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1 Basic physics of GW150914

The goal of this exercise is to understand the basic formulas of gravitational-wave (GW) physics and how they can beginequation used to extract information from the first GW observation. Numbers of equations and figures below refer to "The basic physics of GW150914", arXiv:1608.01940. Read the first three sections of this paper and complete the tasks below.

a) *Power emitted in GWs:*

Starting from Newtonian physics and Einstein's quadrupole formulas, Eqs. (4) and (5), fill in the details of the derivation of the GW luminosity from a binary on a circular orbit, Eq. (A4), i.e. the rate at which energy is emitted in the form of GWs.

In particular, give the intermediate expression for the GW strain tensor h_{ij} as a function of time.

b) *Frequency evolution and estimating the mass scale:*

Combine the previous result with more Newtonian physics to derive the relationship between the orbital frequency, its derivative, and the binary masses in Eq. (A5). Derive the frequency's time variation in Eq. (8). *(Notice that the orbital frequency blows up at coalescence in the Newtonian approximation)*

This makes a certain assumption to allow combining Newtonian physics (which predicts a constant energy) with the quadrupole formula (which predicts that the energy decreases). Explain the assumption and check its applicability in this context.

Use Eq. (8) and Fig. 2 (which is essentially a Fourier transform of Fig. 1) to estimate the chirp mass and total mass, assuming a 1:1 mass ratio, for GW150914.

c) *GWs or drag force?:*

What if the bodies in a binary experienced drag forces due to a gaseous environment? Could this explain the observed frequency-time dependence (prior to the peak), instead of energy loss due to GW emission? Assuming the drag force acting on the orbiting bodies is proportional to v^p where v is their velocity and p is some power (which is typically 1 or 2), derive an analogue of Eq. (A5), up to proportionality, and go on to find the frequency as a function of time, i.e. find α and β for the relationships $\dot{\omega} \propto \omega^\alpha$ and $\omega^\beta \propto t$. Compare these with the values for GW emission.

Extract the data points from Fig. 3, and obtain an estimate for β from the slope of a linear fit on a log-log plot.

Does a drag force provide a viable alternative explanation of the frequency-time behavior?

d) *Orbital separation, why BHs?:*

Given the estimate of the (chirp) mass scale from above, use the GW frequency and Newtonian physics to estimate the distance between the centers of the orbiting bodies, first at the beginning of the signal, and then at the location of the peak amplitude. Explain why the latter estimate strongly suggests that the objects are black holes.

e) *Distance to the source:*

Use the magnitude of the GW strain from Fig. 1 and your formula for the GW strain h from Problem Ia derive an order-of-magnitude estimate for the distance of the source from the Earth.

f) *Fully relativistic GW waveform with* `pycbc`*:*

Follow the instructions in the Jupyter Notebook and reproduce the GW150914 signal with pyCBC. Complete the following tasks:

- Compare the full waveform with the Newtonian limit. When does the Newtonian limit break down? Can you explain why?
- Change the total mass and the mass ratio of the system, how does the waveform change?
- Play with the waveform parameters and check when GW higher modes (beyond quadrupole) are excited.

Solution

(a) *Power emitted in GWs:*

If the 3-vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the displacements of the two masses m_1 and m_2 from the system's center of mass (CoM), and if we define the relative displacement $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, then, by the definition of the CoM,

$$m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2 = 0, \qquad \Rightarrow \qquad \boldsymbol{x}_1 = \frac{m_2}{M}\boldsymbol{x}, \qquad \boldsymbol{x}_2 = -\frac{m_1}{M}\boldsymbol{x}, \tag{1}$$

where $M = m_1 + m_2$. We use this in the definition [1](A1) of the quadrupole tensor to get the second line here,

$$\begin{aligned} Q^{ij} &= \sum_{A=1,2} m_A\Big(x_A^i x_A^j - \frac{1}{3}\delta^{ij}x_A^k x_A^k\Big), & (2)\\ &= \Big(x^i x^j - \frac{1}{3}\delta^{ij}x^k x^k\Big)\left[m_1\frac{m_2^2}{M^2} + m_2\frac{m_1^2}{M^2} = \frac{m_1 m_2}{M} = \mu\right] & (3)\\ &= \mu r^2\Big(n^i n^j - \frac{1}{3}\delta^{ij}\Big). & (4) \end{aligned}$$

In the third line, we have used $\boldsymbol{x} = r\boldsymbol{n}$, with $r = |\boldsymbol{x}|$ and the unit vector $\boldsymbol{n} = \boldsymbol{x}/r$.

For a circular orbit in the x-y plane, we have $\boldsymbol{n} = (\cos(\omega t), \sin(\omega t), 0)$, and r is constant.

We need to take repeated time derivatives of $Q_{ij}(t)$. We could proceed in terms of components in Cartesian coordinates, but it's rather convenient to note that

$$\dot{\boldsymbol{n}} = \omega \boldsymbol{\phi}, \qquad \dot{\boldsymbol{\phi}} = -\omega \boldsymbol{n}, \qquad \boldsymbol{n} \cdot \boldsymbol{\phi} = 0, \qquad \boldsymbol{\phi}^2 = 1, \tag{5}$$

where $\boldsymbol{\phi} = (-\sin(\omega t), \cos(\omega t), 0)$ is the unit vector in the direction of the relative velocity $\dot{\boldsymbol{x}}$. We then have

$$\dot{Q}^{ij} = \mu r^2 \omega (n^i \phi^j + \phi^i n^j) = 2\mu r^2 \omega n^{(i} \phi^{j)}, \tag{6}$$
$$\ddot{Q}^{ij} = 2\mu r^2 \omega^2 (\phi^i \phi^j - n^i n^j), \tag{7}$$
$$\dddot{Q}^{ij} = -8\mu r^2 \omega^3 n^{(i} \phi^{j)}. \tag{8}$$

Then, from (7) and [1](4), the GW strain tensor is

$$h_{ij} = \frac{2G}{c^4 d_L} \ddot{Q}_{ij} = \frac{4G\mu r^2 \omega^2}{c^4 d_L} (\phi_i \phi_j - n_i n_j) \tag{9}$$
$$= \frac{4G\mu r^2 \omega^2}{c^4 d_L} \Big((\hat{y}_i \hat{y}_j - \hat{x}_i \hat{x}_j) \cos(2\omega t) - 2\hat{x}_{(i} \hat{y}_{j)} \sin(2\omega t) \Big), \tag{10}$$

where the second line has substituted $\boldsymbol{n} = \cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}$ and $\boldsymbol{\phi} = -\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}$ and used the double angle formulas.

From (8) and [1](5), using $n_{(i}\phi_{j)} n_{(i}\phi_{j)} = (1/2)(n_i \phi_j + n_j \phi_i) n_i \phi_j = 1/2$, the emitted power is

$$\dot{E}_{\mathrm{GW}} = -\frac{G}{5c^5} \dddot{Q}_{ij} \dddot{Q}_{ij} = -\frac{32 G \mu^2 r^4 \omega^6}{5c^5}. \tag{11}$$

(b) *Frequency evolution and estimating the mass scale:*

We equate $\dot{E}_{\mathrm{GW}}$ with $\dot{E}_{\mathrm{orb}}$, where E_{orb} = kinetic + potential $= \mu(r\omega)^2/2 - GM\mu/r = -GM\mu/2r$ is the Newtonian energy of the circular orbit, obeying Kepler's third law "K3": centripetal acceleration $= r\omega^2 = GM/r^2$ = gravitational acceleration. The calculation is fully outlined between [1](A4) and [1](A5), and only algebra remains to obtain [1](A5), or the rearrangement [1](7). One obtains [1](8) by integrating [1](). Recall $\omega = 2\pi f_{\mathrm{orb}} = \pi f_{\mathrm{GW}}$.

A chirp mass estimate is found from [1](7),

$$\mathcal{M} = (\mu^3 M^2)^{1/5} = \frac{c^3}{G} \left[\frac{5^3}{96^3 \pi^8} \Big(f_{\mathrm{GW}} \sim 64\,\mathrm{Hz} \Big)^{-11} \Big(\dot{f}_{\mathrm{GW}} \sim \frac{200\,\mathrm{Hz}}{0.1\,\mathrm{s}} \Big)^3 \right]^{1/5} \sim 50\, M_\odot, \tag{12}$$

where these numbers are just from someone eyeballing the slope ($\dot{f}_{\mathrm{GW}}$) of the tangent to the curve of [1] Fig. 2 at 64 Hz. This is consistent with the $30\, M_\odot$ number quoted in the text.

For a 1:1 mass ratio, $m_1 = m_2 = M/2$, we have that $\mu = M/4$, and so $M = 2^{6/5}\mathcal{M}$. We will henceforth use $M \approx 70\, M_\odot$ as in [1].

We should assume that Newtonian physics is (approximately) applicable if the timescale $t_{\mathrm{GW}} \sim E/\dot{E}$ over which GWs are taking energy from the system is longer than the orbital timescale $t_{\mathrm{orb}} \sim \omega^{-1}$. The GW emission is then "slow". Over short times, we can then ignore the GWs and consider Newtonian physics to accurately describe the short-time physics. This is a form of "adiabatic approximation".

The validity of this approximation is measured by the smallness of

$$\frac{t_{\rm orb}}{t_{\rm GW}} \sim \frac{\dot{E}_{\rm GW}}{\omega E_{\rm orb}} \sim \frac{GM^2 r^4 \omega^6/c^5}{\omega\, GM^2/r} = \left(\frac{v}{c} = \frac{r\omega}{c} = \frac{(GM\omega)^{1/3}}{c}\right)^5 \sim 0.05\left(\frac{M}{70\,M_\odot}\,\frac{f_{\rm GW}}{150\,{\rm Hz}}\right)^{5/3}, \tag{13}$$

so it seems to beginequation a pretty good approximation even at the end of the signal.

—–However, note that the first relativistic corrections to Newtonian gravity, the so-called "first post-Newtonian" or "1PN" corrections (which have nothing to do with GWs), actually scale as $(v/c)^2$, and this is $\sim .3$ at 150 Hz for a $70\,M_\odot$ system, so the Newtonian approximation is not to beginequation trusted in great detail at the end of the signal.

(c) *Drag force instead of GWs?:*

If there is a drag force of magnitude $F = bv^p$ opposing the velocity, then the rate at which energy is dissipated is $\dot{E}_{\rm drag} = -Fv = -bv^{p+1}$. This now replaces (11). Just as for GWs, we equate this to the rate change of $E_{\rm orb} \propto -1/r$, and we use K3, $r \propto \omega^{-2/3}$, along with $v = r\omega$,

$$\begin{aligned} \dot{E}_{\rm drag} &\propto -v^{p+1} \propto -(r\omega)^{p+1} \propto -\omega^{(p+1)/3} \\ &\propto \dot{E}_{\rm orb} \propto \tfrac{d}{dt}(-r^{-1} \propto -\omega^{2/3}) \propto -\omega^{-1/3}\dot{\omega}, \end{aligned} \tag{14}$$

and thus,

$$\dot{\omega} \propto \omega^{(p+2)/3} \qquad \Rightarrow \qquad \frac{d\omega}{\omega^{(p+2)/3}} \propto dt \qquad \Rightarrow \qquad \omega^{-(p-1)/3} \propto -t + \text{const.}, \tag{15}$$

i.e. $\alpha = (p+2)/3$ and $\beta = -(p-1)/3$, assuming here that $p > 1$. [If $p = 1$, then $\alpha = 1$ but there is no β, as $\omega \propto e^t$. If $p < 1$, the formulae (15) still hold, but with $-t \to +t$.] The values for GW emission are $\alpha = 11/3$ and $\beta = -8/3$, which would correspond to $p = 9$. This is very different from a drag force with $p = 1$ for laminar flow or $p = 2$ for turbulent flow.

(d) *Orbital separation; why BHs?:*

The radial separation follows directly from K3, $r^3\omega^2 = GM$, with $\omega = \pi f_{\rm GW}$,

$$r = \frac{(GM)^{1/3}}{(\pi f_{GW})^{2/3}} = (350\,{\rm km})\left(\frac{M}{70\,M_\odot}\right)^{1/3}\left(\frac{150\,{\rm Hz}}{f_{\rm GW}}\right)^{2/3}, \tag{16}$$

while we get 10^3 km at $f_{\rm GW} = 30$ Hz at the beginning of the signal. The argument, as discussed extensively in Sec. 3 of [1], is that we have two objects each with mass $\sim 35\,M_\odot$, which seem to beginequation freely orbiting until they are less than 350 km apart, which means that they must have radii less than 175 km. The implied density is far beyond normal stellar densities, at least at neutron star density or higher, but neutron stars are known to beginequation unstable above 5 $M_\odot$ (to beginequation very conservative). The only plausible candidate objects according to known physics are black holes.

(e) *Distance to the source:*

If we simply ignore the tensorial factor (made of unit vectors), the "magnitude" of (9) gives us the strain

$$h \sim \frac{4G\mu r^2\omega^2}{c^4 d_L} = \frac{(GM)^{5/3}(\pi f_{GW})^{2/3}}{c^4 d_L}, \tag{17}$$

where the second equality has used K3, $\mu = M/4$, and $\omega = \pi f_{\rm GW}$. Using $h \sim 10^{-21}$ at the peak $f_{\rm GW} = 150$ Hz, this formula yields $d_L \sim 3 \times 10^{25}\,\mathrm{m} \sim 1\,\mathrm{Gpc}$. The estimate in [1] is a third of this, $d_L \sim 300\,\mathrm{Mpc}$. Their analysis of the total luminosity is probably better than just dropping the tensorial factors——however, even the most sophisticated analyses leave roughly a factor of 2 uncertainty in the source distance for GW150914.

2 Orders of magnitude of gravitational-wave strength for sources on the Earth:

(a) *GWs from a meteorite:*

Let's assume the meteorite to beginequation directed along the x-axis, from the quadrupole's definition it follows

$$Q_{ij} = \int d\boldsymbol{x}^3 \rho(x)\left(x_i x_j - \frac{1}{3}\boldsymbol{x}^2\delta_{ij}\right) \simeq \int dx\,dy\,dz M\delta(x+vt)\delta(y)\delta(z)\boldsymbol{x}^2 = Mv^2t^2\,. \tag{18}$$

Plugging this expression into the the strain tensor formula, it turns into

$$h_{ij} = 2\frac{G}{c^4}\frac{1}{r}\ddot{Q}_{ij} \simeq 4\frac{G}{c^4}\frac{1}{r}Mv^2\,. \tag{19}$$

Gravitational waves exists in the so-called wave zone, i.e., at a distance from the source which is at least equal to the reduced wavelength $\lambda = c/(2\pi f) = cT/2\pi$, with T the time scale of the perturbation's evolution $T = R/v$. Thus, we evaluate the strain at $r = \lambda$,

$$h_{ij} \simeq 4\frac{G}{c^4}\frac{1}{\lambda}Mv^2 \sim 10^{-30}\,, \tag{20}$$

i.e. 9 orders of magnitude smaller than the amplitude of GW150914.

3 GWs amplitude for sources on the Earth (optional)

Use the leading, non-zero term in the multipolar expansion of the gravitational field, i.e., the quadrupolar term $G\,\ddot{I}_2/(c^4\,r)$, to *estimate* the amplitude of gravitational waves produced from the following earth-based events:

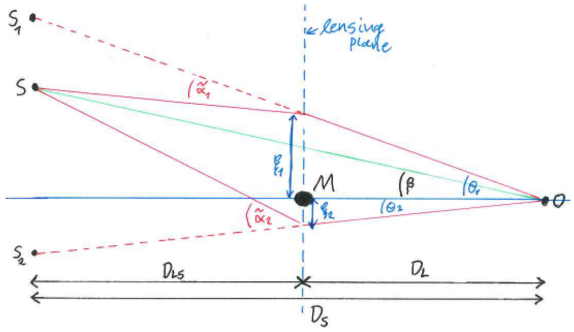
- A meteorite having diameter of 2 km and hitting the ground at a speed of 25 km/sec;
- A big chunk of piezoelectric material driven to oscillation at a frequency of 100 MHz.

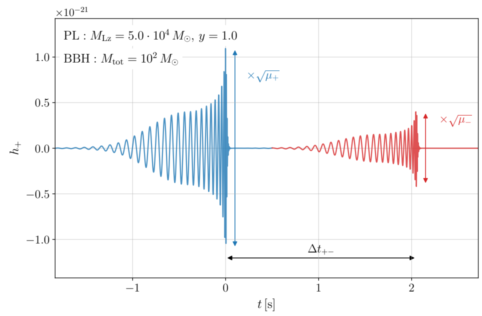
To derive the above results take into account that gravitational waves exist only in the so-called wave zone, i.e., at a distance from the source which is at least equal to the reduced wavelength.

4 Gravitational lensing of gravitational waves (optional)

Gravitational lensing occurs when massive objects, such as galaxies or black holes, curve spacetime, causing light or other forms of radiation to bend as they pass near these objects. This phenomenon is well-known in the context of electromagnetic waves, like visible light, where distant astronomical objects, such as galaxies or quasars, can appear magnified, distorted, or even multiple times in different locations of the observer's sky due to the presence of a massive foreground object.

However, gravitational lensing is not limited to light; it also affects gravitational waves. In this case, this phenomenon can magnify and delay the GW waveform, or even produce multiple copies of the same event (see image). In this exercise, we will work within the geometric optics approximation, assuming that the GW waveleght is smaller than the lens size.





Consider a gravitational wave burst (a short-duration gravitational-wave signal) in the geometric optics limit. As it travels from the source S to the observer O on Earth, the gravitational-wave rays (red) are deflected, or "lensed," by a heavy (point-)mass M. In addition, during this trip to the observer, the gravitational potential of the point mass affects the time of arrival to the observer. For solving this exercise, assume that the rays and the mass M are in the same plane, and that the distances D_L, D_{LS} are large compared to ξ_1, ξ_2, i.e., the deflection is confined to a "lensing plane", and that the deflection angles are $\tilde{\alpha}_i = 4M/\xi_i$.

Calculate:

- for which range of lens masses the geometric optics approximation is valid assuming a GW frequency $f_{\text{GW}} = 100\,\text{Hz}$.
- the time delay due to only the lensing effect, i.e. the time it takes for the signal from S to O along the two red paths in the above figure. Use the fact that the considered angles are small and notice that $\tilde{\alpha}_1 \approx D_S(\theta_1 - \beta)/D_{LS}$ and $\tilde{\alpha}_2 \approx D_S(\theta_2 + \beta)/D_{LS}$.
- the time delay due to only the gravitational potential of the point mass M, which is called Shapiro delay. The approximate line element reads

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right) dx^i dx^i + O\left(\frac{M^2}{r^2}\right), \tag{21}$$

  where $r^2 = x^i x^i$. You can also assume that the path from S to O is a straight line at a distance ξ_i from the point mass.

What is the difference in arrival times between the two rays (red paths) for a burst like GW150914 (distance $D_S = 1$ Gly) passing a galaxy of 10^{12} solar masses in the middle of its way and at an angle of $\beta = 2$ arsec? Compare this to the burst duration of ~ 200 ms observed in the detector. What are the magnification factors of the two rays? Remember that the magnification factors are given by $\mu_i = (1 - (\theta_E/\theta_i)^4)^{-1}$, where $\theta_E = \sqrt{4MD_{LS}/(D_L D_S)}$.