Abstract damped wave equations: The optimal decay rate

Lorenzo Liverani

Friedrich-Alexander-Universität
DYNAMICS, CONTROL, MACHINE LEARNING AND NUMERICS

joint work with F. Dell'Oro and V. Pata

Abstract damped wave equations

- $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ Hilbert space
- $A : \mathfrak{D}(A) \subset H \to H$ strictly positive selfadjoint operator
- $f : \sigma(A) \subset (0, \infty) \to [0, \infty)$ continuous function

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$$\ddot{u}(t) + 2f(A)\dot{u}(t) + Au(t) = 0 \tag{W}$$

- $f(A)$ is the selfadjoint operator constructed via the functional calculus of A

$$f(A) = \int_{\sigma(A)} f(s)dE_A(s)$$

being E_A the spectral measure of A

Examples

- $\Omega \subset \mathbb{R}^n$ bounded domain with smooth boundary $\partial\Omega$
- $A = -\Delta$ with $\mathfrak{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$
- $f(s) = s^\theta$ with $\theta \in \mathbb{R}$

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We get the wave equation with fractional damping

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Beam and plate equations with fractional damping can be obtained in a similar way choosing $A = \Delta^2$ with

$$\mathfrak{D}(\Delta^2) = \{u \in H^2(\Omega) \cap H_0^1(\Omega) : \Delta u \in H^2(\Omega) \cap H_0^1(\Omega)\}$$

The solution semigroup

- Product space $\mathcal{H} = \mathfrak{D}(A^{\frac{1}{2}}) \times H$
- Linear operator $\mathbb{G} : \mathfrak{D}(\mathbb{G}) \subset \mathcal{H} \to \mathcal{H}$ defined as

$$\mathbb{G}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -2f(A)v - Au \end{pmatrix}$$

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$$S(t) = e^{t\mathbb{G}} : \mathcal{H} \to \mathcal{H}$$

$\rightarrow$ for every $\boldsymbol{u}_0 \in \mathcal{H}$ the unique (mild) solution $\boldsymbol{u}(t)$ to (W) with initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ is given by

$$\boldsymbol{u}(t) = S(t)\boldsymbol{u}_0$$

Exponential stability

$S(t)$ is said to be exponentially stable if there exist $\omega > 0$ and $C \geq 1$ such that

$$\|S(t)\|_{L(\mathcal{H})} \leq Ce^{-\omega t}$$

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For the semigroup $S(t) = e^{t\mathbb{G}}$ generated by (W) exponential stability occurs if and only if

$$\inf_{s\in\sigma(A)} f(s) > 0 \qquad \text{and} \qquad \sup_{s\in\sigma(A)} \frac{f(s)}{s} < \infty \tag{EXP}$$

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We define the exponential decay rate as

$$\omega_* = \sup\left\{\omega > 0 : \|S(t)\|_{L(\mathcal{H})} \leq Ce^{-\omega t} \text{ for some } C = C(\omega) \geq 1\right\}$$

The SDG condition

Much easier to detect is the spectral bound of $\mathbb{G}$

$$\sigma_* = \sup_{\lambda \in \sigma(\mathbb{G})} \mathfrak{Re}\,\lambda$$

related to ω_* through the (possibly strict) inequality

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$$\omega_* \leq -\sigma_*$$

$S(t)$ satisfies the spectrum determined growth (SDG) condition if

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$$u(t) = \begin{cases} c_1 e^{-(a-\sqrt{a^2-1})t} + c_2 e^{-(a+\sqrt{a^2-1})t} & a > 1 \\ c_1 e^{-t} + c_2 t e^{-t} & a = 1 \\ c_1 e^{-at} \sin[(\sqrt{1-a^2})t] + c_2 e^{-at} \cos[(\sqrt{1-a^2})t] & a < 1 \end{cases}$$

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when $a = 1$ the norm of the solution reads

$$\|S(t)(u_0, v_0)\|_{\mathcal{H}} = \sqrt{u_0^2 + v_0^2 + 2(u_0^2 - v_0^2)t + 2(u_0 + v_0)^2 t^2}\, e^{-\frac{t}{2}}$$

Problem 12 (R. Nagel)**.** Let $(T(t))_{t\geq 0}$ be a C_0-semigroup whith growth bound

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \|T(t)\| \leq M^{\omega} \cdot e^{t\omega} \text{ for } t \geq 0\}$$

Find condition such that ω_0 is minimum, i.e.,

$$\|T(t)\| \leq M_0 \cdot e^{t\omega_0} \text{ for } t \geq 0$$

Comments. This corresponds to a characterization of boundedness for semigroups.

Source: R. Nagel's list of problems collected in 2003 at the workshop in Bari.

$S(t)$ satisfies the strong spectrum determined growth (SSDG) condition if the decay rate $\omega_* = -\sigma_*$ is attained, that is if

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Previous contributions. Under some assumptions which prevent $f(s)$ to grow at infinity faster than s^θ with $\theta < \frac{1}{2}$ J. Goldstein and coauthors [2012-2014] obtained sharp exponential decay estimates for trajectories originating from sufficiently regular initial data

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Our results. Within the sole assumption (EXP) we show that $S(t)$ fulfills the SSDG condition except in some particular resonant cases where the term $e^{-\omega_* t}$ is penalized by a factor t

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Our results. Within the sole assumption (EXP) we show that $S(t)$ fulfills the SSDG condition except in some particular resonant cases where the term $e^{-\omega_* t}$ is penalized by a factor t

→ **this result is optimal** and the decay rate is the best possible allowed by the theory

The Spectrum of $\mathbb{G}$

For every fixed $s \in \sigma(A)$ we introduce the pair of complex numbers

$$\lambda_s^{\pm} = \begin{cases} -f(s) \pm i\sqrt{s - f^2(s)} & \text{if } f(s) \leq \sqrt{s} \\ -f(s) \pm \sqrt{f^2(s) - s} & \text{if } f(s) > \sqrt{s} \end{cases}$$

which are nothing but the solutions to the second order equation

$$\lambda^2 + 2f(s)\lambda + s = 0$$

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We also consider the (possibly empty) set

$$\Lambda = \left\{ \lambda < 0 : \exists\, s_n \in \sigma(A) \; : \; s_n \to \infty \; \text{ and } \; \lim_{n\to\infty} \frac{f(s_n)}{s_n} = -\frac{1}{2\lambda} \right\}$$

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The spectrum of $\mathbb{G}$ reads

$$\sigma(\mathbb{G}) = \bigcup_{s\in\sigma(A)} \{\lambda_s^{\pm}\} \cup \Lambda$$

We introduce the continuous function $\phi : \sigma(A) \to (0, \infty)$

$$\phi(s) = \begin{cases} f(s) & \text{if } f(s) \le \sqrt{s} \\ f(s) - \sqrt{f^2(s) - s} & \text{if } f(s) > \sqrt{s} \end{cases}$$

along with the number

$$m_* = \inf_{s \in \sigma(A)} \phi(s)$$

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The following hold

- $m_* > 0$
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Definition

$S(t)$ is said to be resonant if there exists $s_* \in \sigma(A)$ such that

$$m_* = \phi(s_*) \qquad \text{and} \qquad f(s_*) = \sqrt{s_*}$$

Statement of the result

Theorem

There exists a constant $C \geq 1$ such that

- $\|S(t)\|_{L(\mathcal{H})} \leq Ce^{-m_* t}$ *if $S(t)$ not resonant*
- $\|S(t)\|_{L(\mathcal{H})} \leq C(1+t)e^{-m_* t}$ *if $S(t)$ resonant*

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Since $\sigma_* = -m_*$ the latter yields

Corollary

- *If $S(t)$ is not resonant then it fulfills the SSDG condition*
- *If $S(t)$ is resonant then it fulfills the SDG condition but not the SSDG one*

Application: wave equations with fractional damping

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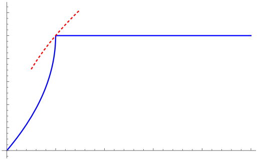
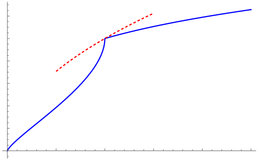


$\boldsymbol{\theta = 0}$



$\boldsymbol{\theta \in (0, \tfrac{1}{2})}$

The function ϕ is increasing and thus

$$m_* = \phi(s_0) \qquad \text{where} \qquad s_0 = \min \sigma(A) > 0$$

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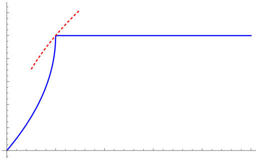

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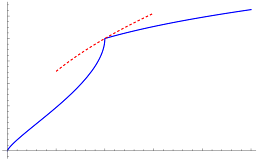

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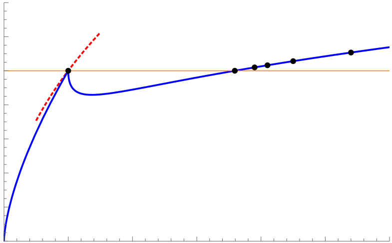
The function ϕ is increasing and thus

$$m_* = \phi(s_0) \qquad \text{where} \qquad s_0 = \min \sigma(A) > 0$$

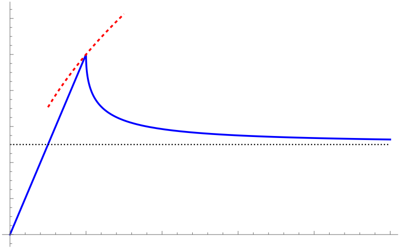
$\rightarrow S(t)$ is resonant if and only if $s_0 = a^{\frac{2}{1-2\theta}}$

$$\boxed{\boldsymbol{\theta \in \left(\tfrac{1}{2}, 1\right)}}$$



- ϕ increasing for $s < a^{\frac{2}{1-2\theta}}$
- ϕ decreasing for $s \in (a^{\frac{2}{1-2\theta}}, s_{\mathrm{m}}),\ s_{\mathrm{m}} = \min\{\phi(s) : s > a^{\frac{2}{1-2\theta}}\}$
- ϕ increasing and diverging to infinity for $s > s_{\mathrm{m}}$

$\boxed{\boldsymbol{\theta = 1}}$



- $\phi(s) = as$ for $s < \frac{1}{a^2}$
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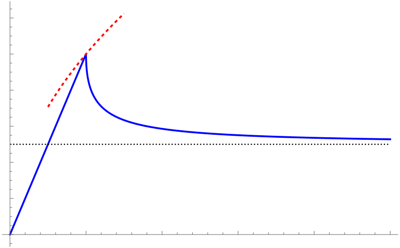

- $\phi(s) = as$ for $s < \frac{1}{a^2}$
- ϕ reaches its maximum value $\frac{1}{a}$ and then it is decreasing and converges to $\frac{1}{2a}$

$\rightarrow$ resonance cannot occur except in the trivial case $\sigma(A) = \{\frac{1}{a^2}\}$

Sketch of the proof

Theorem

There exists a constant $C \geq 1$ such that

- $\|S(t)\|_{L(\mathcal{H})} \leq Ce^{-m_* t}$ *if $S(t)$ not resonant*
- $\|S(t)\|_{L(\mathcal{H})} \leq C(1+t)e^{-m_* t}$ *if $S(t)$ resonant*

Sketch of the proof

For $K \geq 2$ and $\varepsilon \in (0,1)$ we decompose $\sigma(A)$ into the disjoint union

$$\sigma(A) = \sigma_0 \cup \sigma_1 \cup \sigma_2 \cup \sigma_3$$

where

$$\sigma_0 = \left\{ s \in \sigma(A) : \frac{f(s)}{\sqrt{s}} > K \right\}$$

$$\sigma_1 = \left\{ s \in \sigma(A) : \frac{f(s)}{\sqrt{s}} \leq 1 - \varepsilon \right\}$$

$$\sigma_2 = \left\{ s \in \sigma(A) : 1 + \varepsilon \leq \frac{f(s)}{\sqrt{s}} \leq K \right\}$$

$$\sigma_3 = \left\{ s \in \sigma(A) : 1 - \varepsilon < \frac{f(s)}{\sqrt{s}} < 1 + \varepsilon \right\}$$

- Given any trajectory

$$(u(t), \dot{u}(t)) = S(t)(u_0, v_0) \in \mathfrak{D}(\mathbb{G})$$

we define the energy

$$\mathsf{E}(t) = \|A^{\frac{1}{2}} u(t)\|^2 + \|\dot{u}(t)\|^2$$

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$$\mathsf{E}(t) = \sum_{\imath=0}^{3} \mathsf{E}_\imath(t)$$

where

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$\rightarrow \mathsf{E}_\imath \equiv 0$ if $\sigma_\imath = \emptyset$

- For every $K \geq 2$ large enough we have the inequality

$$\mathrm{E}_0(t) \leq 3\mathrm{E}_0(0)e^{-2m_* t}$$

- For every $\varepsilon \in (0,1)$ we have the inequality

$$\mathrm{E}_1(t) \leq \frac{2-\varepsilon}{\varepsilon}\mathrm{E}_1(0)e^{-2m_* t}$$

- For every $\varepsilon \in (0,1)$ and every $K \geq 2$ we have the inequality

$$\mathrm{E}_2(t) \leq \frac{9K^2}{\varepsilon}\mathrm{E}_2(0)e^{-2m_* t}$$

- For every $\varepsilon \in (0, \frac{1}{16})$ such that $\sigma_3 \neq \emptyset$ we have the inequality

$$\mathrm{E}_3(t) \leq \frac{8}{\varepsilon}\mathrm{E}_3(0)e^{-2m_3(1-4\sqrt{\varepsilon})t}$$

where $m_3 = \inf_{s \in \sigma_3} \phi(s)$

$$\|S(t)\|_{L(\mathcal{H})} \leq Ce^{-m_* t} \quad \text{if } S(t) \text{ not resonant}$$

$\|S(t)\|_{L(\mathcal{H})} \leq Ce^{-m_* t}$ if $S(t)$ not resonant

Using the fact that $S(t)$ is not resonant we show that for all ε small

$$m_3 > m_*$$

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We conclude that

$$\mathrm{E}(t) = \sum_{\imath=0}^{3} \mathrm{E}_\imath(t) \leq M\mathrm{E}(0)e^{-2m_* t}$$

for some $M = M(K, \varepsilon) > 0$

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for every $\varepsilon \in (0, \frac{1}{16})$ and thus

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Thank you for your attention