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Controllability and Inverse Problems for Parabolic Systems with Dynamic Boundary Conditions

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Joint work with **L. MANIAR**

Dedicated to the memory of Professor **HAMMADI BOUSLOUS**

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Benasque, August, 18-30, 2024

Let $T > 0$. We consider the heat equation with dynamic boundary condition

$$\begin{cases} \partial_t y - d\Delta y = f(t,x) & \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta \Delta_\Gamma y_\Gamma + d\partial_\nu y = g(t,x) & \text{on } \Gamma_T, \\ y_\Gamma = y_{|\Gamma} & \text{on } \Gamma_T, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \text{in } \Omega \times \Gamma, \end{cases} \tag{1}$$

where

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\Gamma = \partial\Omega$.
- $(y_0, y_{0,\Gamma}) \in \mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma)$ are initial data, $f \in L^2(\Omega_T)$ and $g \in L^2(\Gamma_T)$ are source terms.
- $d > 0$ and $\delta \geq 0$ are constant.

Natural approach: Dynamic Boundary conditions arise naturally as part of the formulation of the problem (Physical laws in $\overline{\Omega}$).

G. R. Goldstein, Derivation and physical interpretation of general boundary conditions, *Adv. Diff. Equ.*, **11** (2006), 457–480.

N. Sauer, Dynamic boundary conditions and the Carslaw-Jaeger constitutive relation in heat transfer, *SN Partial Differ. Equ. Appl.*, **1**, 48 (2020).

Introduction and Motivation

Dynamic boundary conditions models appear in several applications including

- Fluid Dynamics: e.g., precipitation of rainfall infiltration into the soil (L. Jianguo & S. Ning, 1985)
- Heat transfer: e.g., flow of heat for a solid in contact with a fluid (R.E. Langer, 1932)
- Population dynamics: e.g., diffusion of bacteria from a solid to its surface (M. Langlais & F.A. Milner, 2003)

Wellposedness

The abstract form of the system is given by

$$\begin{cases} \partial_t Y = \mathcal{A}Y + F, & 0 < t \leq T \\ Y(0) = Y_0 := (y_0, y_{0,\Gamma}), \end{cases}$$

where $Y := (y, y_\Gamma)$, $F := (f, g)$, and the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{L}^2 \to \mathbb{L}^2$ is given by

$$\mathcal{A} = \begin{pmatrix} d\Delta & 0 \\ -d\partial_\nu & \delta\Delta_\Gamma \end{pmatrix}, \qquad D(\mathcal{A}) = \mathbb{H}^2,$$

where (when $\delta > 0$)

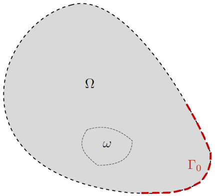
$$\mathbb{H}^2 := \left\{(y, y_\Gamma) \in H^2(\Omega) \times H^2(\Gamma) : y_{|\Gamma} = y_\Gamma\right\}.$$

Null controllability

Consider the problem:

$$\begin{cases} \partial_t y - d\Delta y = \mathbb{1}_\omega(x) v(t,x) & \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta \Delta_\Gamma y_\Gamma + d\partial_\nu y = 0 & \text{on } \Gamma_T, \\ y_\Gamma = y_{|\Gamma} & \text{on } \Gamma_T, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \text{in } \Omega \times \Gamma, \end{cases} \tag{2}$$

where the control region ω is an arbitrary nonempty open subset which is strictly contained in Ω (i.e., $\overline{\omega} \subset \Omega$).



Null controllability

The system is null controllable at time $T > 0$ if for all $(y_0, y_{0,\Gamma}) \in \mathbb{L}^2$, there exists a control $v \in L^2(\omega_T)$ such that the solution satisfies

$$y(T,\cdot) = 0 \;\text{ in } \Omega \quad \text{and} \quad y_\Gamma(T,\cdot) = 0 \;\text{ on } \Gamma. \tag{3}$$

Null controllability

Theorem

Assume that $\delta > 0$. For all $T > 0$, all non-empty open set $\omega \Subset \Omega$ and all initial data $(y_0, y_{0,\Gamma}) \in \mathbb{L}^2$, there exists a control $v \in L^2(\omega_T)$ such that the unique mild solution (y, y_Γ) satisfies

$$y(T,\cdot) = 0 \text{ in } \Omega \quad \text{and} \quad y_\Gamma(T,\cdot) = 0 \text{ on } \Gamma.$$

L. Maniar, M. Meyries, R. Schnaubelt, Null controllability for parabolic problems with dynamic boundary conditions, *Evol. Equat. and Cont. Theo.* **6** (2017), 381-407.

Duality: controllability-observability

Null controllability $\Longleftrightarrow$

$$\forall Y_0 \in \mathbb{L}^2,\ \exists v \in L^2(\omega_T):$$

$$Y(T,\cdot) = e^{T\mathcal{A}} Y_0 + \int_0^T e^{(T-s)\mathcal{A}} (\mathbb{1}_\omega v, 0)\, ds = 0$$

$$\Longleftrightarrow \forall Y_0,\ \exists v:\ Y(T,\cdot) = e^{T\mathcal{A}} Y_0 + \mathcal{T} v = 0 \Longleftrightarrow e^{T\mathcal{A}} Y_0 = -\mathcal{T} v$$

$$\Longleftrightarrow \mathrm{R}\left(e^{T\mathcal{A}}\right) \subset \mathrm{R}(\mathcal{T})$$

$$\overset{Douglas}{\Longleftrightarrow} \exists C > 0: \quad \left\| e^{T\mathcal{A}} \Phi_T \right\|_{\mathbb{L}^2} \leq C \left\| \mathcal{T}^* \Phi_T \right\|_{\mathbb{L}^2} \quad \forall \Phi_T \in \mathbb{L}^2.$$

$\Longleftrightarrow$ **Observability inequality**

Observability

The null controllability is equivalent to the observability inequality for the homogeneous backward system

$$\begin{cases} -\partial_t \varphi - d\Delta\varphi = 0 & \text{in } \Omega_T, \\ -\partial_t \varphi_\Gamma - \delta\Delta_\Gamma \varphi_\Gamma + d\partial_\nu \varphi = 0 & \text{on } \Gamma_T, \\ \varphi_\Gamma = \varphi_{|\Gamma} & \text{on } \Gamma_T, \\ (\varphi, \varphi_\Gamma)|_{t=T} = (\varphi_T, \varphi_{T,\Gamma}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{4}$$

Proposition

Assume that $\delta > 0$. There exists a constant $C > 0$ such that for all $(\varphi_T, \varphi_{T,\Gamma}) \in \mathbb{L}^2$ the mild solution $(\varphi, \varphi_\Gamma)$ of the backward system (4) satisfies the following observability inequality

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} + \|\varphi_\Gamma(0,\cdot)\|^2_{L^2(\Gamma)} \le C \int_{\omega_T} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5}$$

Weight functions

Given a non-empty open set $\omega \Subset \Omega$, there is a function $\eta \in C^2(\bar{\Omega})$ such that (Fursikov-Imanuvilov):

$$\eta > 0 \quad \text{in } \Omega, \quad \eta = 0 \quad \text{on } \Gamma, \quad |\nabla \eta| > 0 \quad \text{in } \overline{\Omega \backslash \omega}.$$

We define the weight functions α and ξ by

$$\alpha(t,x) = \frac{e^{2\lambda \|\eta^0\|_\infty} - e^{\lambda \eta^0(x)}}{t(T-t)} \quad \text{and} \quad \xi(t,x) = \frac{e^{\lambda \eta^0(x)}}{t(T-t)}.$$

The key tool to show the observability inequality is:

Lemma (Carleman estimate)

Assume that $\delta > 0$. Let $T > 0$, $\omega \Subset \Omega$ be nonempty and open. Then there exist constants $C > 0$ and $\lambda_1, s_1 \geq 1$ such that

$$s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha}\xi^3|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha}\xi^3|\varphi_\Gamma|^2 \,\mathrm{d}S\,\mathrm{d}t \\ \leq Cs^3\lambda^4 \int_{\omega_T} e^{-2s\alpha}\xi^3|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{6}$$

for all $\lambda \geq \lambda_1$, $s \geq s_1$ and all $(\varphi, \varphi_\Gamma) \in \mathbb{E}_1$.

Main difficulty

- Absorbing the boundary term

$$s\lambda \int_{\Gamma_T} \rho |\partial_\nu \eta| \, |\nabla_\Gamma \varphi_\Gamma|^2 \, \mathrm{d}S \, \mathrm{d}t \tag{7}$$

is quite challenging, since it appears with the same exponents of the parameters s and λ.

- The assumption $\delta > 0$ is needed to absorb (7) using the equation

$$\Delta_\Gamma \varphi_\Gamma = \frac{1}{\delta}(-\partial_t \varphi_\Gamma + d\partial_\nu \varphi).$$

Boundary controllability

Absorbing the local term:

$$s\lambda \int_{\Gamma_T^0} \rho \left(\partial_\nu \eta^0\right) |\partial_\nu \varphi|^2 \, \mathrm{d}S \, \mathrm{d}t.$$

Key ideas:

- Use of the homogeneous equation $d\partial_\nu \varphi = \partial_t \varphi_\Gamma + \delta \Delta_\Gamma \varphi_\Gamma$.
- The parabolic regularity to estimate

$$\int_0^T \left(\|\varphi\|_{H^4(\Gamma)}^2 + \|\varphi_{tt}\|_{L^2(\Gamma)}^2 \right) \mathrm{d}t$$

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S.E. C, G. El Guermai, A. Khoutaibi, L. Maniar, Boundary null controllability for the heat equation with dynamic boundary conditions, *Evol. Equat. and Cont. Theo.*, **12** (2023), 542–566.

Related literature

R. Lecaros, R. Morales, A. Pérez, S. Zamorano, Discrete Carleman estimates and application to controllability for a fully-discrete parabolic operator with dynamic boundary conditions, *J. Diff. Equ.*, **365**, 832–881 (2023).

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C.G. Gal, L. Tebou, Carleman inequalities for wave equations with oscillatory boundary conditions and application, *SIAM J. Control Optim.*, **55**, 324–364 (2017).

Inverse Problems

Consider the parabolic system with DBC

$$\begin{cases} \partial_t y - d\Delta y = f(t,x) & \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y = g(t,x) & \text{on } \Gamma_T, \\ y_\Gamma(t,x) = y_{|\Gamma}(t,x) & \text{on } \Gamma_T, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \Omega \times \Gamma. \end{cases}$$

Inverse Source Problem: Simultaneous determination of the source terms (f,g) from the measurements

$$Y(T_0,\cdot) := (y(T_0,\cdot), y_\Gamma(T_0,\cdot)) \quad \text{et} \quad y|_{(0,T)\times\omega},$$

where $T_0 \in (0,T)$ and $\omega \Subset \Omega$ is a nonempty open subset.

Inverse Problems

E.M. Ait Ben Hassi, S.E. C, L. Maniar and O. Oukdach, Lipschitz stability for an inverse source problem in anisotropic parabolic equations with dynamic boundary conditions, *Evol. Equat. and Cont. Theo.*, **10** (2021), 837–859.

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- E.M. Ait Ben Hassi, S.E. C, L. Maniar, An inverse problem of radiative potentials and initial temperatures in parabolic equations with dynamic boundary conditions, *J. Inverse Ill-Posed Probl.* **30** (2021), 363-378

- E. M. Ait Ben Hassi, S. E. C, L. Maniar, Stable determination of coefficients in semilinear parabolic system with dynamic boundary conditions, *Inverse Problems*, **38**, 115007 (2022).

Open problem

Let $N \geq 2$ and $\delta = 0$ and consider

$$\begin{cases} \partial_t y - \Delta y = \mathbb{1}_\omega(x) v(t,x) & \text{in } \Omega_T, \\ \partial_t y_\Gamma + \partial_\nu y = 0 & \text{on } \Gamma_T, \\ y_\Gamma = y_{|\Gamma} & \text{on } \Gamma_T, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{8}$$

Using the Carleman estimate approach, the problematic term

$$s\lambda \int_{\Gamma_T} \rho |\partial_\nu \eta| \, |\nabla_\Gamma \varphi_\Gamma|^2 \, \mathrm{d}S \, \mathrm{d}t$$

could be absorbed thanks to the surface diffusion term $\delta \Delta_\Gamma \varphi_\Gamma$ when $\delta > 0$.

Thank you for your attention