# Progress in waveform modeling with the PN-MPM formalism

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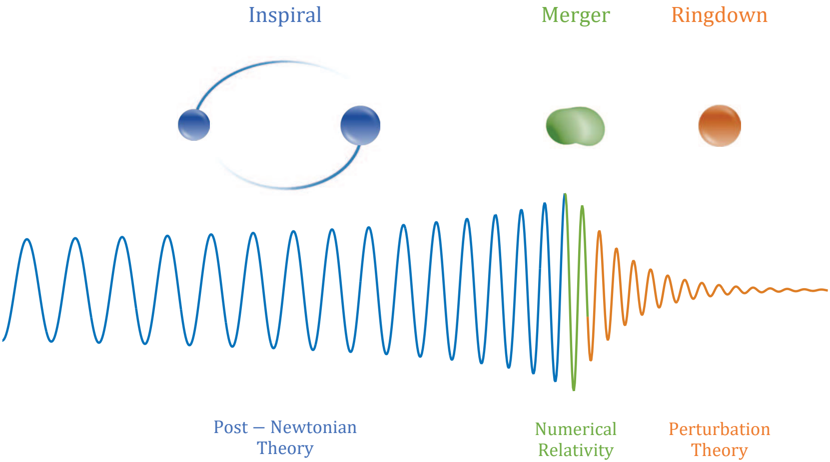
# Introduction

# Inspiral-merger-ringdown



# What are the most important predictions ?

All methods (PN, GSF, NR, ...) lead to a prediction in the form:

$$h_+ - \mathrm{i}h_\times = \frac{1}{r}\sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell}\hat{h}_{\ell m}(t)\,\mathrm{e}^{-\mathrm{i}m\psi(t)}\,Y_{-2}^{\ell m}(\theta,\phi) + \mathcal{O}\left(\frac{1}{r^2}\right)$$

The (half-)phase $\psi$ is that of the $(2,2)$ mode, such that only $\hat{h}_{22} \in \mathbb{R}$. The (half-)frequency of the $(2,2)$ is denoted $\Omega = \mathrm{d}\psi/\mathrm{d}t$, and it is adimensionalized in the PN parameter $\boxed{x = \left(\frac{Gm\Omega}{c^3}\right)^{2/3} \ll 1}$

Since the frequency $x$ is monotonous in the time $t$ (GW chirp !), most gauge invariant results are expressed in terms of $x$

# Post-Newtonian theory in a nutshell

Two-body system obeys virial theorem:

$$\left(\frac{v_{12}}{c}\right)^2 \approx \frac{Gm_{\mathrm{tot}}}{r_{12}c^2}$$

Newton's law of gravitation : $\mathbf{a}_1 = -\frac{Gm_2}{{r_{12}}^2}\mathbf{n}_{12}$.

The quadrupole formula predicts they emit GWs as:

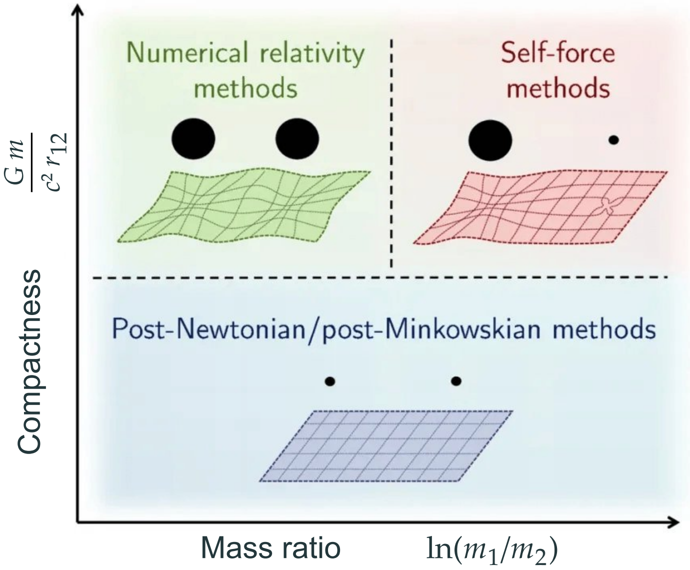
$$h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 R} \perp_{ij,ab}^{\mathrm{TT}} \frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2} \,,$$

where $Q_{ij} = m_1 \left[y_1^i(t) y_1^i(t) - \frac{1}{3}\delta^{ij}|y_1|^2\right] + (1 \leftrightarrow 2)$.

Systems that orbit faster (but not too fast!) can be described with post-Newtonian (PN) corrections in powers of $(v/c)^2 \ll 1$

A correction of order $(v/c)^{2n}$ is said to be of order $n$PN.

# NR, GSF and PN-MPM



# Results in GR at 4.5PN & comparison with GSF

# Flux at 4.5PN

We found [Blanchet, Faye, Henry, Larrouturou & Trestini 2023a]:

$$
\begin{aligned}
\mathcal{F} = {} & \frac{32c^5}{5G}\nu^2 x^5 \\
& \times\Bigg\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
& \qquad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \qquad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
& + \Bigg[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \\
& \qquad - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\Bigg]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}
\end{aligned}
$$

where $\boxed{m = m_1 + m_2}$ is the total mass and $\boxed{\nu = m_1 m_1/(m_1+m_2)^2}$ is the symmetric mass ratio.

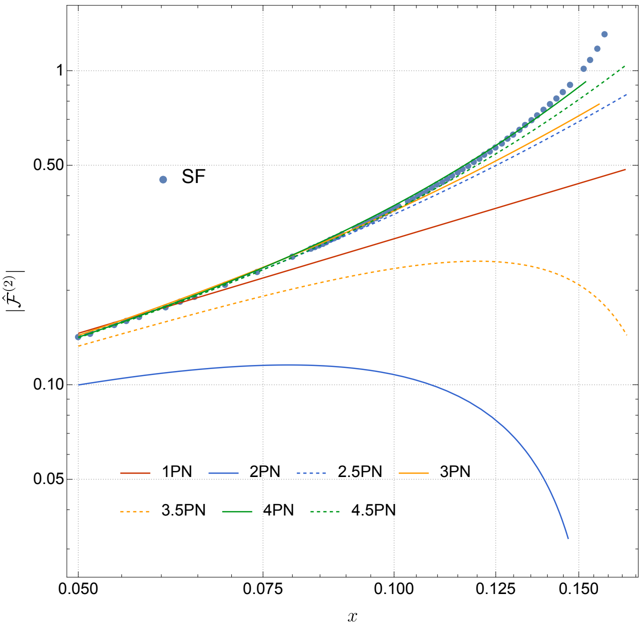
The 4.5N term comes from [Marchand, Blanchet & Faye 2016].

# Comparison with GSF

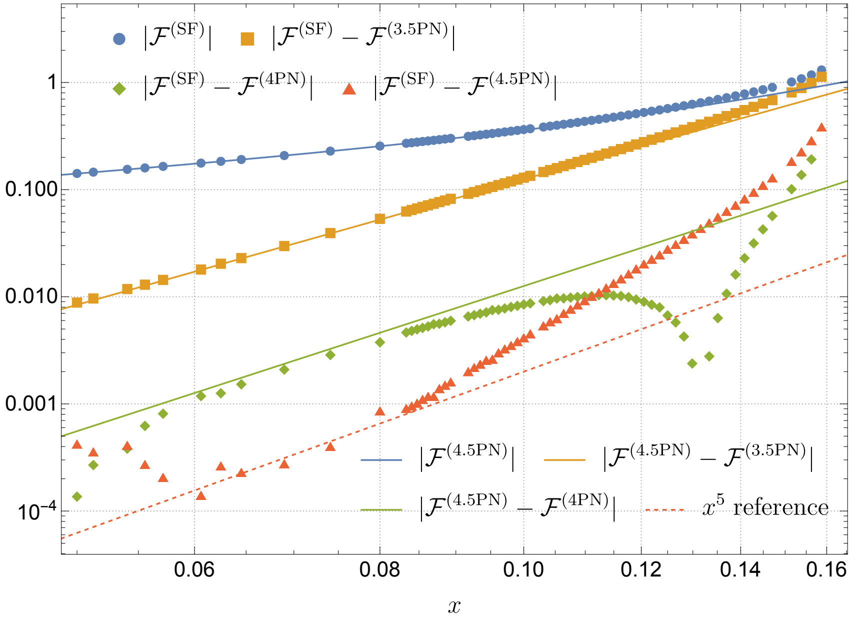
The 1SF result ($\nu^2$ coefficient in the flux) is known analytically to very high PN order [Tagoshi & Sasaki 1994], and we are in perfect agreeement.

The 2SF result ($\nu^3$ coefficient in the flux) was obtained numerically [Warburton, Pound, Wardell, Miller & Durkan 2021], and we have recently found very good agreement for all the coefficients [Warburton, Wardell, Trestini, Henry, Pound, Blanchet, Durkan, Faye & Miller 2024]

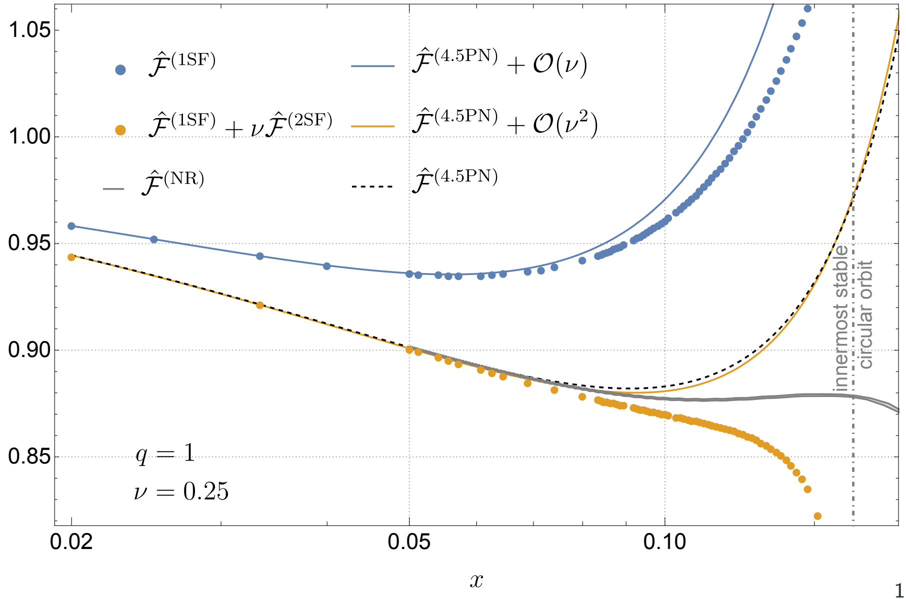
# 2SF flux versus cumulative PN flux



# PN coefficients are individually consistent with 2SF



# Fluxes: PN vs GSF vs NR for $q = 1$



# Mode decomposition of the waveform

Decompose asymptotic waveform into spherical harmonics

$$h_+ - \mathrm{i}h_\times = \frac{1}{r}\sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell}\hat{h}_{\ell m}(t)\,\mathrm{e}^{-\mathrm{i}m\psi(t)}\,Y_{-2}^{\ell m}(\theta,\phi) + \mathcal{O}\left(\frac{1}{r^2}\right)$$

Normalize the modes: $\hat{h}_{\ell m} = \sqrt{\frac{64\pi}{5}}\nu x \hat{H}_{\ell m}$.

Decompose the flux

$$\mathcal{F} = \sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\mathcal{F}_{\ell m}$$

where

$$\mathcal{F}_{\ell m} = \frac{c^3}{8\pi G}|\dot{h}_{\ell m}|^2$$

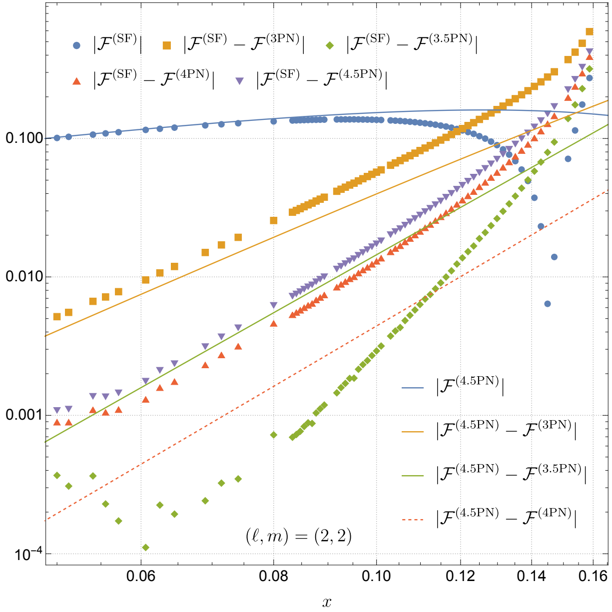
# $(2,2)$ mode at 4PN

We find [Blanchet, Faye, Henry, Larrouturou & Trestini 2023a]:

$$\hat{H}_{22} = 1 + \left(-\frac{107}{42} + \frac{55}{42}\nu\right)x + 2\pi x^{3/2} + \left(-\frac{2173}{1512} - \frac{1069}{216}\nu + \frac{2047}{1512}\nu^2\right)x^2$$
$$+ \left[-\frac{107\pi}{21} + \frac{34\pi}{21}\nu\right] x^{5/2}$$
$$+ \left[\frac{27027409}{646800} - \frac{856}{105}\gamma_E + \frac{2\pi^2}{3} + \left(-\frac{278185}{33264} + \frac{41\pi^2}{96}\right)\nu - \frac{20261}{2772}\nu^2 \right.$$
$$\left. + \frac{114635}{99792}\nu^3 - \frac{428}{105}\ln(16x)\right] x^3$$
$$+ \left[-\frac{2173\pi}{756} - \frac{2495\pi}{378}\nu + \frac{40\pi}{27}\nu^2\right] x^{7/2}$$
$$+ \left[-\frac{846557506853}{12713500800} + \frac{45796}{2205}\gamma_E - \frac{107}{63}\pi^2 + \frac{22898}{2205}\ln(16x) \right.$$
$$+ \left(-\frac{336005827477}{4237833600} + \frac{15284}{441}\gamma_E - \frac{9755}{32256}\pi^2 + \frac{7642}{441}\ln(16x)\right)\nu$$
$$\left. + \left(\frac{256450291}{7413120} - \frac{1025}{1008}\pi^2\right)\nu^2 - \frac{81579187}{15567552}\nu^3 + \frac{26251249}{31135104}\nu^4\right] x^4 + \mathcal{O}(x^{9/2})$$

We exactly recover the analytical 1SF result of [Tagoshi & Sasaki 1994].

# Disagreement 2SF at at level of individual $\mathcal{F}_{\ell m}$ modes



# Energy-flux balance equation

We postulate energy conservation:

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}}$$

where $E$ is the conservative energy of the bound system and $\mathcal{F}$ the energy flux of GWs carried at infinity.

If we have explicit expression in terms of the variable $x$, i.e. $\mathcal{F}(x)$ and $E(x)$, we recast the balance equation as

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathcal{F}}{(\mathrm{d}E/\mathrm{d}x)}} \quad \Longrightarrow x(t) = \dots \qquad \text{(the frequency chirp)}$$

$$\boxed{\frac{\mathrm{d}\psi}{\mathrm{d}x} = -\frac{c^3 x^{3/2}}{Gm}\frac{\mathrm{d}E/\mathrm{d}x}{\mathcal{F}(x)}} \quad \Longrightarrow \psi(x) = \dots \quad \text{where} \quad \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \equiv \Omega(t)$$

# Frequency chirp at 4.5PN

Define dimensionless time $\boxed{\tau = \nu c^3 (t_0 - t)/(5Gm)}$ $[x = (Gm\Omega/c^3)^{2/3}]$

We find [Blanchet, Faye, Henry, Larrouturou & Trestini 2023b]:

$$
\begin{aligned}
x = \frac{\tau^{-1/4}}{4}\Bigg\{ & 1 + \left(\frac{743}{4032} + \frac{11}{48}\nu\right)\tau^{-1/4} - \frac{1}{5}\pi\,\tau^{-3/8} \\
& + \left(\frac{19583}{254016} + \frac{24401}{193536}\nu + \frac{31}{288}\nu^2\right)\tau^{-1/2} + \left(-\frac{11891}{53760} + \frac{109}{1920}\nu\right)\pi\,\tau^{-5/8} \\
& + \Bigg[-\frac{10052469856691}{6008596070400} + \frac{1}{6}\pi^2 + \frac{107}{420}\gamma_\mathsf{E} - \frac{107}{3360}\ln\left(\frac{\tau}{256}\right) \\
& \quad + \left(\frac{3147553127}{780337152} - \frac{451}{3072}\pi^2\right)\nu - \frac{15211}{442368}\nu^2 + \frac{25565}{331776}\nu^3\Bigg]\tau^{-3/4} \\
& + \left(-\frac{113868647}{433520640} - \frac{31821}{143360}\nu + \frac{294941}{3870720}\nu^2\right)\pi\tau^{-7/8} \\
& + \Bigg[-\frac{2518977598355703073}{3779358859513036800} + \frac{9203}{215040}\gamma_\mathsf{E} + \frac{9049}{258048}\pi^2 + \frac{14873}{1128960}\ln 2 + \frac{47385}{1605632}\ln 3 - \frac{9203}{3440640}\ln\tau \\
& \quad + \left(\frac{718143266031997}{576825222758400} + \frac{244493}{1128960}\gamma_\mathsf{E} - \frac{65577}{1835008}\pi^2 + \frac{15761}{47040}\ln 2 - \frac{47385}{401408}\ln 3 - \frac{244493}{18063360}\ln\tau\right)\nu \\
& \quad + \left(-\frac{1502014727}{8323596288} + \frac{2255}{393216}\pi^2\right)\nu^2 - \frac{258479}{33030144}\nu^3 + \frac{1195}{262144}\nu^4\Bigg]\tau^{-1}\ln\tau \\
& + \Bigg[-\frac{9965202491753717}{5768252227584000} + \frac{107}{600}\gamma_\mathsf{E} + \frac{23}{600}\pi^2 - \frac{107}{4800}\ln\left(\frac{\tau}{256}\right) \\
& \quad + \left(\frac{8248609881163}{2746786775040} - \frac{3157}{30720}\pi^2\right)\nu - \frac{3590973803}{20808990720}\nu^2 - \frac{520159}{1634992128}\nu^3\Bigg]\pi\,\tau^{-9/8} + \mathcal{O}(\tau^{-5/4})\Bigg\}
\end{aligned}
$$

# The phase at 4.5PN

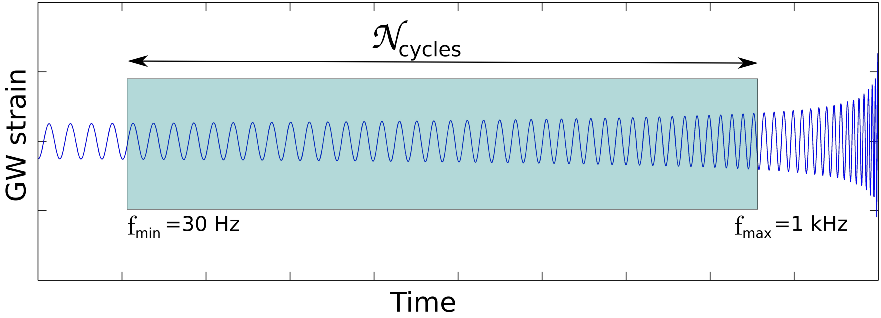
We find [Blanchet, Faye, Henry, Larrouturou & Trestini 2023b]:

$$
\begin{aligned}
\psi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right) \\
& + \bigg[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathsf{E} - \frac{856}{21}\ln(16\,x) \\
& \qquad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\bigg]x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\
& + \bigg[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_\mathsf{E} - \frac{45245}{756}\pi^2 - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 - \frac{9203}{252}\ln x \\
& \qquad + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_\mathsf{E} + \frac{109295}{1792}\pi^2 - \frac{1245514}{1323}\ln 2 + \frac{78975}{392}\ln 3 - \frac{244493}{1323}\ln x\right)\nu \\
& \qquad\qquad + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2\right)\nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4\bigg]x^4 \\
& + \bigg[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_\mathsf{E} + \frac{80}{3}\pi^2 + \frac{856}{21}\ln(16x) \\
& \qquad + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right)\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\bigg]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}
\end{aligned}
$$

Combining with the previous result yields $\psi(t)$ !

# What accuracy do we need?

We count the number of cycles $\mathcal{N}_{\text{cycles}}$ in the detector's bandwidth (e.g. $[f_{\text{min}}, f_{\text{max}}] = [30\,\text{Hz}, 1\,\text{kHz}]$)




Rule of thumb: don't miss a cycle

$\Delta\mathcal{N}_{\text{cycles}} < 1/2$ [Cutler et al. 1993]

But nowadays, we actually need much better accuracy than that!

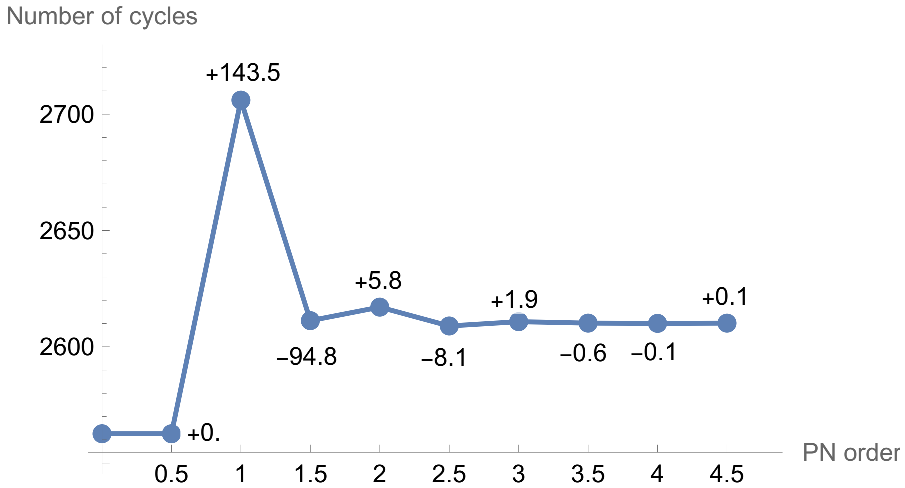
# Behavior of the PN series

Cumulative contribution to the number of cycles

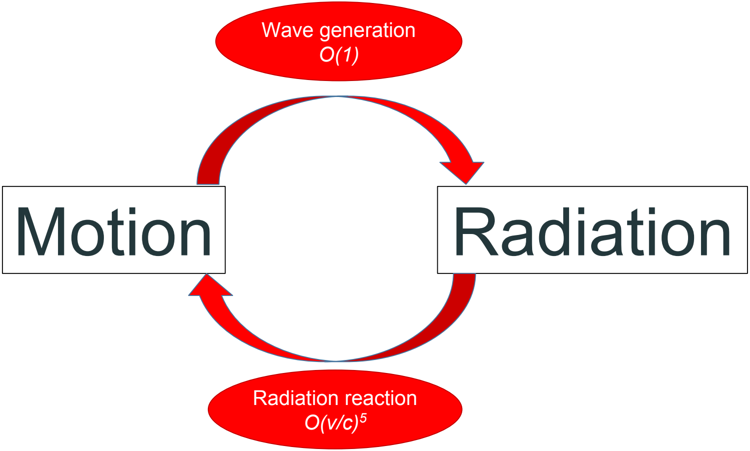
$$\mathcal{N}_{\text{cycles}} = \mathcal{N}^{\text{N}}_{\text{cycles}} + \mathcal{N}^{\text{1PN}}_{\text{cycles}} + \mathcal{N}^{\text{1.5PN}}_{\text{cycles}} + \dots$$

| $\mathcal{N}_{\text{cycles}}$ | LIGO/Virgo | | ET | | LISA | |
|---|---|---|---|---|---|---|
| $f$-band | $[30, 10^3]$ Hz | | $[1, 10^4]$ Hz | | $[10^{-4}, 10^{-1}]$ Hz | |
| $M_\odot$ | $1.4 \times 1.4$ | $10 \times 10$ | $1.4 \times 1.4$ | $500 \times 500$ | $10^5 \times 10^5$ | $10^7 \times 10^7$ |
| N | 2 562.599 | 95.502 | 744 401.36 | 37.90 | 28 095.39 | 9.534 |
| 1PN | 143.453 | 17.879 | 4 433.85 | 9.60 | 618.31 | 3.386 |
| 1.5PN | -94.817 | -20.797 | −1 005.78 | -12.63 | -265.70 | -5.181 |
| 2PN | 5.811 | 2.124 | 23.94 | 1.44 | 11.35 | 0.677 |
| 2.5PN | -8.105 | -4.604 | -17.01 | -3.42 | -12.47 | -1.821 |
| 3PN | 1.858 | 1.731 | 2.69 | 1.43 | 2.59 | 0.876 |
| 3.5PN | -0.627 | -0.689 | -0.93 | -0.59 | -0.91 | -0.383 |
| 4PN | -0.107 | -0.064 | -0.12 | -0.04 | -0.12 | -0.013 |
| 4.5PN | 0.098 | 0.118 | 0.14 | 0.10 | 0.14 | 0.065 |

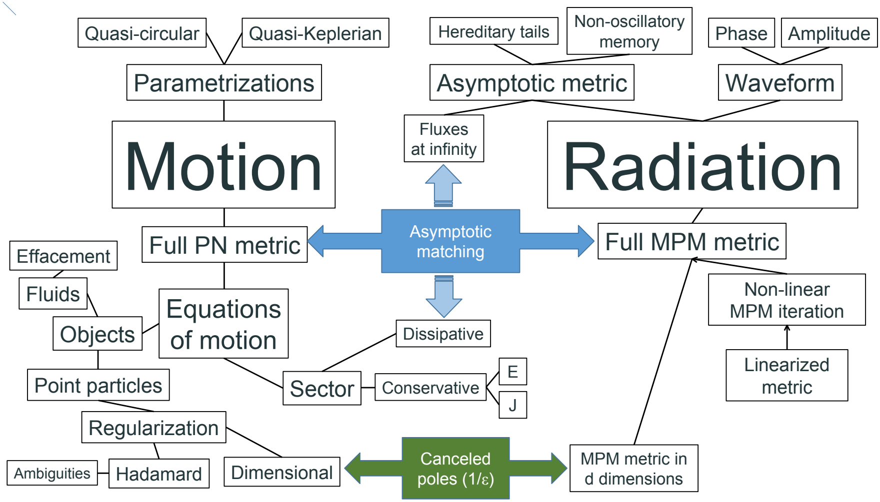
# Behavior of the PN series for a $1.4M_\odot \times 1.4M_\odot$ LVK binary



# Wave generation : big picture



# Wave generation : in practice



# The wave-generation formalism

# Field equations in general relativity

Define the quantity: $h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu}$

Einstein field equations (without $\Lambda$) in the Landau-Lifschitz formulation:

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}(-g)T^{\mu\nu} + \Lambda^{\mu\nu}[h]$$
$$\partial_\nu h^{\mu\nu} = 0$$

Also written

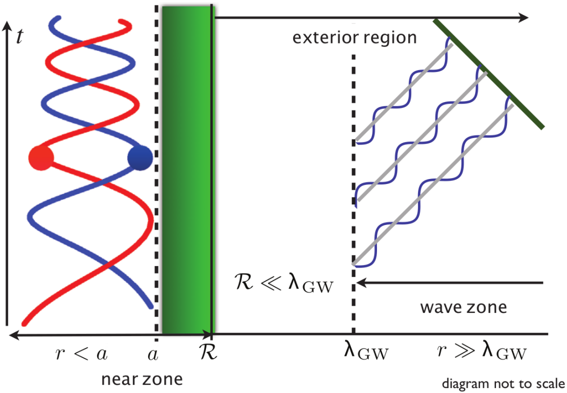
$$\Box h^{\mu\nu} = (16\pi G/c^4)\tau^{\mu\nu}$$

, where we introduced the Landau-Lifschitz pseudo-tensor

$$\tau^{\mu\nu} = (-g)T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}[h]$$

N.B.: $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$

# Near zone vs. exterior vacuum zone




Expand in $\left(\frac{v_{12}}{c}\right)^2 \sim \frac{Gm}{r_{12}c^2} \ll 1$

Valid only for $r \ll \lambda_{\rm GW}$

Post-Newtonian (PN) expansion

Expand in $|h^{\mu\nu}| \ll 1$

Valid only for $T^{\mu\nu} = 0 \Rightarrow r > a$

Multipolar post-Minkowskian (MPM) expansion

# Linearized exterior vacuum solution

Einstein equations in vacuum: $\Box h^{\mu\nu} = \Lambda^{\mu\nu}[h]$ and $\partial_\nu h^{\mu\nu} = 0$

First step: solve the linear vacuum equations

$$\boxed{\Box h_1^{\mu\nu} = 0} \qquad \text{and} \qquad \boxed{\partial_\nu h_1^{\mu\nu} = 0}$$

First, we solve $\Box h_1^{\mu\nu} = 0$ and ignore $\partial_\nu h_1^{\mu\nu} = 0$

$$h_1^{\mu\nu} = \sum_{\ell \geqslant 0} \partial_L \left[ r^{-1} \mathrm{K}_L^{\mu\nu} \left( t - \frac{r}{c} \right) \right]$$

where $\partial_L \equiv \partial_{i_1} ... \partial_{i_\ell}$ and $K_L = K_{i_1 ... i_\ell}$ are contracted together.

# Linearized exterior vacuum solution

We now want the solution to both the wave and gauge equations

$$\boxed{\Box h_1^{\mu\nu} = 0} \qquad \text{and} \qquad \boxed{\partial_\nu h_1^{\mu\nu} = 0}$$

We find that there exists a gauge in which the general solution reads

$$h_1^{00} = -\frac{4}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{M}_L\left(t-\frac{r}{c}\right)\right]$$

$$h_1^{0i} = \frac{4}{c^3}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[r^{-1}\mathrm{M}^{(1)}_{iL-1}\left(t-\frac{r}{c}\right)\right] + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left[r^{-1}\mathrm{S}_{bL-1}\left(t-\frac{r}{c}\right)\right]\right\}$$

$$h_1^{ij} = -\frac{4}{c^4}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[r^{-1}\mathrm{M}^{(2)}_{ijL-2}\left(t-\frac{r}{c}\right)\right] + \frac{2\ell}{\ell+1}\partial_{aL-2}\left[r^{-1}\epsilon_{ab(i}\mathrm{S}^{(1)}_{j)bL-2}\left(t-\frac{r}{c}\right)\right]\right\}$$

where $\mathrm{M}_L$ and $\mathrm{S}_L$ are the mass- and current-type **canonical moments**.

*Ask me about residual gauge freedom*

# The canonical moments

For now, the canonical moments $\mathrm{M}_L$ and $\mathrm{S}_L$ are just free parameters.

They are determined by matching to the near zone metric for a system of two point particles!

*Ask me about UV regularization for point particles*

This can be done systematically and expression are complicated, but the leading order is what we expect:

$$\boxed{\mathrm{M}_{ij} = \int \mathrm{d}^3\mathbf{x}\,\rho(x)\hat{x}_{ij} + \mathcal{O}\left(\frac{1}{c^2}\right)}$$

where $\rho = (T^{00}/c^2)$

*Ask me about the matching procedure*

# Full vacuum PM solution

The linearized metric $h_1^{\mu\nu}$ is the **seed** to construct the full metric.
Assuming $h \ll 1$, write the PM expansion [Blanchet & Damour 1986] of the metric:

$$\boxed{h = Gh_1 + G^2 h_2 + G^3 h_3 + ...}$$

which should solve the **non-linear vacuum equation** $\Box h^{\mu\nu} = \Lambda^{\mu\nu}$.
Thus at each order $n \geq 2$, we find

$$\boxed{\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, ..., h_{n-1}]}$$

For example, at quadratic order, we have $\Box h_2^{\mu\nu} = N^{\mu\nu}[h_1, h_1]$

At cubic order, $\Box h_3^{\mu\nu} = N^{\mu\nu}[h_1, h_2] + N^{\mu\nu}[h_2, h_1] + M^{\mu\nu}[h_1, h_1, h_1]$

# Integrating the MPM iteration in practice (1)

At some order $n$, we what to solve the iterative equations:
$\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, ..., h_{n-1}]$ and $\partial_\mu h_n^{\mu\nu} = 0$. *How to do in practice ?*
First, construct a particular solution to the wave equation. We choose:

$$\boxed{u_n^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}\left[\left(\frac{r}{r_0}\right)^B \Lambda_n^{\mu\nu}\right]}$$

where the retarded inverse d'Alembert operator defined as

$$\Box^{-1} f(\mathbf{x}, t) \equiv -\frac{1}{4\pi}\int \frac{\mathrm{d}\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

and where $\underset{B=0}{\mathrm{FP}}\left[\sum_{k=-n}^{\infty} \alpha_k(\mathbf{x}, t) B^k\right] \equiv \alpha_0(\mathbf{x}, t)$

The finite part operator in $u_n^{\mu\nu}$ :

- is proven to yield a correct particular solution
- cures unphysical $r \to 0$ divergences
- reduces to the usual $\Box^{-1}$ for a regular source.

# Integrating the MPM iteration in practice (2)

Once $u_n^{\mu\nu}$, compute its divergence $w_n^\mu = \partial_\nu u_n^{\mu\nu}$. In general, $w_n^\mu \neq 0$, so $u_n^{\mu\nu}$ does not satisfy the harmonic gauge condition.

From $u_n^{\mu\nu}$, it is always possible to construct a homogeneous solution $v_n^{\mu\nu}$ such that:

$$\boxed{\Box v_n^{\mu\nu} = 0} \qquad \text{and} \qquad \boxed{\partial_\mu v_n^{\mu\nu} = -w_n^\mu}$$

We then define $\boxed{h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu}}$, which satisfies both the wave and harmonic gauge conditions.

We have thus constructed explicitly a MPM solution (note that there is gauge freedom in the choice of $v_n^{\mu\nu}$)

# Asymptotic properties of the metric

In theory, we have now constructed the full metric

$$h = Gh_1 + G^2 h_2 + G^3 h_3 + ...$$

as a complicated functional of the source and gauge moments $(\mathrm{M}_L, \mathrm{S}_L)$

In harmonic coordinates, the $r \to +\infty$ structure (for $t - r/c = \text{const}$) is:

$$h \sim \sum_{p,q} f_{p,q}(t - r/c) \frac{\ln^p(r)}{r^q}$$

Is is possible to go to radiative coordinates $(R, T)$ to obtain the structure

$$h \sim \sum_{p} \frac{f_p(T - R/c)}{R^q}$$

*Ask me about how to construct radiative coordinates*

## Leading order asymptotic metric

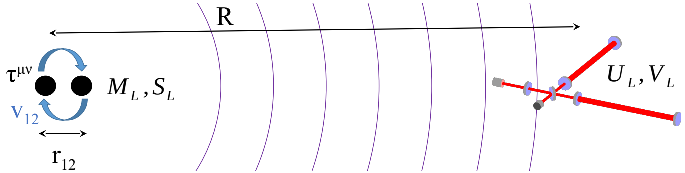
When $R$ is the distance between the GW source and the GW detector, we can only consider the leading order in $1/R$.

In a transverse-traceless gauge, the metric reads:

$$h_{ij}^{\mathrm{TT}} = -\frac{4G}{R}\perp_{ij,ab}^{\mathrm{TT}} \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left[n_{L-2}\mathcal{U}_{abL-2} - \frac{2\ell}{\ell+1}n_{cL-2}\epsilon_{cd(a}\mathcal{V}_{b)dL-2}\right]$$

where $\perp_{ij,ab}^{\mathrm{TT}} = (n_i n_j - \delta_{ij})(n_a n_b - \delta_{ab}) - \frac{1}{2}(n_a n_{(i} - \delta_{a(i})(n_{j)} n_b - \delta_{j)b})$

**The asymptotic metric is thus entirely and gauge-invariantly described by the radiative moment $\mathcal{U}_L$ and $\mathcal{V}_L$.**

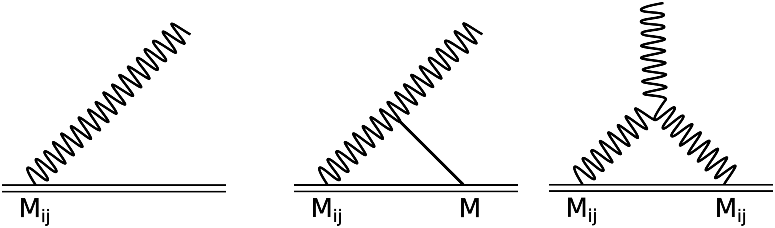


# Non-linear propagation effects: quadratic effects

Goal of MPM construction: find expression for $(\mathcal{U}_L, \mathcal{V}_L)$ in terms of $(\mathrm{M}_L, \mathrm{S}_L)$, which we know in terms of positions $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$.

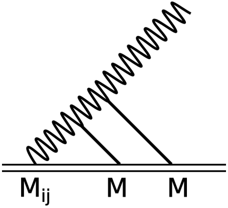
The 2.5PN relation between canonical and radiative moments reads

$$\mathcal{U}_{ij} = \mathrm{M}^{(2)}_{ij}(u) + \frac{2G\mathrm{M}}{c^3}\int_0^\infty \mathrm{d}\tau \left[\ln\left(\frac{\tau}{2b_0}\right) + \frac{11}{12}\right] \mathrm{M}^{(4)}_{ij}(u-\tau)$$
$$- \frac{2G}{7c^5}\left[\int_0^\infty \mathrm{d}\tau\, \mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(3)}_{j\rangle a}(u-\tau) + \text{(instantaneous terms)}\right] + \mathcal{O}\left(\frac{1}{c^6}\right)$$

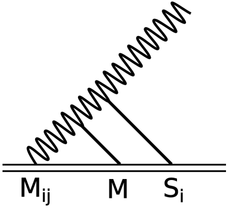



Linear quadrupolar wave | Tail | Memory [Blanchet 1998a]

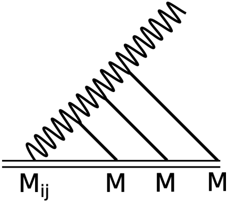
# Non-linear propagation effects: cubic and quartic effects



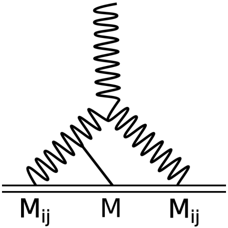

Tail-of-tail
[Blanchet 1998b]

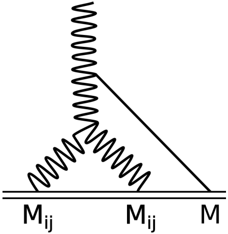


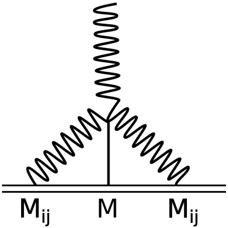

Spin-quadrupole tail
[Trestini & Blanchet 2023]




Tail-of-tail-of-tail
[Marchand, Blanchet & Faye 2016]



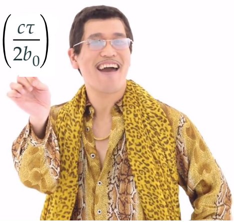



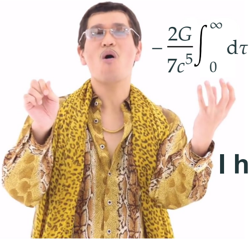




Tails of memory
[Trestini & Blanchet 2023]

$$\frac{2GM}{c^3}\int_0^\infty d\tau\, M^{(4)}_{ij}(u-\tau)\ln\left(\frac{c\tau}{2b_0}\right)$$

**I have a tail**





$$-\frac{2G}{7c^5}\int_0^\infty d\tau\,[M^{(3)}_{a\langle i}M^{(3)}_{j\rangle a}](u-\tau)$$

**I have memory**

**おお！！！**

$$\frac{8G^2M}{7c^8}\int_0^\infty d\rho\, M^{(4)}_{a\langle i}(u-\rho)\int_0^\infty d\tau\, M^{(4)}_{j\rangle a}(u-\rho-\tau)\,\ln\left(\frac{c\tau}{2b_0}\right)$$

**Tails-of-memory !**

*Ask me about the methods used to compute the tails-of-memory*

# Tails of memory: result

After the $\overline{\mathrm{M}}_{ij} \to \mathrm{M}_{ij}$ conversion, we find [Trestini & Blanchet 2023]

$$
\begin{aligned}
\mathcal{U}_{ij}^{\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}} = \frac{8G^2\mathrm{M}}{7c^8}\Bigg\{ & \int_0^{+\infty} \mathrm{d}\rho\, \mathrm{M}^{(4)}_{a\langle i}(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(4)}_{j\rangle a}(u-\rho-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{1613}{270}\right] \\
& - \frac{5}{2}\int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{3}{2}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - 3\int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}^{(2)}_{a\langle i}\mathrm{M}^{(5)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - \frac{5}{2}\int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(6)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{3}{10}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - \int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}_{a\langle i}\mathrm{M}^{(7)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{1}{4}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - 2\mathrm{M}^{(2)}_{a\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(5)}_{j\rangle a}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{27521}{5040}\right] \\
& - \frac{5}{2}\mathrm{M}^{(1)}_{a\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(6)}_{j\rangle a}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{15511}{3150}\right] \\
& + \frac{1}{2}\mathrm{M}_{a\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(7)}_{j\rangle a}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{6113}{756}\right] \Bigg\}
\end{aligned}
$$

Here, $b_0$ and $r_0$ are two arbitrary constants, we checked they cancel out!

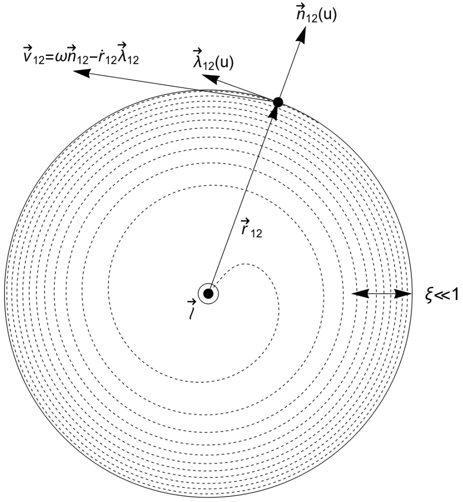
First line: “genuine” tail of memory.

# Compact binaries: quasicircular orbits

Emission of angular momentum *via* GWs $\Rightarrow$ eccentric orbits quickly circularize

Quasicircular approximation measured by adiabatic parameter $\xi \equiv \dot{\omega}/\omega^2 \ll 1$ (of order 2.5PN)

where $\omega(t)$ orbital frequency ; $\phi(t) = \int_{t_0}^{t} \omega(t')\mathrm{d}t'$ orbital phase



# Orbital phase vs. GW phase

At 4PN, we need to distinguish:

| | Orbital | GW |
|---|---|---|
| Phase | $\phi$ | $\psi$ |
| Frequency | $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$ | $\Omega = \frac{\mathrm{d}\psi}{\mathrm{d}t}$ |
| PN variable | $y = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$ | $x = \left(\frac{Gm\Omega}{c^3}\right)^{2/3}$ |

Differ due to **tail terms**. The relation reads: [Blanchet, Faye, Henry, Larrouturou & Trestini 2023a]:

$$\boxed{\psi = \phi - \frac{2GM\omega}{c^3} \ln\left(\frac{\omega}{\omega_0}\right)} \quad \text{where} \quad \omega_0 \equiv \frac{c e^{11/12-\gamma_E}}{4b_0}$$

$$\boxed{x = y\left\{1 - \frac{192}{5}\nu\, y^4\left[\ln\left(\frac{y}{y_0}\right) + \frac{2}{3}\right] + \mathcal{O}(y^5)\right\}} \quad \text{where} \quad y_0 = \left(\frac{Gm\omega_0}{c^3}\right)^{2/3}$$

# Waveforms in scalar-tensor theory

# Generalized Fierz-Pauli-Brans-Dicke theory

Action defined in Jordan frame : $S = S_{\mathrm{ST}}[g_{\alpha\beta}, \phi] + S_{\mathrm{m}}[g_{\alpha\beta}, \mathfrak{m}]$ which reads

$$S_{\mathrm{ST}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right]$$

For the post-Newtonian setup, better to work in Einstein frame. Define

$$\varphi = \frac{\phi}{\phi_0} \qquad \text{and} \qquad \tilde{g}_{\mu\nu} = \frac{\phi}{\phi_0}g_{\mu\nu} \qquad \text{where} \qquad \phi \underset{r\to\infty}{\longrightarrow} \phi_0$$

The action in Einstein frame then reads

$$S = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\Big[\tilde{R} - \frac{3+2\omega(\phi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\Big] + S_{\mathrm{m}}[\varphi^{-1}\tilde{g}_{\alpha\beta}, \mathfrak{m}]$$

# Equivalence to DEF gravity

Our Einstein frame action

$$S = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{3+2\omega(\phi_0\varphi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\right] + S_\mathrm{m}[\varphi^{-1}\tilde{g}_{\alpha\beta}, \mathfrak{m}]$$

is equivalent to DEF gravity [Damour & Esposito-Farèse 1996]:

$$S_\mathrm{DEF} = \frac{c^3}{16\pi G_*}\int \mathrm{d}^4x\sqrt{-g_*}\left[R_* - 2g_*^{\alpha\beta}\partial_\alpha\bar{\varphi}_*\partial_\beta\bar{\varphi}_*\right] + \mathcal{S}_\mathrm{m}\left[\mathcal{A}(\bar{\varphi}_*)g^*_{\alpha\beta}, \mathfrak{m}\right]$$

where $G_* = G/\phi_0$, $\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu}$ and $\bar{\varphi} = \mathcal{T}(\phi)$, where

$$\mathcal{T}(x) = \frac{1}{2}\int^x \mathrm{d}y\sqrt{\frac{3+2\omega(y)}{2y^2}}$$

This was extended to scalar-Gauss-Bonnet by [Shiralilou *et al.* 2021]

# The phase at 1.5PN for quasi-circular orbits

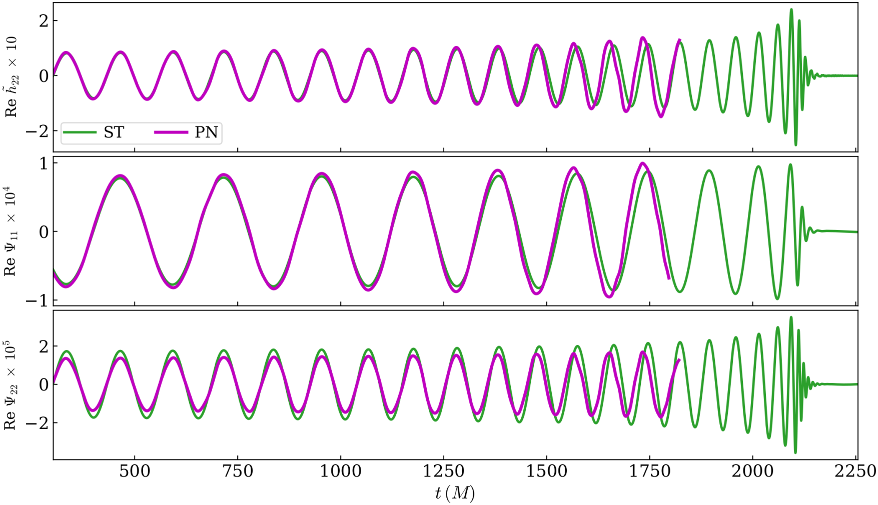
In [Blanchet, Bernard & Trestini 2022], we found:

$$\phi_{\text{circ}} = -\frac{1}{4\zeta\mathcal{S}_-^2\nu x^{1/2}}\Bigg[x^{-1} + \frac{3}{2} + 8\bar{\beta}_+ - 2\bar{\gamma} - 12\bar{\beta}_+\bar{\gamma}^{-1} - \frac{72}{5}\zeta^{-1}\mathcal{S}_-^{-2} - 6\zeta^{-1}\bar{\gamma}\mathcal{S}_-^{-2} - 12\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+ + \delta\Big[-8\bar{\beta}_- + 12\bar{\beta}_-\bar{\gamma}^{-1} + 12\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^{-1}\mathcal{S}_+\Big] + \frac{7}{2}\nu + 3\pi x^{1/2}\log(x)\left(1+\frac{\bar{\gamma}}{2}\right) + x\Big\{\text{complicated expression [Sennett, Marsat \& Buonanno 2016]}\Big\} + \frac{\pi x^{3/2}}{1-\zeta}\Big\{\text{complicated expression}\Big\}\Bigg].$$

**This is the main observable in a GW !**

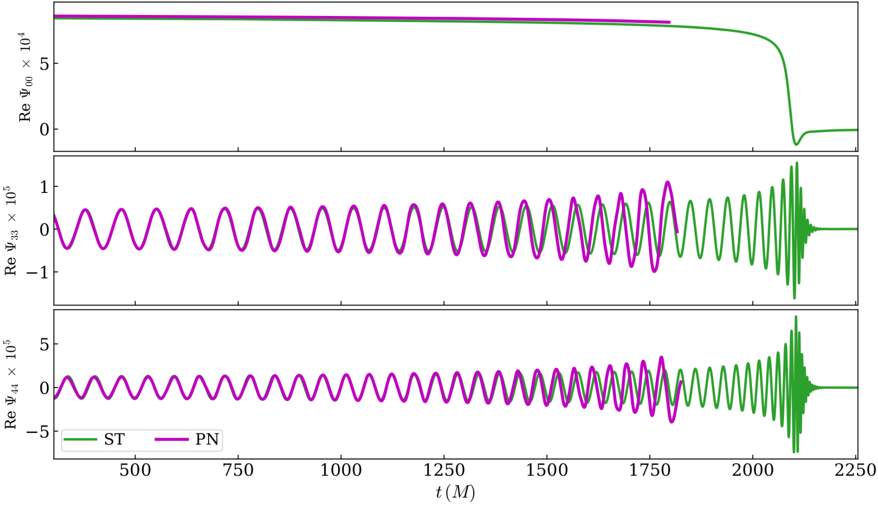
# Comparison to NR

Numerical simulations [Ma *et al.* 2023] found agreement with our results:



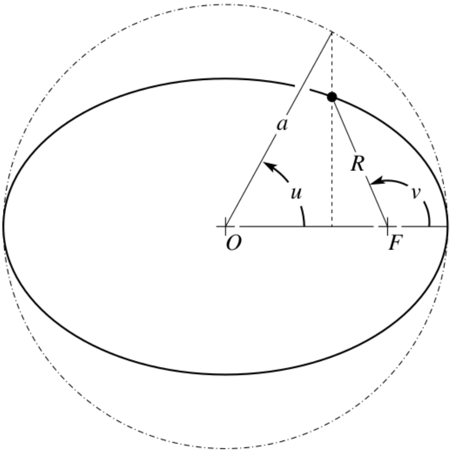
# Comparison to NR (cont'd)

even for the DC memory effect !




See also [Corman & East 2024] for a comparison in scalar Gauss Bonnet

# Extending to eccentric orbits: Kepler solution




$$a^i = -\frac{G_{12} m n^i}{r^2}$$

$$r = a(1 - e\cos u)$$
$$\ell = n(t - t_0) = u - e\sin(u)$$
$$\phi - \phi_0 = v$$
$$v = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$$

Figure from [gr-qc/0407049]

# The quasi-Keplerian solution at 1PN order

What happens if we now want to solve the equations of motion for the 1PN acceleration ? $a^i = -\frac{G_{12} m n^i}{r^2} + \frac{1}{c^2}\,(\text{many terms})^i$

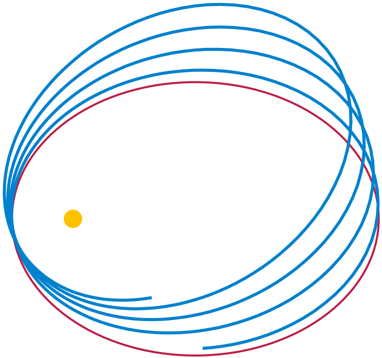
Damour & Deruelle [Ann.IHP.Phys.Th. 43, 1 (1985), p.107] showed that the equations of motion then reads

$$
\begin{aligned}
r &= a_r(1 - e_r \cos u) \\
\phi - \phi_0 &= K v \\
n(t - t_0) &= u - e_t \sin(u) \\
v(u) &= 2 \arctan \left[ \sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan\left(\frac{u}{2}\right) \right]
\end{aligned}
$$

which is the same equation as before, except:

- there are now three eccentricities $e_r$, $e_t$, $e_\phi$
- pericenter precession appears via the factor $K = 1 + k$ (with $k \ll 1$)
- $a_r$ and $n$ acquire post-Newtonian corrections

# Doubly periodic structure of QK motion



The time between two periastrons is the *radial period* denote $P$, so the mean motion $n = 2\pi/P$ is the *radial frequency*.

The time for the angular coordinate $\phi$ to go from $0$ to $2\pi$ is $P/K$, so $\omega = nK$ is the *angular frequency*

Thus, $K = 1 + k$ with $k \ll 1$ is a measure of the pericenter precession

(1)

# The quasi-Keplerian solution at 2PN order

Damour & Schäfer [Nuovo Cim.B 101 (1988) 127] showed that the QK parametrization reads at 2PN

$$
\begin{aligned}
r &= a_r(1 - e_r \cos u) \\
\phi - \phi_0 &= K\Big[v + f_\phi \sin(2v) + g_\phi \sin(3v)\Big] \\
n(t - t_0) &= u - e_t \sin(u) + f_t \sin(v) + g_t(v - u) \\
v(u) &= 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right)\right]
\end{aligned}
$$

Here, the new parameters $f_\phi$, $g_\phi$, $f_t$ and $g_t$ are all of order $\mathcal{O}(1/c^4)$, while all other parameters acquire 2PN corrections.

**I determine these parameters in [Trestini 2024a] at 2PN order**
**⇒ analytically solved the equations of motion !**

# Application: 2.5PN evolution of frequency

In [Trestini 2024b (in prep)], I obtain:

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = \frac{2c^3\zeta\nu x^4}{3\tilde{G}\alpha m}\Bigg\{ \frac{4\mathcal{S}_-^2\left(1+\frac{1}{2}e_t^2\right)}{(1-e_t^2)^{5/2}}$$
$$+ \frac{x}{15(1-e_t^2)^{7/2}}\Big(\mathfrak{X}_1 + e_t^2\mathfrak{X}_2 + e_t^4\mathfrak{X}_3\Big)$$
$$- 8\pi\left(1+\tfrac{1}{2}\bar{\gamma}\right)\mathcal{S}_-^2\varphi_1^s(e_t)\,x^{3/2}$$
$$+ x^2\left(\frac{\mathfrak{X}_4 + e_t^2\mathfrak{X}_5 + e_t^4\mathfrak{X}_6 + e_t^6\mathfrak{X}_7}{(1-e_t^2)^{9/2}} + \frac{\mathfrak{X}_8 + e_t^2\mathfrak{X}_9 + e_t^4\mathfrak{X}_{10}}{(1-e_t^2)^4}\right)$$
$$+ 4\pi\left(1+\tfrac{1}{2}\bar{\gamma}\right)x^{5/2}\Bigg(\mathcal{X}_{11}\varphi_2(e_t) + \mathcal{X}_{12}\varphi_2^s(e_t) + \mathcal{X}_{13}\alpha_1^s(e_t) + \mathcal{X}_{14}\theta_1^s(e_t)$$
$$+ \left(\mathcal{X}_{15} + e_t^2\mathcal{X}_{16}\right)\frac{\varphi_1^s(e_t)}{1-e_t^2} + \mathcal{X}_{17}\frac{\tilde{\varphi}_1^s(e_t)}{(1-e_t^2)^{3/2}} + \mathcal{X}_{18}\varphi_0^s(e_t)\Bigg)\Bigg\}$$

where the enhancement functions of $e_t$ can be obtained numerically or as a small $e_t$ expansion.

# Application: 2.5PN evolution of frequency

In [Trestini 2024b (in prep)], I obtain:

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}e_t}{\mathrm{d}t} \right\rangle = -\frac{c^3\zeta\nu x^3 e_t}{\tilde{G}\alpha m}\Bigg\{ & \frac{2\mathcal{S}_-^2}{(1-e_t^2)^{3/2}} \\
& + \frac{x}{15(1-e_t^2)^{5/2}}\left(\mathfrak{E}_1 + e_t^2\,\mathfrak{E}_2\right) \\
& + \frac{8\pi}{3}\left(1+\tfrac{1}{2}\bar{\gamma}\right)\mathcal{S}_-^2\left(\varphi_1^s(e_t) - \frac{\tilde{\varphi}_1^s}{\sqrt{1-e_t^2}}\right)x^{3/2} \\
& + x^2\left(\frac{\mathfrak{E}_3 + e_t^2\,\mathfrak{E}_4 + e_t^4\,\mathfrak{E}_5}{(1-e_t^2)^{7/2}} + \frac{\mathfrak{E}_6 + e_t^2\,\mathfrak{E}_7}{(1-e_t^2)^3}\right) \\
& + 4\pi\left(1+\tfrac{1}{2}\bar{\gamma}\right)x^{5/2}\Bigg[\mathfrak{E}_8\,\frac{1-e_t^2}{e_t^2}\left(\varphi_2(e_t) - \frac{\tilde{\varphi}_2}{\sqrt{1-e_t^2}}\right) + \mathfrak{E}_9\,\frac{1-e_t^2}{e_t^2}\left(\varphi_2^s(e_t) - \frac{\tilde{\varphi}_2^s}{\sqrt{1-e_t^2}}\right) \\
& \qquad + \mathfrak{E}_{10}\,\frac{1-e_t^2}{e_t^2}\left(\alpha_1^s(e_t) - \frac{\tilde{\alpha}_2^s}{\sqrt{1-e_t^2}}\right) + \mathfrak{E}_{11}\,\frac{1-e_t^2}{e_t^2}\left(\theta_2^s(e_t) - \frac{\tilde{\theta}_2^s}{\sqrt{1-e_t^2}}\right) \\
& \qquad + \frac{\mathfrak{E}_{12}}{e_t^2}\left(\varphi_1^s(e_t) - \frac{\tilde{\varphi}_1^s}{\sqrt{1-e_t^2}}\right) + \mathfrak{E}_{13}\,\varphi_1^s(e_t) + \frac{\mathfrak{E}_{14}}{\sqrt{1-e_t^2}}\,\tilde{\varphi}_1^s(e_t) + \mathfrak{E}_{15}\,\frac{1-e_t^2}{e_t^2}\,\varphi_0^s(e_t)\Bigg]\Bigg\}
\end{aligned}
$$

where the enhancement functions of $e_t$ can be obtained numerically or as a small $e_t$ expansion.

# Conclusion

# Conclusion

In GR:

- flux and phase at 4.5PN for circular orbit [Blanchet, Faye, Henry, Larrouturou & Trestini 2023]
- $(2, 2)$ mode at 4PN for circular orbits [idem]
- perfect agreement with 1SF and 2SF [Warburton, Wardell, Trestini, Henry, Pound, Blanchet, Durkan, Faye & Miller 2024]
- required the computation of tails of memory [Trestini & Blanchet 2023]

In ST theory

- flux, phase and modes at *relative* 2.5PN for circular orbit [Bernard, Blanchet & Trestini 2022]
- QK parametrization for eccentric orbits at 2PN [Trestini 2024a]
- averaged energy and angular momentum fluxes at *relative* 2.5PN [Trestini 2024b (in prep)]
- secular evolution of $x$ and $e_t$ at 2.5PN [Trestini 2024b (in prep)]

# Backup slides

# Solving for the PN metric in the near-zone

Ansatz for the metric in terms of potentials

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_iV_i + V^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$
$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}^i + \mathcal{O}\left(\frac{1}{c^7}\right)$$
$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\right) + \frac{4}{c^4}\hat{W}_{ij} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

Injecting metric into field equations $\Rightarrow$ potentials must satisfy:

$$\Box V = -\frac{4\pi G}{c^2}(T^{00} + T^{kk})$$
$$\Box V_i = -\frac{4\pi G}{c}T^{0i}$$
$$\Box \hat{W}_{ij} = -4\pi G\left(T^{ij} - \delta_{ij}T^{kk}\right) - \partial_i V \partial_i V$$

where $T^{\mu\nu}$ is for two point particles (with $\delta$ functions)

## UV divergences

Solving these equations, we find e.g.

$$V = \frac{Gm_1}{r_1} + \frac{Gm_2}{r_2} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

In various steps of computation (evaluating $T^{\mu\nu}$, equations of motion), need to evaluate the metric at location of particles, e.g.

$$(g_{00})_1 = -1 + \frac{2}{c^2}(V)_1 + \mathcal{O}\left(\frac{1}{c^2}\right)$$

This blows up! We need a prescription to remove "self-energy", as in Newtonian gravity.

# Hadamard regularization

Historically, this was solved with Hadamard regularizaton, which consists in noticing that the structure of any potential $F$ in the vicinity e.g. of particle $1$ reads

$$P(x) = \sum_p r_1^p \underset{1}{f}_p(n_1)$$

When $r_1 \to 0$, the $p \geq 1$ terms vanish, and we discard the divergent $p \leq -1$ terms. The Hadamard regularization consists in keeping only $p = 0$ and averaging over angles:

$$(P)_1 = \frac{1}{4\pi} \int \mathrm{d}\Omega \underset{1}{f}_0(n_1)$$

Problem: in general, $(FG)_1 \neq (F)_1(G)_1$ [this stems from the impossibility to define products in distribution theory]. This leads to ambiguities in PN results at 3PN order.

Solution: dimensional regularization !

# Dimensional regularization

Solution: dimensional regularization was introduced. Rewrite field equations in arbitrary $d$ dimensions, and set $\varepsilon = d - 3$. Modifies definitions of potentials, Green functions, etc. !

Instead of computing an expression the potential $P$ for any field point $\boldsymbol{x}$ and then setting $\boldsymbol{x} = \boldsymbol{y}_1$, we use the fact the the potential $P$ satisfied $\Box P = F$, where the source has the structure

$$F^{(d)}(x) = \sum_p r_1^{p+q\varepsilon} \underset{1}{f}{}^{(d)}_{p,q}(n_1)$$

Solving $\Box P = F$ in a PN sense actually involves the inverse d'Alembert operator

$$\overline{\Box}^{-1}_{\mathrm{ret}} = \frac{1}{\Delta - \frac{1}{c^2}\partial_t} = \sum_{k \geq 0} \frac{1}{c^{2k}} \partial_t^k \Delta^{-k-1}$$

# Dimensional regularization (cont'd)

Our main challenge is to compute

$$Q^{(d)} = \Delta^{-1} F^{(d)} = -\frac{1}{(d-2)\Omega_{d-1}} \int \frac{\mathrm{d}^d \boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^{d-2}} F^{(d)}(x')$$

in the singular limit $\boldsymbol{x} = \boldsymbol{y}_1$, but this immediately well-defined by analytic continuation in $d \in \mathbb{C}$ ! However, impossible to get closed form expression for arbitray $d \Rightarrow$ we can only get the $\epsilon \to 0$ limit.

For this, first compute $(Q)_1$ in 3D in a Hadamard sense. Then, we can compute the difference $\mathcal{D}Q_1 = Q^{(d)}(\boldsymbol{y}_1) - (Q)_1$ using the formula

$$\mathcal{D}Q_1 = -\frac{1}{(d-2)\Omega_{d-1}} \Bigg\{ \sum_q \left( \frac{1}{q\varepsilon + \ln r_1' - 1} \right) \langle \underset{1}{f}{}^{(\varepsilon)}_{-2,q} \rangle$$
$$+ \sum_q \left( \frac{1}{(q+1)\varepsilon} + \ln s_2 \right) \sum_{\ell \geq 0} \frac{(-1)^\ell}{\ell!} \partial_L \left( \frac{1}{r_{12}^{\varepsilon+1}} \langle n_2^L \underset{2}{f}{}^{(\varepsilon)}_{-\ell-3,q} \rangle \right) + \mathcal{O}(\varepsilon) \Bigg\}$$

Individual pieces of the computation have poles in $1/\varepsilon$, but we check that these all disappear in physical observables!

# Linearized exterior vacuum solution

We now want the solution to both the wave and gauge equations

$$\boxed{\Box h_1^{\mu\nu} = 0} \qquad \text{and} \qquad \boxed{\partial_\nu h_1^{\mu\nu} = 0}$$

Imposing harmonic gauge condition $\Rightarrow$ complicated structure described by 6 moments (function of retarded time only) that are STF in their indices $L$

- mass (or electric) source moment $\mathrm{I}_L(t - r/c)$
- current (or magnetic) source moment $\mathrm{J}_L(t - r/c)$
- gauge moments $\mathrm{W}_L$, $\mathrm{X}_L$, $\mathrm{Y}_L$, $\mathrm{Z}_L$

such that $h_1^{\mu\nu} = k_1^{\mu\nu}[\mathrm{I}_L, \mathrm{J}_L] + (\partial\varphi_1)^{\mu\nu}[\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L]$

# Solution to the linearized equations

Explicitly we have $\boxed{h_1^{\mu\nu} = k_1^{\mu\nu}[\mathrm{I}_L, \mathrm{J}_L] + \partial\varphi_1^{\mu\nu}[\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L]}$ where

$$k_1^{00} = -\frac{4}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{I}_L\right]$$

$$k_1^{0i} = \frac{4}{c^3}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left[r^{-1}\mathrm{I}^{(1)}_{iL-1}\right] + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left[r^{-1}\mathrm{J}_{bL-1}\right]\right\}$$

$$k_1^{ij} = -\frac{4}{c^4}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left[r^{-1}\mathrm{I}^{(2)}_{ijL-2}\right] + \frac{2\ell}{\ell+1}\partial_{aL-2}\left[r^{-1}\epsilon_{ab(i}\mathrm{J}^{(1)}_{j)bL-2}\right]\right\}$$

$$\varphi_1^{0} = \frac{4}{c^3}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{W}_L\right]$$

$$\varphi_1^{i} = -\frac{4}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\left\{\partial_{iL}\left[\frac{\mathrm{X}_L}{r}\right] + \partial_{L-1}\left[\frac{\mathrm{Y}_{iL-1}}{r}\right] + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left[\frac{\mathrm{Z}_{bL-1}}{r}\right]\right\}$$

# Why keep $\varphi_1$ free?

The reason why we keep the residual gauge freedom $\varphi_1$ is that it will be determined by matching to a post-Newtonian source. Indeed, there exists explicit expressions which univocally determine all moments (gauge moments as well !) general by a post-Newtonian source, e.g.

$$\mathrm{I}_L(u) = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \sum_{k=0}^{\infty} \left(\frac{r}{c}\right)^{2k} \left\{ \frac{a_{\ell,k}}{c^2} \hat{x}_L \frac{\mathrm{d}^{2k}\left(\overline{\tau}^{00} + \overline{\tau}^{aa}\right)}{\mathrm{d}u^{2k}} + \frac{b_{\ell,k}}{c^3} \hat{x}_{iL} \frac{\mathrm{d}^{2k+1}\overline{\tau}^{0i}}{\mathrm{d}u^{2k+1}} + \frac{c_{\ell,k}}{c^4} \hat{x}_{ijL} \frac{\mathrm{d}^{2k+2}\overline{\tau}_{ij}}{\mathrm{d}t^{2k+2}} \right\}$$

At lowest PN order, we recover the usual Newtonian formulas, e.g.

$$\boxed{\mathrm{I}_{ij} = \int \mathrm{d}^3\mathbf{x}\, \rho(x) \hat{x}_{ij} + \mathcal{O}\left(\frac{1}{c^2}\right)}$$

where $\rho = (T^{00}/c^2)$.

# Canonical construction (1)

Recall that the most general linearized metric reads $h_1^{\mu\nu} = k_1^{\mu\nu}[\mathrm{I}_L, \mathrm{J}_L] + \partial\varphi_1^{\mu\nu}[\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L]$, and that the gauge moments are typically chosen to be nonzero because of matching.

But there are a lot of moments, this is tedious ! What happens if we *insist* that $\boxed{\varphi_1^{\mu\nu} = 0}$ ? We get a metric parametrized by two *canonical* moments $\mathrm{M}_L$ and $\mathrm{S}_L$, namely

$$\boxed{h_{1,\mathrm{can}}^{\mu\nu} = k_1^{\mu\nu}[\mathrm{M}_L, \mathrm{S}_L]}$$

One then iterates the MPM algorithm in the same way, and find that **the two metric are physically equivalent but differ by a (nonlinear) coordinate transformation and a moment redefinition**.

# Canonical construction (2)

The nonlinear transformation under the coordinate transformation $x^\mu \to x'^\mu = x^\mu + \varphi^\mu(x)$ reads :

$$h^{\mu\nu}_{\text{gen}}(x') = \frac{1}{|J|}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\left(h^{\rho\sigma}_{\text{can}} + \eta^{\rho\sigma}\right) - \eta^{\mu\nu}$$

where we can expand $h^{\mu\nu}_{\text{gen}}(x') = \sum_n \frac{1}{n!}\varphi^{\lambda_1}...\varphi^{\lambda_n}\partial_{\lambda_1...\lambda_n} h^{\mu\nu}_{\text{gen}}(x)$.

Specializing to the MPM expansion, we can show that at each order, the two metrics are related by

$$\begin{aligned} h^{\mu\nu}_{\text{n,gen}}[\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, ...] =& h^{\mu\nu}_{\text{n,can}}[\mathrm{M}_L, \mathrm{S}_L] + \partial\varphi^{\mu\nu}_n \\ &+ \Omega^{\mu\nu}_n\left[\varphi_1, ..., \varphi_{n-1}; h_{1,\text{can}}, ..., h_{n-1,\text{can}}\right] \end{aligned}$$

where $\Omega_n$ is an explicitly known functional, and the moments are related by relation such as

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} + 4G\left(\mathrm{W}^{(2)}\mathrm{I}_{ij} - \mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ij}\right) + \mathcal{O}(G^2)$$

## Radiative construction (1)

The canonical construction in harmonic coordinates also exhibits annoying far zone logarithm. Solution: remove them at every order in the *radiative* construction [Blanchet 1987]. Since depart from harmonic coordinates, a divergence term appears in the Einstein equations:

$$\boxed{\partial h^{\mu\nu} - \partial H^{\mu\nu} = \Lambda^{\mu\nu}[h]}$$

where $\boxed{H^{\mu} = \partial_{\rho} h^{\mu\rho}}$. The equivalence with the Einstein equations is now ensured order-by-order by the construction itself.

At linearized order, remove $\ln(r)$ due to tails thanks to "tortoise coordinates".

$$\boxed{h_{1,\mathrm{rad}}[\overline{\mathrm{M}}_L, \overline{\mathrm{S}}_L] = h_{1,\mathrm{harm}}[\overline{\mathrm{M}}_L, \overline{\mathrm{S}}_L] + \partial \xi^{\mu\nu}}$$

where $h_{1,\mathrm{harm}}$ has the same functional form as before, and

$$\boxed{\xi^{\mu} = 2\mathrm{M}\eta^{0\mu} \ln(r/b_0)}$$

# Radiative construction (2)

At every order $n \geq 2$, we want to solve

$$\boxed{\partial h_n^{\mu\nu} - \partial H_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h]}$$

Define $k^\mu = (1, n^i)$. When $r \to +\infty$, we *crucially* have the structure

$$\boxed{\Lambda_n^{\mu\nu} = r^{-2} k^\mu k^\nu \sigma_n(u, \mathbf{n}) + \mathcal{O}(1/r^3)}$$

The $n$-th order is defined by:

$$\boxed{h_n^{\mu\nu} = u_n^{\mu\nu} + v_n^{\mu\nu} + \partial \xi_n^{\mu\nu}}$$

where

$$u_n^{\mu\nu} = \underset{B=0}{\mathrm{FP}} \, \Box^{-1} \left[ (r/r_0)^B \, \Lambda_n^{\mu\nu} \right]$$

$$\Box v_n^{\mu\nu} = 0 \qquad \partial_\nu v_n^{\mu\nu} = -\partial_\nu u^{\mu\nu}$$

$$\boxed{\xi_n^{\mu\nu} \equiv \underset{B=0}{\mathrm{FP}} \, \Box^{-1} \left[ \left( \frac{r}{r_0} \right)^B \frac{c k^\mu}{2 r^2} \int_0^\infty \mathrm{d}\tau \, \sigma_n(u - \tau, \mathbf{n}) \right]}$$

# Relation between harmonic and radiative constructions

Yet again, the canonical metric in the harmonic and radiative construction are related by

- a coordinate transformation, described by a nonlinear gauge transformation and every MPM order:

$$h^{\mu\nu}_{n,\mathrm{rad}}[\overline{\mathrm{M}}_L, \overline{\mathrm{M}}_L] = h^{\mu\nu}_{n,\mathrm{harm}}[\mathrm{M}_L, \mathrm{S}_L] + \partial\varphi^{\mu\nu}_n + \Omega^{\mu\nu}_n\left[\varphi_1, \ldots, \varphi_{n-1}; h_{1,\mathrm{can}}, \ldots, h_{n-1,\mathrm{can}}\right]$$

- a moment redefinition, for example [Trestini et al. 2023]:

$$\overline{\mathrm{M}}_{ij} = \mathrm{M}_{ij} - \frac{26}{15}\frac{G\,\mathrm{M}}{c^3}\,\mathrm{M}^{(1)}_{ij} + \frac{124}{45}\frac{G^2\mathrm{M}^2}{c^6}\,\mathrm{M}^{(2)}_{ij} + \frac{G^2\mathrm{M}}{c^8}\left[-\frac{8}{21}\,\mathrm{M}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a} - \frac{8}{7}\,\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(3)}_{j\rangle a} - \frac{8}{9}\,\epsilon_{ab\langle i}\mathrm{M}^{(3)}_{j\rangle a}\mathrm{S}_b\right].$$

# The different types of moments

| Moment type | Source | Gauge | Canonical (harmonic) | Canonical (radiative) | Radiative |
|---|---|---|---|---|---|
| Notation | $\mathrm{I}_L$, $\mathrm{J}_L$ | $\mathrm{W}_L, \mathrm{X}_L$ $\mathrm{Y}_L$, $\mathrm{Z}_L$ | $\mathrm{M}_L$, $\mathrm{S}_L$ | $\overline{\mathrm{M}}_L$, $\overline{\mathrm{S}}_L$ | $\mathcal{U}_L$, $\mathcal{V}_L$ |
| What does it parametrize | Linearized metric, general gauge | | Linearized metric, harmonic canonical gauge | Linearized metric, radiative canonical gauge | Full metric, asymptotically |
| Value of $\partial_\nu h^{\mu\nu}$ | $= 0$ | $= 0$ | $= 0$ | $\neq 0$ | $= \mathcal{O}(1/r^2)$ |
| How to compute | Matching to PN (stress-energy tensor) | | $\mathrm{M}_L = \mathrm{I}_L + \dots$ | $\overline{\mathrm{M}}_L = \mathrm{M}_L + \dots$ | $\mathcal{U}_L = \overline{\mathrm{M}}_L^{(\ell)} + \dots$ |

$$h_{\mathsf{TT}} \underset{r\to+\infty}{\sim} \frac{1}{r}\sum_{\ell=2}^{\infty} \hat{n}_L \Big(\mathcal{U}_L(u) + \epsilon \mathcal{V}_L(u)\Big)$$

# General integration technique

How to integrate in practice $u_n^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}\left[\left(\frac{r}{r_0}\right)^B \Lambda_n^{\mu\nu}\right]$ ?

We first want to compute the master integral

$$\Psi_L = \Box^{-1}\left[\hat{n}_L\, S(r,u)\right]\,, \tag{2}$$

where $S(r,u) = \mathcal{O}(r^{\ell+5})$ when $r \to 0$ with $u$ kept fixed. We define

$$R_\alpha(\rho,s) \equiv \rho^\ell \int_\alpha^\rho \mathrm{d}\lambda\, \frac{(\rho-\lambda)^\ell}{\ell!}\left(\frac{2}{\lambda}\right)^{\ell-1} S(\lambda,s)$$

where $\alpha$ is an arbitrary constant. Then the solution reads

$$\Psi_L = \int_{-\infty}^{t-r} \mathrm{d}s\, \hat{\partial}_L \left[\frac{R_\alpha\left(\frac{t-r-s}{2},s\right) - R_\alpha\left(\frac{t+r-s}{2},s\right)}{r}\right]$$

If $S(r,u)$ does not converge fast enough, consider $(r/r_0)^B S(r,u)$ instead, and take the finite part when $B \to 0$, denoted $\underset{B=0}{\mathrm{FP}}$.

# Quadratic integration formulas

How to integrate in practice $u_2^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}\left[\left(\frac{r}{r_0}\right)^B \Lambda_2^{\mu\nu}\right]$ ?

The source is typically of the form $r^{-k}\hat{n}_L F(t-r)$. In the case $k=2$, the integral converges so we can discard the finite part:

$$\boxed{\Box^{-1}\left[\frac{\hat{n}_L}{r^2}F(t-r)\right] = -\hat{n}_L \int_1^\infty \mathrm{d}x Q_\ell(x) F(t-rx)}$$

where $Q_\ell(x) = \frac{1}{2}P_\ell(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{\ell}\frac{1}{j}P_{\ell-j}(x)P_{j-1}(x)$ and $P_\ell(x)$ is the Legendre polynomial.

For $k \geq 3$, we can recursively bring ourselves to the case $k=2$.

Asymptotically,

$$\boxed{\int_1^\infty \mathrm{d}x Q_\ell(x) F(t-rx) \underset{r\to\infty}{\sim} \frac{1}{r}\int_0^\infty \mathrm{d}\tau \ln(\tau/r) F(t-r-\tau)}$$

$\Rightarrow$ these are the **tails** (the $\ln(r)$ cancel out in the radiative construction)

# Cubic integration formula

Most difficult integral to compute for the $\mathrm{M} \times \mathrm{M}_{ij} \times \mathrm{M}_{ij}$ interaction:

$$\underset{k,m}{\Psi_L} = \underset{B=0}{\mathrm{FP}} \Box^{-1} \left[ \left( \frac{r}{r_0} \right)^B r^{-k} \, G(t-r) \int_1^{+\infty} \mathrm{d}x \, Q_m(x) F(t-rx) \right]$$

where $F$ and $G$ identically vanish for $t < -\mathcal{T}$ and $Q_m$ is the Legendre function of second kind.

For $k \geq 3$, we can recursively bring ourselves to the case where $k = 1$ and $k = 2$. In the latter case, we don't need the finite parts:

$$\underset{k,m}{\Psi_L} = -\frac{\hat{n}_L}{2r} \int_0^{+\infty} \mathrm{d}\rho \, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau \, F(u-\rho-\tau) \underset{k,m}{K_\ell}(\rho, \tau, r) \,,$$

where the kernel reads

$$\underset{k,m}{K_\ell}(\rho, \tau, r) = \tau^{1-k} \int_{\frac{2\tau}{\rho+2r}}^{\frac{2\tau}{\rho}} \mathrm{d}y \, y^{k-2} Q_m(y+1) \, \Pi_\ell \left( 1 - \frac{\rho y}{\tau}, 1 + \frac{\rho}{r} \right)$$

# Seperating the logarithms

We now know how to integrate all terms, but we want to make sure that the far-zone logarithms explicitly vanish in the radiative construction. To do this, we explicitly extract the logarithmic dependency of the kernels, e.g.

$$\underset{1,m}{K_\ell}(\rho,\tau,r) = \frac{1}{4}\ln^2\left(\frac{r}{r_0}\right) - \frac{1}{2}\ln\left(\frac{r}{r_0}\right)\left[\ln\left(\frac{\tau}{2r_0}\right) + 2H_m\right] + \underset{1,m}{\overline{K}_\ell}(\rho,\tau)$$

This leads to defining elementary functionals

$$\underset{k,m}{\overline{\Psi}_\ell}[F,G] \equiv \int_0^{+\infty} \mathrm{d}\rho\, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, F(u-\rho-\tau)\, \underset{k,m}{\overline{K}_\ell}(\rho,\tau)$$

# Tails-of-memory: raw result

With the two types of elementary functionals and kernels in hand, namely

$$\underset{k,m}{\overline{\Psi}_\ell}[F,G] \equiv \int_0^{+\infty} \mathrm{d}\rho\, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, F(u-\rho-\tau)\, \underset{k,m}{\overline{K}_\ell}(\rho,\tau)$$
$$\underset{k,m}{\overline{\chi}_\ell}[F,G] \equiv \int_0^{+\infty} \mathrm{d}\rho\, G(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, F(u-\rho-\tau)\, \underset{k,m}{\overline{L}_\ell}(\rho,\tau)$$

we find the explicit but untractable result

$$\mathcal{U}_{ij}^{\mathrm{M}\times\overline{\mathrm{M}}_{ij}\times\overline{\mathrm{M}}_{ij}} = \frac{G^2\mathrm{M}}{c^8} \sum_{m,\ell,n} \Big\{ \mathcal{A}^n_{m,\ell}\, \underset{1,m}{\overline{\Psi}_\ell}\big[\overline{\mathrm{M}}^{(n)}_{a\langle i}, \overline{\mathrm{M}}^{(8-n)}_{j\rangle a}\big] + \mathcal{B}^n_{m,\ell}\, \underset{2,m}{\overline{\Psi}_\ell}\big[\overline{\mathrm{M}}^{(n)}_{a\langle i}, \overline{\mathrm{M}}^{(7-n)}_{j\rangle a}\big]$$
$$+ \mathcal{C}^n_{n,\ell}\, \underset{1,m}{\overline{\chi}_\ell}\big[\overline{\mathrm{M}}^{(n)}_{a\langle i}, \overline{\mathrm{M}}^{(8-n)}_{j\rangle a}\big] + \mathcal{D}^n_{m,\ell}\, \underset{2,m}{\overline{\chi}_\ell}\big[\overline{\mathrm{M}}^{(n)}_{a\langle i}, \overline{\mathrm{M}}^{(7-n)}_{j\rangle a}\big] \Big\}$$
$$+ \text{(terms that have a more standard and tractable form)}$$

Impossible to get a simpler integration formula, but one can hope for a simpler end result. **Idea: integrate by parts to have only one derivative combination**

## Simplification method

Introducing a regularizing lower bound $\epsilon$ which we be taken to $0$ at the end, we can integrate by parts. We find

$$\mathcal{U}_{ij}^{\mathrm{M}\times\overline{\mathrm{M}}_{ij}\times\overline{\mathrm{M}}_{ij}} = \mathrm{M}\int_{\epsilon}^{+\infty} \mathrm{d}\rho\, \overline{\mathrm{M}}_{a\langle i}(u-\rho) \int_{0}^{+\infty} \mathrm{d}\tau\, \overline{\mathrm{M}}^{(8)}_{j\rangle a}(u-\rho-\tau)\, \Omega(\rho,\tau) + (\text{surface term})_{\epsilon} + (\text{standard terms})$$

One would expect $\Omega(\rho,\tau)$ to be insanely complicated, with polylogarithms, etc. But actually, all the complicated terms neatly cancel out:

$$\Omega(\rho,\tau) = \frac{7613764}{165375} - \frac{1024076}{18375}\frac{\tau}{\rho} - \frac{2074}{63}\left(\frac{\tau}{\rho}\right)^2 - \frac{104}{15}\left(\frac{\tau}{\rho}\right)^3 + \frac{634076}{55125}\ln\left(\frac{\rho}{2r_0}\right) + \frac{384}{175}\frac{\tau}{\rho}\ln\left(\frac{\rho}{2r_0}\right) - \frac{144}{175}\ln\left(\frac{\rho}{2r_0}\right)^2 + \frac{8}{7}\ln\left(\frac{\tau}{2r_0}\right)$$

This can then be massaged into a tractable form (and finite when $\epsilon \to 0$)

# Tails-of-memory: result

After the $\overline{\mathrm{M}}_{ij} \to \mathrm{M}_{ij}$ conversion, we find [Trestini & Blanchet 2023]

$$
\begin{aligned}
\mathcal{U}_{ij}^{\mathrm{M}\times\mathrm{M}_{ij}\times\mathrm{M}_{ij}} = \frac{8G^2\mathrm{M}}{7c^8}\Bigg\{ & \int_0^{+\infty} \mathrm{d}\rho\, \mathrm{M}^{(4)}_{a\langle i}(u-\rho) \int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(4)}_{j\rangle a}(u-\rho-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{1613}{270}\right] \\
& - \frac{5}{2}\int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{3}{2}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - 3\int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}^{(2)}_{a\langle i}\mathrm{M}^{(5)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - \frac{5}{2}\int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(6)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{3}{10}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - \int_0^{+\infty} \mathrm{d}\tau\, (\mathrm{M}_{a\langle i}\mathrm{M}^{(7)}_{j\rangle a})(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{1}{4}\ln\left(\frac{\tau}{2b_0}\right)\right] \\
& - 2\mathrm{M}^{(2)}_{a\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(5)}_{j\rangle a}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{27521}{5040}\right] \\
& - \frac{5}{2}\mathrm{M}^{(1)}_{a\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(6)}_{j\rangle a}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{15511}{3150}\right] \\
& + \frac{1}{2}\mathrm{M}_{a\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(7)}_{j\rangle a}(u-\tau) \left[\ln\left(\frac{\tau}{2r_0}\right) - \frac{6113}{756}\right] \Bigg\}.
\end{aligned}
$$