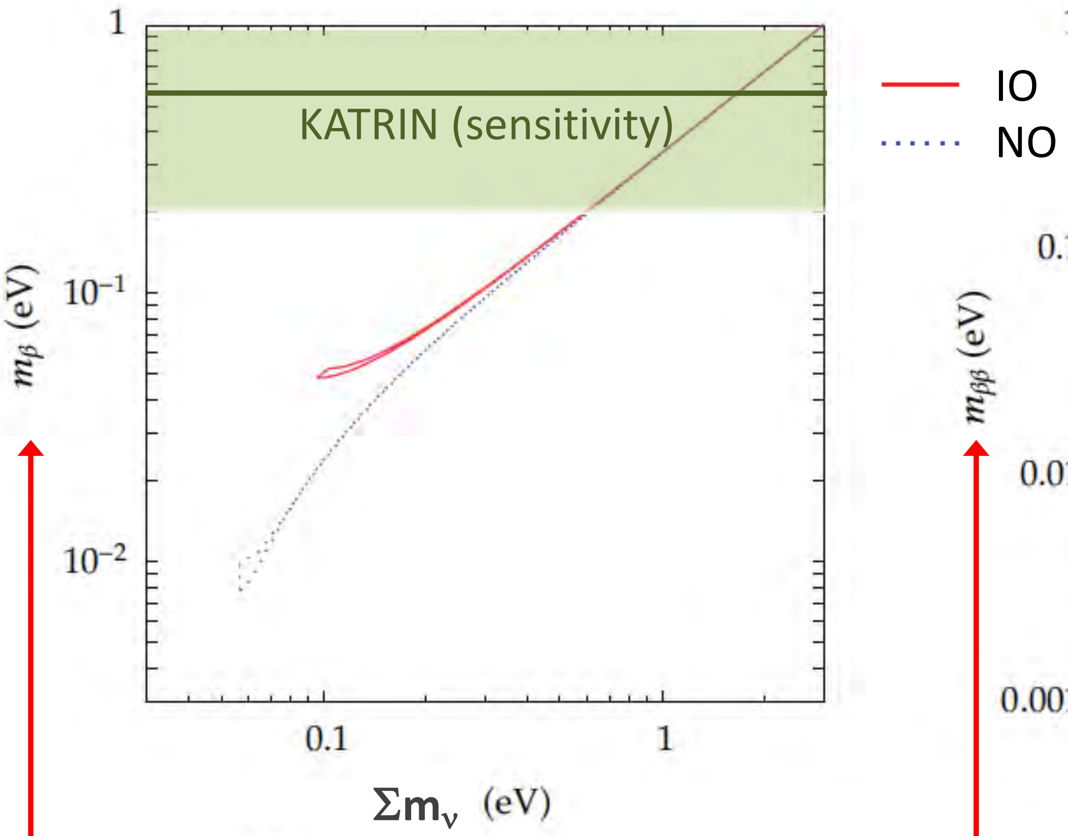
$$\langle m_{\beta\beta} \rangle = \left| \sum_{i=1}^{3} U_{ei}^2 m_i \right|$$

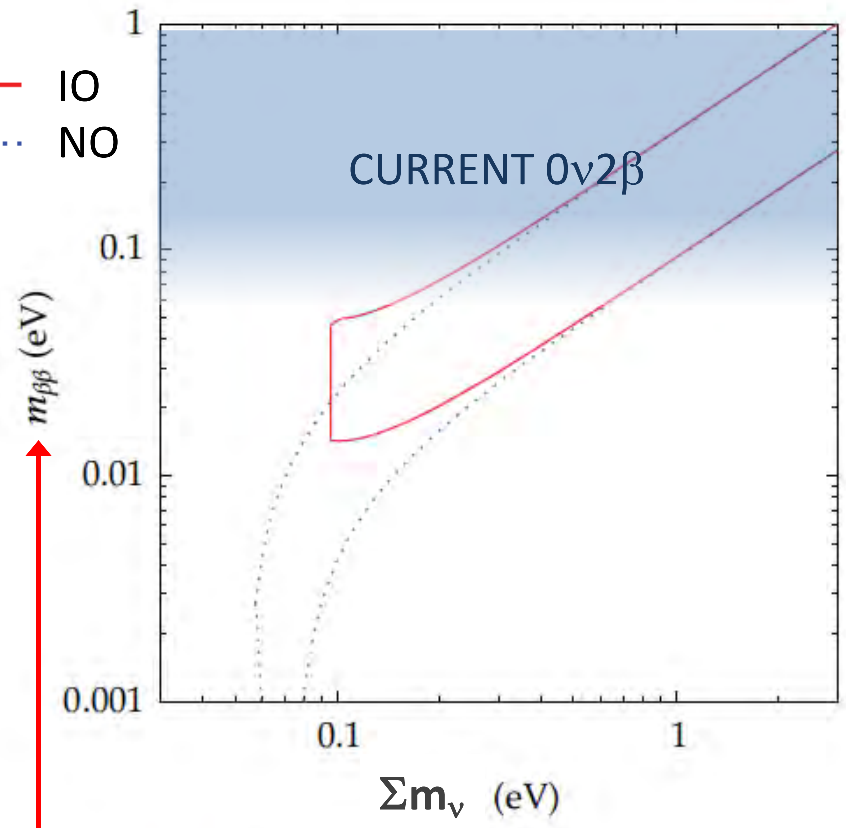




$$m_{\beta\beta} < 36 - 156\,\mathrm{meV}$$

Tritium β decay, 0ν2β and Cosmology



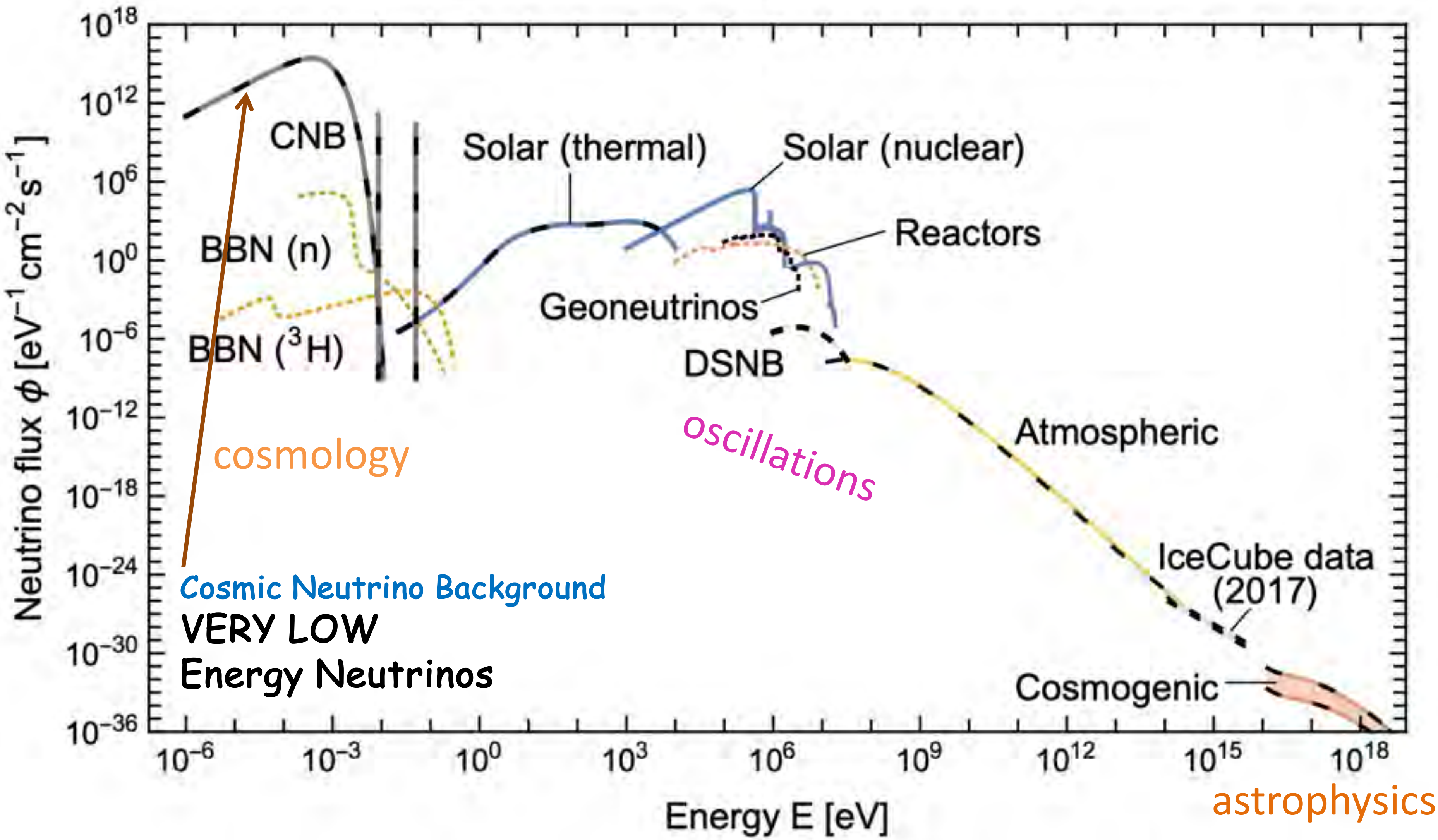


$$\left[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2\right]^{1/2}$$

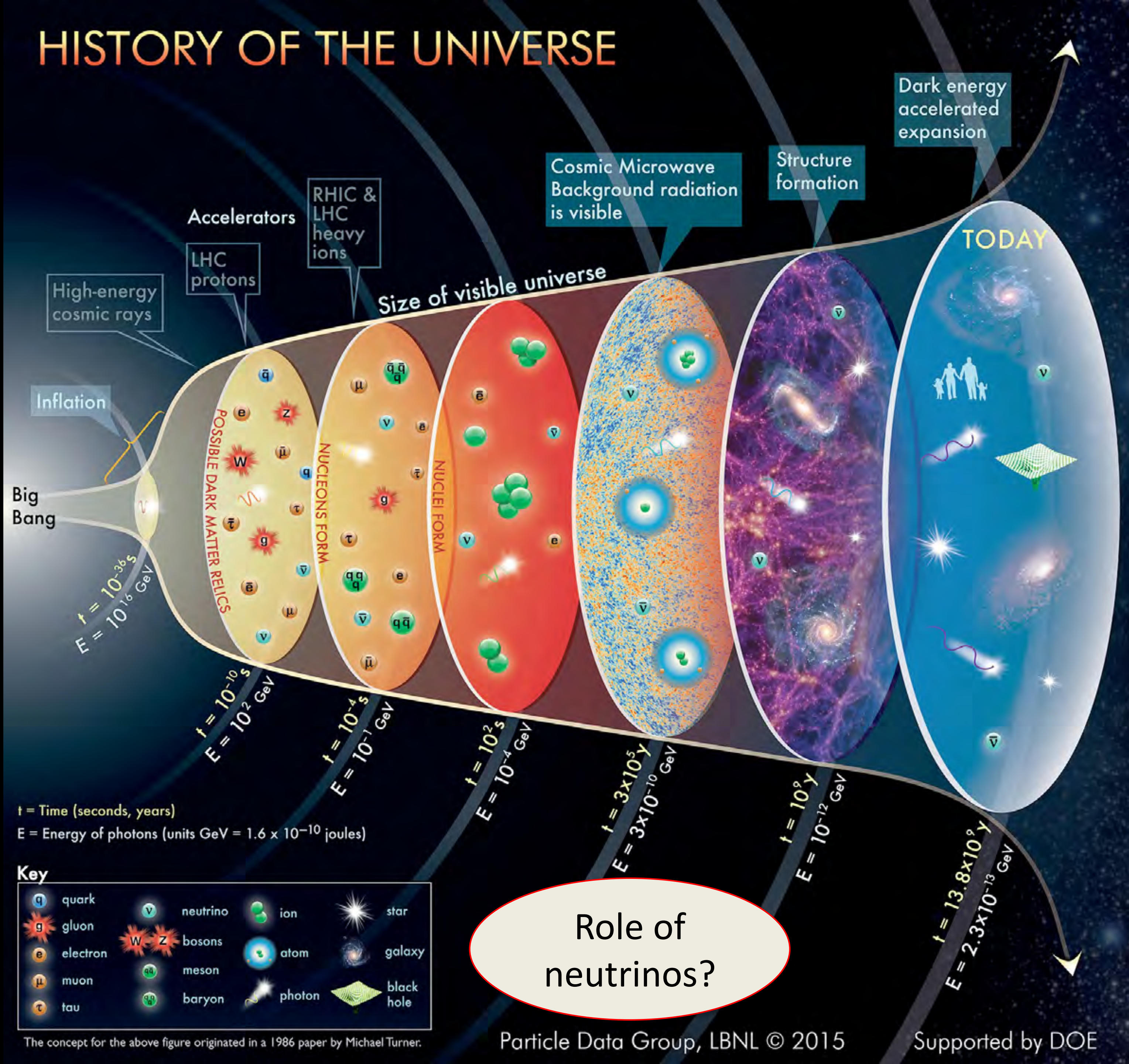
$$\left|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}\right|$$

Introduction to neutrino cosmology

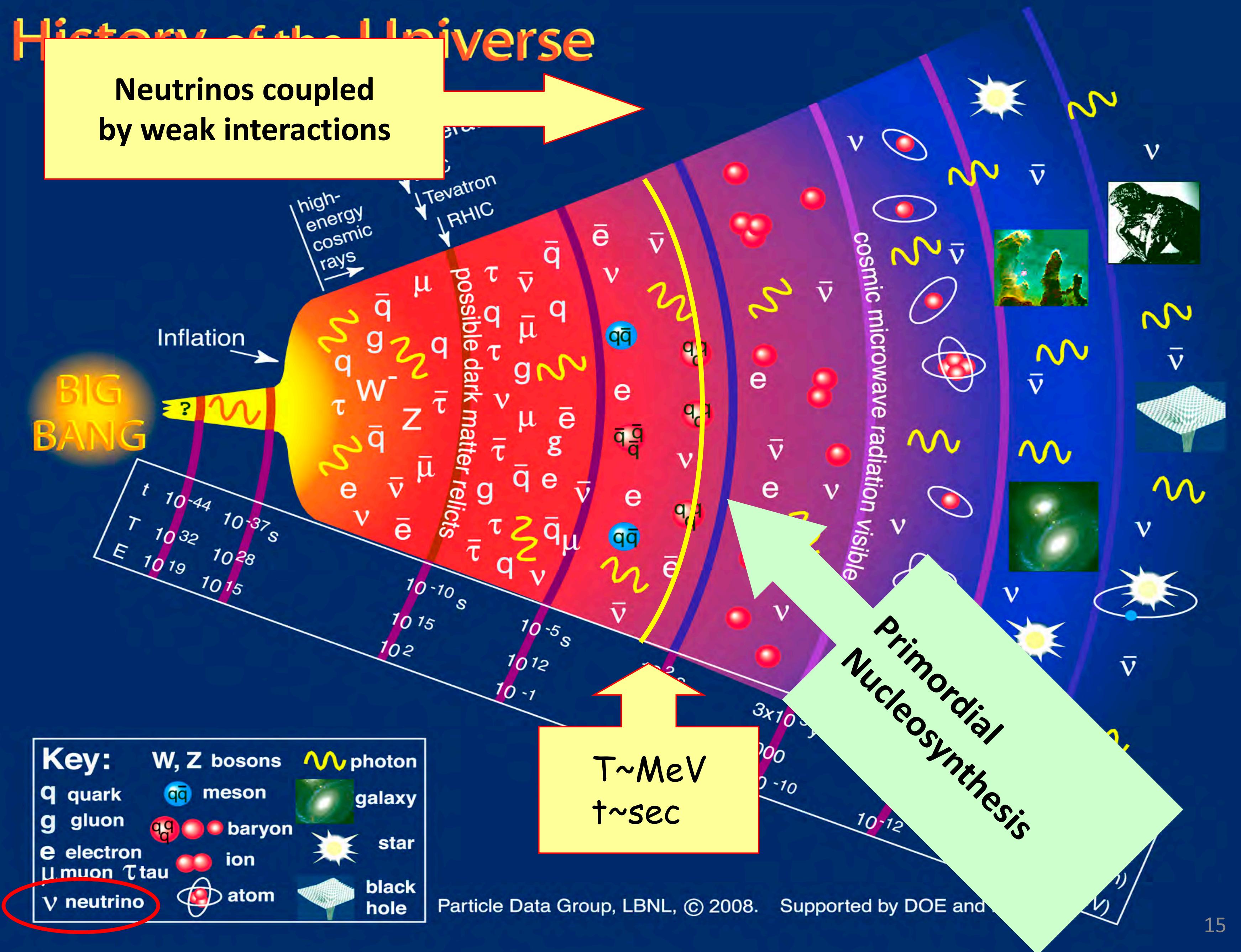
Grand Unified Neutrino Spectrum at Earth



Vitagliano, Tamborra, and Raffelt, Rev Mod Phys **92** (2020) 45006



Role of neutrinos?

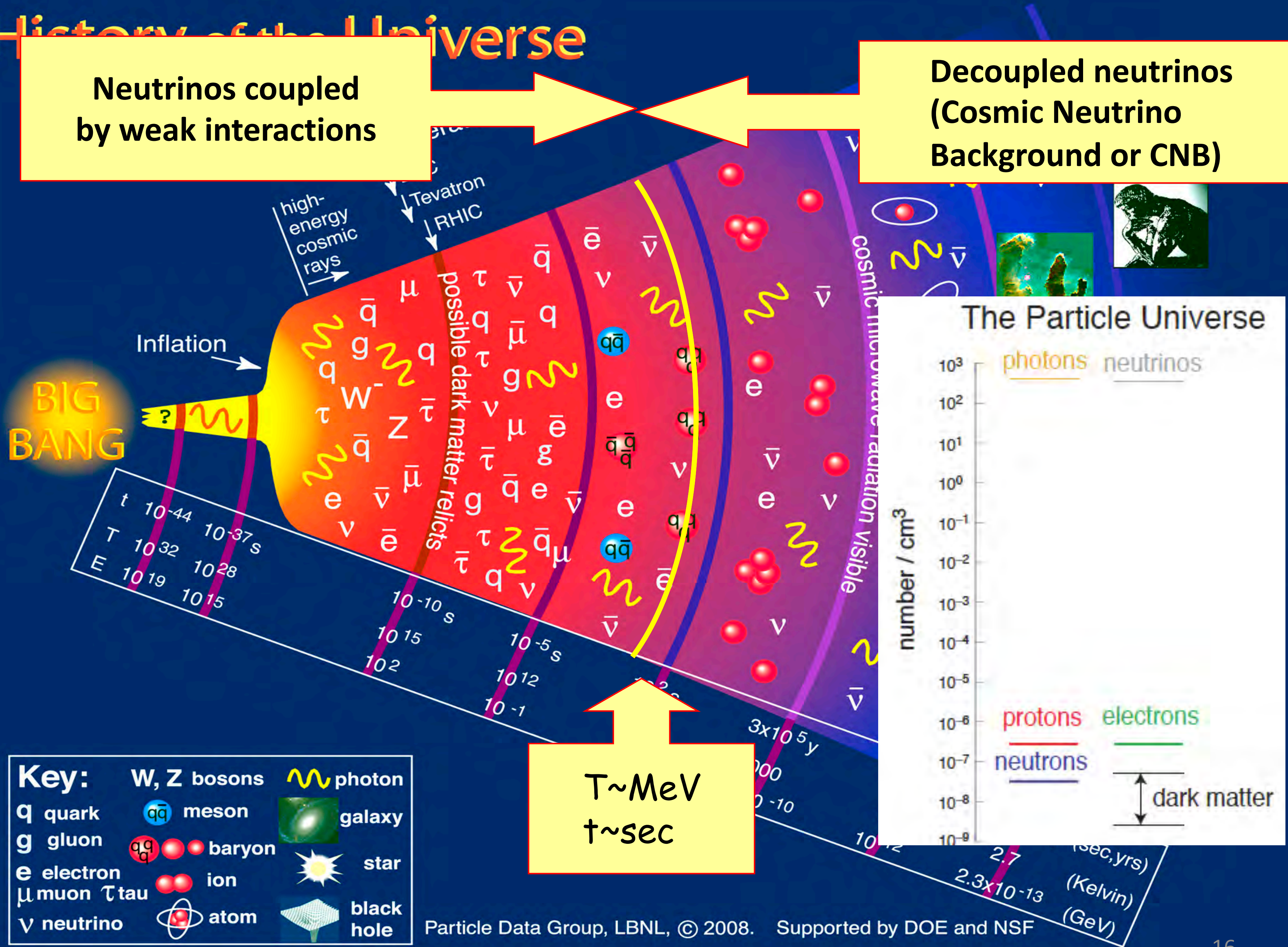

History of the Universe
Neutrinos coupled
by weak interactions
Tevatron
RHIC
high-energy cosmic rays
Inflation
BIG BANG
possible dark matter relicts
cosmic microwave radiation visible
T~MeV
t~sec
Primordial Nucleosynthesis
Key:
q quark
g gluon
e electron
μ muon τ tau
ν neutrino
W, Z bosons
meson
baryon
ion
atom
photon
galaxy
star
black hole
Particle Data Group, LBNL, © 2008. Supported by DOE and

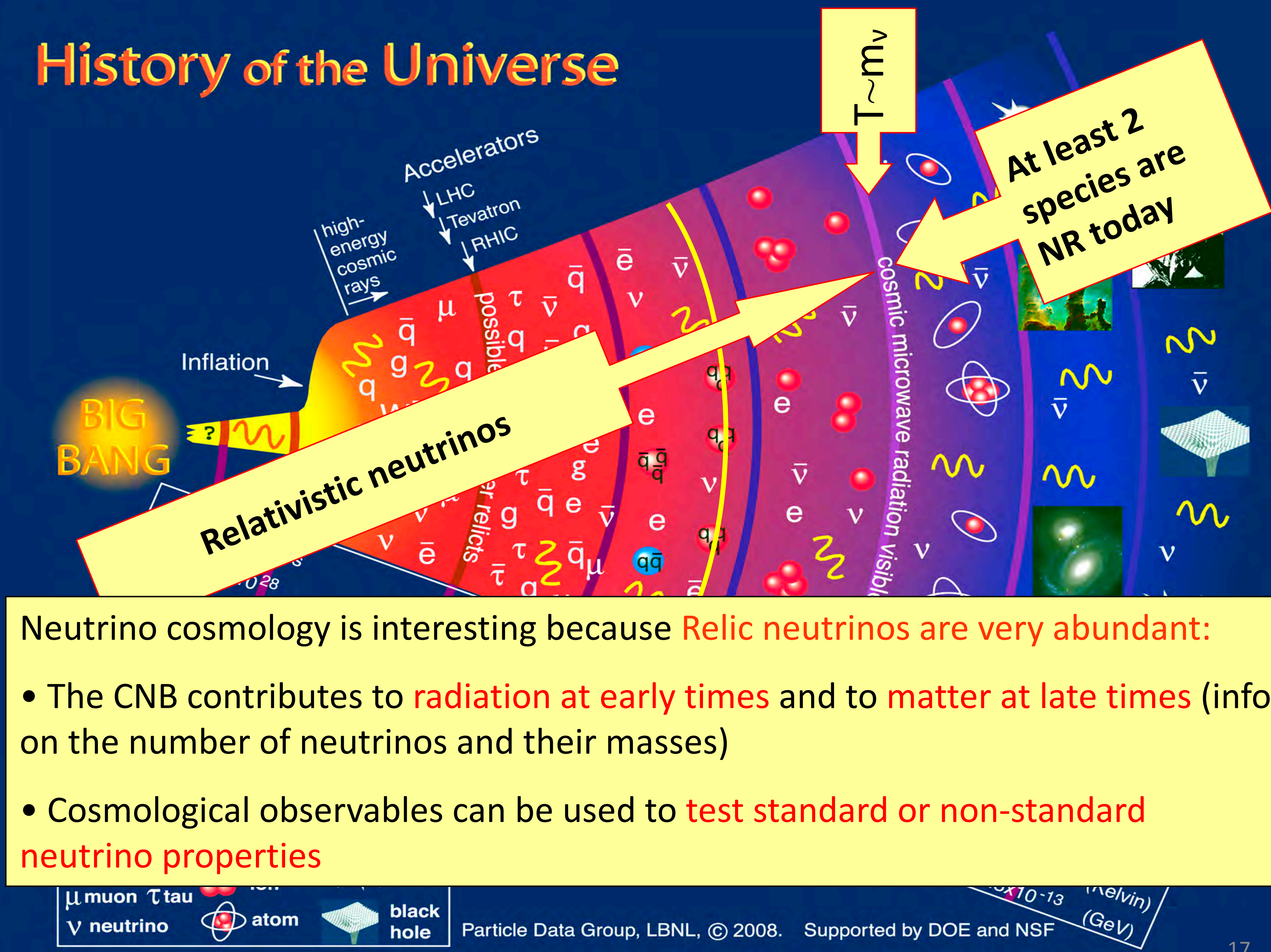
History of the Universe

Neutrinos coupled
by weak interactions

Decoupled neutrinos
(Cosmic Neutrino
Background or CNB)

T~MeV
t~sec

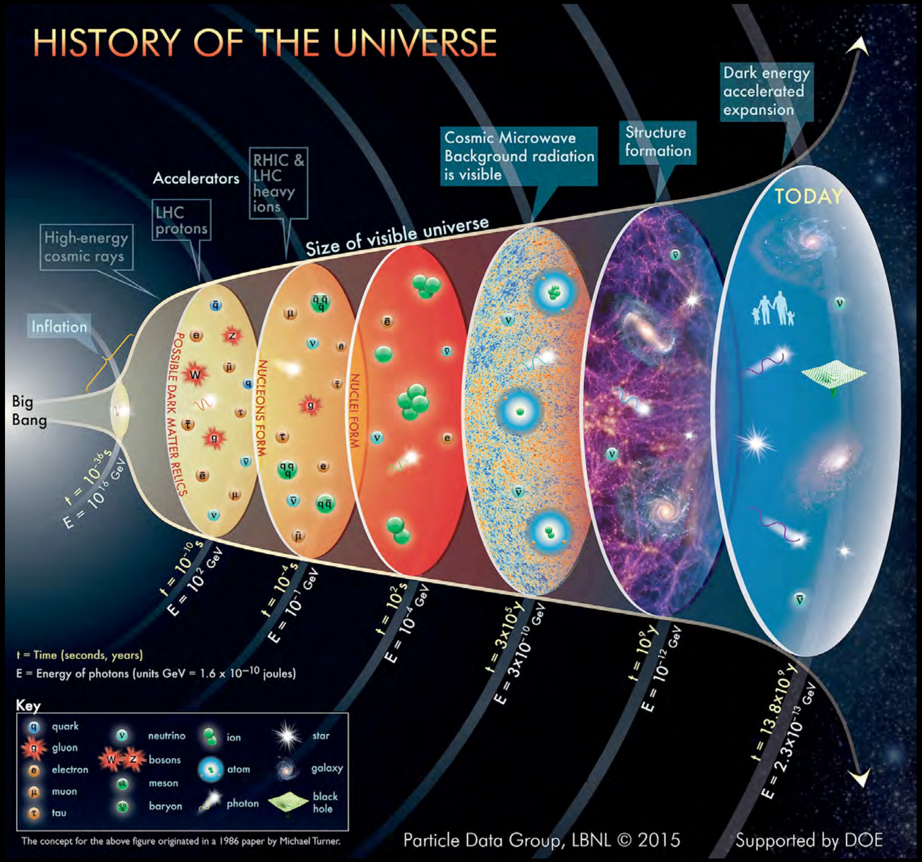




Neutrino cosmology is interesting because Relic neutrinos are very abundant:

- The CNB contributes to radiation at early times and to matter at late times (info on the number of neutrinos and their masses)
- Cosmological observables can be used to test standard or non-standard neutrino properties

J. Asorey Lectures on cosmology

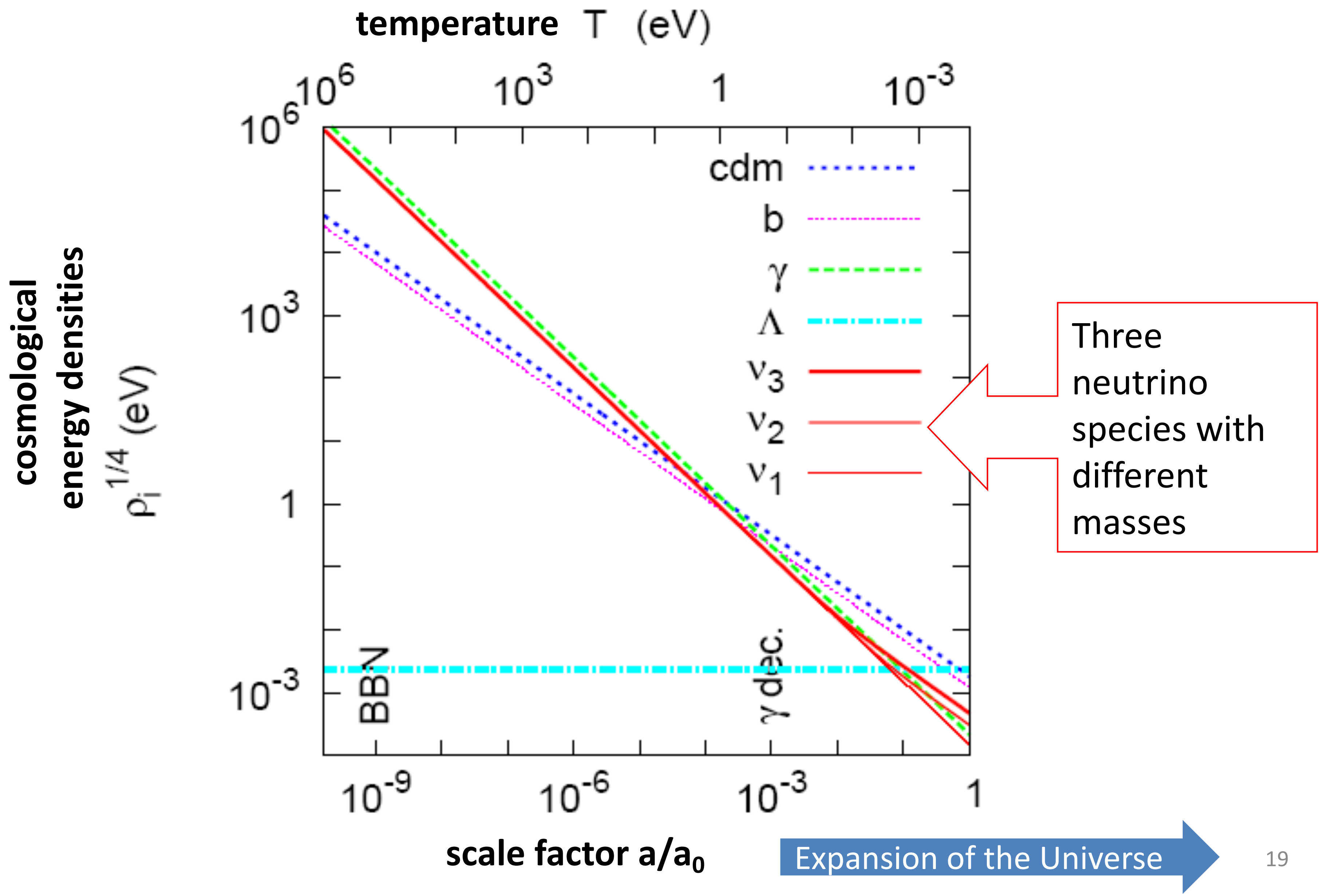


scale factor a/a_0

energy density: $\rho(a) = a^{-3(1+w)}$

$$\rho_R \sim a^{-4} \quad , \quad w = 1/3 \quad \text{(Radiation)}$$
$$\rho_M \sim a^{-3} \quad , \quad w = 0 \quad \text{(Matter)}$$
$$\rho_\Lambda \sim \text{const.} \quad , \quad w = -1 \quad \text{(Cosmological constant)}$$

Evolution of the background densities: 1 MeV → now



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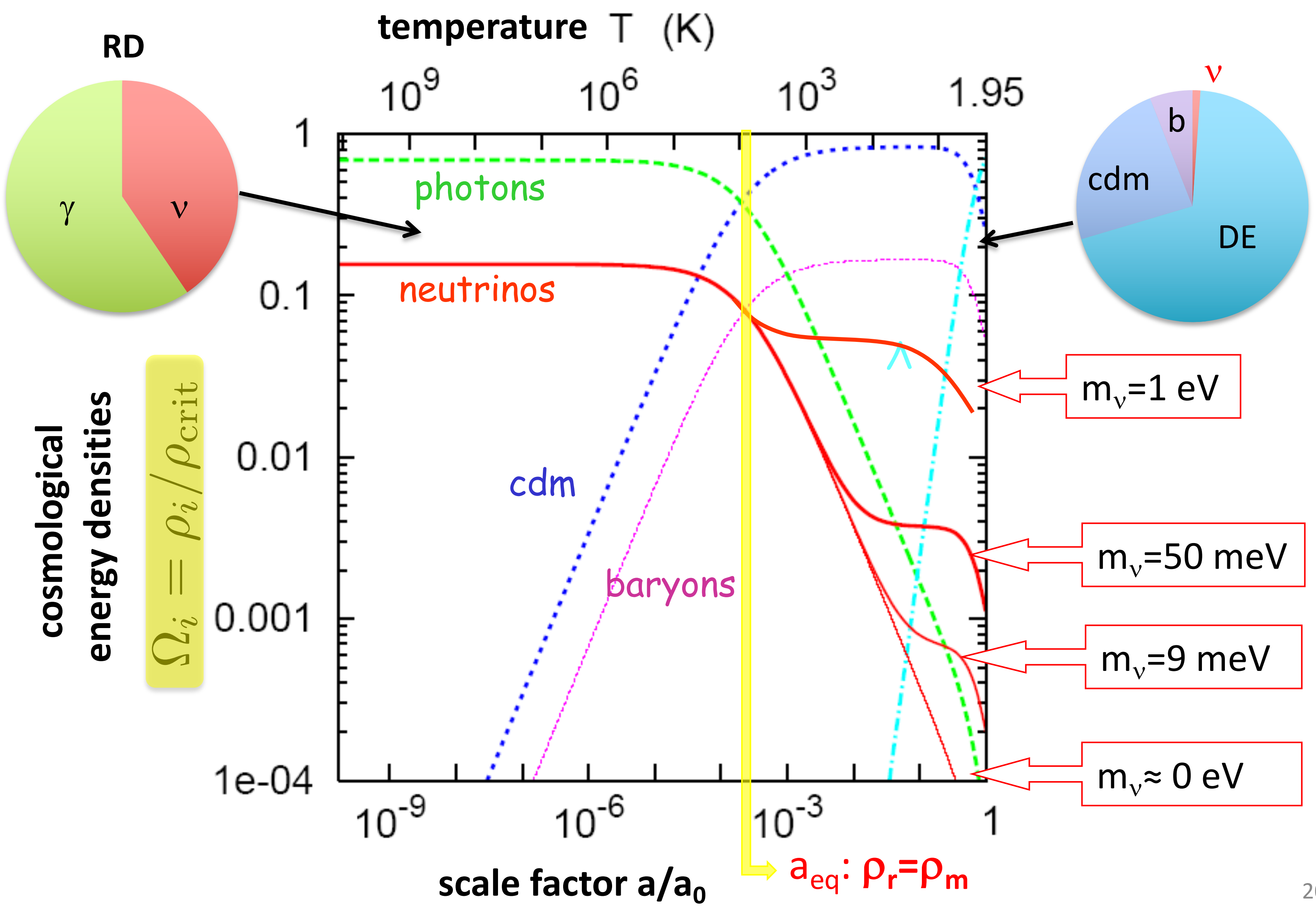
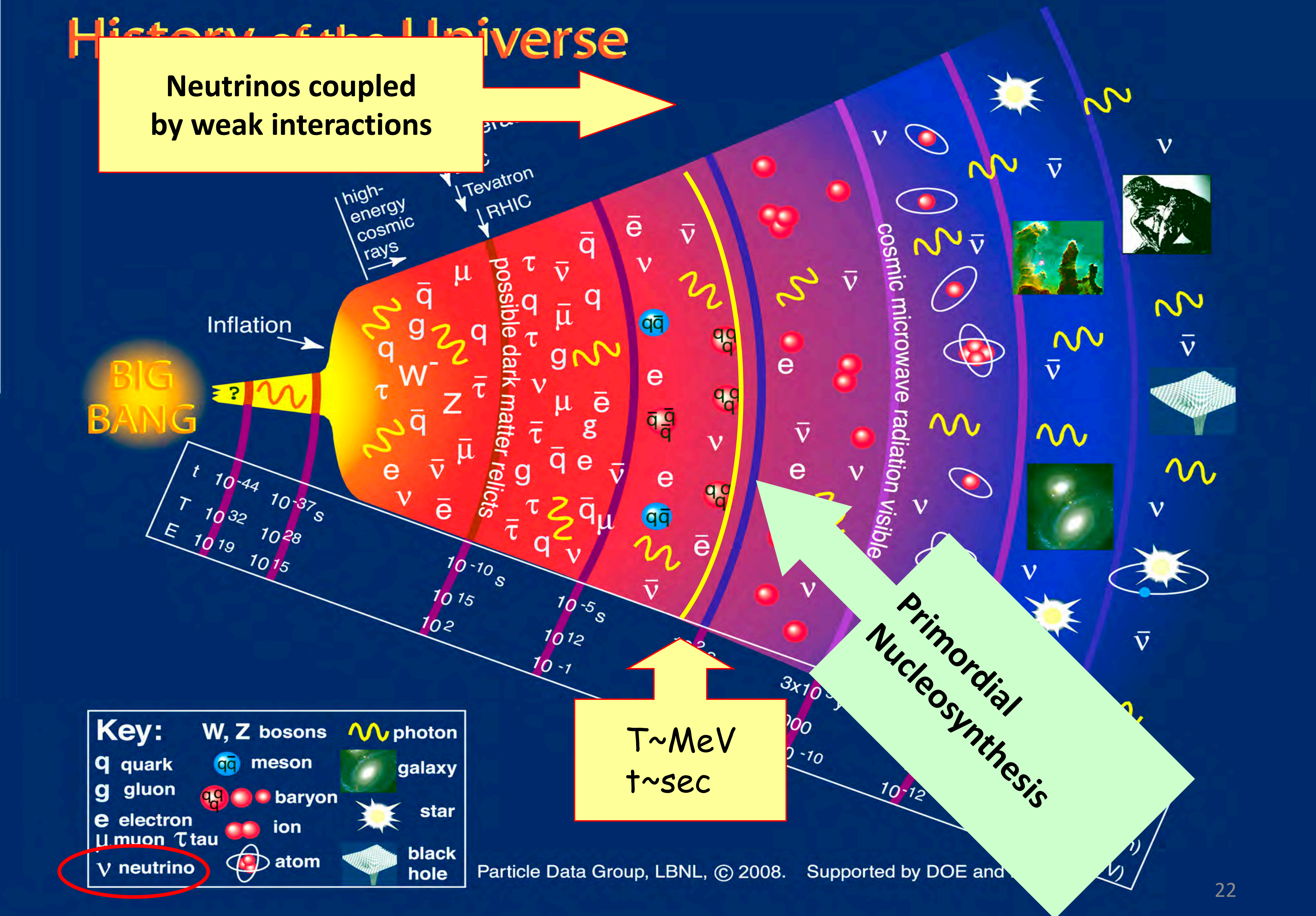


Production and decoupling of relic neutrinos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Neutrinos coupled
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high-energy cosmic rays
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T~MeV
t~sec
Primordial Nucleosynthesis
t 10-44 10-37 s
T 10 32 10 28
E 10 19 10 15
10 -10 s
10 15
10 2
10 -5 s
10 12
10 -1
Key:
q quark
g gluon
e electron
μ muon τ tau
ν neutrino
W, Z bosons
meson
baryon
ion
atom
photon
galaxy
star
black hole
Particle Data Group, LBNL, © 2008. Supported by DOE and

Neutrinos in Equilibrium

$$1\ \mathrm{MeV} \lesssim \mathrm{T} \lesssim \mathrm{m}_\mu$$

$$T_\nu = T_{e^\pm} = T_\gamma$$

$$\nu_\alpha \nu_\beta \leftrightarrow \nu_\alpha \nu_\beta$$

$$\nu_\alpha \bar{\nu}_\beta \leftrightarrow \nu_\alpha \bar{\nu}_\beta$$

$$\nu_\alpha e^\pm \leftrightarrow \nu_\alpha e^\pm$$

$$\nu_\alpha \bar{\nu}_\alpha \leftrightarrow e^+ e^-$$

Equilibrium thermodynamics

Distribution function of particle momenta in equilibrium

$$f_i^{eq}(p,T) = \left[\exp\left(\frac{E_i - \mu_i}{T}\right) \mp 1\right]^{-1}$$

Thermodynamical variables

VARIABLE	RELATIVISTIC		NON REL.
	BOSE	FERMI	
n	$\frac{\zeta(3)}{\pi^2} g T^3$	$\frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3$	$g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$
ρ	$\frac{\pi^2}{30} g T^4$	$\frac{7}{8}\frac{\pi^2}{30} g T^4$	mn
p	$\frac{\rho}{3}$		$nT \ll \rho$
$\langle E \rangle$	$2{,}701T$	$3{,}151T$	$m + \frac{3}{2}T$

$$n = g_i \int \frac{d^2\vec{p}}{(2\pi)^3} f_i(p,T) \qquad \rho = g_i \int \frac{d^2\vec{p}}{(2\pi)^3} E_i \; f_i(p,T)$$

$$p = g_i \int \frac{d^2\vec{p}}{(2\pi)^3} \frac{p^2}{3E_i} f_i(p,T) \qquad \langle E \rangle = \rho/n$$

Particles in equilibrium when T are high and interactions effective

T~1/a(t)

Cosmological energy densities: radiation

Energy density of **relativistic particles** with $f_i(p)$

$$\rho_i = g \int \frac{d^3p}{(2\pi)^3} \frac{p}{e^{p/T_i} \pm 1}$$

$$\rho_i = 3P_i = \begin{cases} \frac{\pi^2}{30} g\, T_i^4 \ , & \text{boson} \\ \frac{7}{8} \frac{\pi^2}{30} g\, T_i^4 \ , & \text{fermion} \end{cases}$$

$$\rho_\gamma = \frac{\pi^2}{15} T_\gamma^4$$

$$\rho_\nu = 3 \times \frac{7}{8} \times \frac{\pi^2}{15} T_\nu^4$$

Neutrino decoupling

As the Universe expands, particle densities are diluted and temperatures fall. Weak interactions become ineffective to keep neutrinos in good thermal contact with the e.m. plasma

Rough, but quite accurate estimate of the decoupling temperature

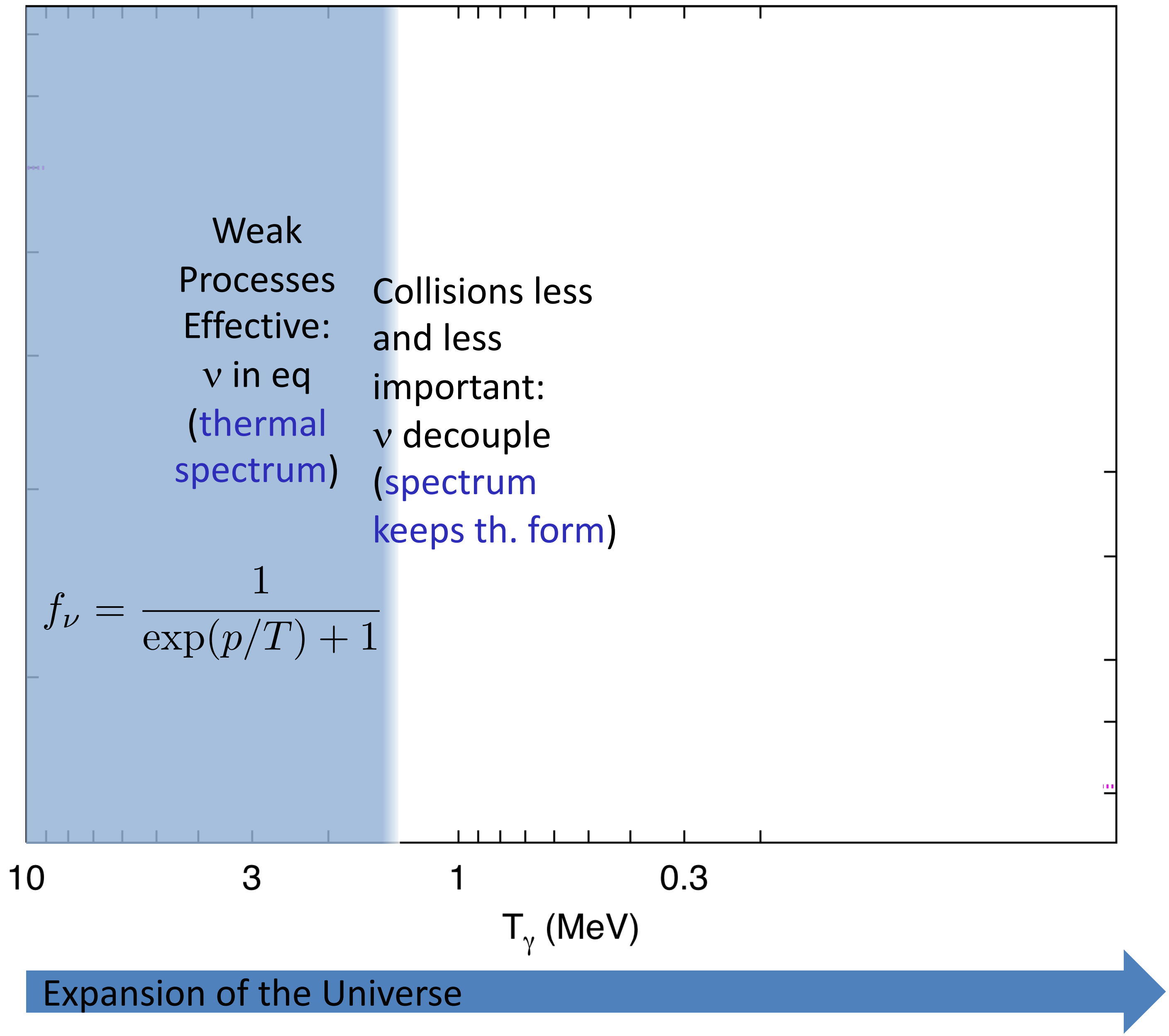
Rate of weak processes ~ Hubble expansion rate

$$\Gamma_W \approx \sigma_W |v| n, \;\; H^2 = \frac{8\pi\rho_{\rm rad}}{3M_P^2} \;\; \rightarrow \;\; G_F^2 T^5 \approx \sqrt{\frac{8\pi\rho_{\rm rad}}{3M_P^2}} \;\; \rightarrow \;\; T_{\rm dec}(\nu) \approx 1 \text{ MeV}$$

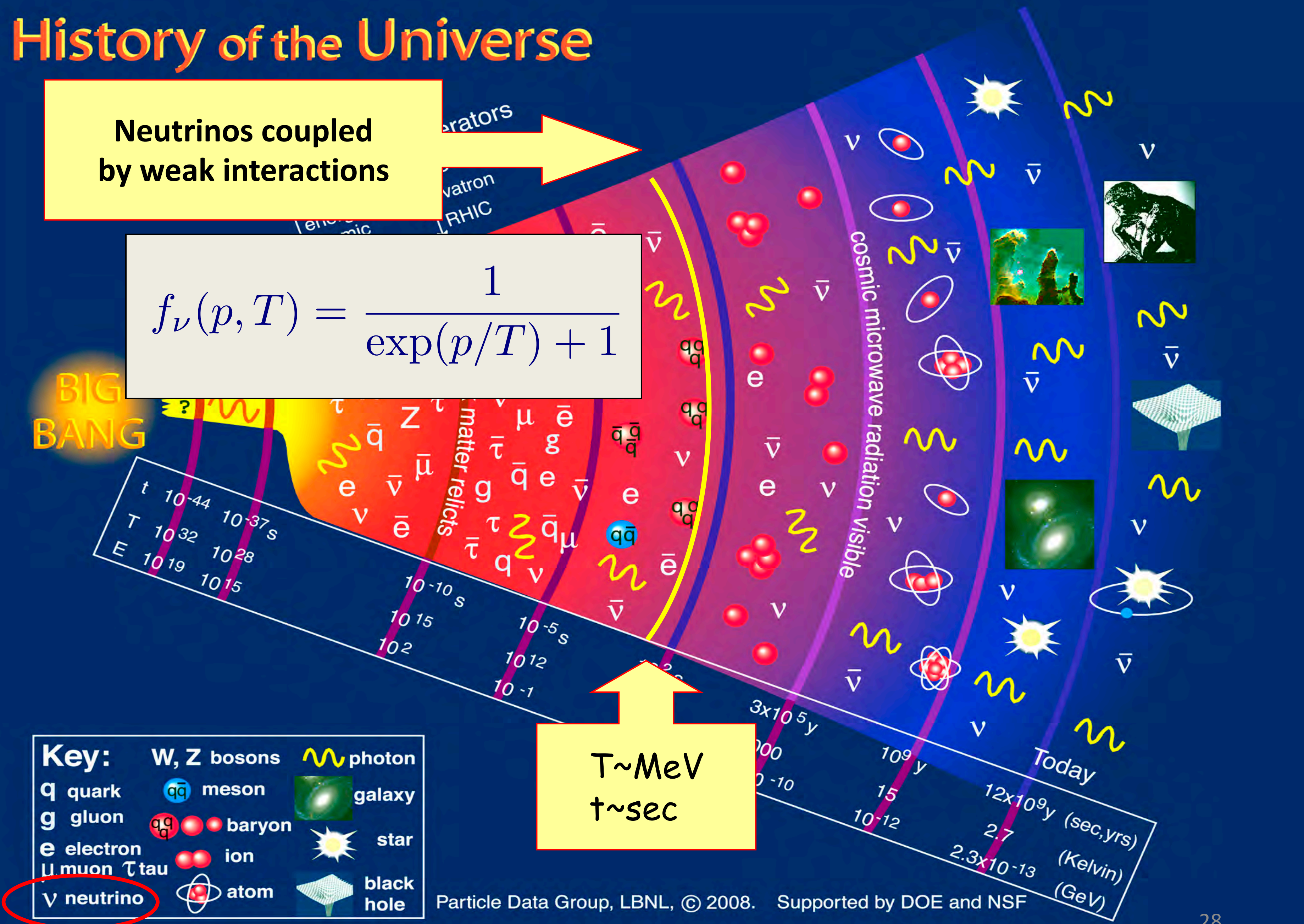
Since ν_e have both CC and NC interactions with $e^{\pm}$

$$T_{\rm dec}(\nu_e) \simeq 2 \text{ MeV} \quad T_{\rm dec}(\nu_{\mu,\tau}) \simeq 3 \text{ MeV}$$

Neutrino decoupling



History of the Universe



Neutrinos coupled by weak interactions

$$f_\nu(p,T) = \frac{1}{\exp(p/T)+1}$$

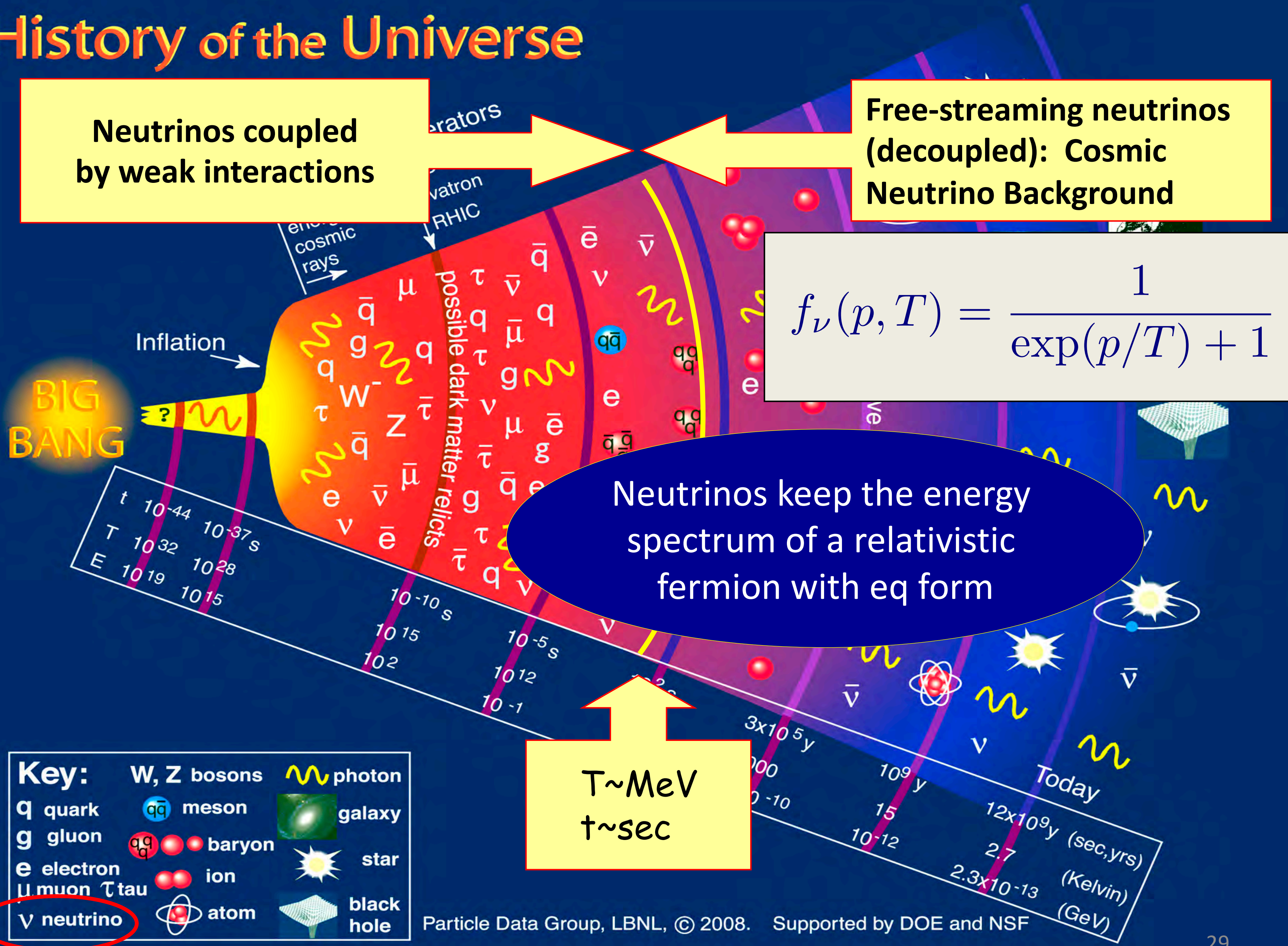
T~MeV
t~sec

History of the Universe

Neutrinos coupled by weak interactions

Free-streaming neutrinos (decoupled): Cosmic Neutrino Background

$$f_\nu(p,T) = \frac{1}{\exp(p/T)+1}$$



Neutrinos keep the energy spectrum of a relativistic fermion with eq form

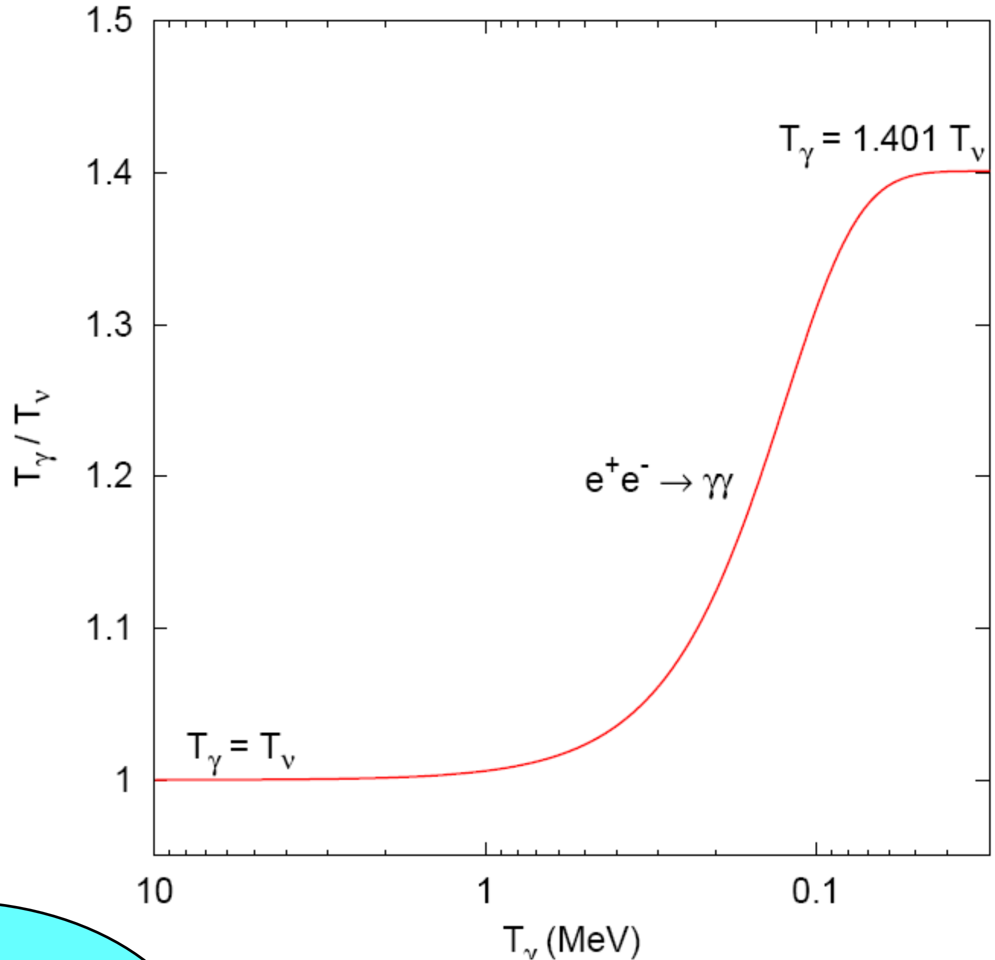
T~MeV
t~sec

Neutrino and photon (CMB) temperatures

At T~m_e, electron-positron pairs annihilate

$$e^+e^- \to \gamma\gamma$$

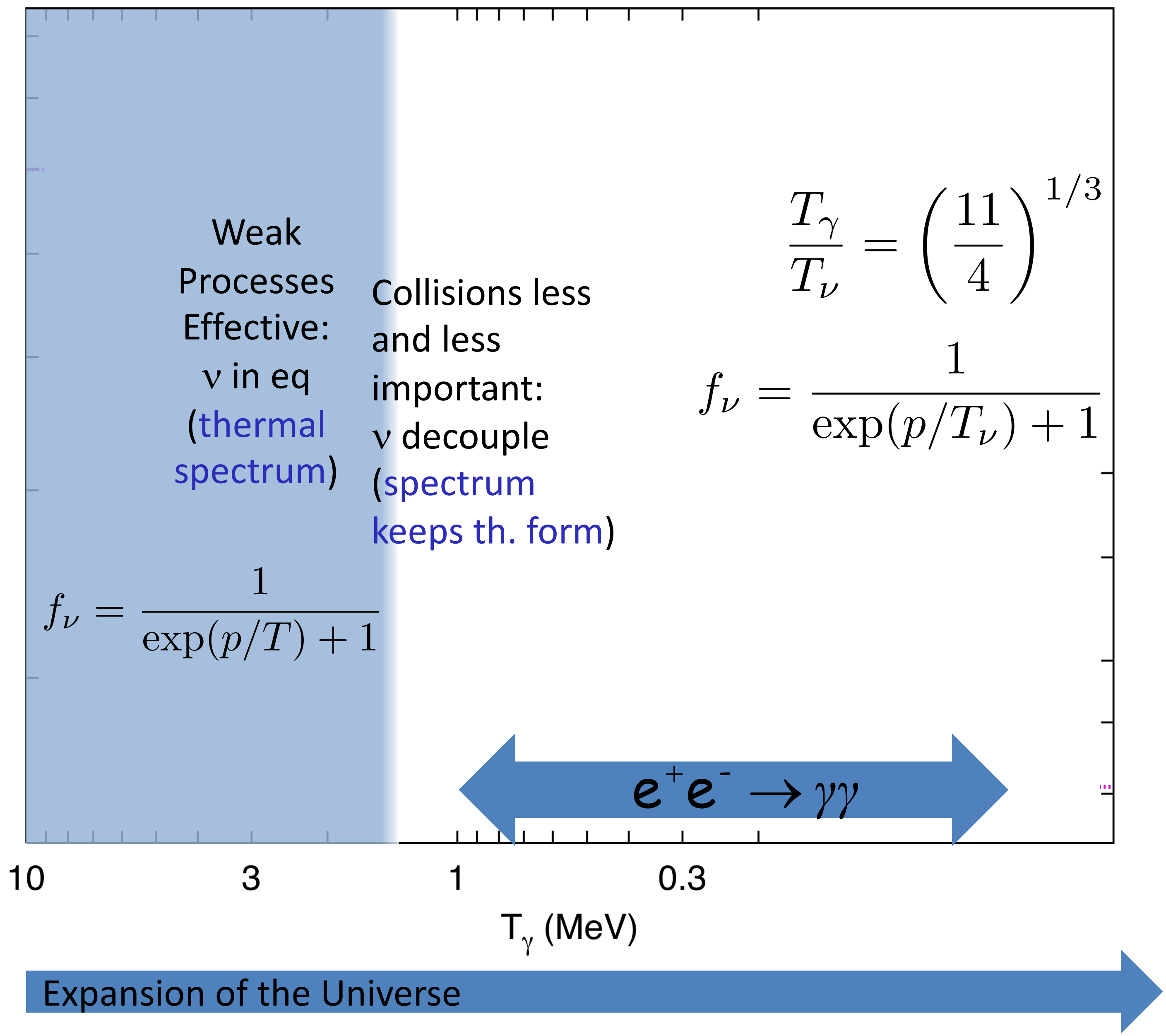
heating photons but not the decoupled neutrinos



$$\frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3}$$

$$f_\nu(p,T) = \frac{1}{\exp(p/T_\nu)+1}$$

Neutrino decoupling and e$^{\pm}$ annihilations



The Cosmic Neutrino Background

Neutrinos decoupled at T~MeV, keeping a spectrum as that of a relativistic species

$$f_\nu(p,T) = \frac{1}{\exp(p/T_\nu)+1}$$

- Number density

$$n_\nu = \int \frac{d^3p}{(2\pi)^3} f_\nu(p,T_\nu) = \frac{3}{11} n_\gamma = \frac{6\zeta(3)}{11\pi^2} T^3_{\mathrm{CMB}}$$

- Energy density

$$\rho_{\nu_i} = \int \sqrt{p^2 + m^2_{\nu_i}} \frac{d^3p}{(2\pi)^3} f_\nu(p,T_\nu) \to \begin{cases} \frac{7\pi^2}{120}\left(\frac{4}{11}\right)^{4/3} T^4_{\mathrm{CMB}} & \text{Massless} \\ m_{\nu_i} n_\nu & \text{Massive } m_\nu \gg T \end{cases}$$

The Cosmic Neutrino Background

Neutrinos decoupled at T~MeV, keeping a spectrum as that of a relativistic species

$$f_\nu(p,T) = \frac{1}{\exp(p/T_\nu)+1}$$

- Number density

At present 112 $(\nu + \bar{\nu})$ cm^{-3} per flavour

- Energy density

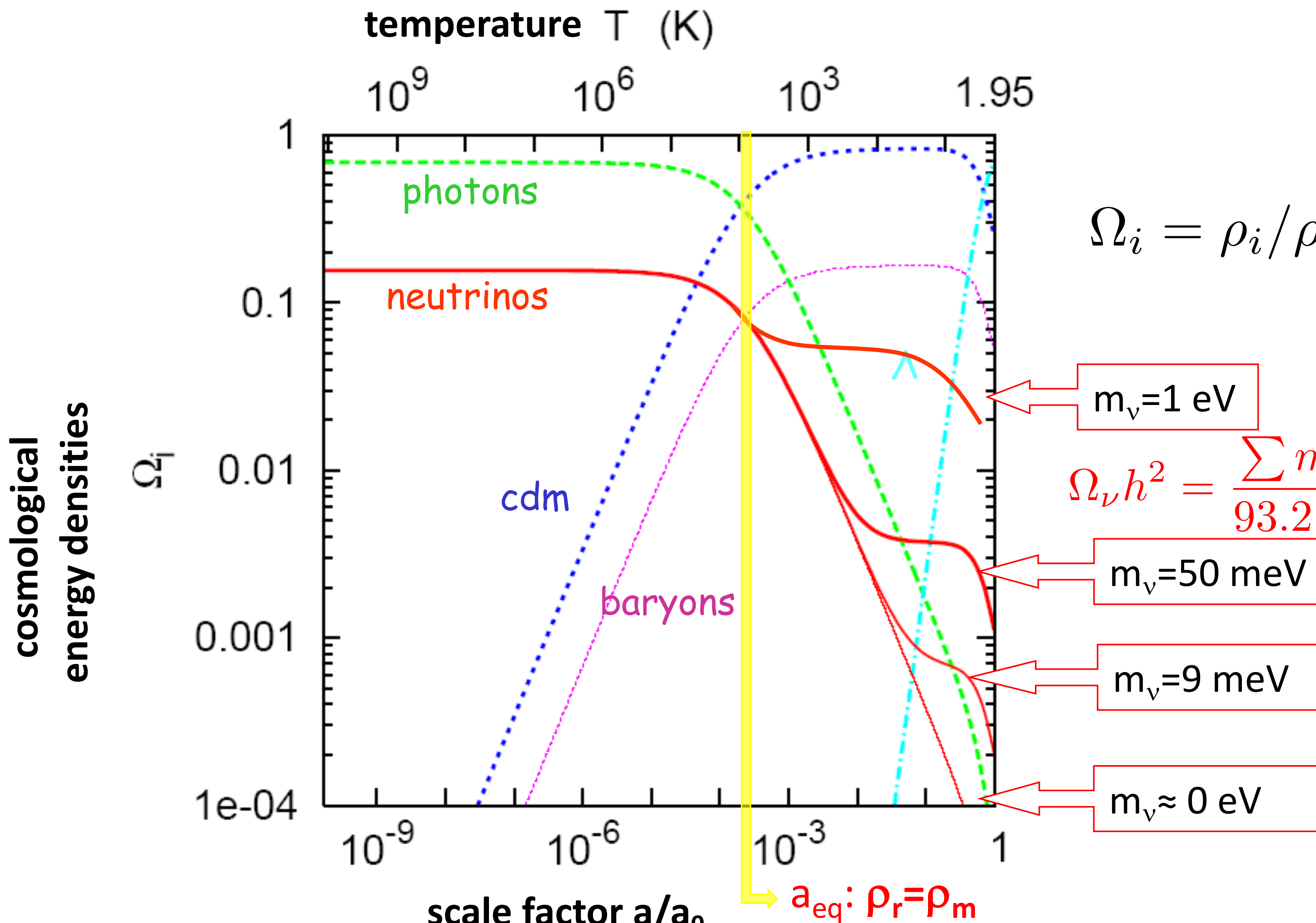
Contribution to the energy density of the Universe

$$\Omega_\nu h^2 \simeq 1.7 \times 10^{-5}$$ Massless

$$\Omega_\nu h^2 = \frac{\sum_i m_i}{93.2\ \mathrm{eV}}$$ Massive

$m_\nu >> T$

Evolution of the background densities: 1 MeV → now



$$\Omega_i = \rho_i / \rho_{\rm crit}$$

$$\Omega_\nu h^2 = \frac{\sum m_{\nu_i}}{93.2\,\mathrm{eV}}$$

The radiation content of the Universe (N_{eff})

Relativistic particles in the universe

At $T<m_e$, the radiation content of the Universe is

$$\rho_{\rm rad} = \rho_\gamma + \rho_\nu = \rho_\gamma \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times 3\right]$$

Valid for standard neutrinos in the
instantaneous decoupling approximation

Relativistic particles in the universe

At T<m_e, the radiation content of the Universe is

$$\rho_{\rm rad} = \rho_\gamma + \rho_\nu + \rho_x = \rho_\gamma \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\rm eff}\right]$$

effective number of relativistic neutrino species
(effective number of neutrinos)

N_{eff} is a way to measure the ratio $\frac{\rho_\nu + \rho_x}{\rho_\gamma}$

1960s-1970s : $N_{eff} = N_\nu$, **extra neutrinos** would enhance the cosmological expansion

>1980s: N_{eff} = **additional relativistic particles**

Number of light neutrino types (LEP data) $N_\nu = 2.984 \pm 0.008$

Relativistic particles in the universe

At $T<m_e$, the radiation content of the Universe is

$$\rho_{\rm rad} = \rho_\gamma + \rho_\nu + \rho_x = \rho_\gamma \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\rm eff}\right]$$

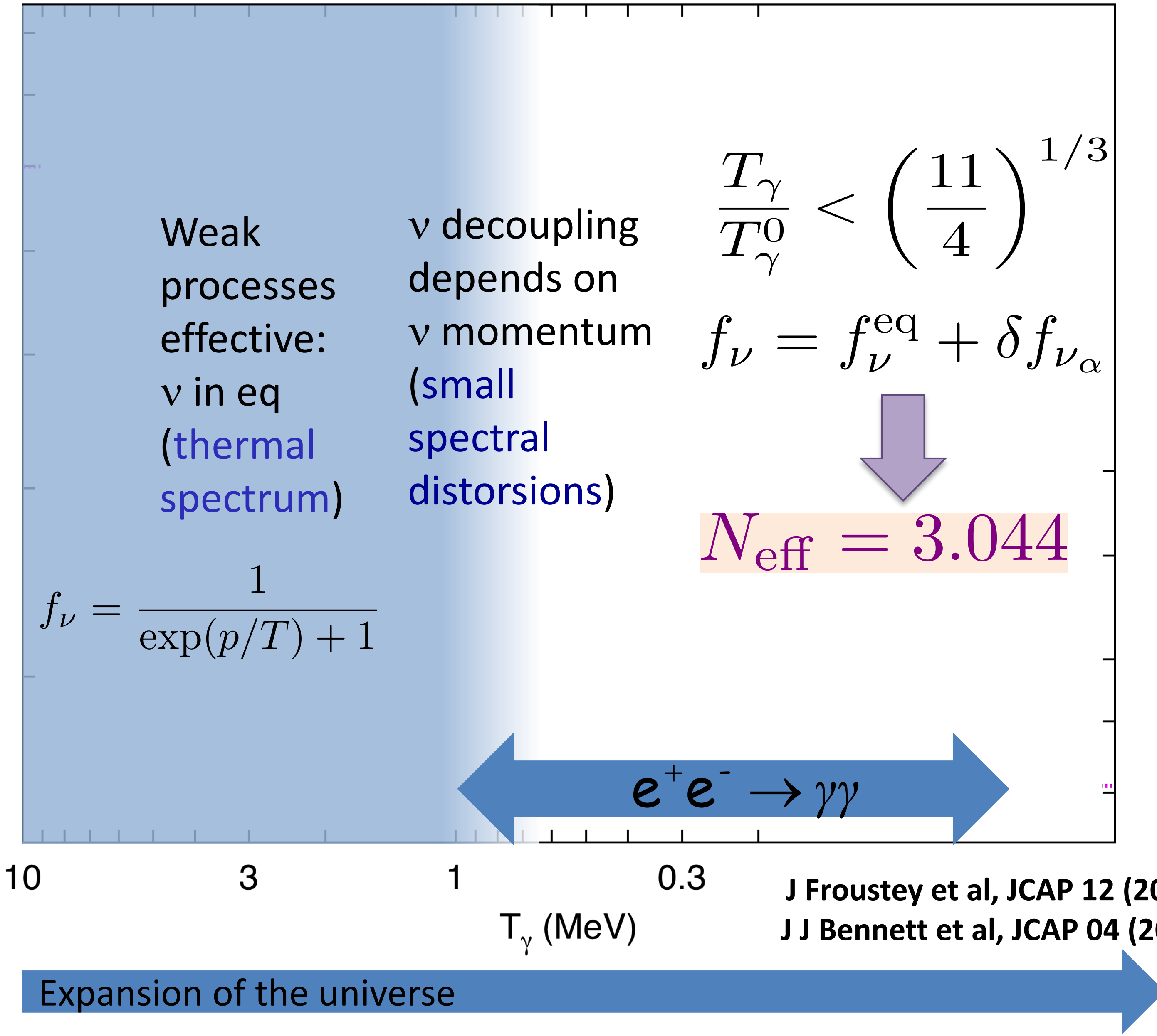
effective number of neutrinos

$N_{\rm eff} \neq 3$

- **additional relativistic particles** (scalars, pseudoscalars, decay products of heavy particles,…)
- **non-standard neutrino physics** (primordial neutrino asymmetries, totally or partially thermalized light sterile neutrinos, non-standard interactions with electrons,…)

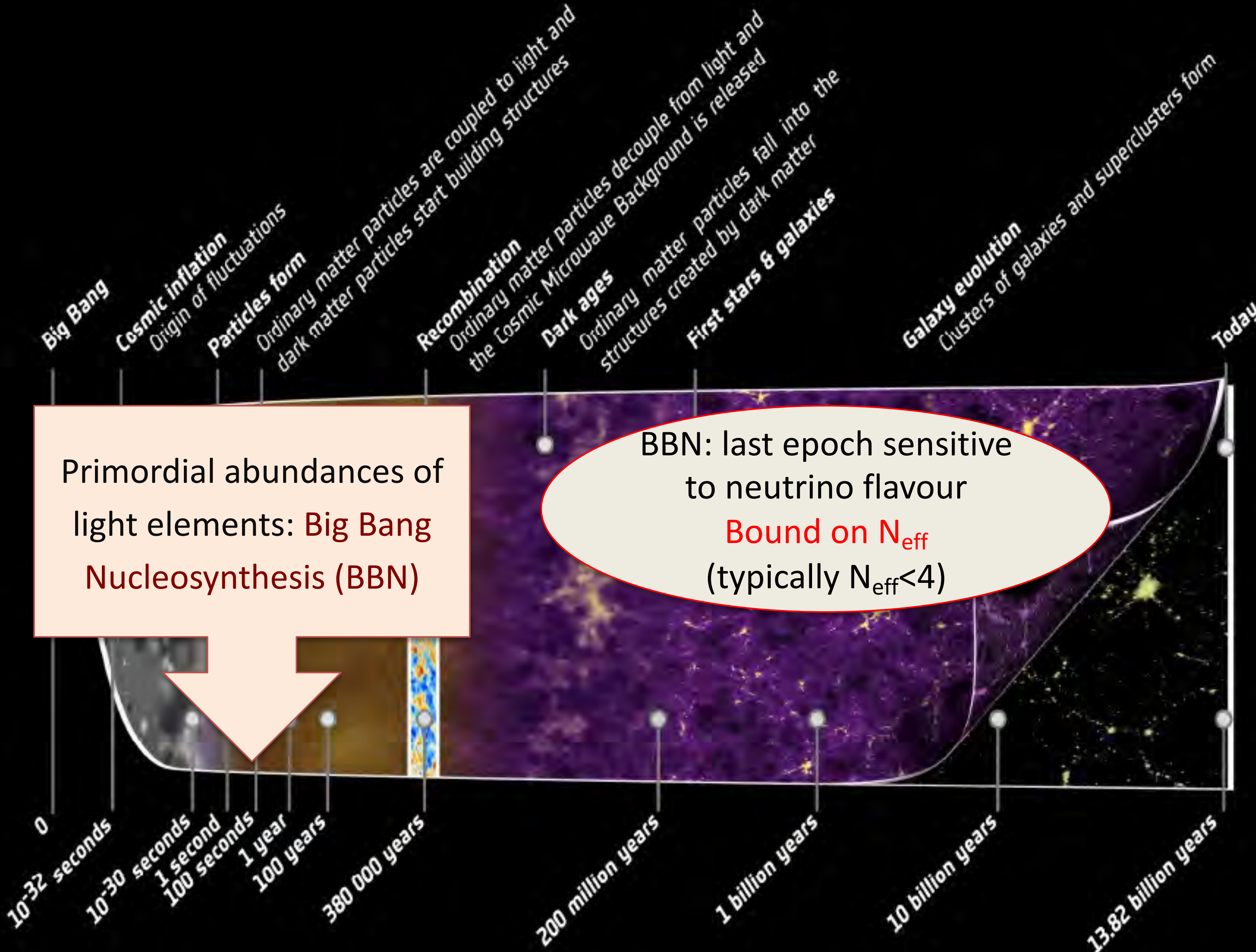
$N_{eff} \neq 3$ in the standard case

$N_{eff} > 3$: small neutrino heating



J Froustey et al, JCAP 12 (2020) 015
J J Bennett et al, JCAP 04 (2021) 073

Neutrinos and Primordial Nucleosynthesis



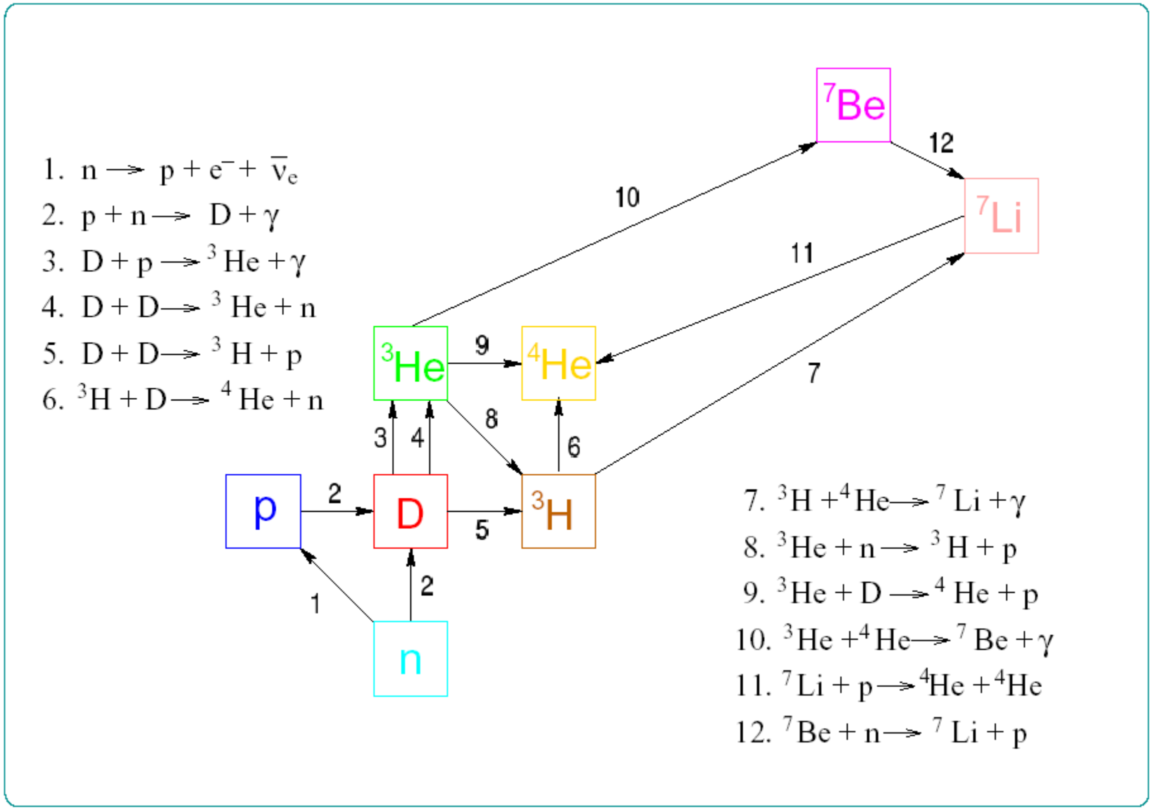
Primordial abundances of light elements: Big Bang Nucleosynthesis (BBN)

BBN: last epoch sensitive to neutrino flavour

Bound on N_{eff}

(typically $N_{eff}<4$)

BBN: Creation of light elements



Produced elements: D, ^{3}He, ^{4}He, ^{7}Li and small abundances of others

Theoretical inputs:

- τ_n, the neutron lifetime;
- G_N, the Newton gravitational constant;
- η, the baryon to photon number density ratio;
- the nuclear rates.

$$\eta_{10} = \frac{n_B/n_\gamma}{10^{-10}} \simeq 274\,\Omega_B h^2$$

BBN: Creation of light elements

Range of temperatures: from 0.8 to 0.01 MeV

$$t \simeq 0{,}74 \left(\frac{\text{MeV}}{T}\right)^2 \text{ sec}$$

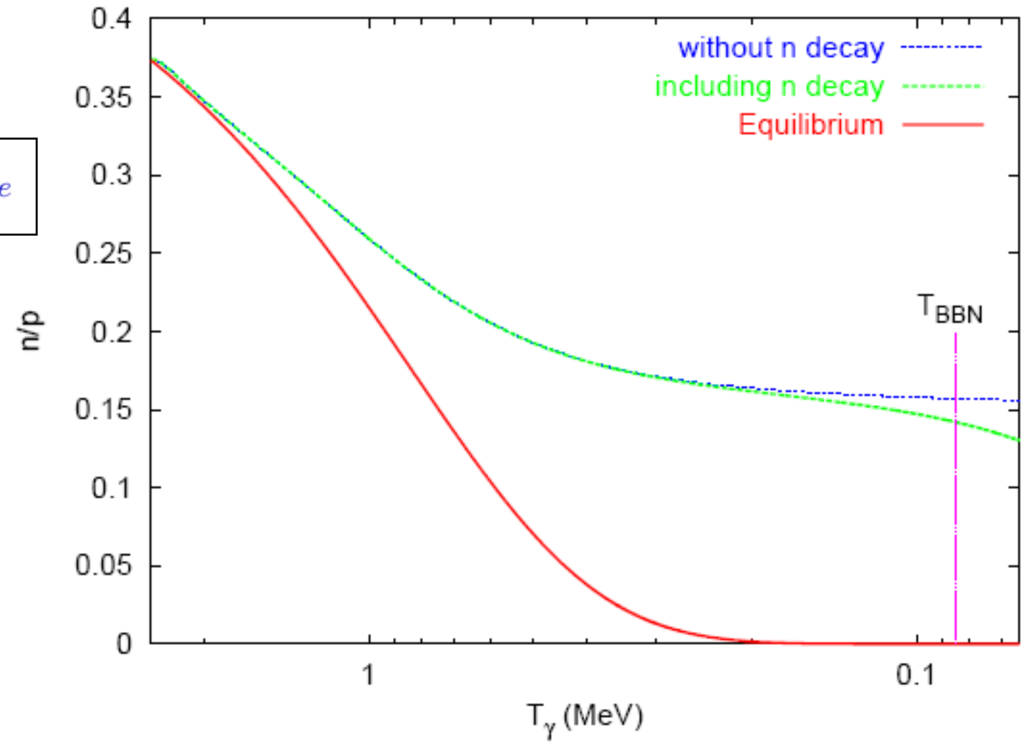
Phase I: 0.8-0.1 MeV

n-p reactions

$$\nu_e + n \longleftrightarrow p + e^- \qquad e^+ + n \longleftrightarrow p + \bar{\nu}_e$$

n/p freezing and neutron decay

$$n \longleftrightarrow p + e^- + \bar{\nu}_e$$

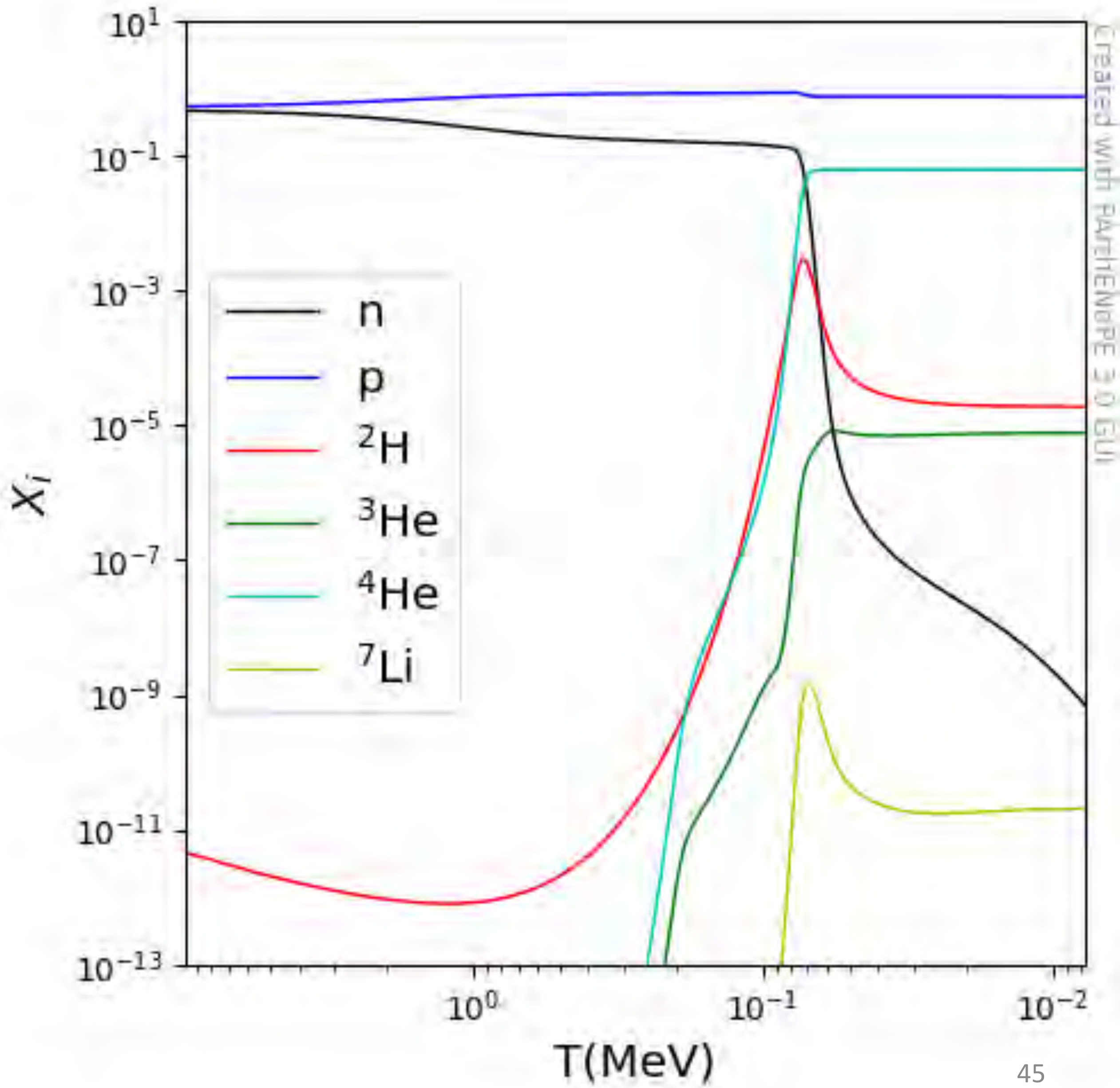


$$\left(\frac{n}{p}\right)_{eq} \simeq \exp\left(-\frac{m_n - m_p}{T_\gamma}\right) = \exp\left(-\frac{1{,}293 \text{ MeV}}{T_\gamma}\right)$$

BBN: Creation of light elements

Phase II: 0.1-0.01 MeV
Formation of light nuclei starting from D

Photodesintegration prevents earlier formation for temperatures closer to nuclear binding energies

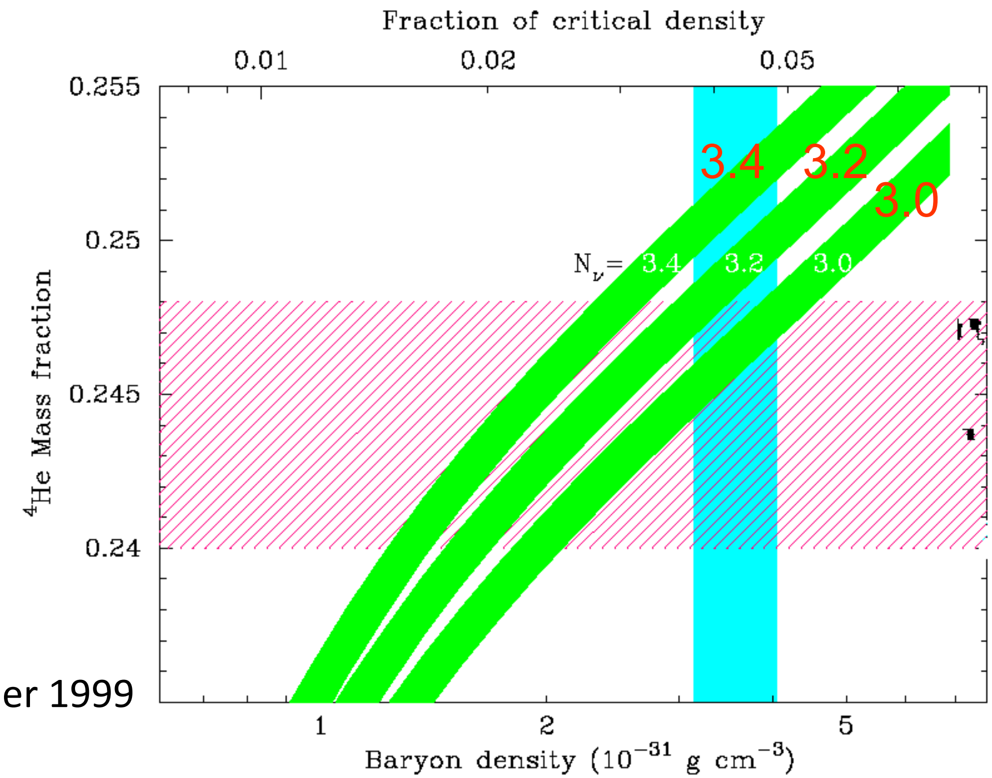


Effect of neutrinos on BBN

1. $\mathbf{N_{eff}}$ fixes the **expansion rate** during BBN

$$H = \sqrt{\frac{8\pi\rho}{3M_p^2}}$$

$$\rho(N_{eff}) > \rho_0 \rightarrow \uparrow {}^4\mathrm{He}$$



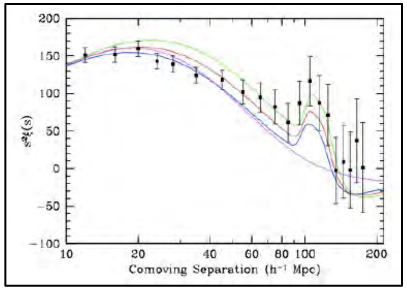
Burles, Nollett & Turner 1999

2. Direct effect of **electron** neutrinos and antineutrinos on the **n-p reactions**

$$\nu_e + n \leftrightarrow p + e^- \qquad e^+ + n \leftrightarrow p + \bar{\nu}_e$$

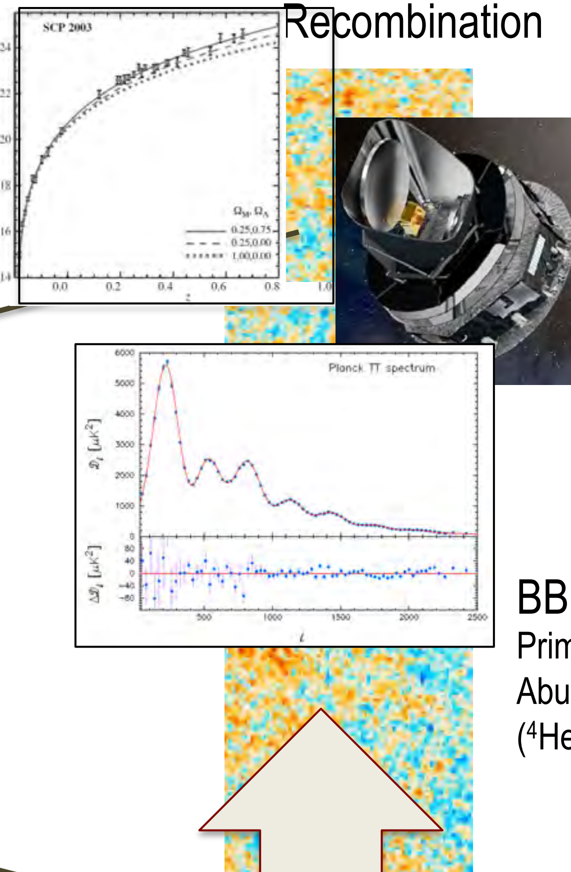
Bounds on N_{eff}

Cosmological Observables



Hubble constant H_0 & cosmic distances measurements: SN Ia and Baryon Acoustic Oscillations (BAO)

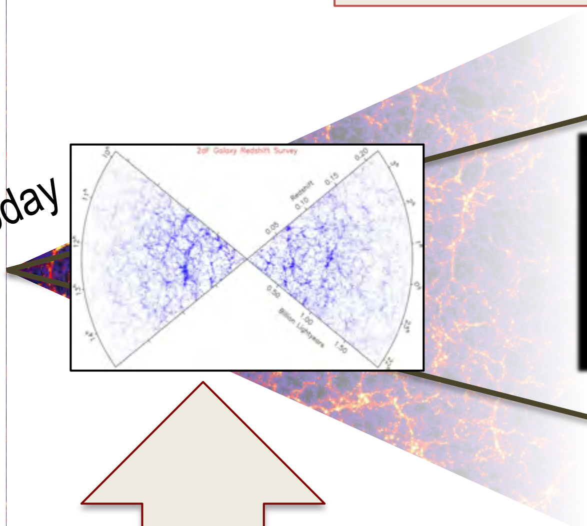
Recombination

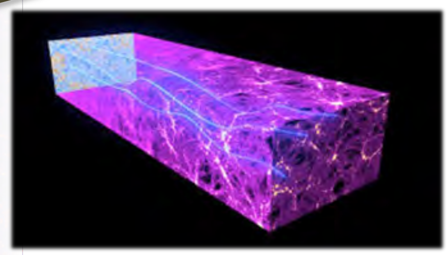


BBN

Primordial Abundances (^{4}He, D, …)

Today



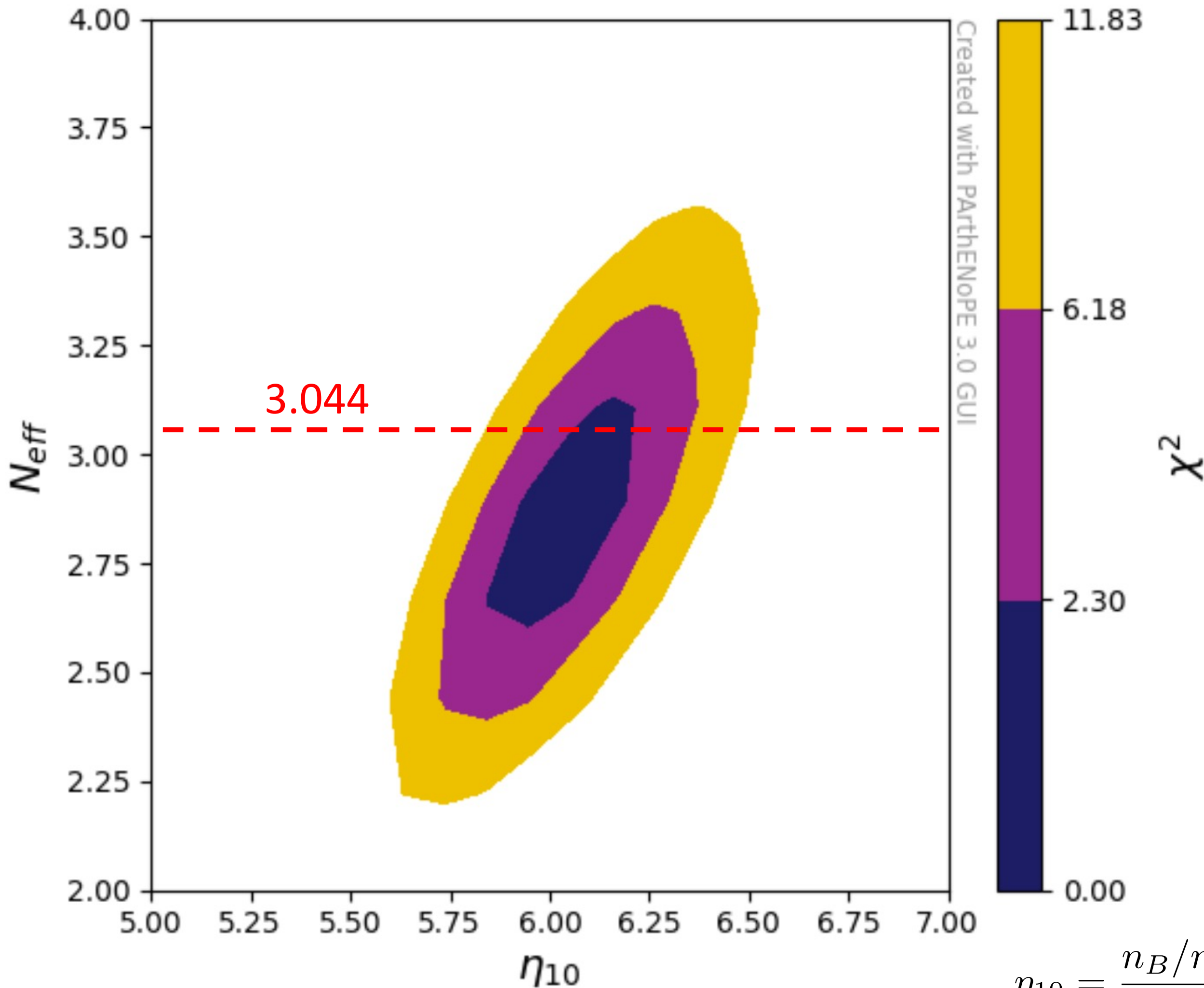


matter density fluctuations
Large-Scale Structures [galaxy / cosmic shear / Lyα] LSS spectrum

Photon momentum after decoupling
CMB secondary anisotropy spectrum

Photon density fluctuations before decoupling
CMB primary anisotropy spectrum (temp+pol)

BBN: allowed ranges for N_{eff}



$$5.6 < \eta_{10} < 6.6$$

$$2.3 < N_{\rm eff} < 3.4$$

(95% CL)

$$\eta_{10} = \frac{n_B/n_\gamma}{10^{-10}} \simeq 274\,\Omega_B h^2$$

^{4}He and D bounds

PArthENoPE BBN code, S Gariazzo et al, Comp Phys Comm 271 (2022) 108205

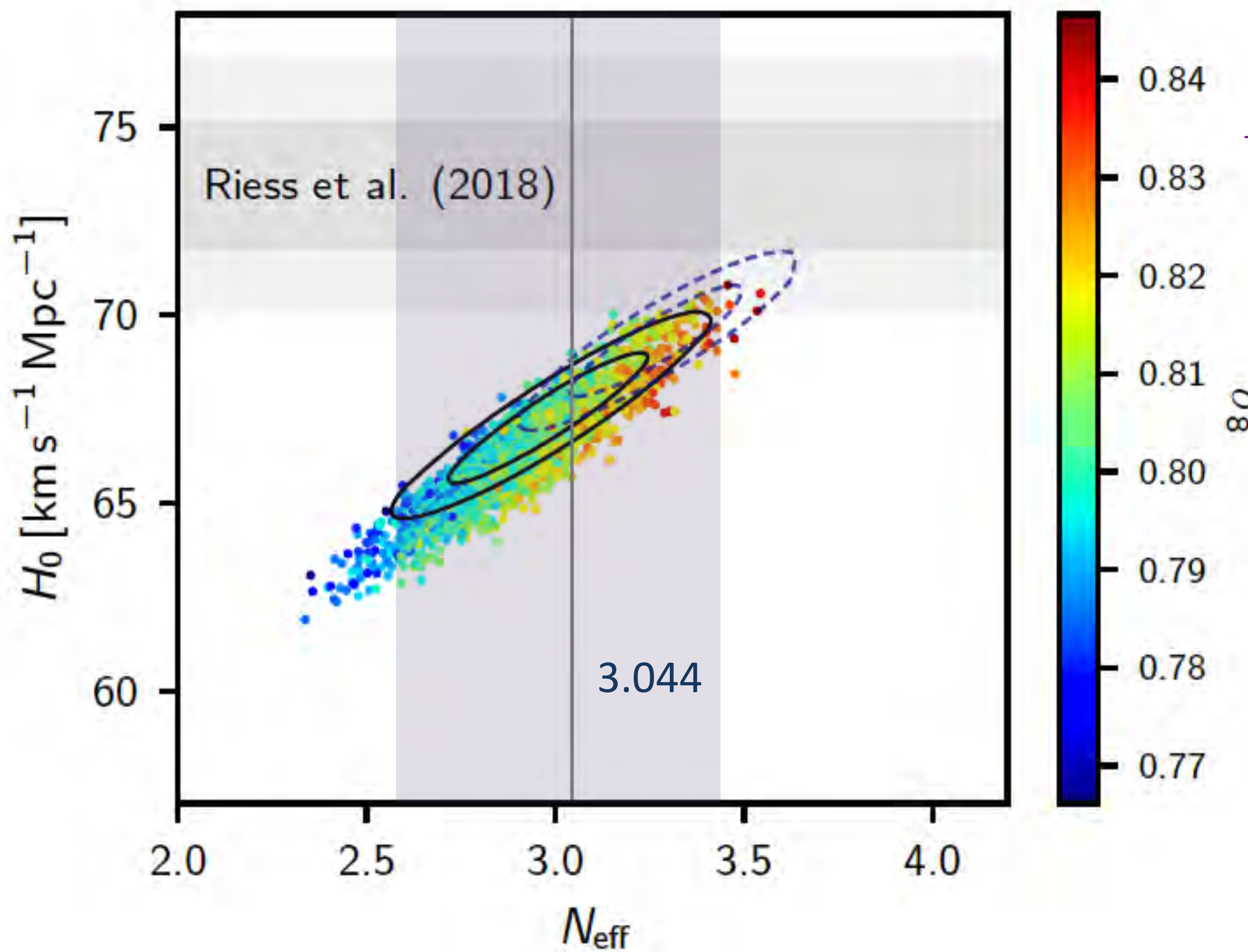
CMB anisotropies + other data

$N_{\text{eff}} \lesssim 17$ (2001) early CMB data

$N_{\text{eff}} = 4.2^{+1.2}_{-1.7}$ (2005) WMAP+…

$N_{\text{eff}} = 2.99^{+0.34}_{-0.33}$ (2018) Planck

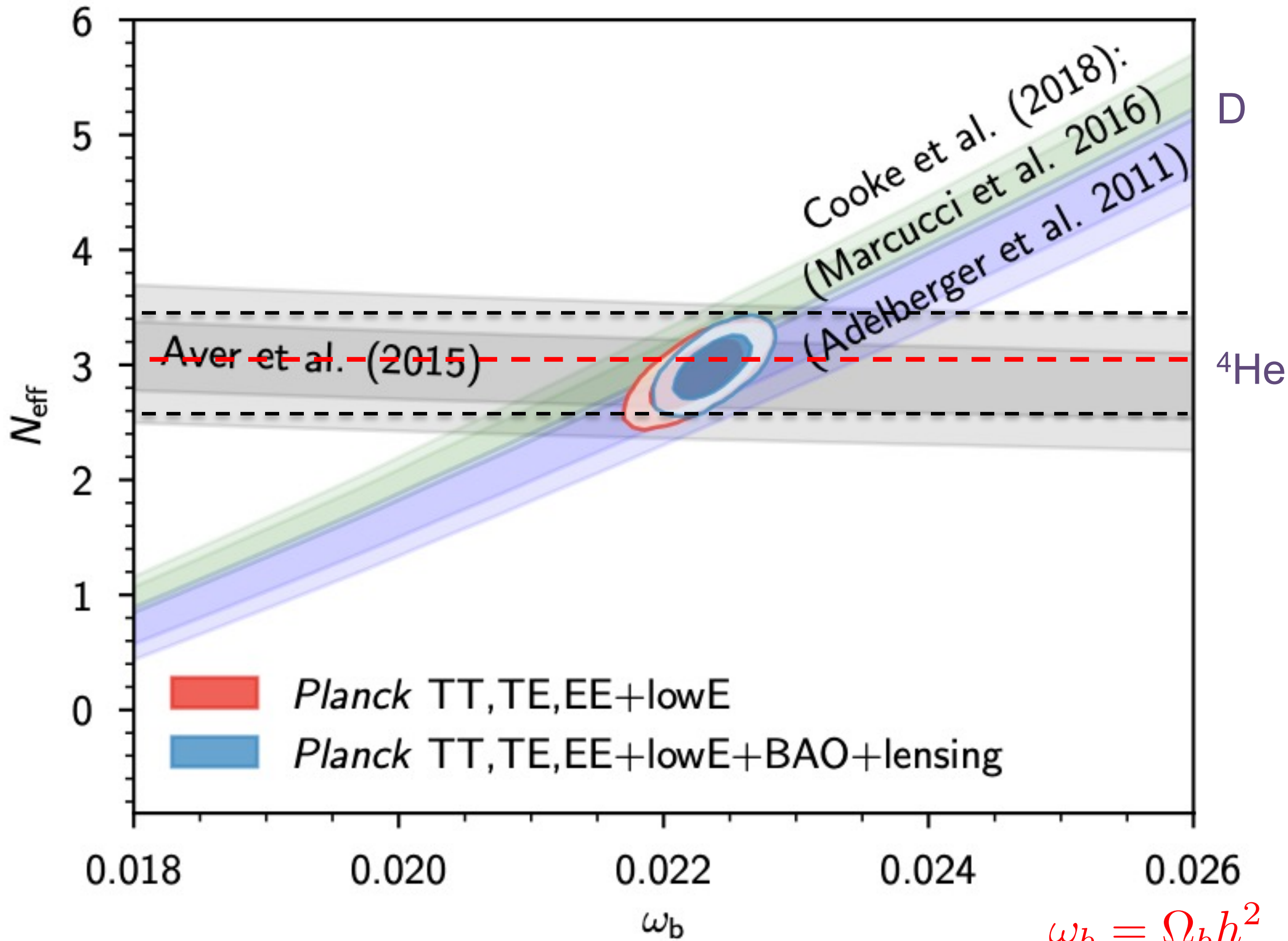
(95%, TT,TE,EE+lowE+lensing+BAO)



$N_{\text{eff}} = 3.10 \pm 0.17$ (68%)

(2024) CMB+DESI

Comparison: allowed ranges for N_{eff} and BBN



$\omega_b = \Omega_b h^2$

[Planck Coll, A&A **641** (2020) A6]

Neutrinos as Dark Matter

The Cosmic Neutrino Background

Neutrinos decoupled at T~MeV, keeping a spectrum as that of a relativistic species

$$f_\nu(p,T) = \frac{1}{\exp(p/T_\nu)+1}$$

- Number density

At present 112 $(\nu + \bar{\nu})$ cm^{-3} per flavour

- Energy density

Contribution to the energy density of the Universe

$$\Omega_\nu h^2 \simeq 1.7 \times 10^{-5}$$ Massless

$$\Omega_\nu h^2 = \frac{\sum_i m_i}{93.2\ \mathrm{eV}}$$ Massive

$m_\nu \gg T$

Neutrinos as Dark Matter

- Neutrinos are natural DM candidates

$$\Omega_\nu h^2 = \frac{\sum_i m_i}{93.2\ \text{eV}} \qquad \Omega_\nu < 1 \rightarrow \sum_i m_i \lesssim 46\ \text{eV}$$

$$\Omega_\nu < \Omega_m \simeq 0.3 \rightarrow \sum_i m_i \lesssim 15\ \text{eV}$$

- They stream freely until non-relativistic (collisionless phase mixing) → **Neutrinos are HOT Dark Matter** (large thermal motion)

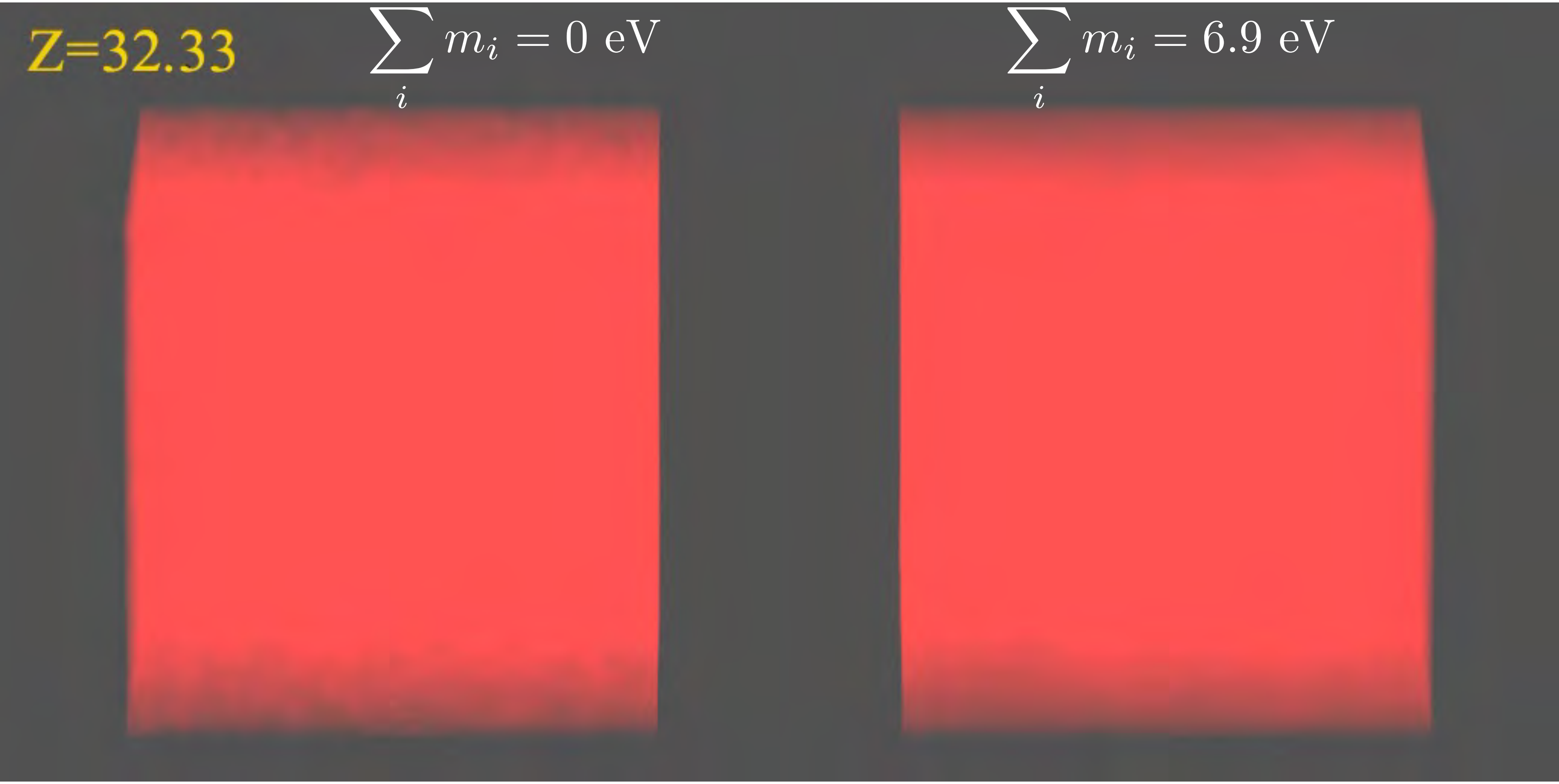
- First structures to be formed when Universe became matter –dominated are very large

- Ruled out by structure formation → CDM

Massive Neutrinos can still be subdominant DM: limits on m_ν from Structure Formation (combined with other cosmological data)

Neutrinos as Hot Dark Matter

Massive Neutrinos can still be subdominant DM: limits on m_ν from Structure Formation (combined with other cosmological data)



S. Hannestad, Cosmology Group, Univ. Aarhus

Cosmological bounds on neutrino mass(es)

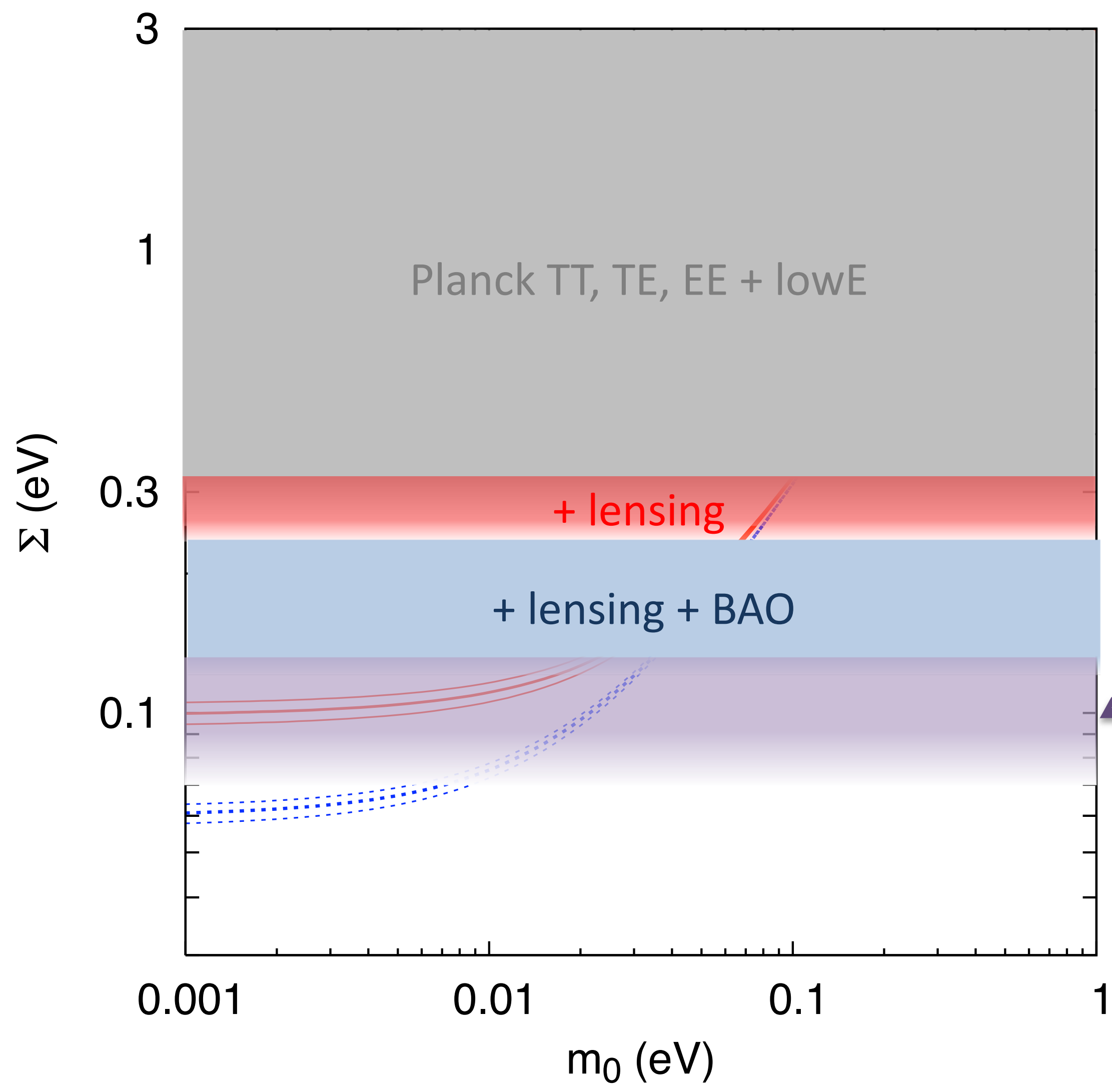
A unique cosmological bound on m_ν DOES NOT exist !

Different analyses have found upper bounds on neutrino masses, since they depend on

- The combination of cosmological data used
- The assumed cosmological model: number of parameters (problem of parameter degeneracies)
- The properties of relic neutrinos

Bounds on Σm_ν from Planck (+other cosmo data)

Cosmological upper limits on the sum of neutrino masses

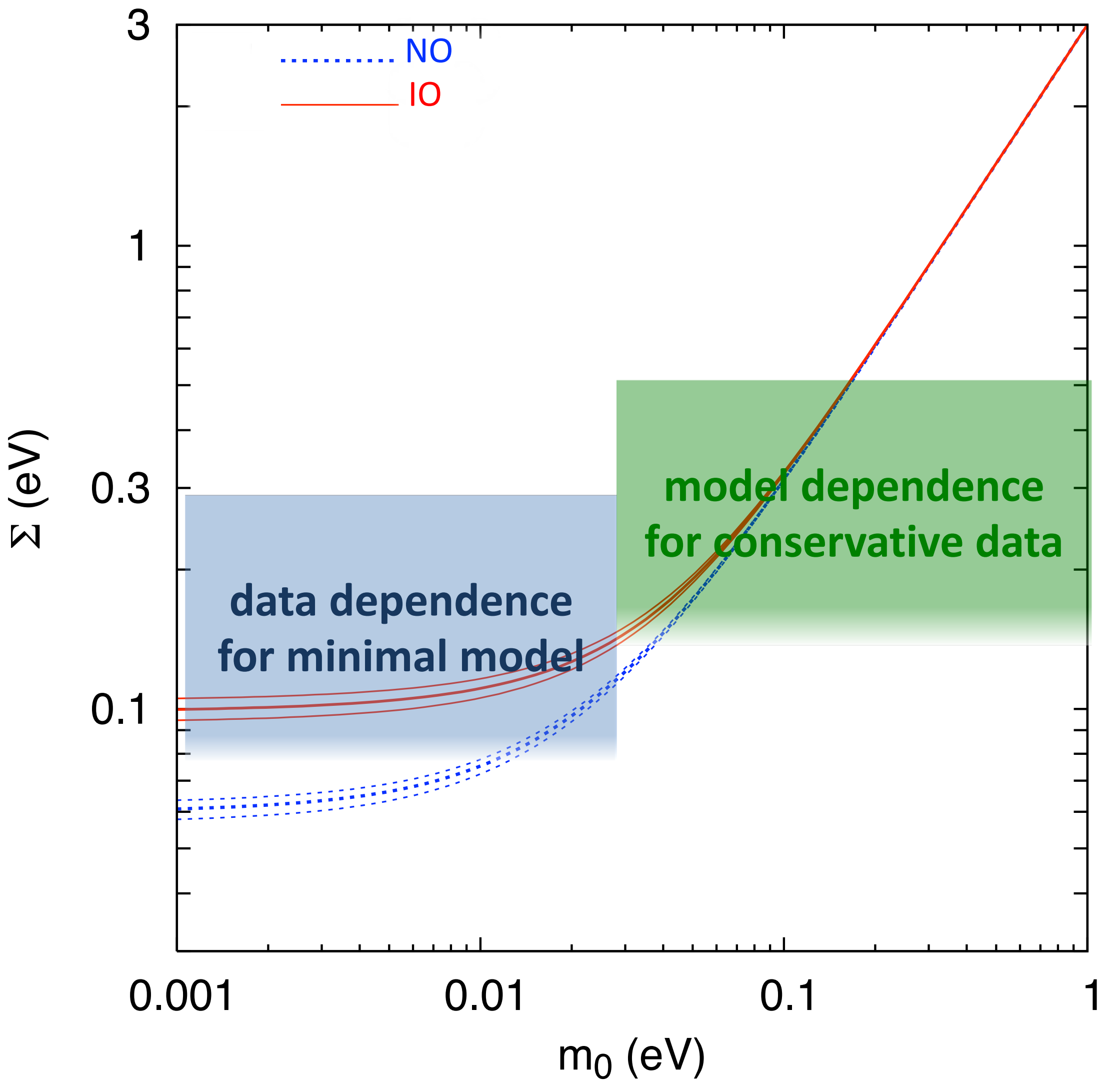


Latest analyses with DESI BAO data

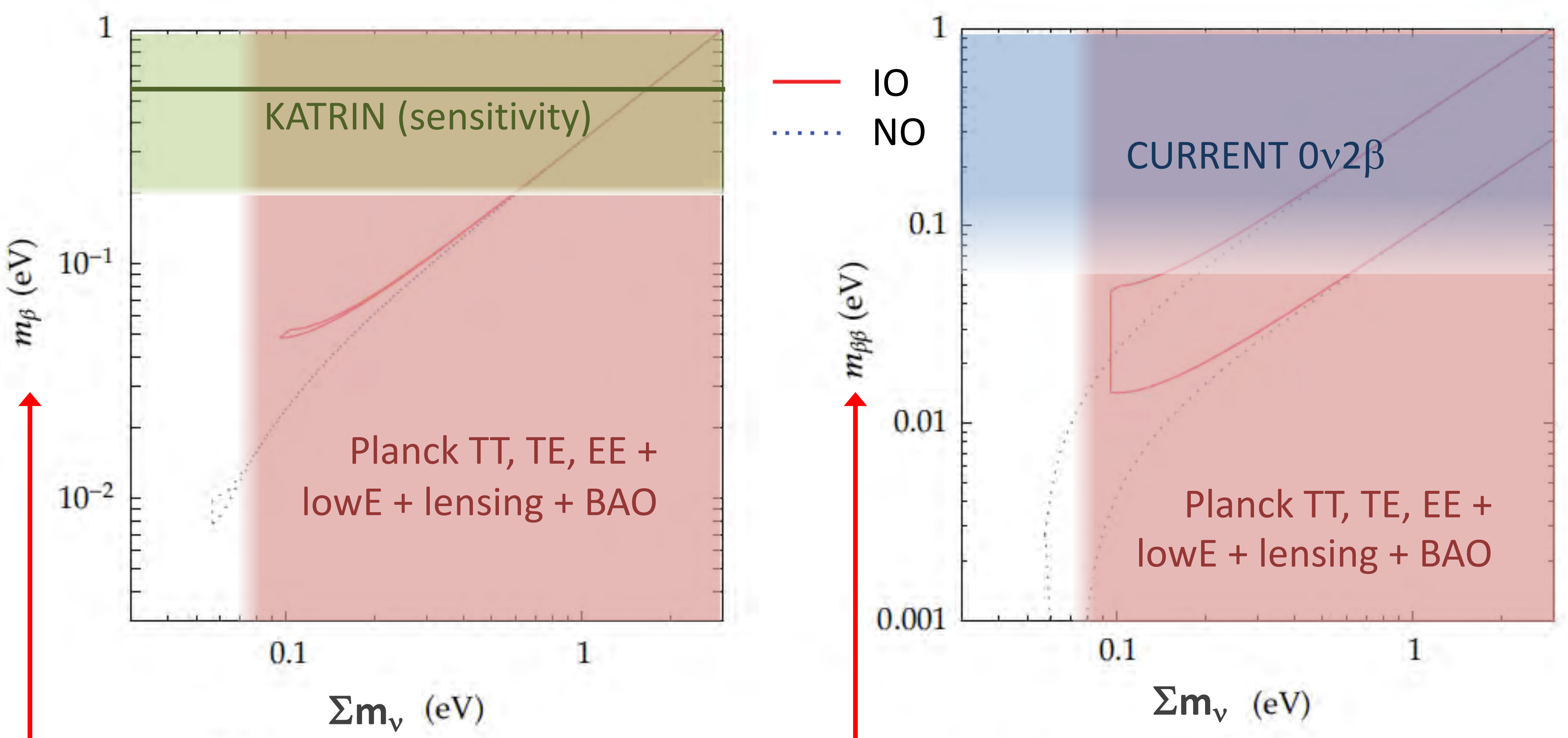
see e.g.
D. Wang et al
arXiv:2405.03368

Bounds on Σm_ν from Planck (+other cosmo data)

Cosmological upper limits on the sum of neutrino masses



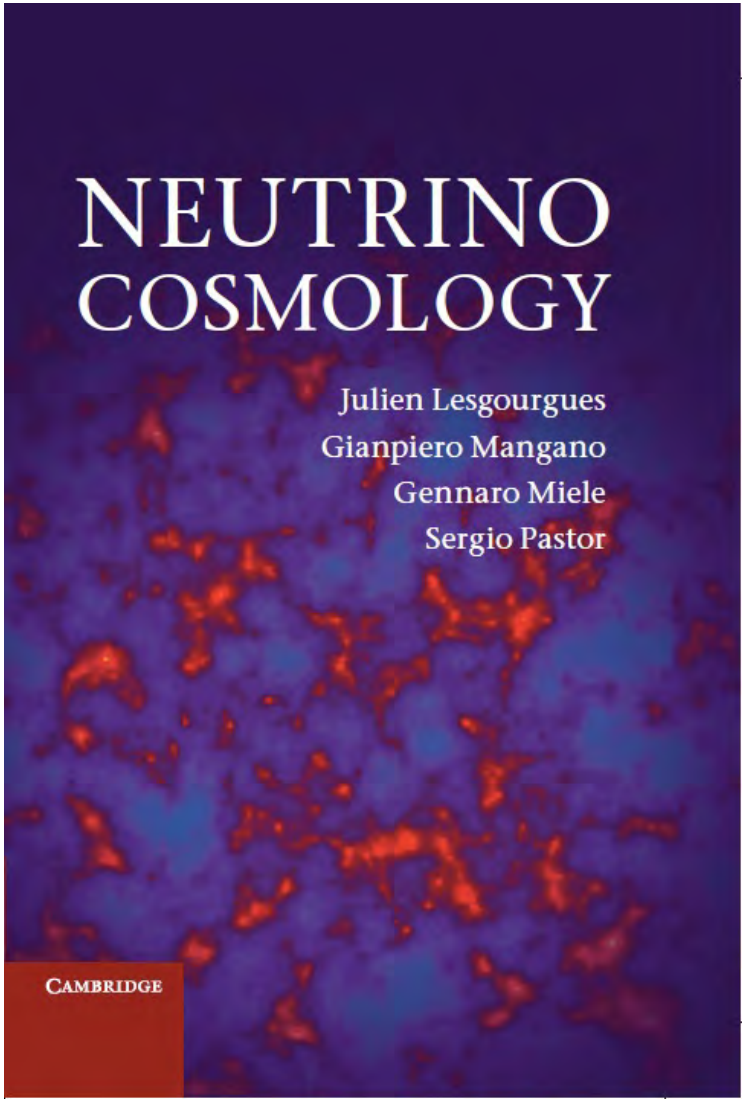
Tritium β decay, 0ν2β and Cosmology



$$[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2]^{1/2}$$

$$|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}|$$

For more details…



Ed. Cambridge Univ. Press, 2013

End of 2nd lecture