Equations of state for neutron matter derived from ladder diagrams and its extrapolation from basic principles

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Alarcón,JAO, PRC106(2022), AP107(2022)
Lope Oter *et al.*, JPG46(2019)

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Introduction

Energy/fermion= $E/A = \bar{\mathscr{E}}$

- E/A in a many-body system formed by fermions is a fundamental Basic Problem
- Particle-Nuclear physics: Symmetric nuclear matter, neutron matter, neutron stars, quark-gluon plasma, etc
- Condensed matter: Cooper pairs, BCS, Hubbard Hamiltonian (contact constant interaction), BEC, ...
- AMO: Trapped ultra-cold atomic gases, unitary limit ($a_s \to \infty$), BEC–BCS crossover,...

$\mathscr{E}$ in the ladder approximation Alarcón,JAO,Ann.Phys.437,168741(2022)

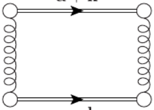
Master Equation for the Energy Density $\mathscr{E}_{\mathscr{L}}$

$$\mathscr{E}_{\mathscr{L}} = \frac{i}{2}\mathrm{Tr}\left(\sum_{d=1}^{\infty}\frac{(t_m L_d)^d}{d}\right) = -\frac{i}{2}\mathrm{Tr}\log\left[I - t_m L_d\right]$$

$$= -\frac{i}{2}\mathrm{Tr}\log e^{2iV_{\mathrm{eff}}} = \mathrm{Tr}\,V_{\mathrm{eff}}$$

- **The trace is taken in the three-momentum, isospin and spin spaces**

- **t_m is the in-medium T matrix. Calculated from the vacuum T matrix, t_V.**

$L_d(p, \boldsymbol{a}) =$ 

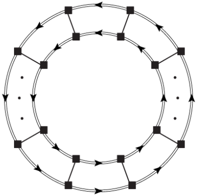
(a)

$$= i\frac{mp}{8\pi^2}\int d\hat{\boldsymbol{k}}\,\theta(k_F - |\boldsymbol{a} + p\hat{\boldsymbol{k}}|)\theta(k_F - |\boldsymbol{a} - p\hat{\boldsymbol{k}}|)\,|p\hat{\boldsymbol{k}}, \sigma_1, \sigma_2\rangle\langle p\hat{\boldsymbol{k}}, \sigma_1, \sigma_2|$$

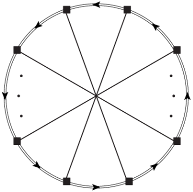
Resummation of ladder diagrams

$t_m \equiv$ Squares joined by inward lines

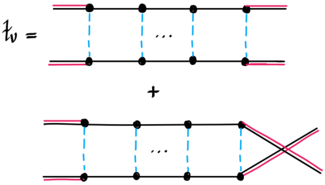
t_V arises by expanding the Γ vertices



Hartree



Fock



It is an interesting combinatorial problem to get $\mathscr{E}_{\mathscr{L}} = -\frac{i}{2}\mathrm{Tr}\log\left[I - t_m L_d\right]$

Alarcón,JAO,Ann.Phys.437,168741(2022)

- The reformulation of many-body field-theory in JAO,PRC65,025204(2002) drives to an adequate arrangement of the diagrams and solves their resummation

- For arbitrary vacuum fermion-fermion interaction
- For arbitrary regularization method

In-medium many-body formalism: One-nucleon sector

Vacuum-vacuum transition amplitude

$$e^{i\mathscr{Z}[J]} = \langle 0_{\rm out}|0_{\rm in}\rangle_J$$

In QCD/ChPT, typically $J = \{v, a, s, p\}$

$$\mathscr{L}_{\rm ext} = \mathscr{L}^0_{\rm QCD} + \bar{q}\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right) q - \bar{q}\left(s - ip\right) q$$

Gasser,Sainio,Svarc,NPB307(1988)

One-nucleon-one-nucleon transition amplitude

$$\mathscr{F}(\boldsymbol{p}', \boldsymbol{p})[J] = \langle \boldsymbol{p}'_{\rm out}|\boldsymbol{p}_{\rm in}\rangle_J$$

$$\mathscr{L}_{\pi N} = -\bar{\psi} D\psi$$
$$\mathscr{L} = \mathscr{L}_{\pi\pi} + \mathscr{L}_{\pi N} + \bar{\eta}\psi + \bar{\psi}\eta$$

η is a fermionic source

In-medium many-body formalism

JAO, PRC65(2002): Similar ideas to many-body field theory:

$$|\Omega\rangle_{\mathrm{in;\,out}} = \prod_{n}^{N} a(\boldsymbol{p}_n \sigma_n)^{\dagger}_{\mathrm{in;\,out}} |\mathrm{vacuum}\rangle$$

$$\mathscr{L} = \underbrace{\mathscr{L}_\phi}_{\text{Pure Mesonic}} + \underbrace{\bar{\psi}(D_0 - V)\psi}_{\text{Bilinear}} + \bar{\eta}\psi + \bar{\psi}\eta$$

Mesons exchanged:
Light, Heavier ones ($M_H \to \infty$ contact interactions)

$$e^{i\mathscr{Z}[J]} = \langle \Omega_{\mathrm{out}} | \Omega_{\mathrm{in}} \rangle_J$$

Generating Functional: Fermions in the Fermi sea are integrated out

$$e^{i\mathscr{Z}[J]} = \int [dU] \exp\Big\{ i \int dx \mathscr{L}_\phi$$
$$+ \mathrm{Tr} \int \frac{d^3p\, \mathbf{n}(\mathbf{p})}{(2\pi)^3} \int d^3x d^3y\, e^{-i\boldsymbol{px}} \log \mathscr{F}(\mathbf{x},\mathbf{y}) e^{i\boldsymbol{py}} \Big\}$$

Tr Trace over discrete internal indices

$$n(p) = \begin{pmatrix} \theta(k_F - |\boldsymbol{p}|) & 0 \\ 0 & \theta(k_F - |\boldsymbol{p}|) \end{pmatrix}$$

$$\mathscr{F}(\mathbf{x},\mathbf{y})_{\alpha\beta} = \delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$$
$$- i \int dt \int dt' e^{iH_0 t} \gamma^0 \left[V[I - D_0^{-1} V]^{-1} \right] (\boldsymbol{k}, \mathbf{y})_{\alpha\beta} e^{-iH_0 t'}$$

Γ is a non-local vertex: Geometric series

Vacuum fermion propagator: $D_0^{-1} = \dfrac{i\delta_{\sigma\sigma'}}{(p^0 - \boldsymbol{p}^2/(2m) + i\epsilon)}$

Perturbative Expansion

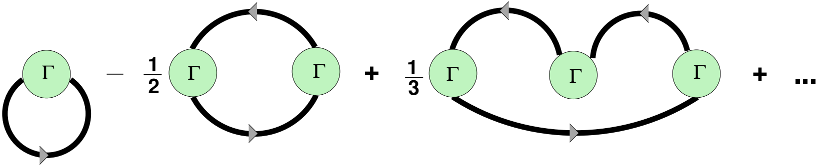
- Series expansion of $\log(1+\epsilon) = -\sum(-1)^d \epsilon^d / d$
- d is the number of Fermi-sea insertions
- Each of them provides an extra $(-)$ sign
- Notice the symmetry factor $1/d$

$$e^{i\mathscr{Z}[J]} = \int [dU] \exp\Big[i \int dx\, \mathscr{L}_\phi$$

$$-i \int \frac{d\boldsymbol{p}}{(2\pi)^3} \int \mathrm{Tr}\left(V[I - D_0^{-1}V]^{-1}|_{(x,y)} n(p) \right) dx\, dy\, e^{ip(x-y)}$$

$$-\frac{1}{2} \int \frac{d\boldsymbol{p}}{(2\pi)^3} \int \frac{d\boldsymbol{q}}{(2\pi)^3} \int \mathrm{Tr} e^{ip(x-y)} e^{-iq(x'-y')} dx\, dx'\, dy\, dy'$$

$$\times \left(V[I - D_0^{-1}V]^{-1}|_{(x,x')} n(q) V[I - D_0^{-1}V]^{-1}|_{(y',y)} n(p) \right) + \ldots \Big]$$

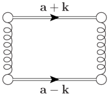


In-medium Generalized Vertices (IGV's)

Integral equation (IE) for $t_m(\boldsymbol{a})$

- $\boldsymbol{a} = \frac{1}{2}(\boldsymbol{k}_1 + \boldsymbol{k}_2)$, $\boldsymbol{p} = \frac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2)$

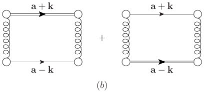
Vacuum intermediate states $G(p)$

$G(p)=$ 

$$= -m \int \frac{d^3 k}{(2\pi)^3} \frac{|p\hat{\boldsymbol{k}}, \sigma_1, \sigma_2\rangle\langle p\hat{\boldsymbol{k}}, \sigma_1, \sigma_2|}{k^2 - p^2 - i\epsilon}$$

Vacuum T matrix t_V $\quad t_V = V - V\,G(p)t_V$

Mixed intermediate states $L_m(p, \boldsymbol{a})$

$L_m(p, \boldsymbol{a})=$ 

$$= -m\int \frac{d^3 k}{(2\pi)^3} \frac{\theta(k_F - |\boldsymbol{a} + \boldsymbol{k}|) + \theta(k_F - |\boldsymbol{a} - \boldsymbol{k}|)}{\boldsymbol{k}^2 - \boldsymbol{p}^2 - i\epsilon} |p\hat{\boldsymbol{k}}, \sigma_1, \sigma_2\rangle\langle p\hat{\boldsymbol{k}}, \sigma_1, \sigma_2|$$

In-medium T matrix $t_m(\boldsymbol{a})$

$$t_m(\boldsymbol{a}) = t_V + t_V L_m(p, \boldsymbol{a}) t_m(\boldsymbol{a})$$

- $|\boldsymbol{k}|$ is bounded in L_m and L_d

No extra divergences stem from L_m

$$t_m(\boldsymbol{a}) = V + V\left(G(p) - L_m(p, \boldsymbol{a})\right) t_m(\boldsymbol{a})$$

The formula for $\mathscr{E}_{\mathscr{L}}$ gives real values

$$\mathscr{E}_{\mathscr{L}} = -\frac{i}{2}\mathrm{Tr}\log\left[I - t_m L_d\right]$$

$$= -\frac{i}{2}\mathrm{Tr}\log\left[t_m(t_m^{-1} - L_d)\right]$$

$$t_m(\boldsymbol{a})^{-1} = V^{-1} + G - L_m(p, \boldsymbol{a})$$

$$\mathscr{E}_{\mathscr{L}} = -\frac{i}{2}\mathrm{Tr}\log\left[A^{-1}B\right]$$

$$A = V^{-1} + G - L_m(p, \boldsymbol{a})$$

$$B = V^{-1} + G - L_m(p, \boldsymbol{a}) - L_d(p, \boldsymbol{a})$$

Within the Fermi seas, where the Tr is taken: $A = B^\dagger$

$I - t_m L_d$ is a unitary matrix

We diagonalize it and sum over its eigenvalues for calculating $\mathrm{Tr}\log[I - t_m L_d]$

Alarcón,JAO,Ann.Phys.437,168741(2022)

Partial waves

- Low energies: One typically characterizes t_V by summing over partial-wave amplitudes (PWAs)
- But in the medium the scattering depends on the total three-momentum ($2\boldsymbol{a}$)

PWAs mix

Notice,

$$\int d\hat{\boldsymbol{k}}\, Y_\ell^m(\hat{\boldsymbol{k}}) Y_{\ell'}^{m'}(\hat{\boldsymbol{k}})^* \theta(\xi - |a\hat{\boldsymbol{z}} \pm \boldsymbol{k}|) \neq 0$$

Transformation under a rotation R

$$L_m(p, R\boldsymbol{a}) = RL_m(p, \boldsymbol{a})R^\dagger$$
$$L_d(p, R\boldsymbol{a}) = RL_d(p, \boldsymbol{a})R^\dagger$$
$$t_V = Rt_V R^\dagger$$
$$t_m(R\boldsymbol{a}) = Rt_m(\boldsymbol{a})R^\dagger$$

$$\chi(S\ell I) = \frac{1-(-1)^{\ell+S+I}}{\sqrt{2}} \, , \; \ell + S + I = \mathrm{odd}$$

$$\mathcal{E}_{\mathscr{L}} = -\frac{2i}{m\pi^3} \sum_{\substack{J,\mu,\ell \\ S,I,i_3}} \chi(S\ell I)^2 \int_0^\infty p dp \int_0^\infty a^2 da \langle J\mu\ell S I i_3 p | \log\left[I - t_m(a\hat{\boldsymbol{z}}) L_d(p, a\hat{\boldsymbol{z}})\right] | J\mu\ell S I i_3 p\rangle$$

Mixing of $J's$ for $a\hat{\boldsymbol{z}}$. IE for $t_m(a\hat{\boldsymbol{z}})$ in PWAs $\quad [v]$, $[t_V]$, $[t_m]$

matrices in the space of coupled PWAs

$$[t_m(a\hat{\boldsymbol{z}})](p',p) = [t_V](p',p) + \frac{m}{(2\pi)^2} \int_0^\infty \frac{k^2 dk}{k^2 - p^2 - i\epsilon} [t_V](p',k) \cdot \mathscr{B} \cdot [t_m(a\hat{\boldsymbol{z}})](k,p)$$

$$[t_V](p',k)_{J'\ell',J_2\ell_2} = \delta_{J'J_2} \langle J'\mu\ell' S p' | t_V | J_2 \mu \ell_2 S k\rangle \, ,$$

$$\mathscr{B}_{J_2\mu\ell_2, J_1\mu\ell_1} = -2\chi(S\ell_2)\chi(S\ell_1) \sum_{m_3 s_3} (m_3 s_3 \mu | \ell_2 S J_2)(m_3 s_3 \mu | \ell_1 S J_1) \times \int d\hat{\boldsymbol{k}} \, Y_{\ell_2}^{m_3}(\hat{\boldsymbol{k}})^* Y_{\ell_1}^{m_3}(\hat{\boldsymbol{k}}) \theta(k_F - |\boldsymbol{k} - a\hat{\boldsymbol{z}}|) \, .$$

Explicit solution: Contact-interaction potential JAO'18

n coupled PWAs

$$v_{\alpha\beta}(k,p) = k^{\ell_\alpha} p^{\ell_\beta} \sum_{i,j=1}^{N} v_{\alpha\beta;ij} k^{2(i-1)} p^{2(j-1)}$$

$$v_{\alpha\beta}(k,p) = [k_\alpha]^T \cdot [v] \cdot [p_\beta]$$

$$[v] = \begin{pmatrix} [v_{11}] & [v_{12}] & \dots & [v_{1n}] \\ [v_{21}] & [v_{22}] & \dots & [v_{2n}] \\ \dots & \dots & \dots & \dots \\ [v_{n1}] & [v_{n2}] & \dots & [v_{nn}] \end{pmatrix}$$

$$[k_\alpha]^T = (\underbrace{0,\dots,0}_{N(\alpha-1)\text{ places}}, k^{\ell_\alpha}, k^{\ell_\alpha+2}, \dots, k^{\ell_\alpha+2(N-1)}, 0, \dots, 0)$$

$$t_V(k,p) = [k_\alpha]^T \cdot [\hat{t}_v(p)] \cdot [p_\beta] \ ,$$

$[\hat{t}_V(p)]$ is given by the *algebraic* equation

$$[\hat{t}_V(p)] = [v] - [v]\cdot[\mathscr{G}(p)]\cdot[\hat{t}_V(p)]$$

$$[\hat{t}_V(p)] = (I + [v]\cdot[\mathscr{G}(p)])^{-1}\cdot[v]$$

$$[\mathscr{G}(p)_{\alpha\beta}] = -\frac{m}{(2\pi)^2}\int_0^\infty \frac{k^2 dk}{k^2 - p^2 - i\epsilon}[k_\alpha][k_\beta]^T$$

Divergent integrals [General cutoff regularization van Kolck'99]

$$L_n = -\frac{m}{2\pi^2}\int_0^\infty dk k^{n-1} = \theta_n \Lambda^n \ ,$$

θ_n depends on the scheme (DR is $\theta_n = 0$)

Renormalization: Matching with the
Effective Range Expansion (ERE) in vacuum ($k_F = 0$)

$$\frac{4\pi}{m}(p^\ell)t_V(p,p)^{-1}(p^\ell) + i(p^\ell)^2(p) = -(a)^{-1} + \frac{1}{2}(r)p^2 + \sum_{i=2}^{M}(v_\ell^{(2i)})p^{2i}$$

(a), (r) and $(v_\ell^{(2i)})$: $n\times n$ matrices , $(p)^\ell = \mathrm{diag}(p^{\ell_1},\ldots,p^{\ell_n})$

Once renormalized with $\theta_n \neq 0$ and $\Lambda \to \infty$, the solution is

$$t_V(k,q) = \frac{4\pi}{m}(k)^\ell \tau(p) q^\ell \ , \ \text{Off-shell}$$

$$\tau(p)^{-1} = -(a)^{-1} + \frac{1}{2}(r)p^2 + \sum_{i=2}^{M}(v_\ell^{(2i)})p^{2i} - i(p^\ell)^2(p) \ , \ \text{on-shell}$$

Reason: $p, k \leq k_F$ $(k_F/\Lambda \to 0)$

Uncoupled: $\tau(p) = \dfrac{1}{p\cot\delta}$ We can take $M \to \infty$

Coupled: $\tau(p) = (p)^{-\ell}\dfrac{1}{2ip}(S_{JS}-1)(p)^{-\ell}$.

$$t_V(q,q') = \frac{(qq')^\ell}{p^{2\ell+1}\cot\delta_\ell - ip^{2\ell+1}}$$

$$t_m(q,q') = \frac{(qq')^\ell}{p^{2\ell+1}\cot\delta_\ell - ip^{2\ell+1} - q'^\ell L_m q'^\ell}$$

Applications: S waves, $v(k,p) = c_0$ Alarcón,JAO,Ann.Phys.437,168741(2022)

$$\tau_m(p) = \left(-\frac{1}{a_0} - ip + \mathscr{G}_m(p)\right)^{-1}$$

$$\mathscr{E}_{\mathscr{L}} = \frac{8k_F^5}{m\pi^3}\int_0^1 s^2 ds \int_0^{\sqrt{1-s^2}} \kappa d\kappa \arctan\left(\frac{a_0 k_F I}{1 - a_0 k_F R/\pi}\right)$$

$s = a/k_F$, $\kappa = p/k_F$, $R + i\pi I = (\mathscr{G}_m - ip)\pi/k_F$ Kaiser'11

$$\bar{\mathscr{E}} = \frac{3{k_F}^2}{10m}\left\{\xi + \frac{\zeta}{a_0 k_F} - \frac{5\nu}{3(a_0 k_F)^2} + \ldots\right\}$$

Bertsch parameter

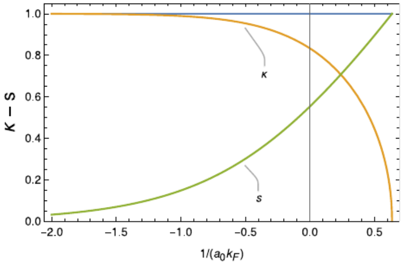
$$\xi = 1 - \frac{80}{\pi}\int_0^1 dss^2 \int_0^{\sqrt{1-s^2}} d\kappa\kappa \arctan\left(\frac{\pi I}{R}\right) = 0.5066.$$

- The experimental actual value in the superfluid phase is $\xi = 0.370(5)(8)$ Ku *et al.*,Science 335 (2012) 563
- The experimental value of ξ for normal matter at the unitary limit is $\xi \approx 0.45$ Navon *et al*, Science 328 (2010) 729

In-medium poles in the S-wave amplitude at the Fermi surface $\kappa = \sqrt{1-s}$ and $s \in [0,1]$

$$\tau_m(p) = \left(-\frac{1}{a_0} - ip + \mathscr{G}_m(p)\right)^{-1}$$

$$\kappa = \tanh\left(\frac{1}{\kappa}\left[1 - \frac{\pi}{2a_0 k_F}\right]\right)\ , \ \kappa \in [0,1]$$

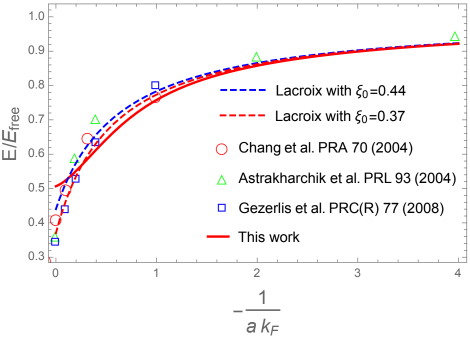


Smooth transition as a function of $1/a_0 k_F$

Cooper pairs for $1/(a_0 k_F) \to -\infty$
Total momentum, $P = 2k_F s \to 0$, and relative momentum $p = k_F \kappa \to k_F$

Unitary limit: $P = 2sk_F = 1.11 k_F$ and $p = \kappa k_F = 0.88 k_F$

Molecular side: $2/\pi > a_0 k_F > 0$



Lacroix: Density functionals

Rest: Quantum Montecarlo [**QM**] calc.

Including r_0 and v_2

$$v(k,p) = c_0 + \frac{1}{2}c_2(k^2+p^2)$$

$$t_m(k,p) = \frac{4\pi/m}{-\frac{1}{a}+\frac{1}{2}r_0p^2 - ip + \mathscr{G}_m(p)}$$

Including r_0 and v_2

$$\mathcal{E}_{\mathscr{L}} = \frac{8k_F^5}{m\pi^3}\int_0^1 dss^2 \int_0^{\sqrt{1-s^2}} d\kappa\kappa \arctan\left(\frac{a_0 k_F I}{1 - a_0 r_0 k_F^2 \kappa^2/2 - a_0 k_F R/\pi}\right)$$

Bertsch parameter in the unitary limit $\xi(r_0 k_F)$

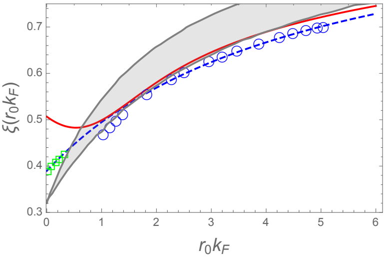
$$\xi(k_F) = 1 - \frac{80}{\pi}\int_0^1 dss^2 \int_0^{\sqrt{1-s^2}} d\kappa\kappa \arctan\left(\frac{\pi I}{\pi r_0 k_F \kappa^2/2 + R}\right)$$

$$\xi(k_F) = \xi(r_0 = 0) + \eta_e r_0 k_F + \delta_e (r_0 k_F)^2 + \dots$$

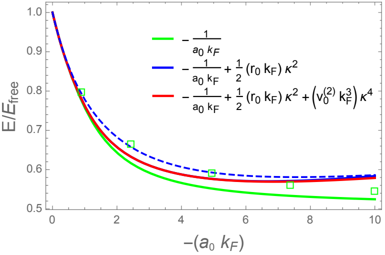
Our results [normal matter]: $\eta_e = -0.0859$, $\delta_e = 0.0644883$

QMC: $\eta_e = 0.127$, $\delta_e = -0.055$ Forbes,Gandolfi,Gezerlis'12

Dependence of ξ with r_0



Blue dashed: DFT Lacroix *et al.*'17
Circles: Schwenk,Pethick'05
Squares: QMC Forbes *et al.*'11;'12
Area: Schäfer *et al.*'05



Blue dashed: DFT Lacroix'19

Squares: QMC Gezerlies,Carlson'10

Inclusion of v_2

$$v(k,p) = c_0 + c_2(k^2+p^2) + c_4(k^4+p^4)$$

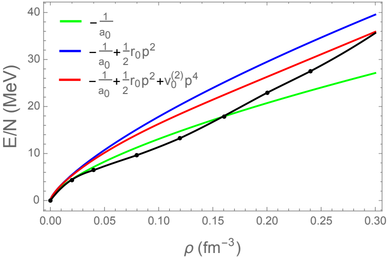
$$\mathcal{E}_L = \frac{8k_F^5}{m\pi^3}\int_0^1 dss^2 \int_0^{\sqrt{1-s^2}} d\kappa\kappa \arctan\left(\frac{a_0 k_F I}{1 - a_0 r_0 k_F^2 \kappa^2/2 - a_0 v_0^{(2)} k_F^4 \kappa^4 - a_0 k_F R/\pi}\right)$$

$$\xi(k_F) = 1 - \frac{80}{\pi}\int_0^1 dss^2 \int_0^{\sqrt{1-s^2}} d\kappa\kappa \arctan\left(\frac{I}{r_0 k_F \kappa^2/2 + v_0^{(2)} k_F^3 \kappa^4 + R/\pi}\right)$$

$$\xi(k_F) = \xi(r_0 = 0) + \eta_e r_0 k_F + \gamma_e v_0^{(2)} k_F^3 + \delta_e (r_0 k_F)^2 + \ldots$$

Our result: $\gamma_e = -0.164$

For neutron matter $a_0 = -18.95 \pm 0.40$ fm Chen *et al.*'08, $r_0 = 2.75$ fm and $v_2 = -0.50$ fm^3 Navarro *et al.*'16



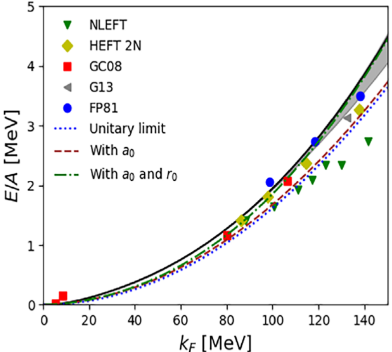
Dots: Variational calc. Akmar *et al.*'84

This curve was not well calculated algebraically until our work

It supports the Unitary-Gas Conjecture Tews *et al.*, Astrophys.J.848(2017)105

Applications: Neutron matter [PNM] Alarcón,JAO, forthcoming

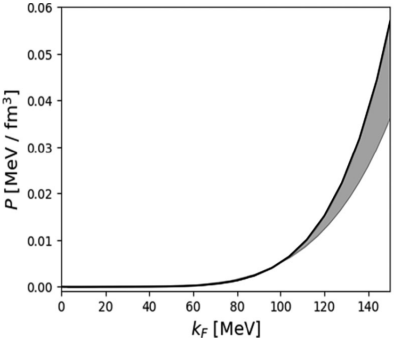
- Our results for the resummation of ladder diagrams are renormalized
- t_m and $\bar{\mathscr{E}}$ are expressed directly in terms of experimental phase shifts and mixing angles
- We show them up to $k_F \lesssim 150$ MeV. The off-shell part $\propto q^\ell p^{\ell'}$ reflects the contact-interaction nature assumed as starting point. LC $p^2 < -m_\pi^2/4$.
- In density up to $\rho \lesssim 1.5 \cdot 10^{-2}$ fm$^{-3} \approx 0.1\,\rho_s$



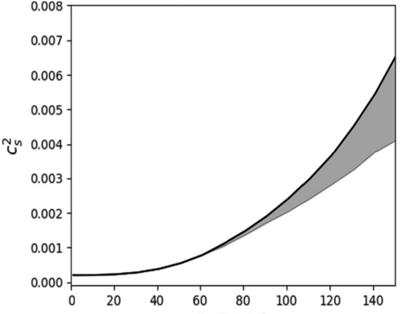
Our results: Solid line
Unitary limit: $a_0 \to \infty$, dotted line
S-**wave:** $a_0 = -18.95$ fm, dashed line
S-**wave:** $a_0 + r_0 (= 2.75)$ fm, dash-dotted line
NLEFT: NLO Chiral-EFT on the lattice Epelbaum *et al.*'09
FP81: Variational calculation Friedman,Pandharipande'81
G13: Auxiliary-Field QMC: N2LO χ *NN* potential Gezerlies *et al.*'13
HEFT 2N: Auxiliary-Field QMC: $V_{\text{low}-k}$ N3LO χ *NN* potential Wlazlowski *et al*'14
GC08: QMC calculations Gezerlis, Carlson'08

P, c_s^2, $S(\rho)$, L

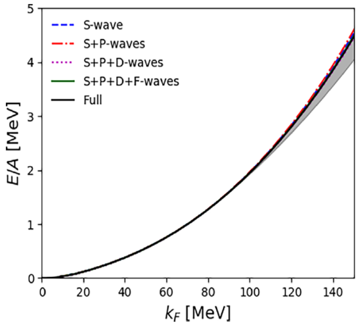
$$P(\rho) = \rho^2 \frac{\partial \bar{\mathscr{E}}}{\partial \rho}$$



$$c_s^2(\rho) = \frac{1}{m}\frac{\partial P}{\partial \rho}$$
$$= \frac{2\rho}{m}\frac{\partial \bar{\mathscr{E}}}{\partial \rho} + \frac{\rho^2}{m}\frac{\partial^2 \bar{\mathscr{E}}}{\partial \rho^2}$$



Partial-wave decomposition



It is dominated by the S wave– $^1S_0$ PWA

Regulator: $\exp(-(q - M_\pi/2)^2/\Lambda^2)$ for off-shell $q > M_\pi/2$ [entering in L_m]

For $k_F < 120$ MeV there is no significant impact for $\Lambda \gtrsim M_\pi$

Its effect increases for higher k_F, as expected

Symmetry energy: $S(\rho) = \bar{\mathscr{E}}_{\mathrm{PNM}}(\rho) - \bar{\mathscr{E}}_{SNM}(\rho)$

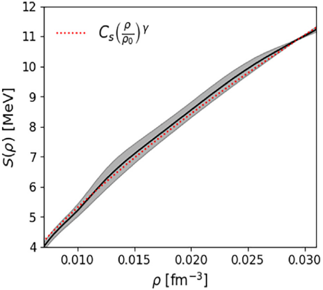
L Slope of $S(\rho)$ at ρ_0 $L = 3\rho_0 \left.\frac{dS(\rho)}{d\rho}\right|_{\rho_0}$

Fit: S_0, L Simple parameterization

$\bar{\mathscr{E}}(\rho, x_p),\ x_p \equiv \rho_p/\rho$ Gandolfi *et al.*'18

$$\bar{\mathscr{E}}(\rho, x_p) = \bar{\mathscr{E}}(\rho, \frac{1}{2}) + C_s \left(\frac{\rho}{\rho_0}\right)^{\gamma_s} (1 - 2x_p)^2$$

$$L = 3C_s\gamma_s$$

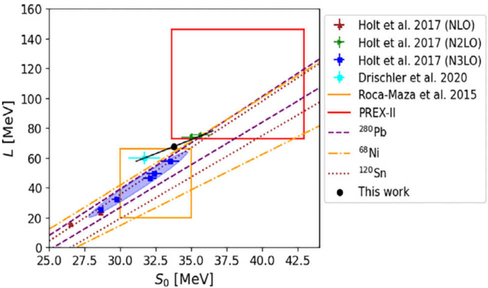


Fit: $C_s = 34.77 \pm 0.15$ MeV,

$\gamma_s = 0.667 \pm 0.003$

Empirical bands extracted from static dipole polarizability in nuclei ^{280}Pb, ^{68}Pb, ^{120}Sn Roca-Maza *et al.*'15

Perturbative χ-EFT at NLO,N2LO,N3LO Holt,Kaiser'17



Correlation ellipse at 95% C.L.

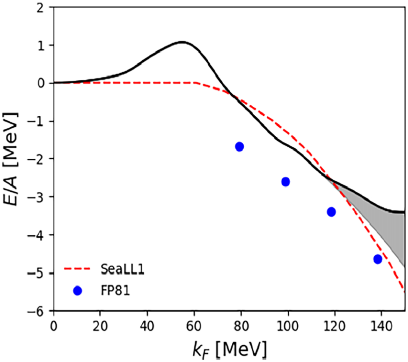
31 MeV $< S_0 <$ 38 MeV

57 MeV $< L <$ 84 MeV

Symmetric Nuclear Matter [SNM]

Alarcón,JAO, forthcoming

- Up to $k_F < 150$ MeV or $\rho = 3 \cdot 10^{-2}$ fm^{-3}

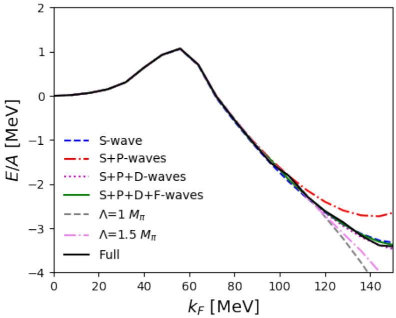


FP81: MC variational calc. Friedman,Pandharipande'91
SeaLL1: Density Functional calc. Bulgrac *et al.*'18

Spinodal instability

$\bar{\mathscr{E}} > 0$ up to $k_F < 70$ MeV. SNM is no the most favorable phase, α and heavy nuclei form Shen *et al.*'98

Partial-wave decomposition



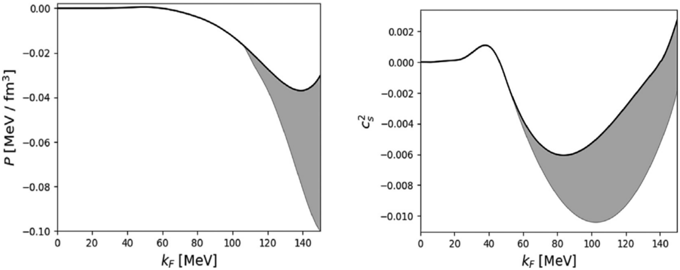
Dominance of S-waves
P-wave contribution is repulsive,
but cancelled by higher waves

Regulator: $\exp(-(q - M_\pi/2)^2/\Lambda^2)$ for off-shell $q > M_\pi/2$
[entering in L_m]

For $k_F < 120$ MeV there is not impact on $\Lambda \gtrsim M_\pi$

It keeps growing for $k_F > 120$ MeV, as expected

P and c_s^2



$P > 0$ up to $k_F = 59$ MeV

$c_s^2 < 0$ up to $k_F = \xi_c = 140$ MeV or $\rho = 2.4 \cdot 10^{-2}$ fm^{-3} $\quad [K = \frac{c_S^2}{m\rho}]$

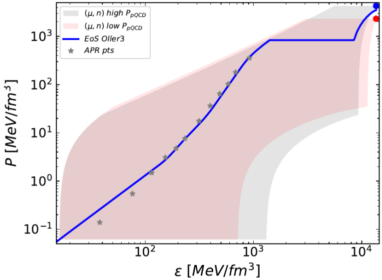
$\xi_c \approx 190$ MeV Shen *et al.*'98 and $\xi_c \approx 200$ MeV Machleidt *et al.*

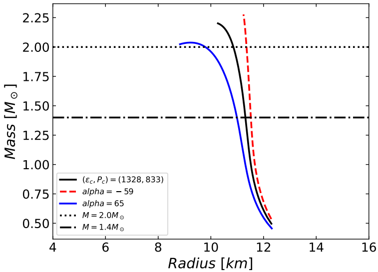
Conclusions

1. The ladder diagrams to calculate $\bar{\mathscr{E}} \equiv E/A$ is resummed
2. This is done for arbitrary vacuum fermion-fermion interactions
3. And also for arbitrary regularization method
4. The case of contact interactions is fully solved for arbitrary cutoff regularization
5. DR regularization is disregarded [for finite order ERE]
6. Renormalized $\bar{\mathscr{E}}$ is obtained and expressed in terms of experimental scattering data. No free parameters, no scale dependence.

8. Application to S waves: Including a_0, r_0, v_2
9. PNM and SNM are studied for $k_F < 150$ MeV
10. $\bar{\mathscr{E}}$, P, c_s^2, K, $S(\rho)$, S_0 and L are provided
11. Restrict EOS parameterizations for low ρ
12. Then, extrapolate these results to larger ρ

From Eva Lope Oter's talk





NICER's constraint $R = 11.8 - 13.1$ Km at $1.4 M_\odot$ Kostas Glampedakis' talk

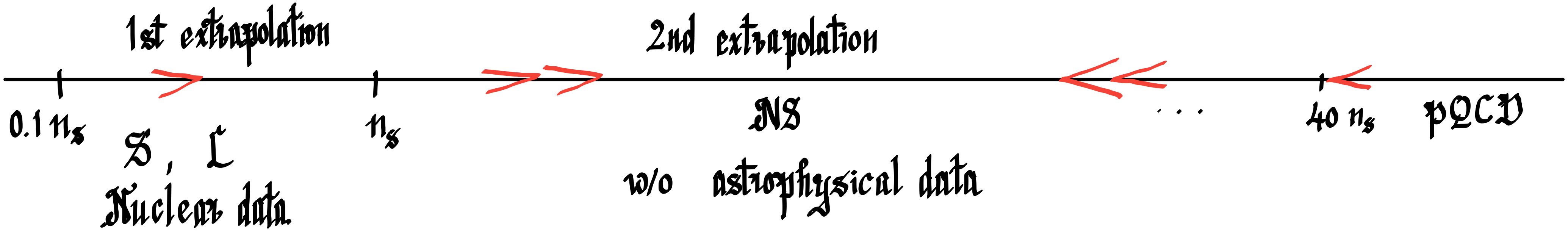
Extrapolation to higher densities of the EoS OlA

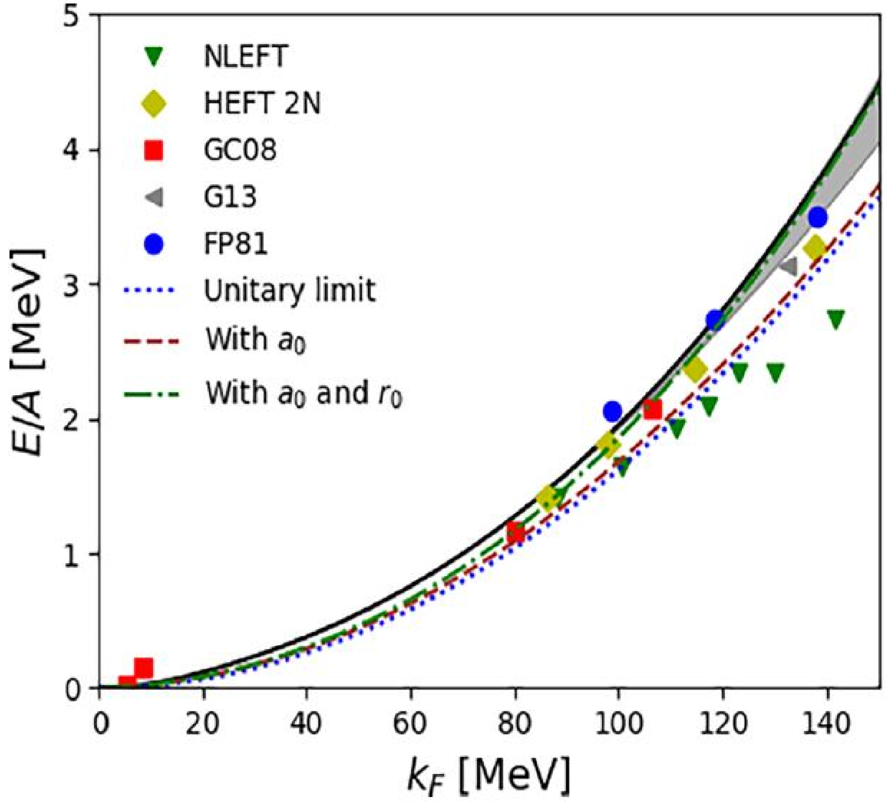
Alarcón, JAO, PRC 107 (2023)

From $n \simeq 0.1\, n_s$ up to pQCD $n \sim 45\, n_s$, $n_s = 0.16\ fm^{-3}$

N^3LO pQCD Gorda et al, PRL 127 (2021)
$\mathcal{O}(\alpha_s^3)$

In between we have the NS density region $n_{c,max} \lesssim 6.5\, n_s$





OIA EoS

1. Renormalized

   No regulator dependence

2. Directly expressed in terms of experimental data, $\delta({}^{2S+1}L_J)$, $\epsilon_{L-L'}$

3. Specific in-medium power counting.

   LO calculation → Ladder Resummation

   Nonperturbative calculation

   $^3S_1 - {}^3D_1$: Deuteron

   $^1S_0$: Antibound state

Many-body perturbation calculations in chiral EFT

1. Strong cutoff dependence

   Small variation 450-500 MeV

   – 10% –

2. A potential with many free parameters fitted to data is required

   Intrinsic difficulties

3. Naive used of vacuum power counting

   They neglect this and show results only for $n > 0.5\, n_S$

3. was solved for contact interactions, $k_F \lesssim 150$ MeV

The upper limit $n = 0.1\, n_s$ in OLA is in the crust region

E/A relative to uniform nuclear matter in β equilibrium (npe) Sharma et al. A&A 584 (2015)

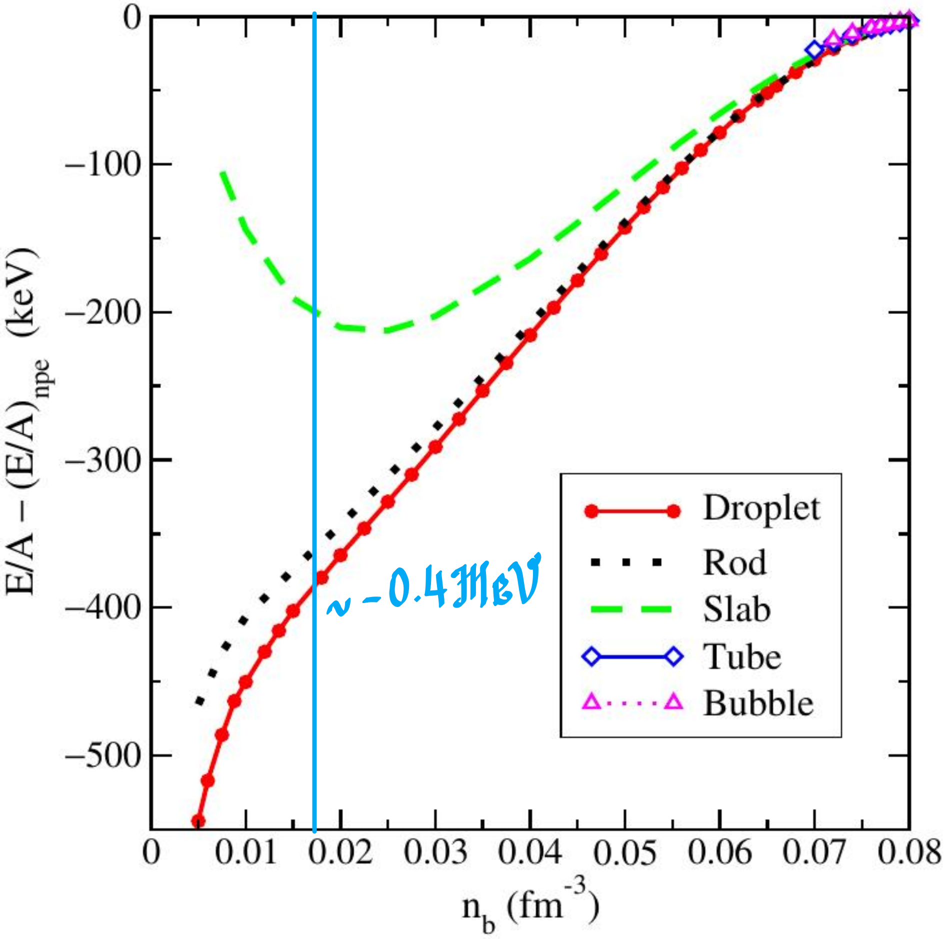
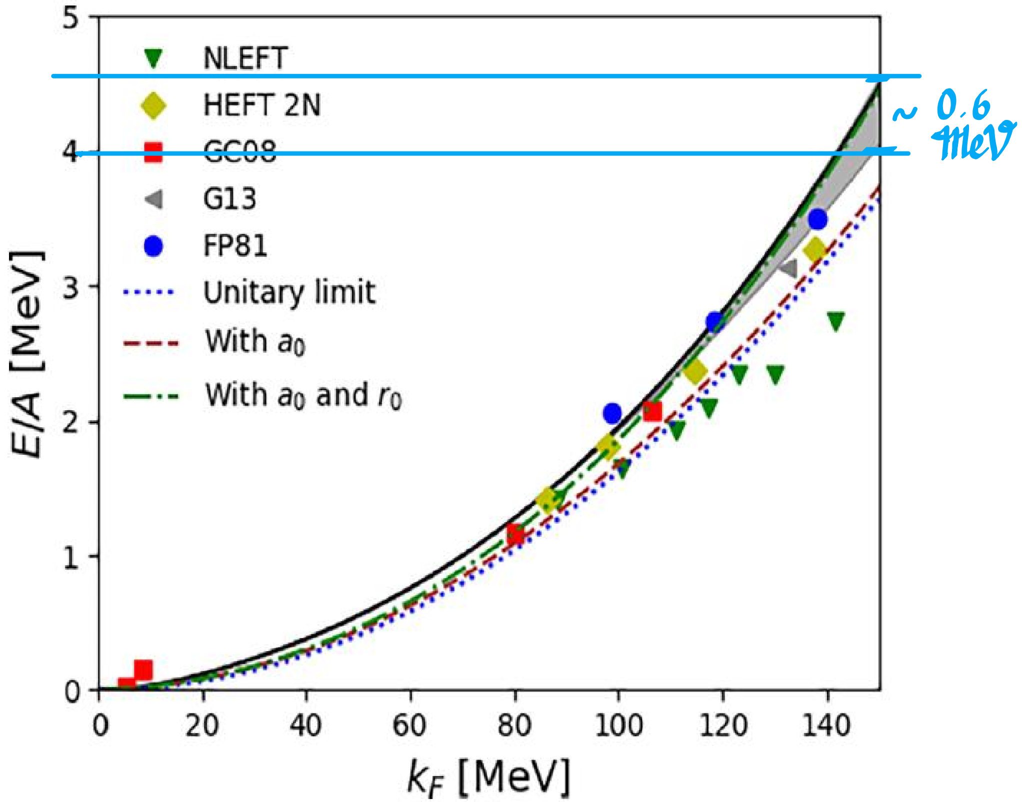


Fig. 3. Energy per baryon of different shapes relative to uniform *npe* matter as a function of baryon density in the inner crust.



It is perfectly inside our uncertainty band

Proton fraction < 3%

Extrapolation method based on Lope Oter et al., JPG 46 (2019)
Komoltsev, Kurkela, PRL 128 (2022)

We mostly proceed in the (ε, P) plane
and to a lesser extent in the (μ, n) plane

$\varepsilon = \frac{U}{V}$ $\quad n = \frac{N}{V}$

Lowest density part : OlA : $E/A(n)$

Back and forth from different along trajectories of extrapolation planes

$\varepsilon = n\left(M_N + E/A\right)$

$P = n^2 \frac{d\,E/A}{dn}$

And viceversa
Discretizing

$n_{i+1} = \frac{\varepsilon_{i+1}}{M_N + (E/A)_{-1}}$

$P_i = n_i^2 \frac{(E/A)_{i+1} - (E/A)_{i-1}}{n_{i+1} - n_{i-1}}$

E/A_{i+1}, n_{i+1} known at i, $i-1$

$\mu_i = \frac{\varepsilon_i + P_i}{n_i}$ Euler equation

In the extrapolation the causality conditions are fulfilled

$(\varepsilon, \mathcal{P})$ plane: $c_s^2 = \dfrac{d\mathcal{P}(\varepsilon)}{d\varepsilon}$ $\boxed{0 \leq c_s^2 \leq 1}$

This delimits an area in $(\varepsilon, \mathcal{P})$ plane between the extremes in the extrapolation. A $10^3 \times 10^3$ grid is constructed.

(μ, n) plane: $c_s^{-2} = \dfrac{\mu}{n} \dfrac{dn(\mu)}{d\mu}$ $\boxed{c_s^2 \geq 1}$

Simultaneously with the $(\varepsilon, \mathcal{P})$ causal constraint.

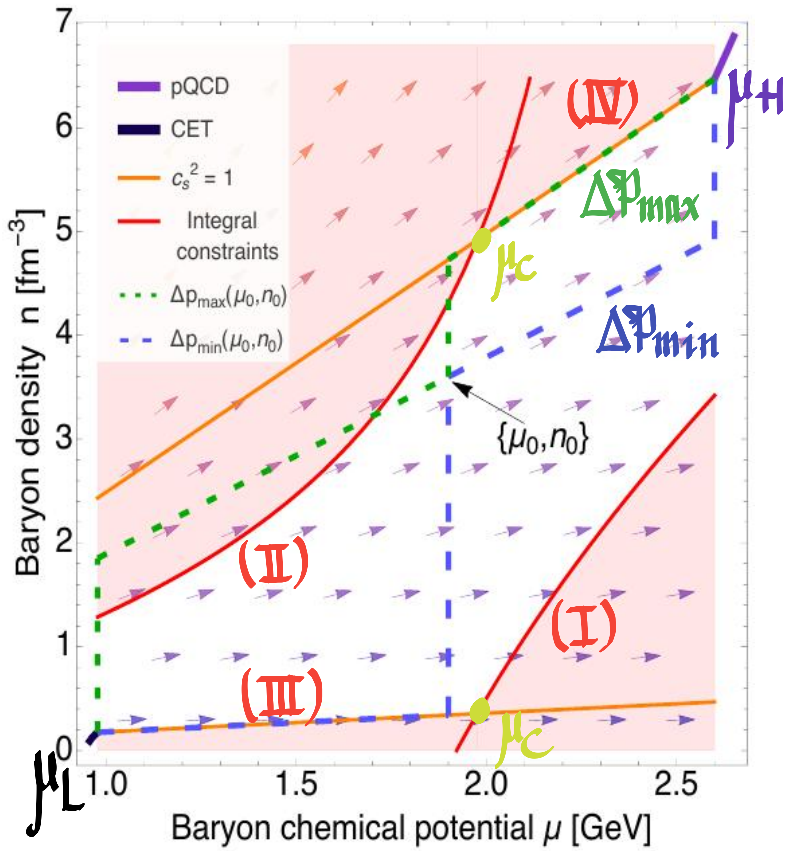
The causal area is further limited by

$$\frac{dP(\mu)}{d\mu} = n(\mu) \longrightarrow P_H - P(n) = \int_{\mu(n)}^{\mu_H} d\mu\, n(\mu)$$

This result and

$$c_s^{-2} = \frac{\mu}{n}\frac{dn(\mu)}{d\mu} \geq 1 \longrightarrow \frac{dn(\mu)}{d\mu} \geq \frac{n}{\mu}$$

are heavily used by Komoltsev and Kurkela PRL 128 (2022)



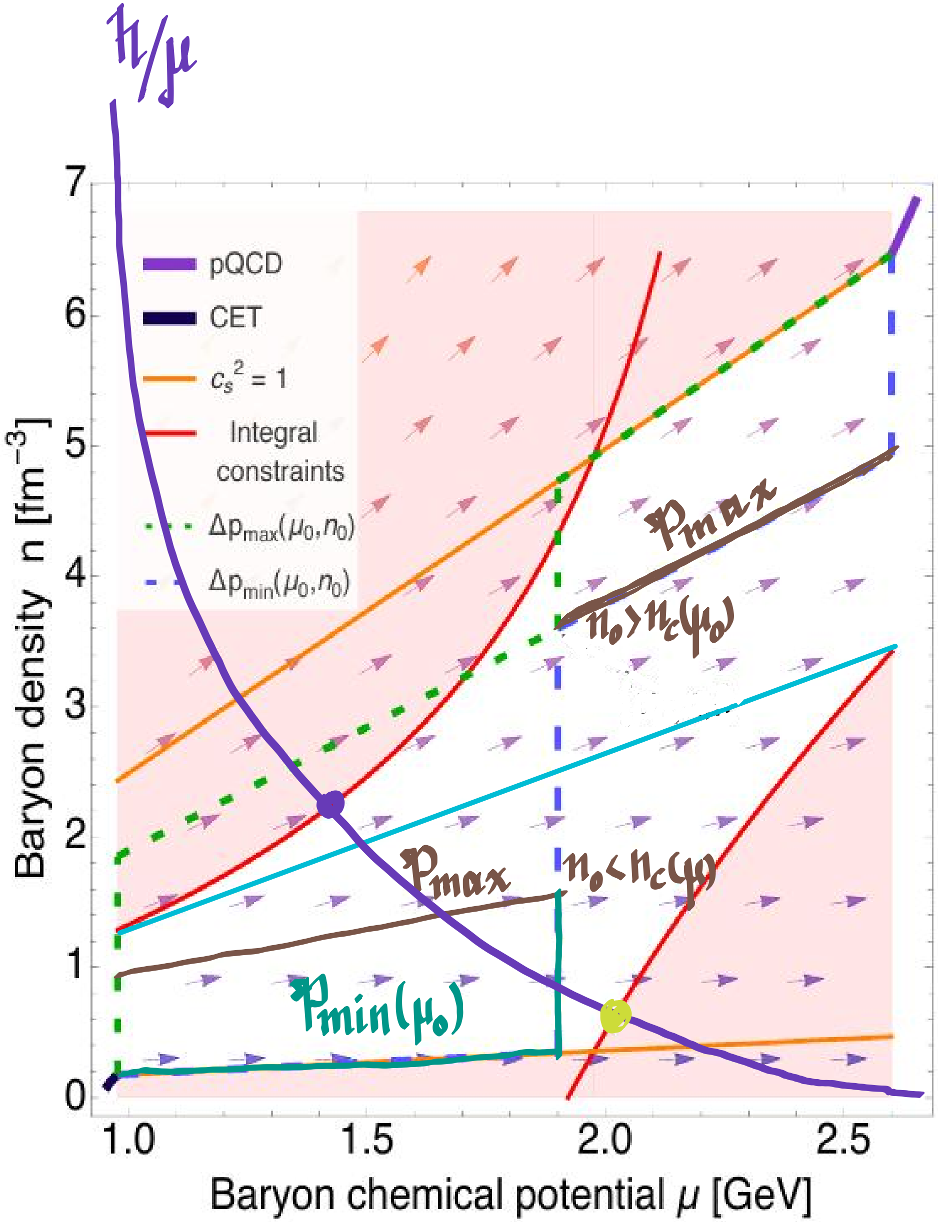
$$\Delta P_{max} = P_H - P_L = \Delta P_{min}$$

(I) (II)

$$\frac{dn(\mu)}{d\mu} = \frac{n}{\mu}$$ (III), (IV)

$$\mu_c = \sqrt{\frac{\mu_L \mu_H (\mu_H n_H - \mu_L n_L - 2\Delta P)}{\mu_L n_H - \mu_H n_L}}$$

This construction in the (μ, n) is taken to the (ε, p) plane by using the Euler equation $\varepsilon = -p + \mu n$



1. Determine $p_{min}(\mu_0, n_0)$, $p_{max}(\mu_0, n_0)$ at each allowed (n_0, μ_0)

2. Fixed-enthalpy lines $h = \varepsilon + p = \mu n$

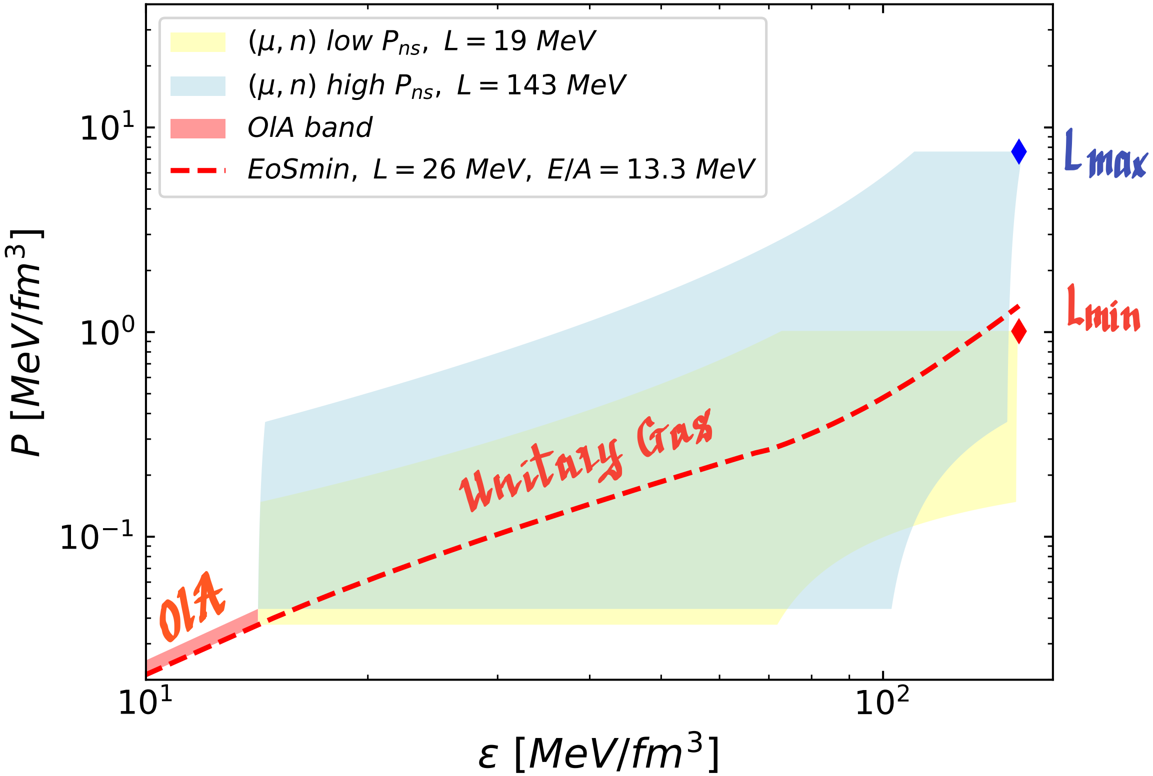
$$\left.\begin{array}{ll} (\varepsilon, p): & p(\varepsilon) = h - \varepsilon \\ (\mu, n): & n(\mu) = \dfrac{h}{\mu} \end{array}\right\}$$

p is maximized/minimized along the same isenthalpic line in both planes

- $\{\varepsilon_{max}(\mu), p_{min}(\mu)\}$
  Lower boundary in (ε, p) plane

- $\{\varepsilon_{min}(\mu), p_{max}(\mu)\}$
  Upper Boundary in (ε, p) plane

Extrapolation: $0.1\, n_s \le n \le n_s$ of OlA EoS



$$\mu_s = M_N + \frac{E}{A}\Big|_s + \frac{P_s}{n_s}$$

Needed for the construction in the (μ, n) plane H=S

♦ $P_s = 7.6$ MeV fm^{-3}, $\mu_s = 1013.5$ MeV

♦ $P_s = 1.0$ MeV fm^{-3}, $\mu_s = 962$ MeV

Experimental Information:

$$\frac{E}{A}\Big|_{SNM} = -16.0 \pm 0.5 \text{ MeV}$$

$$n_s = 0.16 \pm 0.01 \text{ fm}^{-3}$$

Bender, Heenen, Reinhard, RMP 75 (2003)

$$S_0 = \frac{E}{A}\Big|_{PNM}(n_s) - \frac{E}{A}\Big|_{SNM}(n_s)$$

$$L = 3 n_s \frac{d}{dn} S(n)\Big|_{n_s} = \frac{3}{n_s} P(n_s)$$

PREX II Reed et al. PRL 126 (2021)

$S_0 = 38.1 \pm 4.7$ MeV

$L = 106 \pm 37$ MeV

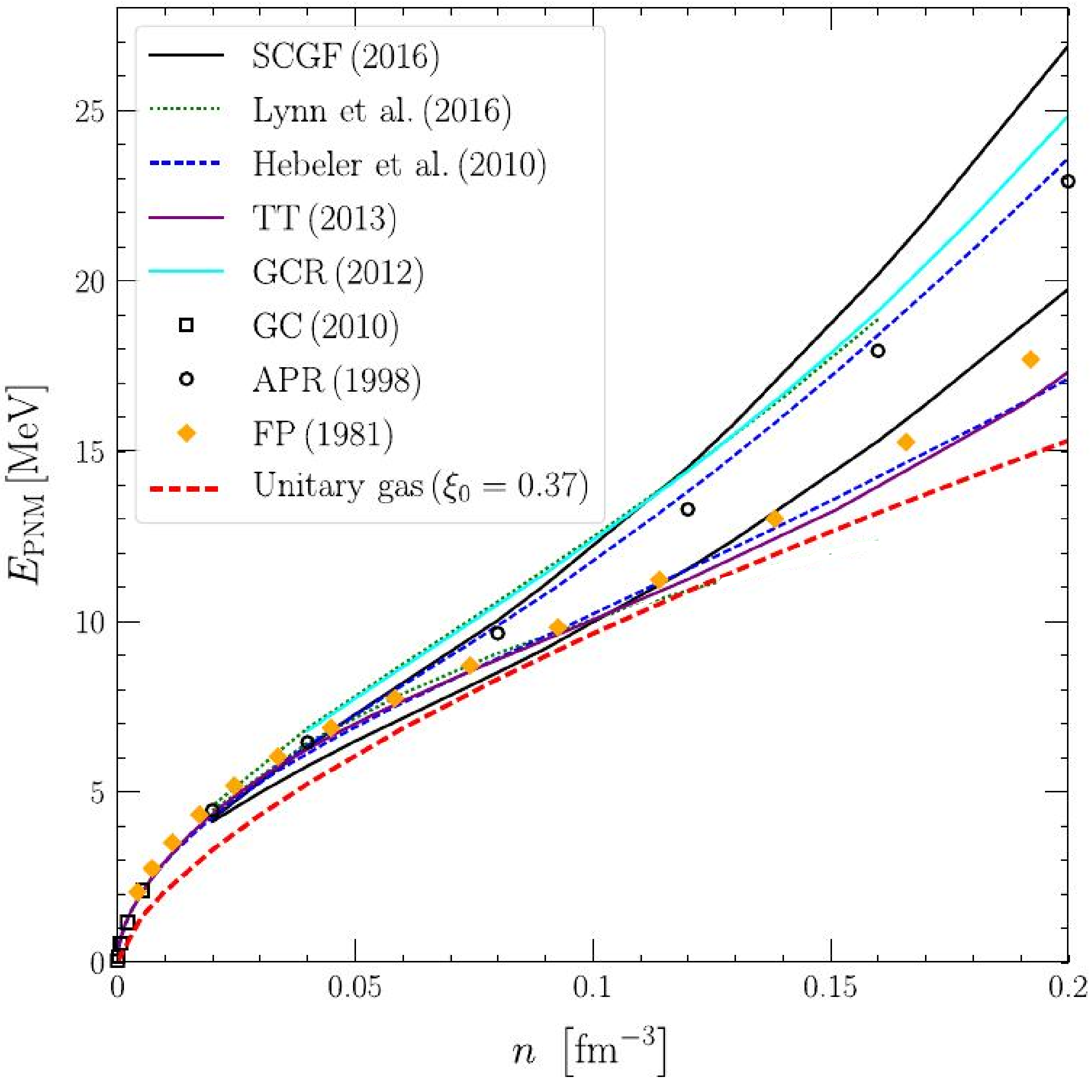
CREX Reed et al. PRC 109 (2024)

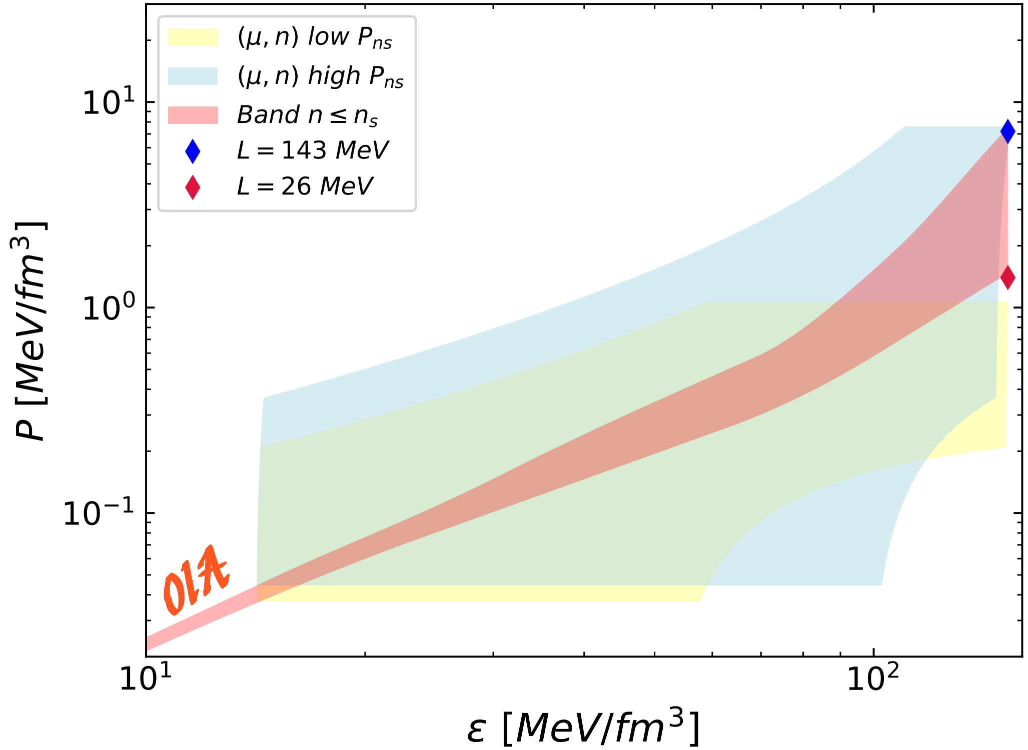
1.- $L = 110 \pm 40$ MeV

2.- $L = 19 \pm 19$ MeV

Instead of the controversial minimum value for L we limit P_s by using the Unitary Gas Conjecture (UGC): Unitary gas sets a lower bound on E/A

Tews et al. Ast.Phys.J 848 (2017)

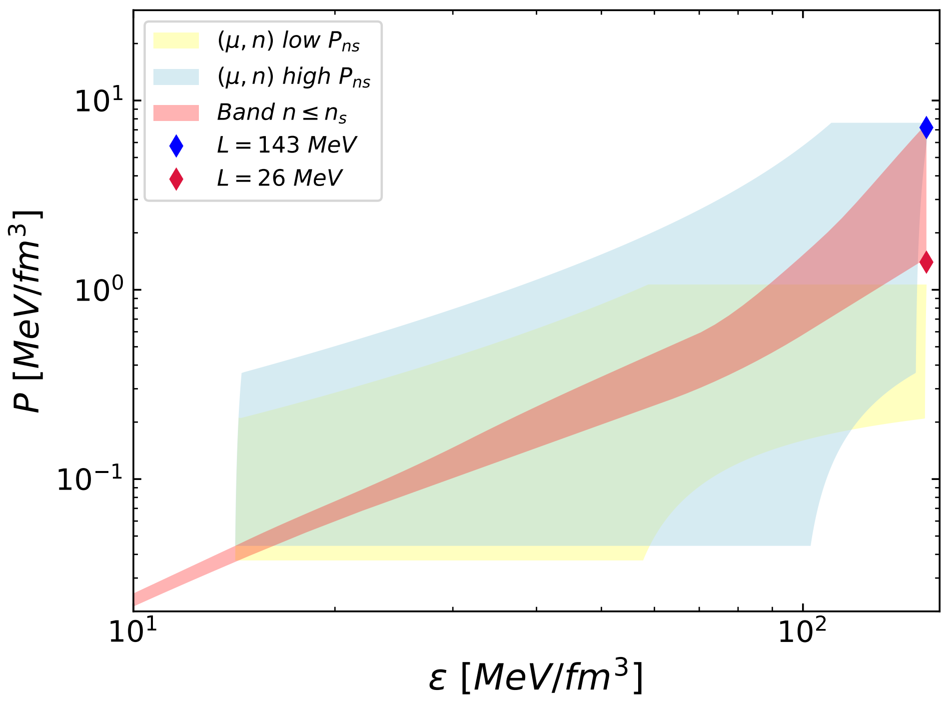




$P_s = 1.4$ MeV fm^{-3}
$L = 26$ MeV } UGC

$E/A_{CREX+SNM} = 16.9$ MeV

$E/A|_{UGC} = 13.2$ MeV



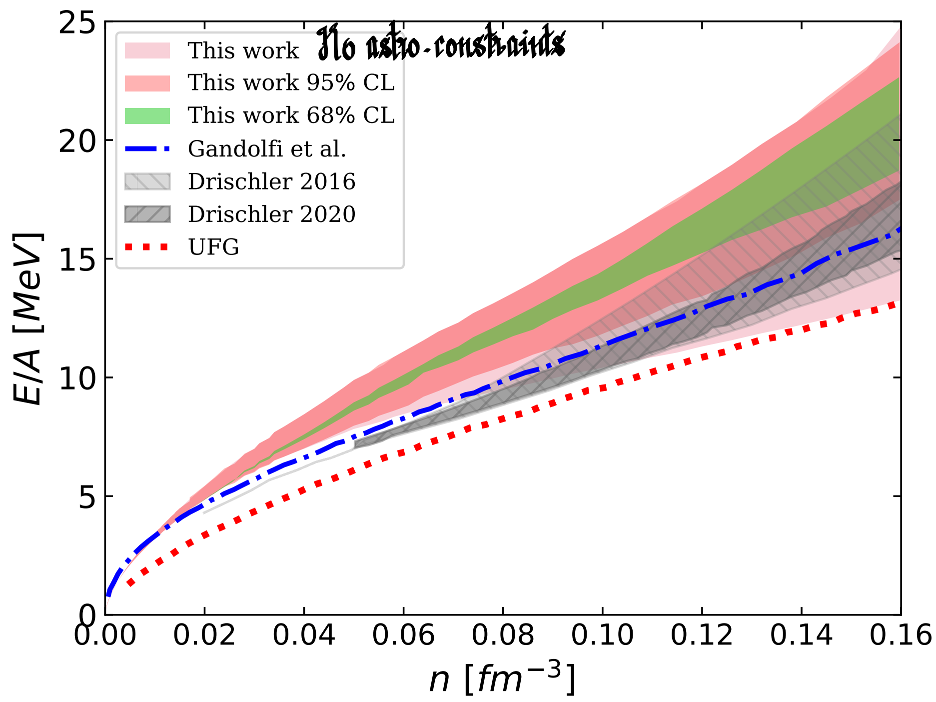
♦ $P_s = 7.62$ MeV fm^{-3}

♦ $P_s = 1.41$ MeV fm^{-3}

In the grid: Proceed from lower to higher densities.

c_s^2 from point to point raises softly with $dc_s^2/d\varepsilon > 0$

Avoid too early phase transitions



Our values taken as input from phenomenology and UFG

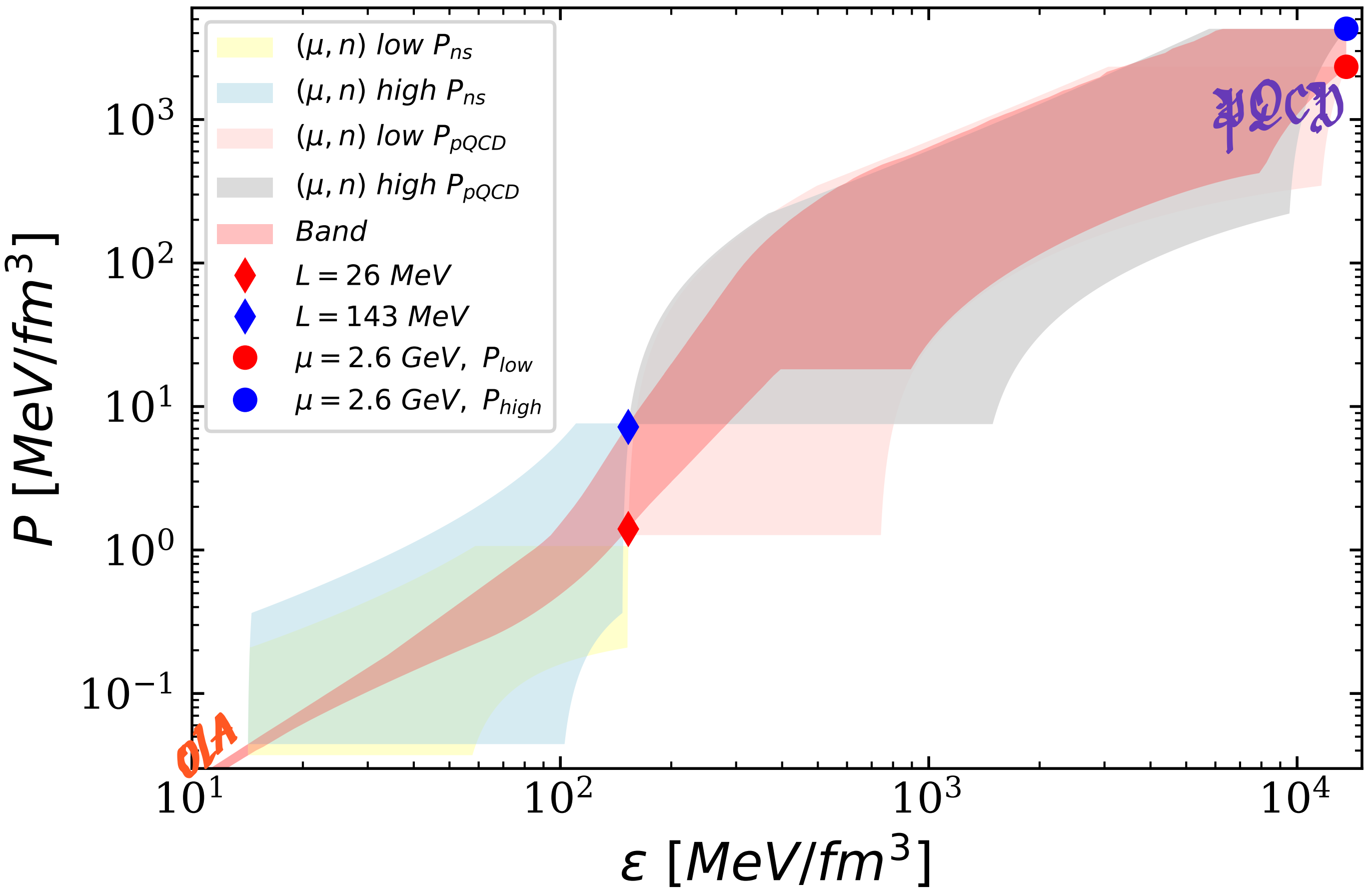
$30.5 \leq S \leq 43.5$ MeV

$26.5 \leq L \leq 143$ MeV

Extrapolation from n_s up to pQCD ≈ 40 n_s

Highest density region from N^3LO pQCD $P(\mu)$

Gorda et al. PRD 104 ('21)



$\mu_H = 2.6$ GeV

4384 }
2334 } → $X \in [1, 4]$

Renormalization scale

$X \approx \frac{3\Lambda}{\mu}$

$n_s < n < 2.5\, n_s$

c_s^2 softly increases

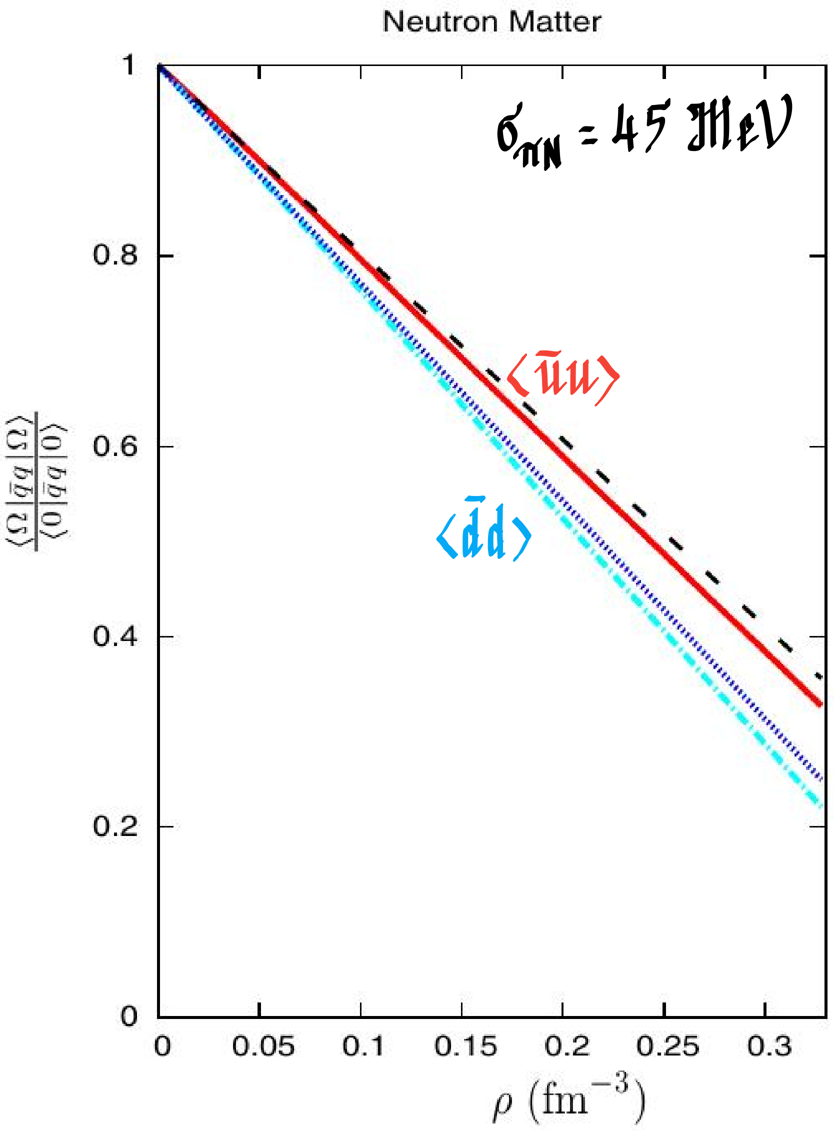
$dc_s^2/d\varepsilon > 0$

$n > 2.5\, n_s$

Phase transitions (PT) are allowed

Why?

Because of the behavior of the quark condensate $\langle \Omega | \bar{q} q | \Omega \rangle (n)$



Vanishing almost linearly in $\sigma_{\pi N}$

$2.5\, n_s$ for $\sigma_{\pi N} \approx 50$ MeV

Largest $\sigma_{\pi N} \approx 60$ MeV

Alarcon et al. PRD 85 ('10)
Hofenrichter et al. PRL 115 ('15)

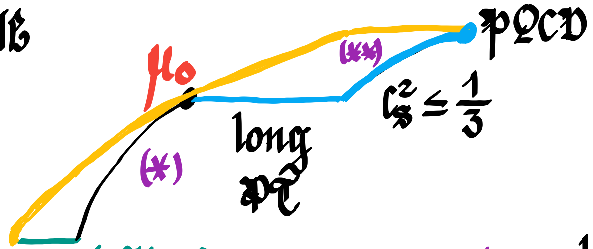
} Chiral EFT, Dispersion relations, Exp data πN

Lattice QCD : Smaller values
43.7 ± 3.6 MeV
Agadjanov et al. PRL 131 ('23)

$n > 2.5\, n_s$ $\quad \frac{dc_s^2}{d\varepsilon} \geq 0$; PT $\frac{dc_s^2}{d\varepsilon} = 0$ is allowed ; Jump in ε with $P = ct$, $c_s^2 = 0$

$c_s^2 = 0$ along PT

(ε, P) Plane



short PT

[Not in all EoS]

Only for $M_{NS} \geq 2.18\, M_\odot$

(*) $\frac{dc_s^2}{d\varepsilon} > 0$ smooth rising until $c_s = c_s^2|_{max}$

(**) Highest density

Practical realization showing explicit compatibility with pQCD

Brandes, Weise Symmetry 16('24)

statistical analysis conclude against PT $M_{NS} \leq 2.1\, M_\odot$

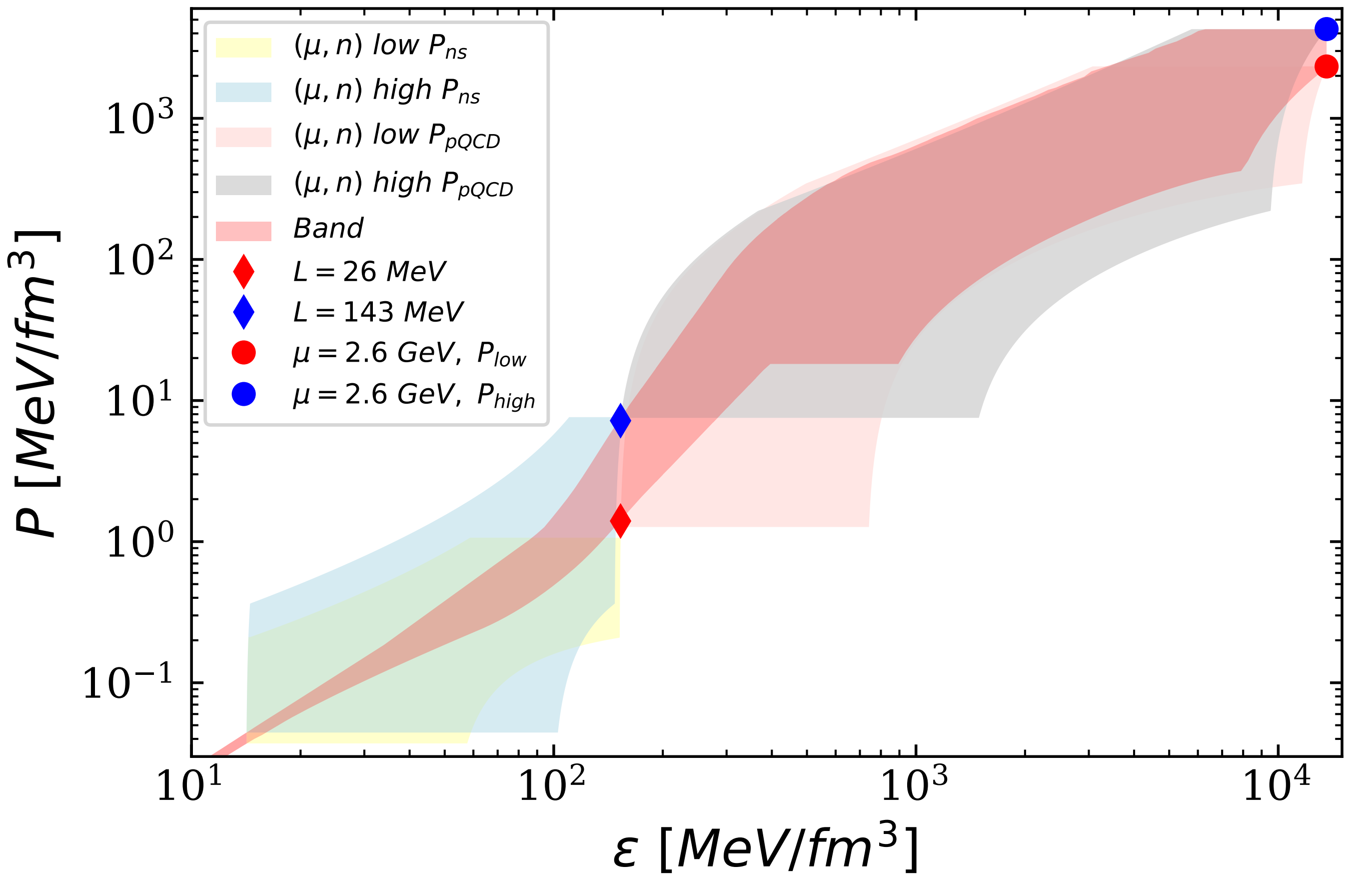
$c_{s,max}^2 = 1$ $\quad \mu_0 = 2250$ MeV $< \mu_c \approx 2500$ MeV

$c_{s,max}^2 = 0.781$ $\quad \mu_0 = 2040$ MeV

Hippert et al., 2402.14085

Tang et al., 2404.09563

$\approx$ Central neutron star density



Constrained by

Causality

Thermodynamical consistency

Thermodynamical stability

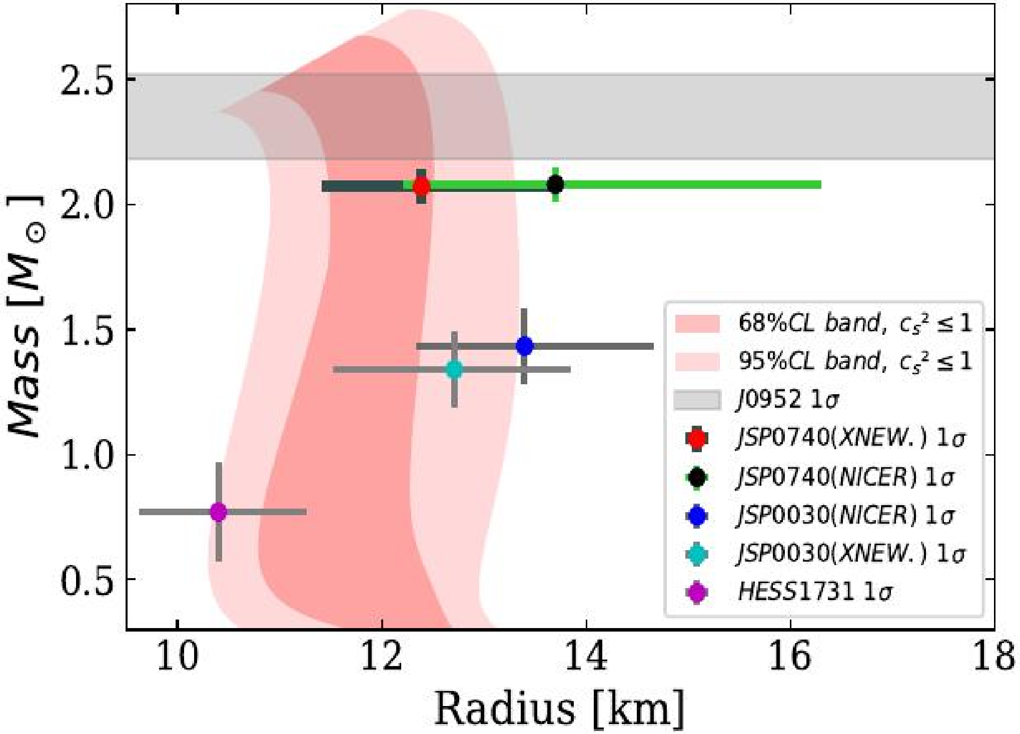
pQCD

Empirical nuclear data

No astrophysical constraints

Astrophysical constraints included and calculated with GR.

6 NS masses, 5 radii, 3 tidal deformabilities. Augmented TOV system of equations are solved



$c_s^2 \le 1$

S_0, L are further restricted

$35.9 \le S_0 \le 39.7$ MeV

$41.0 \le L \le 68.0$ MeV

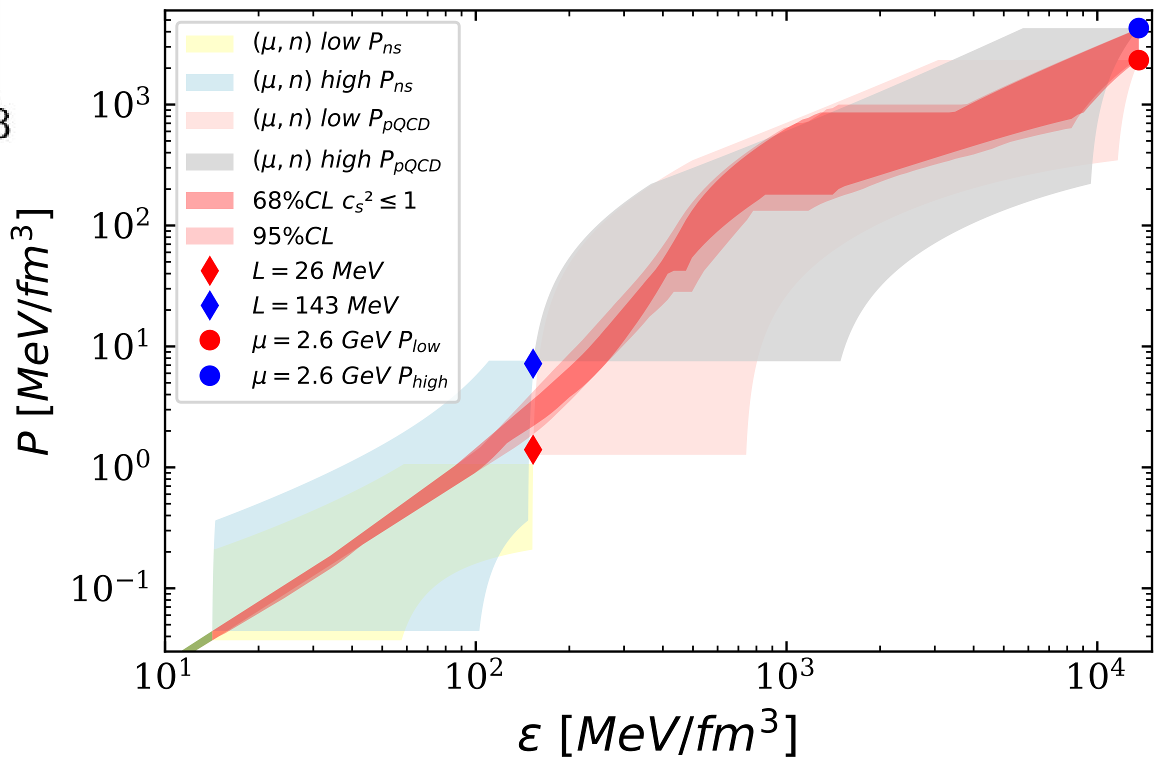
$5.0\, n_s \le n_{c,max} \le 6.1\, n_s$

$2230 \le \mu_{c,max} \le 2240$ MeV

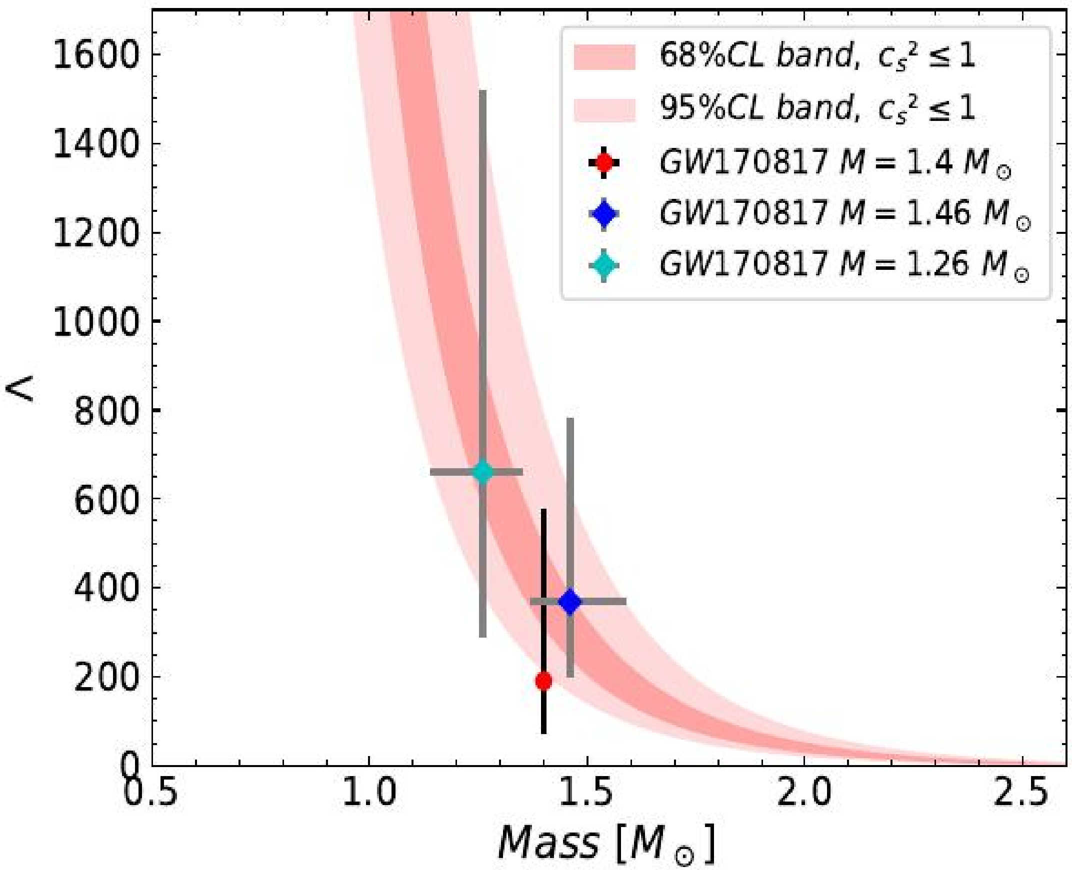
PTs for $M > 2.18\, M_\odot$

$3.2\, n_s \le n_c \le n_{c,max}$

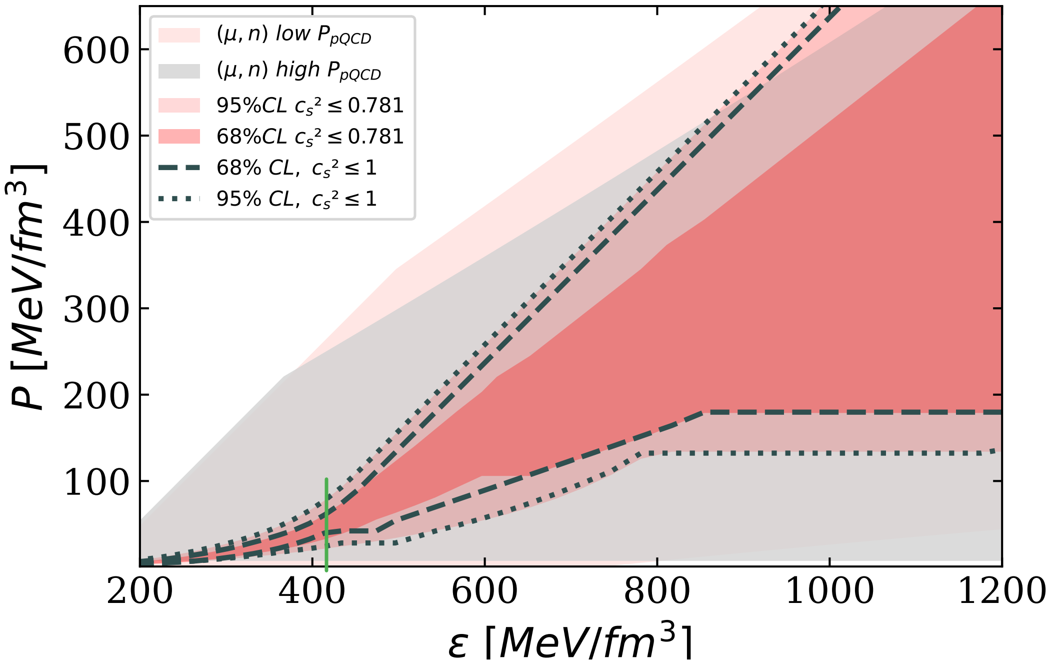
1st order PT $\Delta n/n > 1$



Tidal Deformability



Upper limit: $c_s^2 \leq 0.781$ Hippert et al., 2402.14085 ; Tang et al., 2404.09563



$\varepsilon \lesssim 450\, MeV fm^{-3}$

The same band as with $c_s^2 \leq 1$

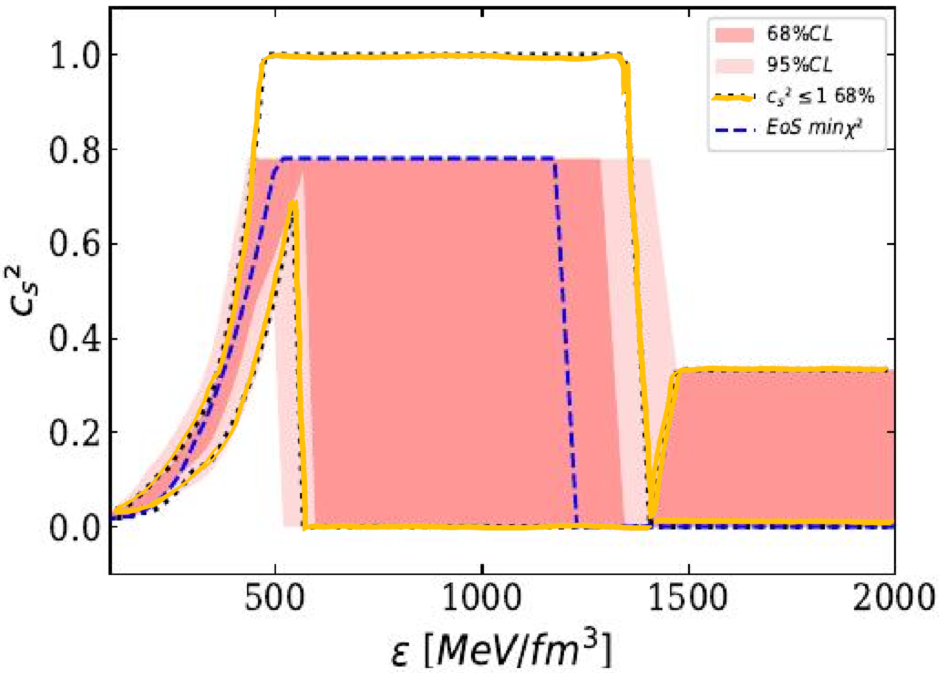
Larger $5.2\, n_s \leq n_{c,max} \leq 6.52\, n_s$

Smaller $\mu_{c,max} \approx 2020$ MeV

PTs $M \geq 2.18\, M_\odot$, $3.3\, n_s \leq n \leq n_{c,max}$

1st order PT $\Delta n/n > 1$

Speed of sound squared



Conclusions

1.- Renormalized and cutoff independent EoS for nuclear matter, $n \leq 0.1\, n_s$
$k_F \lesssim m_\pi \approx \frac{k_{F,s}}{2}$

2.- By resumming the ladder series, LO in-medium corrections
Already a nonperturbative calculation

3.- Directly expressed in terms of nucleon-nucleon scattering data.

4. Extrapolation until n_s, $n_{c,max}$, pQCD
Causality, Thermodynamical consistency and stability (μ, n), (ε, P) planes
pQCD, Nuclear and Astrophysical data

5. $35.9 \leq S_0 \leq 39.7$ MeV
$41.0 \leq L \leq 68.0$ MeV

$5.0\, n_s \leq n_{c,max} \leq 6.1\, n_s$
$2230 \leq \mu_{c,max} \leq 2240$ MeV

PTs for $M > 2.18\, M_\odot$
$3.2\, n_s \lesssim n_c \leq n_{c,max}$
1st order PT $\Delta n/n > 1$