Observability of the heat equation from very small sets

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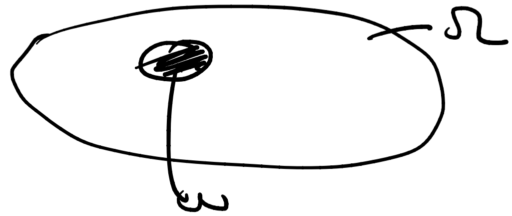
$T>0$, $\Omega \subset \mathbb{R}^d$ a bounded connected open set of class C^+ and $\omega \subset\subset \Omega$ be a measurable set



Control problem

$$\begin{cases} \partial_t y - \Delta y = h 1_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases}$$

Null-controllability (in L^2)

$\forall y_0 \in L^2(\Omega)$, $\exists h \in L^2((0,T)\times\omega)$, s.t. $y(T,\cdot)=0$ in Ω.

$\Longleftrightarrow$ (duality) **observability (in L^2)** $\exists C(\Omega,\omega,T)>0$ s.t.

$$\|e^{T\Delta}u_0\|^2_{L^2(\Omega)} \le C\int_0^T\int_\omega |e^{t\Delta}u_0|^2\,dt\,dx, \quad \forall u_0 \in L^2(\Omega)$$

Here, we treat situation where possibly $|\omega|=0$ where $|\cdot|$ is the d-dimensional Lebesgue measure, we focus on the following modification of observability

$$\|e^{T\Delta}u_0\|_{L^\infty(\Omega)} \le C\int_0^T \sup_{x\in\omega}|u(t,x)|\,dt \quad \forall u_0 \in L^2(\Omega) \qquad (O)$$

Question "Geometrical measure" condition on ω ensuring (O) ?

Bibliographical comments — When (a) holds?

* Fattorini, Russell (1971); $d=1$, $\omega=(a,b)\subset\subset\Omega$.
* Lebeau, Robbiano (1995) / Fursikov, Imanuvilov (1996); $d\geq 2$, ω open
* Apraiz, Escauriaza, Wang, Zhang (2014); $d\geq 1$, $|\omega|>0$

Can we go further? Can we observe with $|\omega|=0$?

Hausdorff content and dimension, for $s\geq 0$, $E\subset\mathbb{R}^d$,

$$\mathcal{C}^s(E)=\inf\left\{\sum r_i^s\ ;\ E\subseteq\bigcup B(x_i,r_i)\right\}.$$

$$\dim(E)=\inf\{s\geq 0,\ \mathcal{C}^s(E)>0\}$$

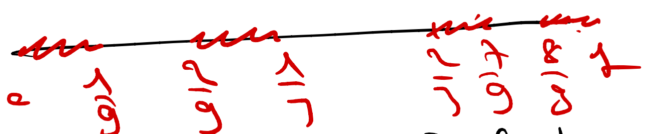
This is a generalization of the standard Lebesgue measure adapted to "measure" for instance fractal sets.

examples:

- $\mathcal{C}^d(\Omega)\simeq|\Omega|>0$
- K Cantor set, $K=\bigcap_{n\in\mathbb{N}}K_n$ where

  $K_0=[0,1]$, $K_1=[0,1/3]\cup[2/3,1]$

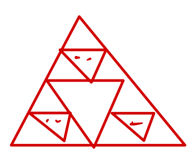
$K_2=$



$\hookrightarrow$ $|K|=0$ and $\dim(K)=\frac{\log(2)}{\log(3)}=s_K$,

then $\mathcal{C}^s(K)>0$ $\forall s<s_K$.

- T Sierpinski triangle in $\mathbb{R}^2$



$$\dim(T)=\frac{\log(3)}{\log(2)}$$

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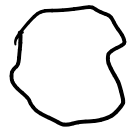
* Burq, Moyano (2023); $d\geq 1$, $\mathcal{C}^{d-1+\delta}(\omega)>0$ where $\delta\in(0,1)$ is close to 1 a priori ($0<<\delta<1$).
* Zhu (2023), $d=1$, $\mathcal{C}^{\delta}(\omega)>0$ $\forall\,\delta\in(0,1)$

Thm (Green, Lo B., Merton, Orsoni, 2024) Let $\delta \in (0,1)$. Assume $\mathcal{H}^{d-1+\delta}(\omega) > 0$. Then (C) holds.

App: $\Omega =$ , $\omega =$ Sierpinski triangle $\Rightarrow$ (C) holds.

Sharpness: The result is sharp with respect to the scale of Hausdorff dimension. Indeed, take $\begin{cases} -\Delta\Phi = \lambda\Phi \\ \Phi = 0 \end{cases}$

$\|\Phi\|_2 = 1$, $\mathcal{H}^{d-1}(Z) \geq c\sqrt{\lambda}$ by Logunov, Malinnikova, Nadirashvili, Nazarov (2021), where $Z = \{x \in \Omega;\ \Phi(x) = 0\}$.

(C) with $\omega = Z$ and $u_0 = \Phi$

$$e^{-\lambda T} \lesssim C \int_0^T \sup_{x \in Z} |e^{-\lambda t} \underbrace{\Phi(x)}_{=0}|\, dt = 0 \Rightarrow \perp\perp$$

proof (based on a spectral inequality and the so-called Lebeau–Robbiano method)

$$\ast \quad \Big\| \sum_{\lambda_k \leq \Lambda} a_k \varphi_k \Big\|_{L^2(\Omega)} \leq e^{C\sqrt{\Lambda}} \Big\| \sum a_k \varphi_k \Big\|_{L^\infty(\omega)} \quad (S)$$

where $\mathcal{H}^{d-1+\delta}(\omega) > 0$

We actually give two proofs of (S).

1) Based on Malinnikova's result (2004): quantitative propagation of smallness for gradients of harmonic functions and the standard adding variable trick, $\tilde{u}(x,t) = \sum a_k \varphi_k(x) \frac{\sinh(\sqrt{\lambda_k}\, t)}{\sqrt{\lambda_k}}$

$$\sup_K |\nabla \tilde{u}| \leq C \Big(\sup_E |\nabla \tilde{u}|\Big)^\alpha \Big(\sup_{\tilde{\Omega}} |\nabla \tilde{u}|\Big)^{1-\alpha}$$

2) Direct proof.

OPEN PROBLEM $\quad \partial_t u - \mathrm{div}\,(\underbrace{A(x)\nabla_x u}_{\in W^{1,\delta}(\Omega)}) = 0$

$\delta > 0$, $\mathcal{H}^{d-1+\delta}(\omega) > 0$,

Does (0) holds? (Conjecture: Yes...).

Another problematic that we explore in our paper is: "Can we construct particular measurable sets ω such that (0) holds?"

For instance, in every dimension $d \geq 1$, we build a ω with $\dim_{\mathcal{H}}(\omega) = 0$ such that (0) holds ...