Unitarization of one-loop graviton-graviton scattering

Study of the graviball

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References:

M. Peláez, J.A.O. to be sent for publication (soon)
J.A.O. PLB 835 (2022)
D. Blas, J. Martín-Camalich, J.A.O. PLB 827 (2022); JHEP08 (2022)

New Frontiers in Strong Gravity
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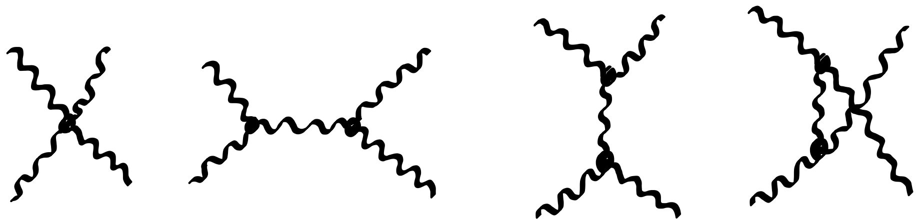
§1. Introduction:

One-loop graviton-graviton scattering was calculated by Dunbar and Norridge NPB 433 (1995) in Dimensional Regularization

Ultraviolet finite (no counterterms)   Nieuwenhuizen, Wu, JMP 18 (1972)

Infrared (IR) infinity

Leading order A_0



Quantum gravity as an Effective Field Theory $s \ll M_p^2 = G^{-1}$

Donoghue PRL 72 (1994)

Interactions are organized in a derivative expansion

Linearization

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa h_{\alpha\beta}$$

LO: $O(Gs)$

NLO: $O((Gs)^2)$

$R_{\mu\nu\alpha\beta} \sim p^2$

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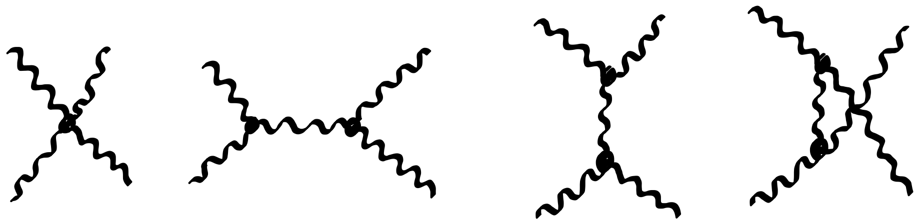
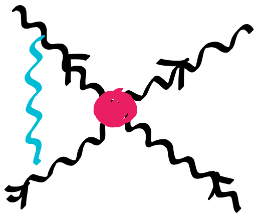
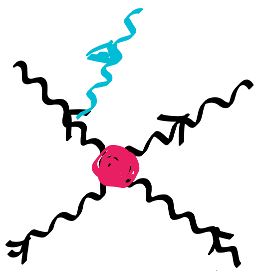


Structure of IR divergences   Weinberg, PR 140 (1965)



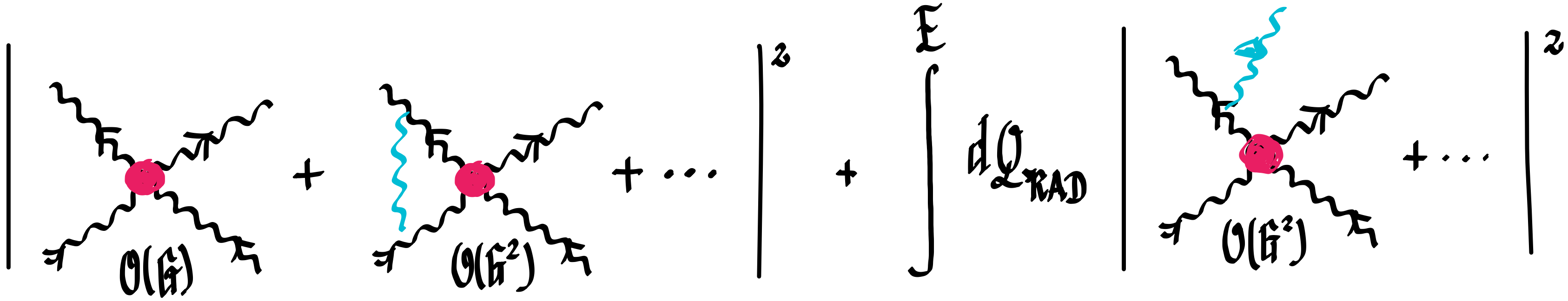
Virtual soft gravitons



Radiated real soft gravitons

Cancellation of IR divergences in transition rates

Weinberg, PR 140 (1965)



$$\alpha \longrightarrow \beta \; : \qquad \Gamma_{\beta\alpha}(\leq E) = \left[\frac{E}{\Lambda}\right]^{B} b(B)\, \Gamma^{0}_{\beta\alpha}$$

$\Gamma^{0}_{\beta\alpha}$ No soft gravitons

$$\boxed{|\vec{q}\,| > \Lambda} \qquad \text{Weinberg condition}$$

Weinberg resums soft virtual and real gravitons (photons).

Cancellation of IR divergences in the scattering amplitudes

Weinberg, PR 140 (1965) – Much less used!

For the S-matrix:

$$S_{\beta\alpha} = S^0_{\beta\alpha}\,\exp(\Phi_{\beta\alpha}) \qquad \Phi_{\beta\alpha} = -i4\pi G \sum_{n,m} \eta_n \eta_m \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{(p_n\cdot p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{(q^2+i\varepsilon)(p_n\cdot q + i\eta_n \varepsilon)(p_m\cdot q - i\eta_m \varepsilon)}$$

$$\Phi_{\beta\alpha} = \frac{1}{2\pi}\left(\frac{1}{\epsilon} - \ln\Lambda^2\right)\sum_{n,m} \eta_n \eta_m\, p_n\cdot p_m \ln\left(-\eta_n\eta_m\, 2 p_n\cdot p_m\right) \qquad m_n \to 0\,, \quad \eta_n = \pm 1 \ \begin{matrix}\text{out}\\ \text{in}\end{matrix}\text{going}$$

Calculated by us in DR

$$S^0_{\beta\alpha} = S_{\beta\alpha}\,\exp(-\Phi_{\beta\alpha})$$

$$T^0_{\beta\alpha} = T_{\beta\alpha}\,\exp(-\Phi_{\beta\alpha})$$

$$S = I + iT$$

Partial-wave amplitudes (PWA) require properly handling $\delta(\beta-\alpha)$

Blas, Camalich
IAO '22

QFT perturbative knowledge

$$\mathcal{T}_{\beta\alpha} = \mathcal{T}^{(0)}_{\beta\alpha} + \mathcal{T}^{(1)}_{\beta\alpha} + \dots$$ Loop expansion.

$\mathcal{T}^{(n)}$ with n loops.

Free of Infrared divergences: ←

$$\mathcal{S}^{0}_{\beta\alpha} = \mathcal{S}_{\beta\alpha} \exp(-\Phi_{\beta\alpha})$$

$$\mathcal{T}^{0}_{\beta\alpha} = \mathcal{T}_{\beta\alpha} \exp(-\Phi_{\beta\alpha})$$

$$A_{\beta\alpha} = \mathcal{T}^{(0)}_{\beta\alpha} + \mathcal{T}^{(1)}_{\beta\alpha} - \Phi_{\beta\alpha}\mathcal{T}^{(0)}_{\beta\alpha} + \mathcal{O}(\hbar^2)$$

Four-graviton scattering amplitudes

$\mathcal{T}^{(0)}$: De Witt PR162('67)

$\mathcal{T}^{(1)}$: Dunbar, Norridge, NPB433('95)

$\mathcal{T}^{(2)}$: Abreu et al., PRL124('20)

Three independent scattering amplitudes: Crossing, Parity, Time reversal

(+ + ; + +) (+ − ; + +) (− − ; + +)

IR divergent at $O(\hbar)$ — IR finite at $O(\hbar)$

Tree level:

$$T^{(0)}_{++;++}(s,t,u) = \frac{8\pi G s^4}{stu}$$

$$T^{(0)}_{+-;++}(s,t,u) = 0$$

$$T^{(0)}_{--;++}(s,t,u) = 0$$

$O(\hbar)$ IR finite amplitudes:

$$T^{(1)}_{+-;++}(s,t,u) = \frac{G^2(s^2+t^2+u^2)}{90}$$

$$T^{(1)}_{--;++}(s,t,u) = -\frac{G^2(s^2+t^2+u^2)}{30}$$

Dunbar, Norridge '95

$$T^{(1)}_{++;++}(s,t,u) = \frac{8G^2 s^4}{stu}\left[f_1(s,t,u) + f_2(s,t,u)\right] + 4(Gs)^2 h(s,t,u)$$

IR divergent

Dimensionfull log arguments

$$f_1(s,t,u) = \frac{1}{\epsilon}\left[s\ln(-s) + t\ln(-t) + u\ln(-u)\right]$$

$$f_2(s,t,u) = s\ln(-t)\ln(-u) + t\ln(-u)\ln(-s) + u\ln(-s)\ln(-t)$$

$$h(s,t,u) = \frac{1922t^4 + 9143t^3u + 14622t^2u^2 + 9143tu^3 + 1922u^4}{180s^4} + \frac{(t+2u)(2t+u)(2t^4 + 2t^3u - t^2u^2 + 2tu^3 + 2u^4)}{s^6}\left(\ln^2\frac{t}{u} + \pi^2\right) + \frac{(t-u)(341t^4 + 1609t^3u + 2566t^2u^2 + 1609tu^3 + 341u^4)}{30s^5}\ln\frac{t}{u}$$

We have evaluated in DR: $\Phi = \frac{G}{\pi}\left(\frac{1}{\epsilon} - \ln\Lambda^2\right)\left[s\ln(-s) + t\ln(-t) + u\ln(-u)\right]$

$$A^{(1)}_{++;++} = T^{(1)}_{++;++} - T^{(0)}_{++;++}\,\Phi = \left[\ln\Lambda^2 f_1 + f_2 + 4(Gs)^2 h\right]\frac{8Gs^4}{stu}$$

$\frac{1}{\epsilon}$ is cancelled

Free of infrared divergences:

$$A_{\beta\alpha} = T^{(0)}_{\beta\alpha} + T^{(1)}_{\beta\alpha} - \Phi_{\beta\alpha}\,T^{(0)}_{\beta\alpha} + O(\hbar^2)$$

$$\mathcal{A}^{(1)}_{++;++} = \mathcal{T}^{(1)}_{++;++} - \mathcal{T}^{(0)}_{++;++}\,\Phi = \left(\ln\Lambda^2 f_1 + f_2 + 4(Gs)^2 h\right)\frac{8Gs^4}{stu}$$

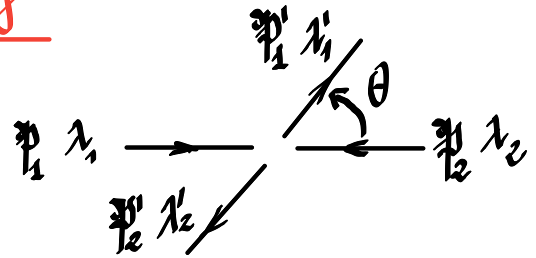
$$\ln\Lambda^2 f_1 + f_2 = s\ln\frac{-t}{\Lambda^2}\ln\frac{-u}{\Lambda^2} + t\ln\frac{-u}{\Lambda^2}\ln\frac{-s}{\Lambda^2} + u\ln\frac{-t}{\Lambda^2}\ln\frac{-s}{\Lambda^2} = \hat{f}_2$$

Dimensionless log arguments

$$A^{(1)}_{++;++}(s,t,u) = \frac{8G^2s^4}{stu}\left(s\ln\frac{-t}{\Lambda^2}\ln\frac{-u}{\Lambda^2} + t\ln\frac{-u}{\Lambda^2}\ln\frac{-s}{\Lambda^2} + u\ln\frac{-t}{\Lambda^2}\ln\frac{-s}{\Lambda^2}\right)$$
$$+ 4(Gs)^2\Bigg(s^{-6}(t+2u)(2t+u)(2t^4+2t^3u-t^2u^2+2tu^3+2u^4)\left(\ln^2\frac{t}{u}+\pi^2\right)$$
$$+\frac{s^{-5}}{30}(t-u)(341t^4+1609t^3u+2566t^2u^2+1609tu^3+341u^4)\ln\frac{t}{u}$$
$$+\frac{s^{-4}}{180}(1922t^4+9143t^3u+14622t^2u^2+9143tu^3+1922u^4)\Bigg)$$

PWAs are IR finite and satisfy perturbative unitarity

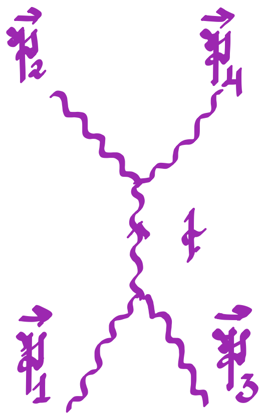
$$t = -s\,\sin^2\frac{\theta}{2} \qquad u = -s\,\cos^2\frac{\theta}{2} \qquad s = 4p^2$$



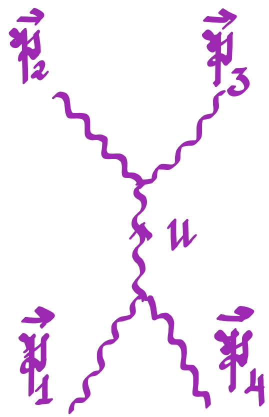
$$T^{J}_{\lambda'_1\lambda'_2;\lambda_1\lambda_2}(s) = \frac{1}{8\pi^2}\int_{-1}^{1} d\cos\theta\; d^{J}_{\lambda\lambda'}(\theta)\; A_{\lambda'_1\lambda'_2;\lambda_1\lambda_2}(s,t_\theta,u_\theta)$$

For instance: J=0 Leading Order

$$T^{J=0}_{22;22}(s) = \frac{2Gs}{\pi}\int_{-1}^{+1}\frac{d\cos\theta}{\sin^2\frac{\theta}{2}} = \frac{8Gs}{\pi}\lim_{\theta\to 0}\log\sin\frac{\theta}{2}$$



t-channel



u-channel

Ending with IR finite PWAs

Blas, Martin-Camalich, JAO, PLB 835 ('22)

$$\mathcal{S}^0_{\beta\alpha} = \mathcal{S}_{\beta\alpha} \exp(-\Phi_{\beta\alpha})$$

$$\mathcal{S}_{\beta\alpha} = \delta_{\beta\alpha} + i\,\mathcal{T}_{\beta\alpha}$$

Identity

1.- Redefinition of the Scattering Operator : $\mathcal{T}_{\beta\alpha} \to \mathcal{T}_{\beta\alpha}\, e^{-\Phi_{\beta\alpha}}$

2.- Redefinition of the Identity : $\delta_{\beta\alpha} \to \delta_{\beta\alpha}\, e^{-i\,\mathrm{Im}\,\Phi_{\beta\alpha}}$

Graviton-Graviton Scattering

$$-i\,\mathrm{Im}\,\Phi = 2iGs \log\frac{\mu}{\Lambda}$$

μ : Graviton mass

Change in the PWA because of this redefinition $\quad S_{\beta\alpha} \to S_{\beta\alpha}\, e^{-iGm\Phi_{\beta\alpha}}$

Tree level:

$$-iGm\Phi_{\beta\alpha}\langle\beta|\alpha\rangle_J = i\,2Gs\ln\frac{\mu}{\Lambda}\cdot\underbrace{\frac{2}{\pi}\left(\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}+(-1)^J\delta_{\lambda_1'\lambda_2}\delta_{\lambda_2'\lambda_1}\right)}_{\langle J\mu\,\lambda_1'\lambda_2'|\mathbb{1}|J\mu\,\lambda_1\lambda_2\rangle}$$

$$\mathcal{A}^{J(0)}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(s) = \mathcal{T}^{J(0)}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}(s) + \frac{4Gs}{\pi}\ln\frac{\mu}{\Lambda}\left(\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}+(-1)^J\delta_{\lambda_1'\lambda_2}\delta_{\lambda_2'\lambda_1}\right)$$

Example: $J=0$

$$\mathcal{T}^{0(0)}_{22,22}(s) = -\frac{Gs^2}{\pi}\int_{-1}^{1}\frac{d\cos\theta}{t-\mu^2} = \frac{4Gs}{\pi}\log\left(1+\frac{s}{\mu^2}\right)\xrightarrow[\mu\to 0]{}\frac{4Gs}{\pi}\log\frac{s}{\mu^2}$$

$$\mathcal{A}^{0(0)}_{22,22}(s) = \frac{4Gs}{\pi}\log\frac{s}{\mu^2}+\frac{4Gs}{\pi}\log\frac{\mu^2}{\Lambda^2} = \frac{4Gs}{\pi}\log\frac{s}{\Lambda^2}$$

1-Loop level:

Problematic piece in $A^{(1)}_{++;++}(s,t,u)$:

$$\frac{8G^2s^3}{tu}\left(s\ln\frac{-t}{\Lambda^2}\ln\frac{-u}{\Lambda^2}+t\ln\frac{-u}{\Lambda^2}\ln\frac{-s}{\Lambda^2}+u\ln\frac{-t}{\Lambda^2}\ln\frac{-s}{\Lambda^2}\right)$$

Real Part:

$[t,u<0\ ,\ s>0]$ $$\frac{8G^2s^3}{tu}\left(s\ln\frac{-t}{\Lambda^2}\ln\frac{-u}{\Lambda^2}+t\ln\frac{-u}{\Lambda^2}\ln\frac{s}{\Lambda^2}+u\ln\frac{-t}{\Lambda^2}\ln\frac{s}{\Lambda^2}\right)$$

$\theta\to 0$ $u\to -s$, $t\to 0$ $\theta\to\pi$ $t\to s$, $u\to 0$

It has a finite limit

As expected: No IR divergent.

Imaginary part: $\log(-s)=\log(s)-i\pi$

$s+i0^+$

$$-8\pi G^2s^3\left[\frac{1}{u}\ln\frac{-u}{\Lambda^2}+\frac{1}{t}\ln\frac{-t}{\Lambda^2}\right]$$

Divergent for $t\to 0$, $u\to 0$

1-loop :

$$-\frac{1}{2}\left(\mathrm{Im}\,\Phi_{\beta\alpha}\right)^2 \cdot \langle\beta|\alpha\rangle_J = -(2\,G s)^2 \ln^2\frac{\mu}{\Lambda}\cdot\frac{1}{\pi}\left(\delta_{\lambda_1'\lambda_1}\delta_{\lambda_2'\lambda_2}+(-1)^J\delta_{\lambda_1'\lambda_2}\delta_{\lambda_2'\lambda_1}\right)$$

Example : $J=0$

$$\mathrm{Im}\,T^{J=0\,(1)}_{22;22}(s) = -\frac{2G^2s^3}{\pi}\int_{-1}^{+1} d\cos\theta\,\frac{\ln(t-\mu^2)}{t-\mu} = \frac{2(Gs)^2}{\pi}\ln\frac{s}{\mu^2}\ln\frac{s\mu^2}{\Lambda^4}$$

$$\mathrm{Im}\,A^{J=0\,(1)}_{22;22}(s) = \frac{8(Gs)^2}{\pi}\ln^2\frac{\mu}{\Lambda} + \frac{2(Gs)^2}{\pi}\ln\frac{s}{\mu^2}\ln\frac{s\mu^2}{\Lambda^4} = \frac{2(Gs)^2}{\pi}\ln^2\frac{s}{\Lambda^2}$$

Perturbative unitarity is fulfilled

$$\mathrm{Im}\,A^{J=0\,(1)}_{22;22}(s) = \frac{\pi}{8}\left(A^{J=0\,(0)}_{22;22}(s)\right)^2$$

IR-finiteness for arbitrary J:

$$\mathrm{Im}\,A^{J(1)}_{22;22}(s) = \frac{8(G s)^2}{\pi} \ln^2\frac{\mu}{\Lambda} - \frac{2G^2 s^3}{\pi}\int_{-1}^{+1} d\cos\theta\ \frac{P_J(\cos\theta)}{t-\mu^2} \ln\frac{-t+\mu^2}{\Lambda^2}$$

$$= \frac{8(G s)^2}{\pi} \ln^2\frac{\mu}{\Lambda} - \frac{2G^2 s^3}{\pi}\int_{-1}^{1} d\cos\theta\ \frac{\ln(-t+\mu^2)/\Lambda^2}{t-\mu^2} - \frac{2G^2 s^3}{\pi}\int_{-1}^{1} d\cos\theta \left[P_J(\cos\theta)-1\right] \ln\frac{-t}{\Lambda^2}$$

$$= \frac{2(G s)^2}{\pi} \ln^2\frac{s}{\Lambda^2} - \frac{2G^2 s^3}{\pi}\int_{-1}^{1} d\cos\theta \left[P_J(\cos\theta)-1\right] \ln\frac{-t}{\Lambda^2}$$

Like J=0 — Infrared finite

UV-finite PWAs for complex s

Looking for poles in the complex s-plane : Resonances

Key Point : $\ln(-s) = \ln(s) - i\sigma\pi$ $\qquad \arg(s) \in (-\pi, \pi]$

$\sigma = \text{sign}(\text{Im}(s))$

Delicate term :

(A) $$\frac{8G^2 s^3}{tu}\left\{ s \ln\frac{-t}{\Lambda^2}\ln\frac{-u}{\Lambda^2} + t\ln\frac{-u}{\Lambda^2}\ln\frac{s}{\Lambda^2} + u\ln\frac{-t}{\Lambda^2}\ln\frac{s}{\Lambda^2}\right\}$$

(B) $$-i\sigma\pi\, 8G^2 s^3\left\{\frac{1}{u}\ln\frac{-u}{\Lambda^2} + \frac{1}{t}\ln\frac{-t}{\Lambda^2}\right\}$$

(A) : The same analysis as for the real part of the PWA when $s \in \mathbb{R}$

(B) : The same analysis as for the imaginary part of the PWA when $s \in \mathbb{R}$

Weinberg's ressummation PR 140 ('65)

$$S_{\beta\alpha} = S^0_{\beta\alpha} \exp(\Phi_{\beta\alpha})$$

$$\Phi_{\beta\alpha} = -i4\pi G \sum_{n,m} \eta_n \eta_m \int^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{(p_n \cdot p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{(q^2 + i\varepsilon)(p_n \cdot q + i\eta_n \varepsilon)(p_m \cdot q - i\eta_m \varepsilon)}$$

$$\Phi_{\beta\alpha} = \frac{1}{2\pi}\left(\frac{1}{\varepsilon} - \ln\Lambda^2\right) \sum_{n,m} \eta_n \eta_m \, p_n \cdot p_m \ln(-\eta_n \eta_m 2 p_n \cdot p_m) \qquad m_n \to 0.$$

$$S^0_{\beta\alpha} = S_{\beta\alpha} \exp(-\Phi_{\beta\alpha})$$

The factors E^A in (2.51) and E^B in (2.52) correctly represent the shape of the energy spectrum for E ranging from zero (where Γ vanishes) up to some maximum smaller (though not necessarily much smaller) than any energy characterizing the process $\alpha \rightarrow \beta$.

Section II.5

Weinberg PR 140 ('65)

$$\Gamma_{\beta\alpha}(\leq E) = \left[\frac{E}{\Lambda}\right]^B b(B)\, \Gamma^0_{\beta\alpha}$$

IV. GRAVITATIONAL RADIATION IN NON-RELATIVISTIC COLLISIONS

The rate of emission of energy in soft gravitational radiation during collisions is

$$P(\leqq\Lambda)=\int_0^\Lambda E\,d\Gamma(\leqq E)\,. \qquad (4.1)$$

Here "soft" means that the emitted energy E is less than some cutoff Λ chosen smaller than the energies characteristic of the collision process. The rate $\Gamma(\leqq E)$

Section IV

Weinberg PR 140 ('65)

$\Lambda^2 = \frac{s}{a^2}$, $a > 1$, $\ln a = \mathcal{O}(1)$

Blas, Gamalich, JAO, PLB, JHEP (2022)

a : Emergent number characteris of the theory

Eg : $\ln(a) = \gamma_E$ for Coulomb scattering

Blas, Gamalich, JAO, JHEP (2022)

JAO, PLB (2022)

JAO, PLB (2022)

$$A_J = \frac{e^2}{2p^2}\ln a + \frac{i m e^4}{8\pi p^3}(\ln a)^2 + \mathcal{O}(\hbar^2)$$

Known exact solution for nonrelativistic Coulomb scattering

$$S_J = \frac{\Gamma(J+1-i\gamma)}{\Gamma(J+1+i\gamma)} = 1 + i\frac{mp}{\pi}A_J \qquad , \quad \gamma = \frac{m\alpha}{p}$$

$$A_J = \frac{e^2}{2p^2}\gamma_E + \frac{i m e^4}{8\pi p^3}\gamma_E^2 + \mathcal{O}(\hbar^2)$$

This is also the case in QED by applying the small t (momen transfer squared) expansion (going beyond the eikonal approximation)

Bazhanov, Fronko, Solov'ev, Yushin, Theor. Math. Phys 33 (1977)

Complete formula for calculating PWAs , $s \in \mathbb{C}$

$$A^{J(1)}_{++;++}(s) = \frac{G^2 s^2}{2\pi^2} \int_{-1}^{+1} d\cos\theta \, P_J(\cos\theta) \left[\frac{8s}{tu} \left\{ s \ln\left(a \sin\frac{\theta}{2}\right) \ln\left(a\cos\frac{\theta}{2}\right) + t \ln\left(a \cos\frac{\theta}{2}\right) \ln a + u \ln\left(a\sin\frac{\theta}{2}\right) \ln a \right\} \right.$$
$$+ \frac{1922t^4 + 9143t^3u + 14622t^2u^2 + 9143tu^3 + 1922u^4}{180s^4}$$
$$+ \frac{(t+2u)(2t+u)(2t^4+2t^3u - t^2u^2 + 2tu^3 + 2u^4)}{s^6} \left(\ln^2 \tan^2\frac{\theta}{2} + \pi^2 \right)$$
$$\left. + \frac{(t-u)(341t^4 + 1609t^3u + 2566t^2u^2 + 1609tu^3 + 341u^4)}{30s^5} \ln\tan^2\frac{\theta}{2} \right] + i\sigma\frac{\pi}{8}\left(A^{J(0)}_{++;++}(s)\right)^2$$

$$A^{J(0)}_{++;++}(s) = \frac{8Gs}{\pi} \ln a - \frac{2Gs^2}{\pi} \int_{-1}^{1} d\cos\theta \, \frac{P_J(\cos\theta) - 1}{t - \mu^2}$$

Unitarization :

Unitarity :

$$\mathrm{Im}\, A^{J}_{++;++}(s+i0^+) = \frac{\pi}{8}\, |A^{J}_{++;++}(s)|^2 \quad , \quad s>0$$

OK,

$$\mathrm{Im}\, \frac{1}{A^{J}_{++;++}(s+i0^+)} = -\frac{\pi}{8} \quad , \quad s>0$$

Perturbative Unitarity:

$$\mathrm{Im}\, A^{J(1)}_{++;++}(s+i0^+) = \frac{\pi}{8}\left(A^{J(0)}_{++;++}(s)\right)^2 \quad , \quad s>0$$

Unitarization Methods

Reviews JAO Symmetry ('20)
PPNP ('20)

Inverse Amplitude Method (IAM)

$$A^J_{IAM}(s) = \frac{A^{J(0)}(s)^2}{A^{J(0)}(s) - A^{J(1)}(s)}$$

Lehmann ('72)
Truong ('88); Dobado et al ('90)

Algebraic N/D Method (N/D)

Chew, Mandelstam ('60)
JAO, Oset ('99)

$$A^J_{ND}(s) = \frac{K^J(s)}{1 + K^J(s)\, g(s)}$$

$$g(s) = \frac{1}{8} \ln \frac{-s}{Q^2}$$

Unitarity 2-point function $\quad \operatorname{Im} g(s) = -\frac{\pi}{8}\,, \; s>0$

$$A^J_{ND} = K^J(s) - K^J(s)^2\, g(s) + \mathcal{O}(\hbar^2)$$

$$\mathcal{K}^{J(0)} = A^{J(0)}(s)$$

$$\mathcal{K}^{J(1)} = A^{J(1)}(s) + A^{J(0)}(s)^2\, g(s)$$

Q^2 : Natural size $\qquad \left|\frac{NLO}{LO}\right| = O\left(\frac{s}{Q^2}\right)$

$$\left|\frac{A^{(0)2}\, g(s)}{A^{0(0)}}\right| \sim \frac{G}{\pi} \ln \Lambda \cdot s \qquad Q_0^2 = \pi\,(G \ln \Lambda)^{-1} \qquad A^J_{ND} = A^J_{IAM} + O(G^3)$$

We also consider $\frac{1}{2} Q_0^2 ,\ 2 Q_0^2$

IAM does not depend on Q^2

$$\frac{1}{A^J_{ND}} = \frac{1}{\mathcal{K}^J} + g(s) = \frac{1}{A^{J(0)}} - \frac{A^{J(1)}}{A^{J(0)}} + O(\hbar^2) = \frac{1}{A^J_{IAM}} + O(\hbar^{-2})$$

In all cases $A^{J(0)} + A^{J(1)}$ is reproduced

ND0 : $\dfrac{\mathcal{A}^{J(0)}}{1+\mathcal{A}^{J(0)}\,g(s)}$ Unitarized LO study Blas, Martin-Camalich, JAO ('22)

ND1 : $\dfrac{\mathcal{A}^{J(1)}+\mathcal{R}^{J(1)}}{1+(\mathcal{A}^{J(1)}+\mathcal{R}^{J(1)})\,g(s)}$ Unitarized NLO study

These methods have been extensively used in Hadron Physics

σ or $f_0(500)$: lightest resonance in QCD

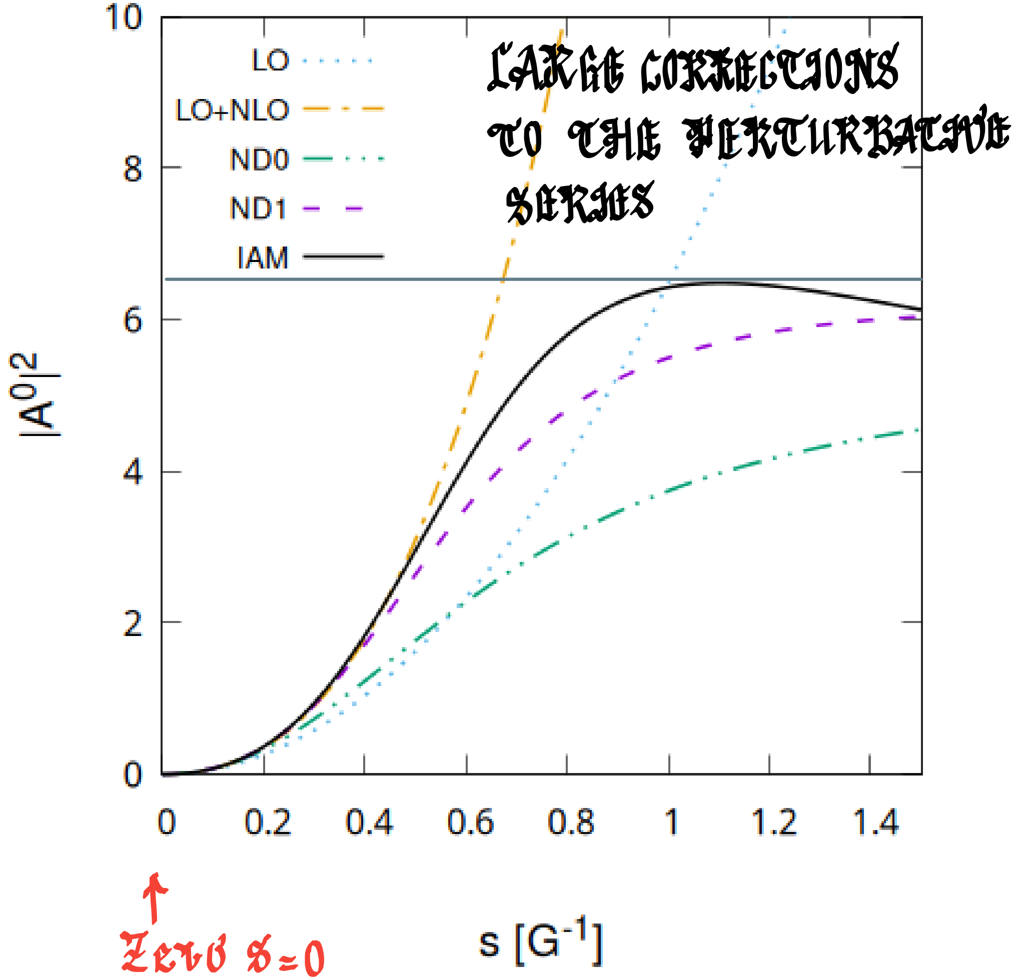
and many others $\rho(770)$, $f_0(980)$, $a_0(980)$, $\Lambda(1405)$, ...

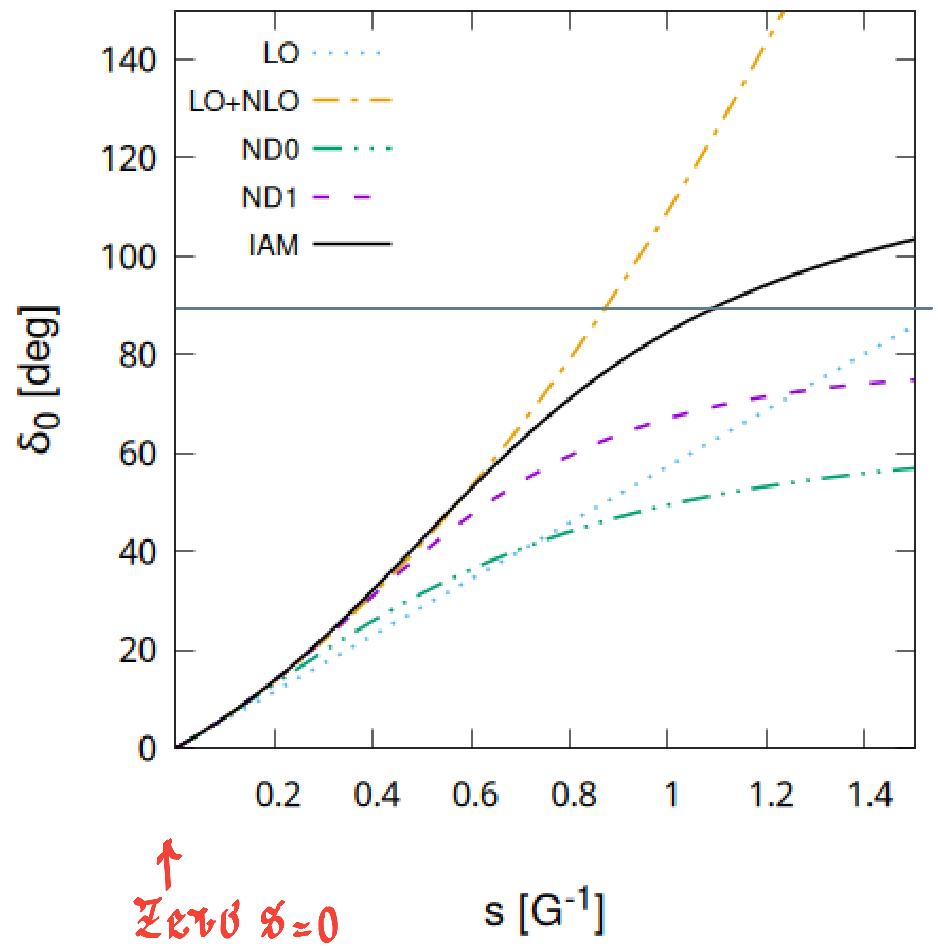
Electroweak sector : $W_L W_L$ scattering and possible resonances

J=0 graviton-graviton scattering: [Now ln a = 1]

$$|A^0(s)|^2 \leq \frac{64}{\pi^2} \simeq 6.49$$ Unitarity Bound

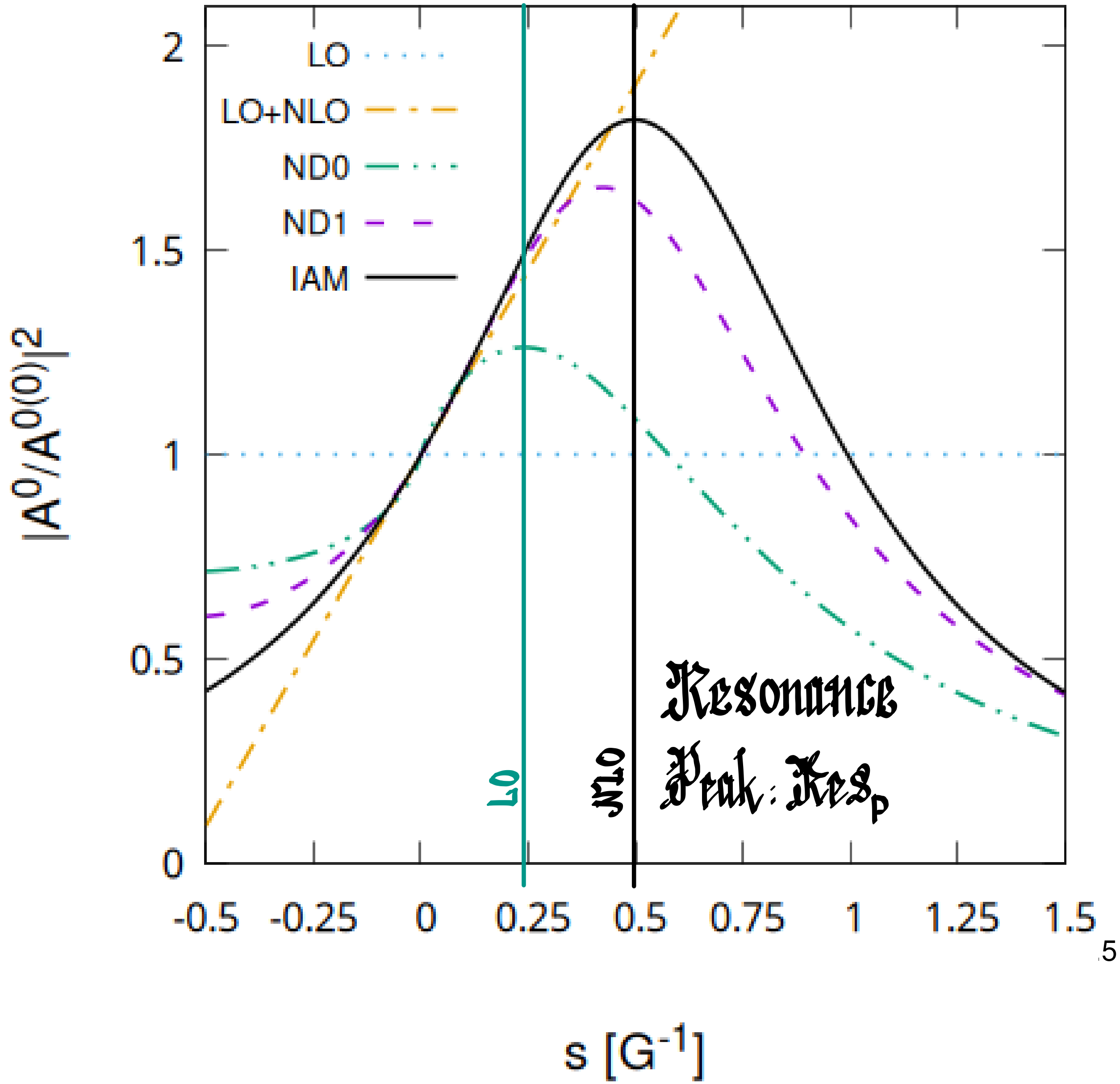
Phase $A^0(s) = \frac{8}{\pi} \sin\delta_0 \, e^{i\delta_0}$





Graviball $I=0$ Resonance from the physical s-axis:

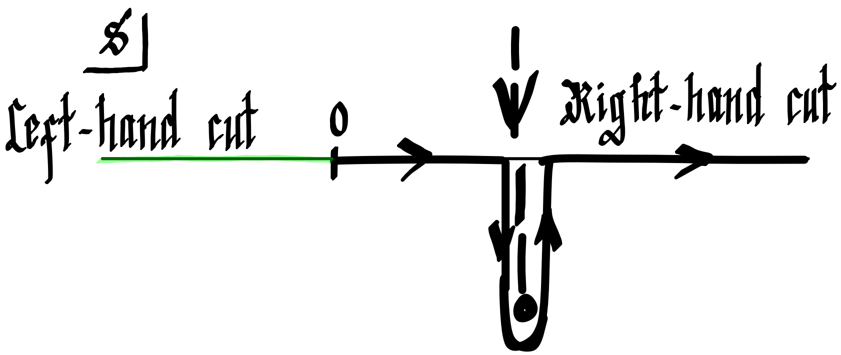
$$|A^0/A^{0(0)}|^2$$



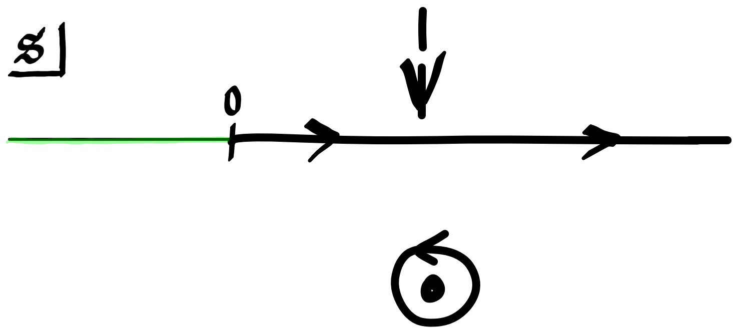
Graviball: Lightest resonance in pure gravity

Resonance is a pole in the 2nd Riemann sheet

Crossing smoothly the real axis above threshold $\longrightarrow$ 2nd RS.



$\equiv$



$$\frac{1}{\mathcal{A}^J_{II}(s)} = \frac{1}{\mathcal{A}^J_{I}(s)} + i\sigma\frac{\pi}{4}$$

Pole Positions:

Method	s_P $[G^{-1}]$ $\lvert s_P\rvert$ $[G^{-1}]$	$\sqrt{s_P}$ $[G^{-\frac{1}{2}}]$	s_P $[\pi G^{-1}]$ $\lvert s_P\rvert$ $[\pi G^{-1}]$	Residue $\frac{\pi}{8}\lvert\gamma^2\rvert$ $[G]$
$\ln a = 1$				
IAM	$0.497 - i\,0.549$ 0.741	$0.787 - i\,0.349$	$0.158 - i\,0.175$ 0.236	0.549 (0%)
$Q^2 = \pi G^{-1}$				
ND0	$0.224 - i\,0.639$ 0.677	$0.378 - i\,0.268$	$0.071 - i\,0.203$ 0.216	0.483 (24%)
ND1	$0.424 - i\,0.522$ 0.673	$0.740 - i\,0.352$	$0.135 - i\,0.166$ 0.214	0.402 (23%)

Residue: $\gamma^2 = \lim_{s \to s_P} (s - s_P)\, A^{I=0}_{++;++}(s)$

Narrow-resonance limit + Unitarity: $\mathrm{Im}\, s_P \cong -\frac{\pi}{8} |\gamma|^2$

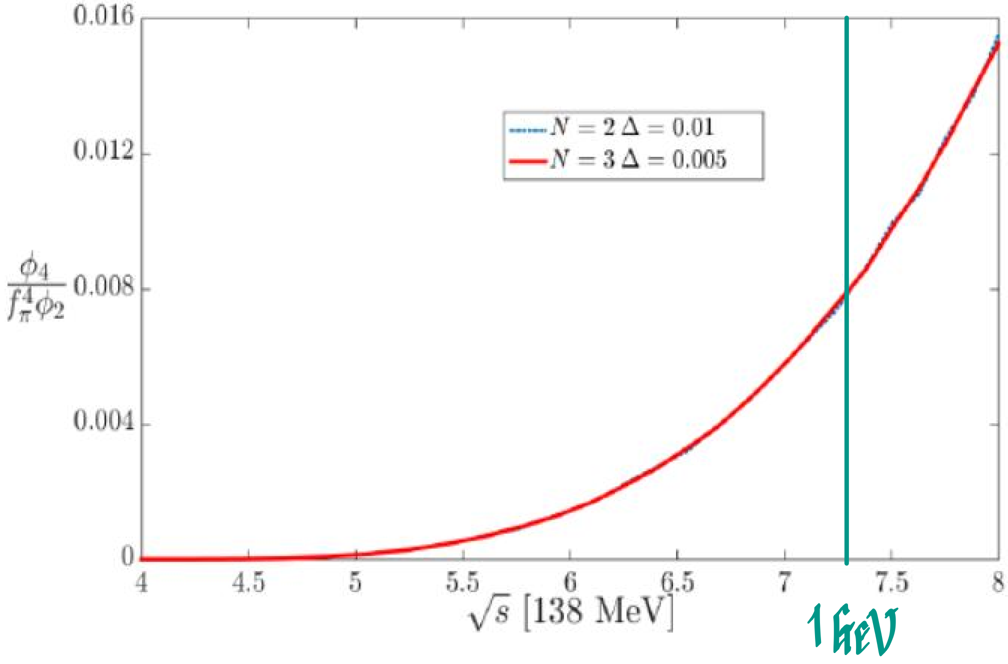
As discussed in Blas, Camalich, JAO ('22)

Suppression of phase space for massless multi-particle states

Salas-Bernández, Llanes-Estrada, JAO, Escudero-Pedrosa, SciPost Phys.11,020(2021)

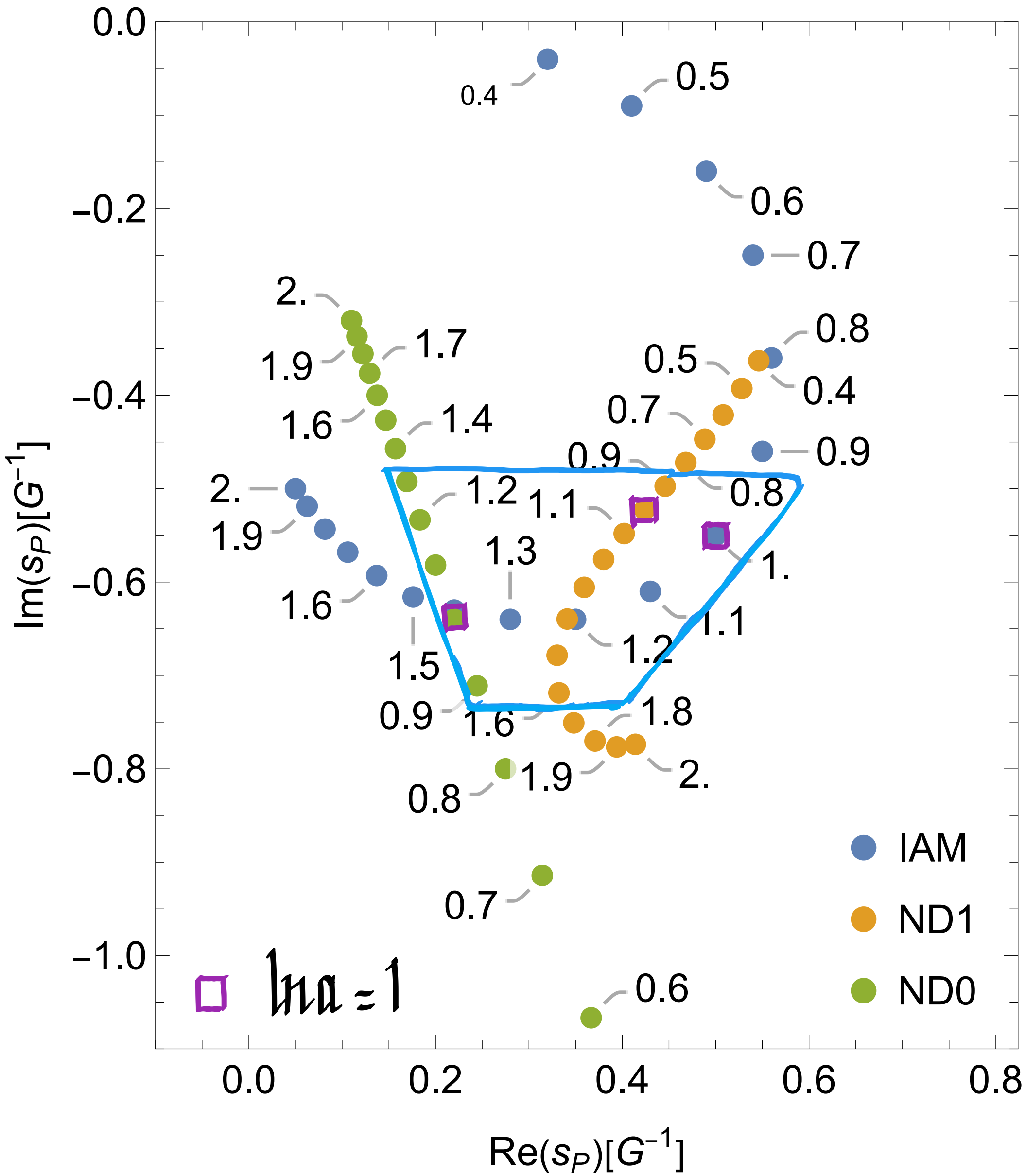
Phase space of n massless particles pions

$$\phi_n = \frac{s^{n-2}}{2(4\pi)^{2n-3}(n-1)!(n-2)!}$$



$\sqrt{s} \in [0.55, 1.1]$ GeV , 4π & 2π

Varying lna: Pole Positions



Naked eye inspection:

lna 0.9 - 1.4

Concentration of Poles

For the same lna

ND1 - IAM lie quite close to each other [they cross for lna = 1.2]

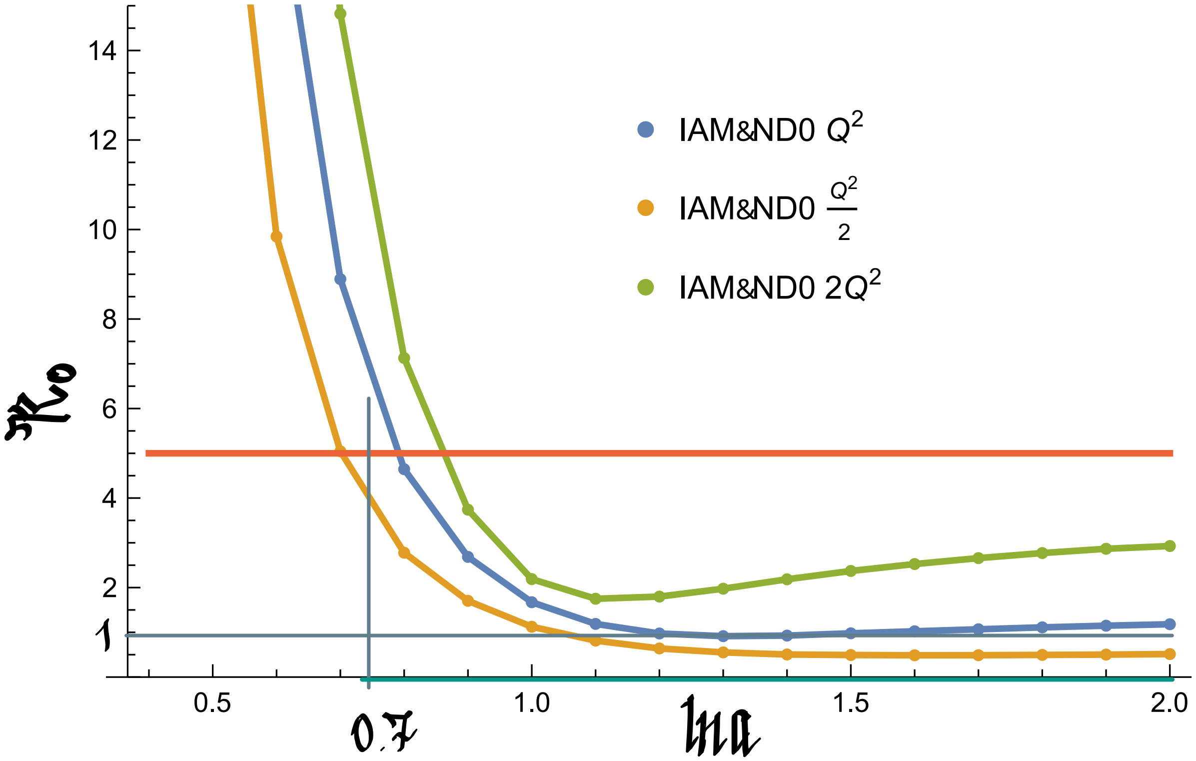
ND0 - IAM $\sim \mathcal{O}\left(\frac{s_P}{Q_0^2}\right)$ Reasonably Close

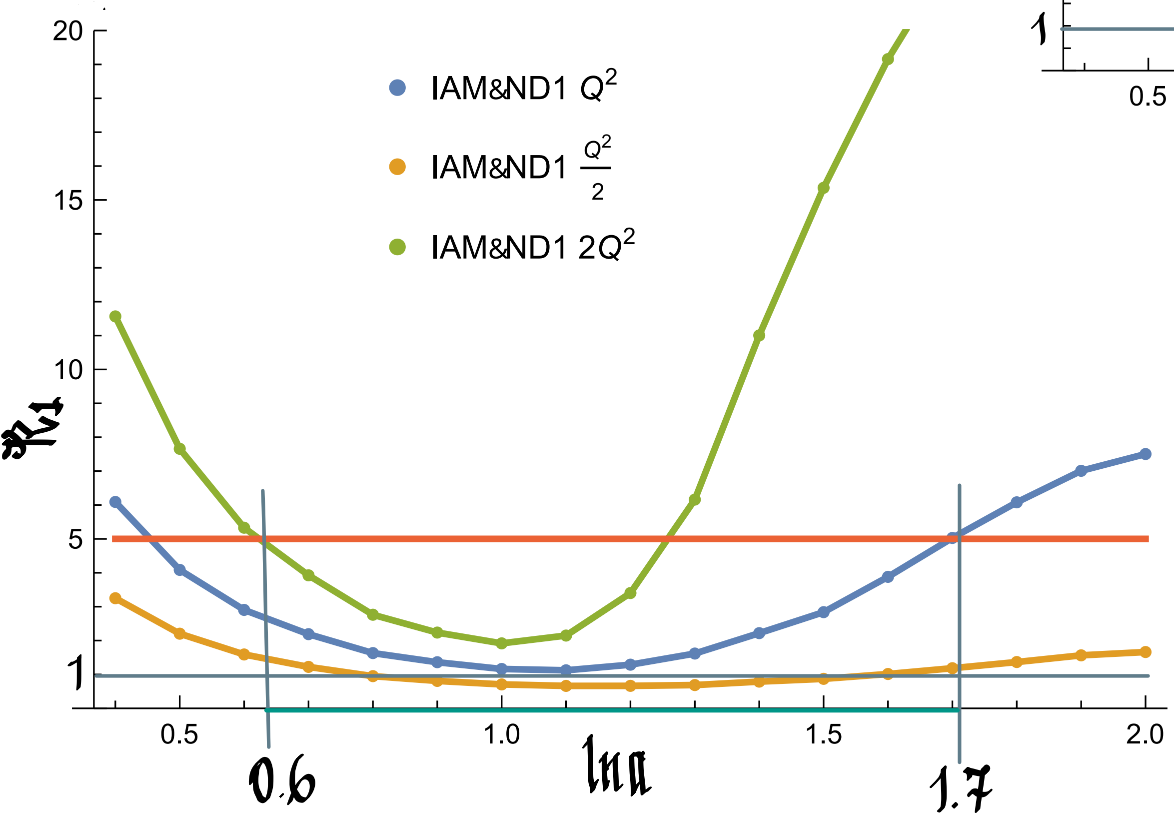
Being more quantitative:

$$R_0 = \left| \frac{s_{P;\text{IAM}} - s_{P;\text{ND0}}}{s_{P;\text{IAM}}} \right| \Big/ \left| \frac{s_{P;\text{IAM}}}{2^\tau Q_0^2} \right|$$

$$R_1 = \left| \frac{s_{P;\text{IAM}} - s_{P;\text{ND1}}}{s_{P;\text{IAM}} - s_{P;\text{ND0}}} \right| \Big/ \left| \frac{s_{P;\text{IAM}}}{2^\tau Q_0^2} \right|$$

$\tau = -1, 0, 1$





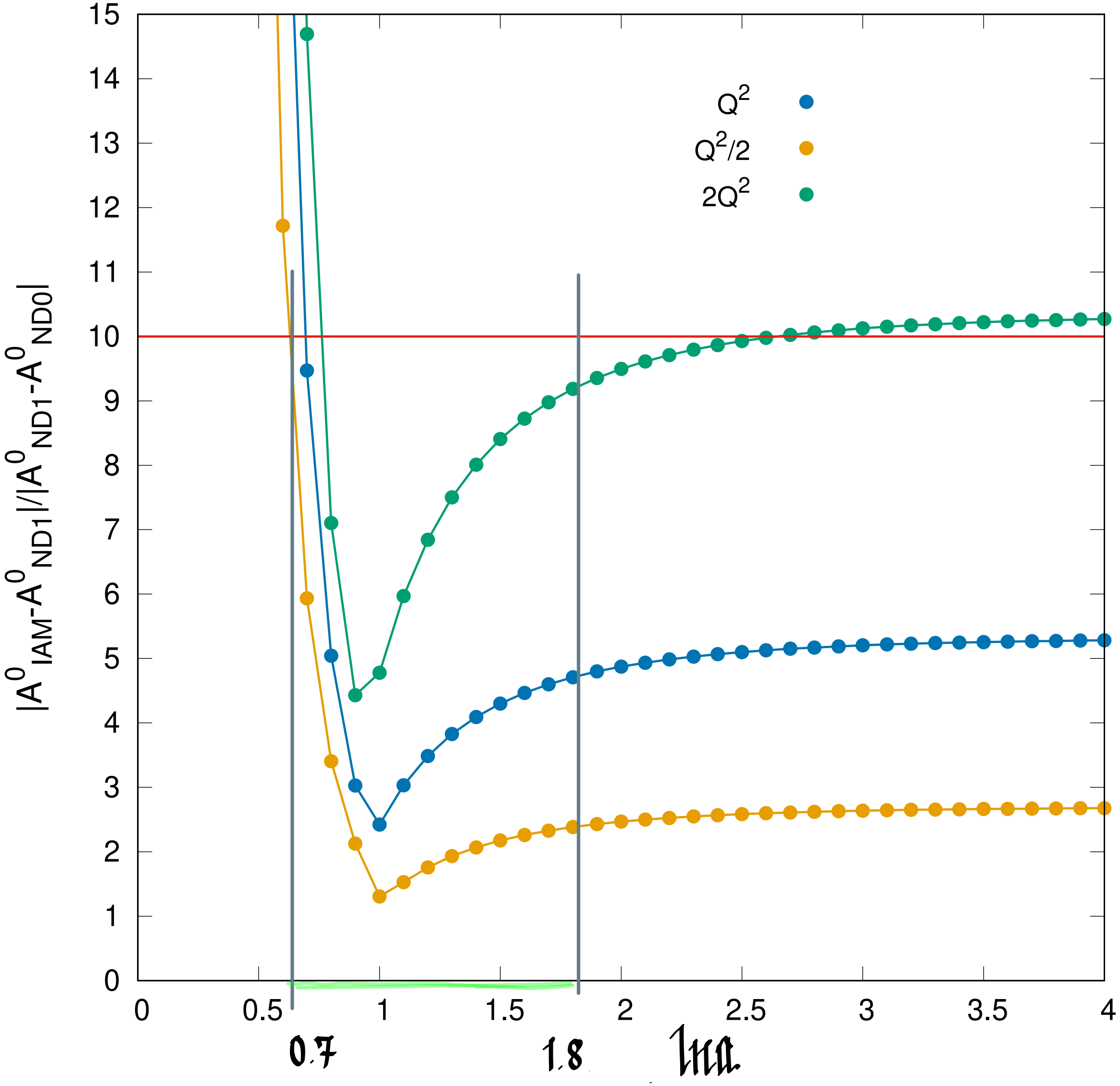
Differences along the physical s-axis

$$\left| \frac{A^J_{IAM} - A^J_{ND1}}{A^J_{IAM} - A^J_{ND0}} \right| = \mathcal{O}\left(\frac{s}{Q^2}\right)$$

$s \in [0, 0.4]$ 2τ Q_0^2

All in all

$0.5 \lesssim \ln a \lesssim 1.7$



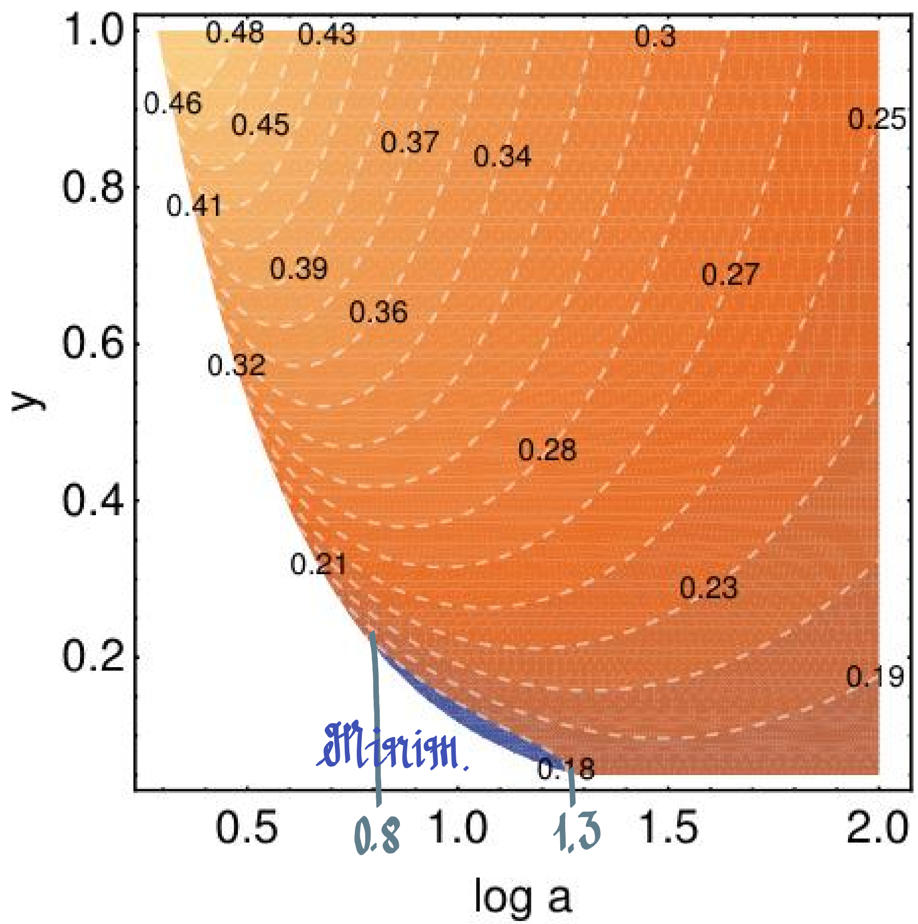
Maximal-stability estimate of $\ln a$

from the unitarized LO study [NDO] Blas, Camalich, JAO JHEP ('22)

By minimizing the dependence in d of the unitarity scale Q_d^2 , $4 \le d \le 5$

EFT : 0.5 - 1.7
Gs-expansion
NLO study

d-expansion
LO study 0.8 - 1.3



Remarkably close range for preferred values of $\ln a$.

The two methods are independent

Connection with the Kulish-Faddeev formalism:

Key references: Chung, PR 140 ('65)

Kulish, Faddeev, Theor. Math. Phys. ('70)

[KF]

Ending with IR finite scattering amplitudes in QED

1.- Redefiniton of the asymptotic states

2.- Redefinition of the S-matrix operator.

1.- Redefiniton of the asymptotic states

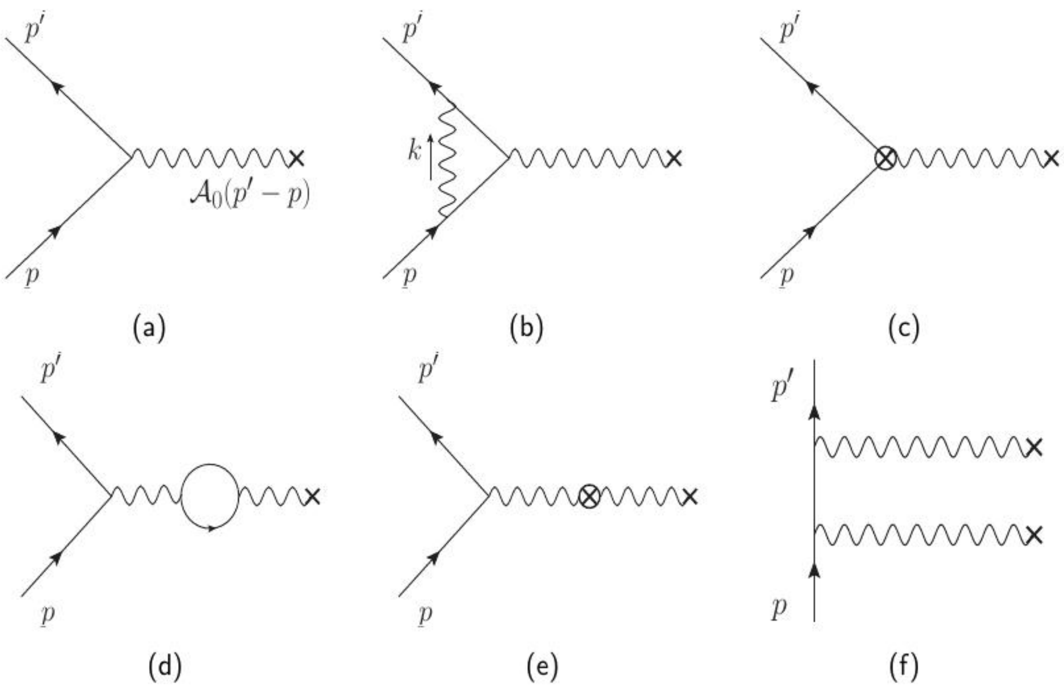
$$|\vec{p},\sigma\rangle = \exp\left[(-e)\sum_{h=1}^{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3 2k^0}\left(\frac{p\cdot\varepsilon^{(h)}(k)}{p\cdot k}\,a^{(h)}(\vec{k})^\dagger - \frac{p\cdot\varepsilon^{(h)}(k)^*}{p\cdot k}\,a^{(h)}(k)\right)\right] c_\sigma(\vec{p})^\dagger|0\rangle$$

Real photons

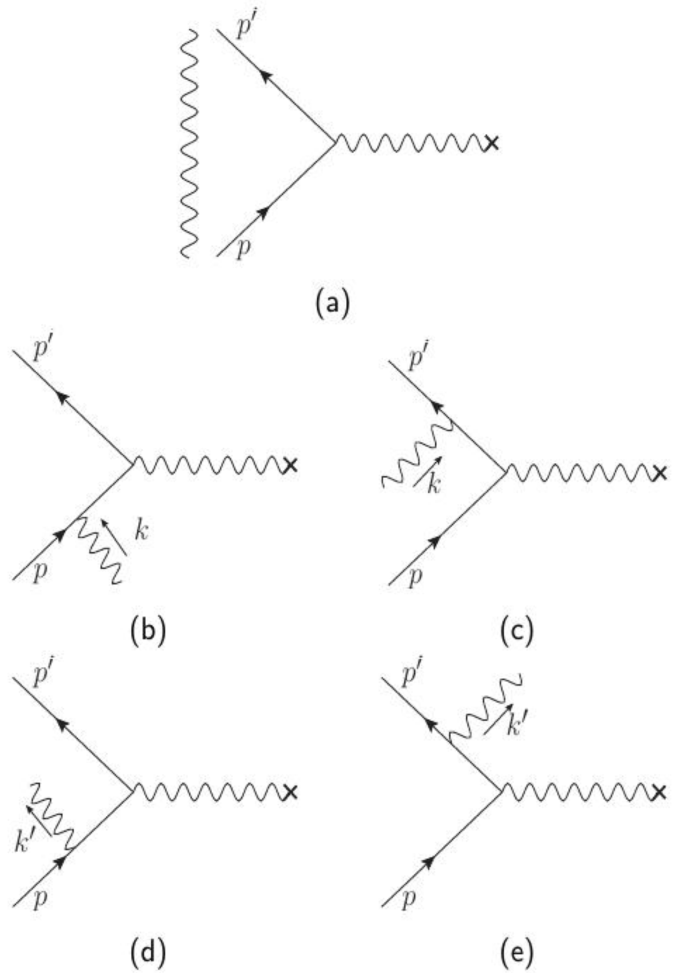
$$\mathcal{S}^{(h)}(\vec{k}) = -e\,\frac{p\cdot\varepsilon^{(h)}(\vec{k})}{p\cdot k}$$

$$|\vec{p},\sigma\rangle = \left(1 - \frac{1}{2}\sum_{h=1}^{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3\,2k^0}\,|\mathcal{S}^{(h)}(\vec{k})|^2 + \sum_{h=1}^{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3\,2k^0}\,\mathcal{S}^{(h)}(\vec{k})\,a^{(h)}(\vec{k})^\dagger + \mathcal{O}(e^2)\right) c_\sigma(\vec{p})^\dagger\,|0\rangle$$

Marcela Peláez, JAO: We have studied Coulomb scattering at one-loop in QED:

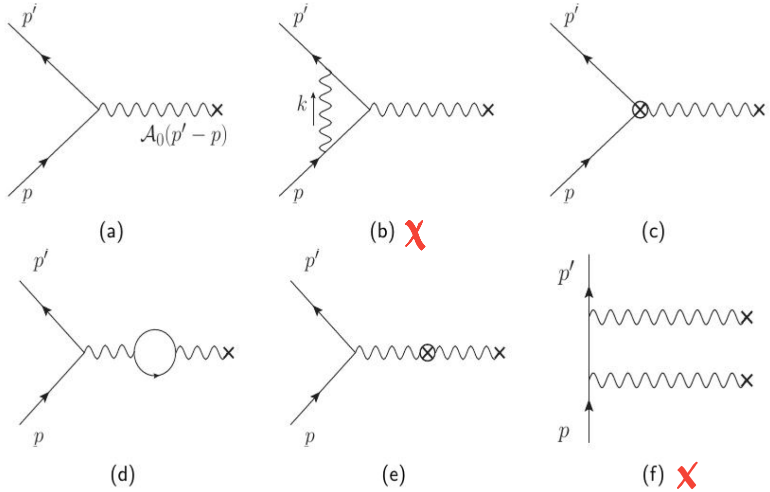


standard QED diagrams



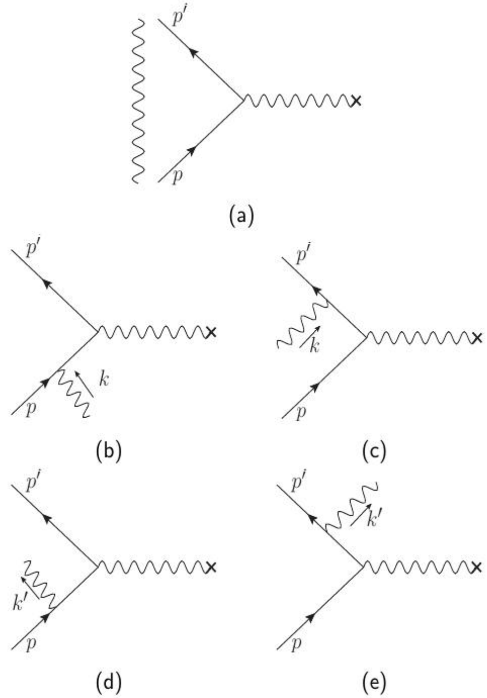
Chung's diagrams

standard QED diagrams



× IR divergent

Chung's diagrams



$$A_4 = -A_0(\bar{q}^2)\,\frac{e^2}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3\,2k^0}\left[\frac{p}{p\cdot k}-\frac{p'}{p'\cdot k}\right]^2 = \frac{A_0(\bar{q}^2)\,\alpha}{\pi}\left(-1+\frac{2m^2-t}{t\,\sigma(t)}\ln\frac{\sigma(t)-1}{\sigma(t)+1}\right)\left(\ln\frac{2\Lambda}{\mu}-1\right)$$

✓ It cancels the IR divergence from the vertex-correction diagram (b)

× But not the IR divergence from once-iteration diagram

2.- Redefinition of the S-matrix :

$$S^0_{\beta\alpha} = S_{\beta\alpha} \exp\left(-i\,\mathrm{Im}\,\Phi_{\beta\alpha}\right)$$

$$\Phi_{\beta\alpha} = -i4\pi G \sum_{n,m} \eta_n \eta_m \int \frac{d^D q}{(2\pi)^D} \frac{(p_n \cdot p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{(q^2 + i\varepsilon)(p_n \cdot q + i\eta_n \varepsilon)(p_m \cdot q - i\eta_m \varepsilon)}$$

$$i\,\mathrm{Im}\,\Phi = i \frac{e e'}{4\pi v} \ln \frac{\mu^2}{\Lambda^2} \quad , \quad v = \frac{p}{E}$$

As before with Weinberg's formula

$$T^0_{\beta\alpha} = T_{\beta\alpha} \exp\left[-i\,\mathrm{Im}\,\Phi_{\beta\alpha}\right]$$

✓ This removes the IR divergence in the iteration diagram

2.- Redefinition of the S-matrix :

$$S^0_{\beta\alpha} = S_{\beta\alpha} \exp\left(-i\,\mathrm{Im}\,\Phi_{\beta\alpha}\right) \qquad \Phi_{\beta\alpha} = -i4\pi G \sum_{n,m} \eta_n \eta_m \int \frac{d^D q}{(2\pi)^D} \frac{(p_n \cdot p_m)^2 - \frac{1}{2} m_n^2 m_m^2}{(q^2+i\varepsilon)(p_n\cdot q + i\eta_n \varepsilon)(p_m \cdot q - i\eta_m \varepsilon)}$$

$$i\,\mathrm{Im}\,\Phi = i \frac{e e'}{4\pi v} \ln \frac{\lambda^2}{\Lambda^2} \quad , \quad v = p/E$$

As before with Weinberg's formula

$$T^0_{\beta\alpha} = T_{\beta\alpha} \exp\left[-i\,\mathrm{Im}\,\Phi_{\beta\alpha}\right]$$

√ This removes the IR divergence in the iteration diagram

At the end the KF formalism and ours (more handy) based on $S^0_{\beta\alpha} = S \exp(-\Phi_{\beta\alpha})$ are equivalent

Weinberg ('65)

Change of perspective on Weinberg's formula

$$S^{0}_{\beta\alpha} = S_{\beta\alpha} \exp(-\Phi_{\beta\alpha})$$

I. Redefinition of the S-matrix : It is the same $\exp(-i\,\mathrm{Im}\,\Phi_{\beta\alpha})$

II. $T_{\beta\alpha} \exp(-\mathrm{Re}\,\Phi_{\beta\alpha})$

- Accounts for the same IR divergences as the redefinition of the asymptotic states in the KF formalism.

$\mathrm{Re}\,\Phi_{\beta\alpha}$ and $\int^{\Lambda} \frac{d^3k}{(2\pi)^3\, 2k^0} S^{(h)}_{\beta}(\vec{k}) \cdot S^{(h')}_{\alpha}(\vec{k})$ give the same basic loop integral

III. We also take care of $\delta(\beta-\alpha) \exp(-i\,\mathrm{Im}\,\Phi_{\beta\alpha})$ for the calculation of the PWAs

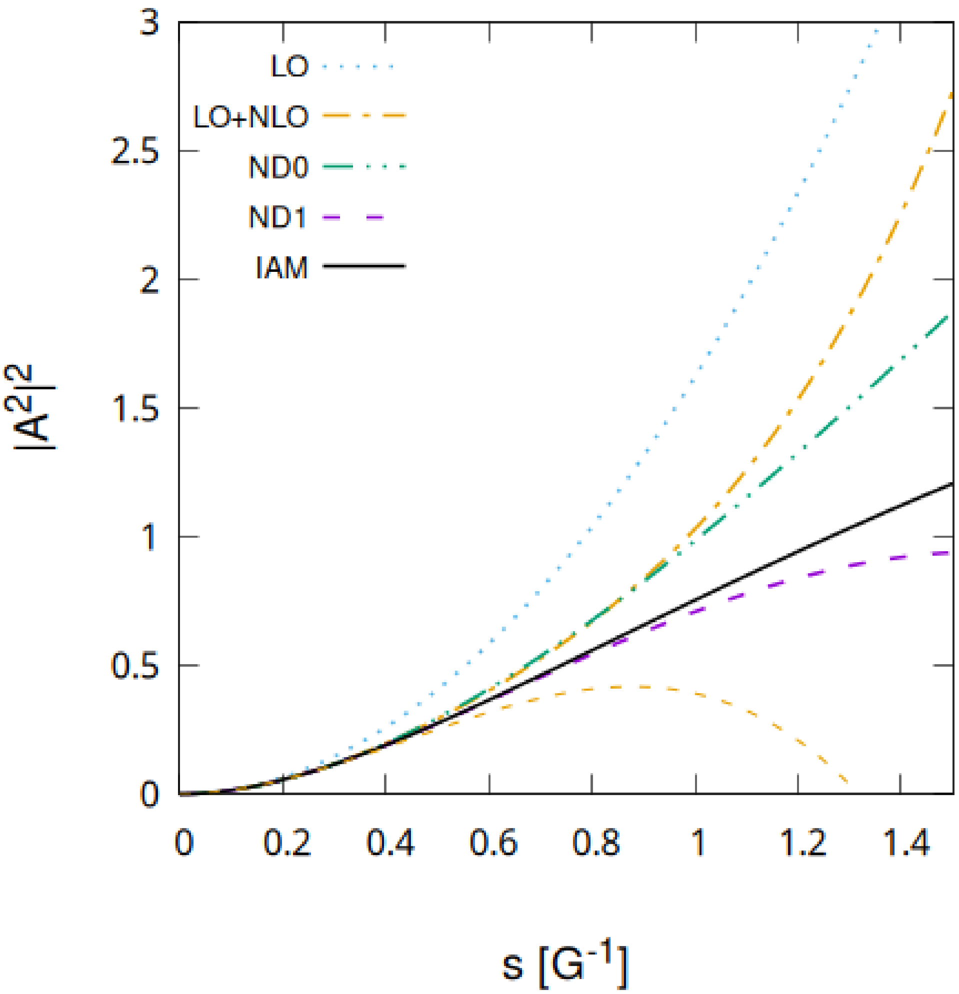
IV. Standard Transition Rates: $\Gamma^{ph}_{\beta\alpha}(\leq E) = \left[\frac{E}{\Lambda}\right]^{B} b(B)\, \Gamma^{0}_{\beta\alpha}$ Weinberg ('65)

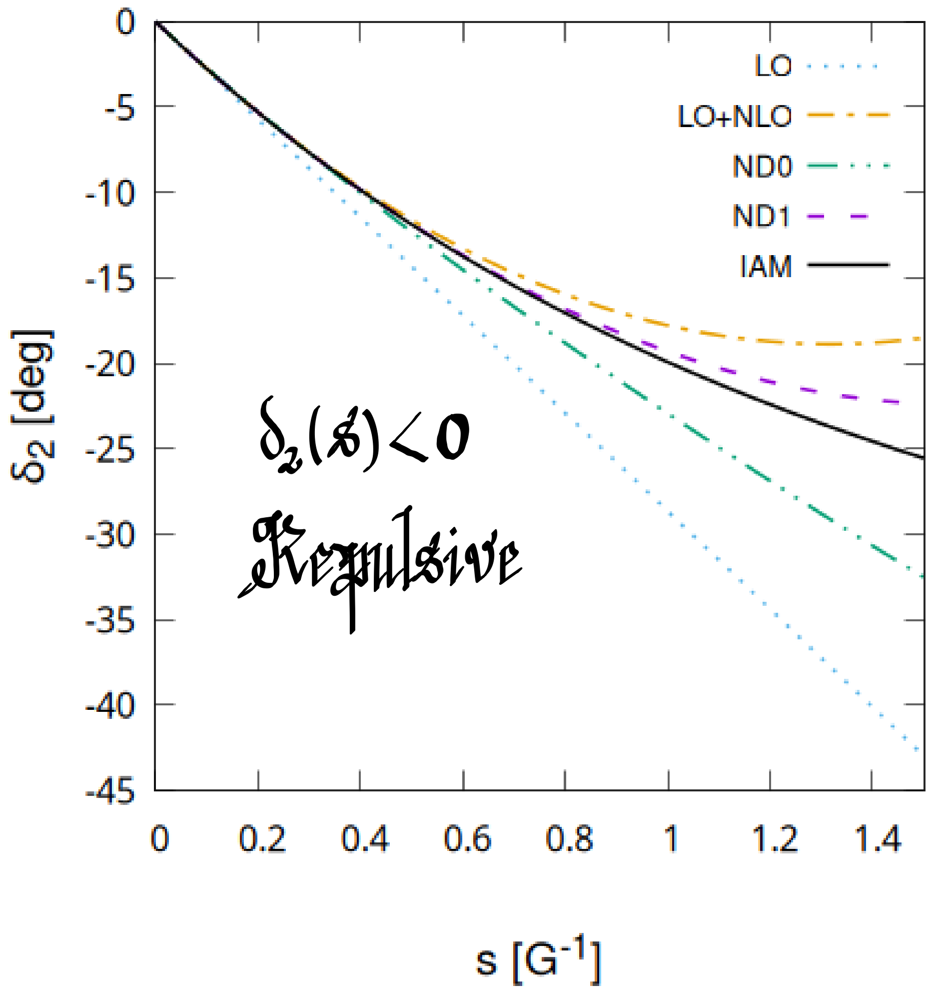
We also consider PWAs, resonances and bound states

V: Our approach can be directly taken to gravity

D-wave I=2 : $A^2_{++;++}$

$1 n_a = 1$





It is much more perturbative than I=0
weaker

No resonance poles

$$A^{2(0)} = -\frac{4s}{\pi}(3 - 2\ln a)$$

It becomes attractive for $\ln a > \frac{3}{2}$

$\ln a = 1.4 - 1.6$ there is large sensitivity to $\ln a$

$\ln a = 3$

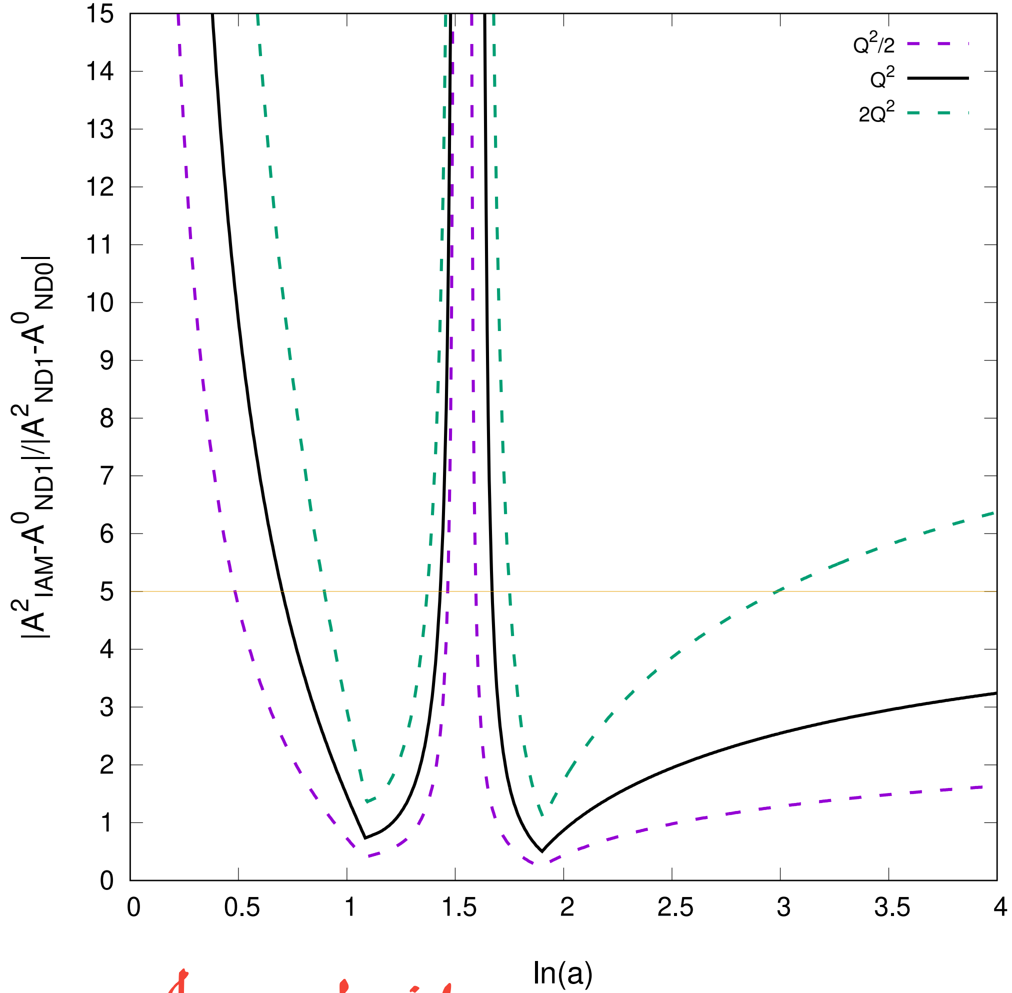
ND0: $0.08 - i\,0.45\ G^{-1}$,

ND1: $0.31 - i\,0.83\ G^{-1}$,

IAM: $0.03 - i\,0.66\ G^{-1}$,

ND1: $-0.29 - i\,0.51\ G^{-1}$.

Large relative variations between the different methods



$\ln a \approx 1$ is again the most reasonable

Conclusions

1. Graviton-graviton scattering amplitudes up to one loop
2. IR-finite scattering amplitudes
3. IR-finite PWAs. Perturbative unitarity.
4. Unitarization: IAM, algebraic N/D method
5. Confirmation of the Graviball: The lightest resonance in pure gravity
6. Equivalence with the Kulish-Faddeev formalism for QED
7. Straightforward extension to gravity

Unitarization of two-loop graviton-graviton scattering amplitude Abreu et al (2020)

Including matter light fields. Graviball $g_p \sim \frac{1}{N}$ Han, Willenbrock (2005), Dvali et al

Connection of graviball with the inflaton

Connection of graviball and QM picture of BHs as bound states of gravitons

Dvali (2010), Dvali, Gómez (2011)