FernUniversität in Hagen

Observability of Stochastic Evolution Equations in Infinite Dimensions

Fatima Zahra Lahbiri
Joint work with: Said Hadd

FernUniversität in Hagen

Outline

Observed stochastic linear systems

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ filtered probability space.
- $(\mathrm{W}(\mathrm{t}))_{\mathrm{t}\geq 0}$ one dimensional Brownian motion.
- $\mathrm{A} : \mathrm{D}(\mathrm{A}) \subset \mathrm{H} \to \mathrm{H}$ a generator of a $\mathcal{C}_0$-semigroup $(\mathrm{T}(\mathrm{t}))_{\mathrm{t}\geq 0}$ on a Hilbert space H.
- $\mathcal{M} : \mathrm{H} \to \mathrm{H}$ is a linear bounded operator.
- $\mathrm{C} \in \mathcal{L}(\mathrm{D}(\mathrm{A}), \mathrm{Y})$ is the observation operator (not necessary closed).

For any $\xi \in \mathrm{L}^2_{\mathcal{F}_0}(\Omega, \mathrm{H})$, consider the observed stochastic linear system:

$$\begin{cases} \dot{\mathrm{X}}(\mathrm{t}) = \mathrm{A}\mathrm{X}(\mathrm{t}) + \mathcal{M}(\mathrm{X}(\mathrm{t}))\mathrm{d}\mathrm{W}(\mathrm{t}), & \mathrm{X}(0) = \xi, \quad \mathrm{t} \geq 0, \\ \mathrm{Y}(\mathrm{t}) = \mathrm{C}\mathrm{X}(\mathrm{t}), & \mathrm{t} \geq 0. \end{cases} \tag{1}$$

Observed stochastic linear systems

☞ The mild solution of system (1):

$$X(t) = T(t)\xi + \int_0^t T(t-s)\mathscr{M}(X(t))dW(s) \in H, \quad \mathbb{P}-\text{a.s},$$

for any $t \geq 0$ and $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$.

☞ First problem

– For any $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$, how to give a sense to the following process

$$\begin{aligned} Y(t) &= CX(t) \\ &= CT(t)\xi + C\int_0^t T(t-s)\mathscr{M}(X(t))dW(s). \end{aligned}$$

Main Assumption

- Assume that (C,A) is admissible, i.e, for some $\tau > 0$, there exists a constant $\gamma := \gamma(\tau) > 0$

$$\int_0^\tau \|CT(t)x\|^2 dt \leq \gamma^2 \|x\|^2, \quad \forall x \in D(A).$$

Yosida extension: (Weiss 1989)

We can associate with each $C \in \mathcal{L}(D(A), Y)$, the operator

$$D(C_\Lambda) := \left\{ x \in H : \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x \text{ exists in } Y \right\},$$
$$C_\Lambda x := \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x.$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ and $\lambda \in \rho(A)$.

Observed stochastic Linear systems

The stochastic version of Weiss's result

Assume that (C, A) is admissible. Then

- For $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$, $T(t)\xi \in D(C_\Lambda)$, $\quad$ a.e.t > 0, $\quad \mathbb{P}-$ a.s.
- For $\alpha > 0$, there exists $\kappa > 0$ such that

$$\mathbb{E}\int_0^\alpha \|C_\Lambda T(t)\xi\|_Y^2 dt \leq \kappa^2 \mathbb{E}\|\xi\|^2, \quad \xi \in L^2_{\mathcal{F}_0}(\Omega, H).$$

- We replace the term $CT(t)\xi$ for $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$ by $C_\Lambda T(t)\xi$.

Main Result

Proposition: Hadd-L 2021

Assume that (C, A) is admissible. Then for any $\alpha > 0$ and $\zeta \in \mathrm{L}^2_{\mathbb{F}}([0,\alpha];\mathrm{H})$, we have for some constant $\gamma := \gamma(\alpha) > 0$

$$(\mathrm{T} \diamond \zeta)(\mathrm{t}) := \int_0^{\mathrm{t}} \mathrm{T}(\mathrm{t}-\mathrm{s})\zeta(\mathrm{s})\mathrm{dW}(\mathrm{s}) \in \mathrm{D}(\mathrm{C}_\Lambda) \quad \text{a.e. } \mathrm{t} \geq 0, \quad \mathbb{P}-\text{a.s}, \quad \text{and}$$

$$\mathbb{E}\int_0^{\alpha} \|\mathrm{C}_\Lambda(\mathrm{T}\diamond\zeta)(\mathrm{t})\|^2_{\mathrm{Y}}\mathrm{dt} \leq \gamma^2 \mathbb{E}\int_0^{\alpha} \|\zeta(\mathrm{s})\|^2_{\mathrm{H}}\mathrm{ds}.$$

☞ Consequence: $\mathrm{X}(\mathrm{t}) \in \mathrm{D}(\mathrm{C}_\Lambda)$, a.e $\mathrm{t} \geq 0$, $\mathbb{P}-$a.s, and there is a constant $\mathrm{c}_\alpha > 0$,

$$\mathbb{E}\int_0^{\alpha} \|\mathrm{C}_\Lambda \mathrm{X}(\mathrm{t})\|^2_{\mathrm{Y}}\mathrm{dt} \leq \mathrm{c}_\alpha^2 \mathbb{E}\|\xi\|^2_{\mathrm{H}}, \quad \forall \xi \in \mathrm{L}^2_{\mathcal{F}_0}(\Omega;\mathrm{H}).$$

Observed Stochastic Semilinear Systems

For $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$. Consider the following observation semilinear system

$$\begin{cases} dX(t) = (AX(t) + G(X(t)))dt + F(X(t))dW(t), & X(0) = \xi, \quad t \geq 0, \\ Y(t) = CX(t), & t \geq 0, \end{cases} \tag{2}$$

where $C \in \mathcal{L}(D(A), Y)$ and $G, F : H \to H$ satisfying

$$\|G(x) - G(y)\| + \|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in H, \tag{3}$$

for $L > 0$.

Daprato-Zabczyk 2014

Under (3), there exists $X(\cdot) \in \mathcal{C}_{\mathbb{F}}(0, +\infty; L^2(\Omega, H))$ such that

$$X(t) = T(t)\xi + \int_0^t T(t-s)G(X(s))ds + \int_0^t T(t-s)F(X(s))dW(s).$$

Observed stochastic semilinear systems

☞ The problem is

– For any $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$, how to give a sense to the following process

$$Y(t) = CT(t)\xi + C\int_0^t T(t-s)G(X(s))ds + C\int_0^t T(t-s)F(X(s))dW(s).$$

Theorem (Hadd, L 2021)

Let (C, A) be admissible and condition (3) be satisfied. Then the semilinear stochastic system (2) is well-posed, i.e, $X(t;\xi) \in D(C_\Lambda)$ for a.e. $t > 0$ and $\mathbb{P}$-a.s and

$$\mathbb{E}\int_0^\alpha \|C_\Lambda (X(t;\xi_1) - X(t;\xi_2))\|_Y^2 dt \leq c^2 \mathbb{E}\|\xi_1 - \xi_2\|^2, \quad \xi_1, \xi_2 \in L^2_{\mathcal{F}_0}(\Omega, H).$$

Observed stochastic semilinear systems

For the well-posedness we use Yosida extensions

Let the condition (3) be satisfied, and (C, A) admissible. Then

(i) The mild solution of the system (2) satisfies for all $\xi \in \mathrm{L}^2_{\mathcal{F}_0}(\Omega, \mathrm{H})$

$$\mathrm{X}(\mathrm{t}, \xi) \in \mathrm{D}(\mathrm{C}_\Lambda), \quad \text{for a.e.}, \mathrm{t} > 0, \mathbb{P} - \text{a.s.}$$

(ii) For any $\xi_1, \xi_2 \in \mathrm{L}^2_{\mathcal{F}_0}(\Omega, \mathrm{H})$, there is $\upsilon_\alpha > 0$

$$\mathbb{E}\|\mathrm{X}(\mathrm{t}, \xi_1) - \mathrm{X}(\mathrm{t}, \xi_2)\|^2_{\mathrm{H}} \leq \upsilon_\alpha \mathbb{E}\|\xi_1 - \xi_2\|^2_{\mathrm{H}}, \qquad \mathrm{t} \in [0, \alpha].$$

Observed stochastic semilinear systems

(iii) For a constant $\delta := \delta(\gamma, \alpha, \mathrm{L}, \upsilon_\alpha)$

$$\begin{aligned} \mathbb{E}\int_0^\alpha \|\mathrm{C}_\Lambda(\mathrm{X}(\mathrm{t},\xi_1) - \mathrm{X}(\mathrm{t},\xi_2))\|_{\mathrm{Y}}^2 \mathrm{dt} &\leq 3\gamma^2 \mathbb{E}\|\xi_1 - \xi_2\|_{\mathrm{H}}^2 \\ + \quad 3\gamma^2\alpha\mathbb{E}\int_0^\alpha \|\mathrm{G}(\mathrm{X}(\mathrm{s},\xi_1)) - \mathrm{G}(\mathrm{X}(\mathrm{s},\xi_2))\|_{\mathrm{H}}^2 \mathrm{ds} \\ + 3\gamma^2\mathbb{E}\int_0^\alpha \|\mathrm{F}(\mathrm{X}(\mathrm{s},\xi_1)) - \mathrm{F}(\mathrm{X}(\mathrm{s},\xi_2))\|_{\mathrm{H}}^2 \mathrm{ds} \\ \leq \delta^2 \mathbb{E}\|\xi_1 - \xi_2\|_{\mathrm{H}}^2 \end{aligned}$$

☞ Remark: This well-posedness establishes a foundation for further investigations into the exact observability properties of the said system.

Contribution IV: Main Result

Definition: Stochastic case

Let (C, A) be admissible. For some $\tau > 0$, we say that the system (2) is τ-exactly observable if there exists a constant $\kappa_\tau > 0$ such that for any $\xi_1, \xi_2 \in L^2_{\mathcal{F}_0}(\Omega, H)$,

$$\|C_\Lambda X(\cdot;\xi_1) - C_\Lambda X(\cdot;\xi_2)\|_{L^2_{\mathbb{F}}(0,\tau;Y)} \geq \kappa_\tau \|\xi_1 - \xi_2\|_{L^2_{\mathcal{F}_0}(\Omega,H)}.$$

Theorem: Hadd, L 2021

Assume that the condition (3) is satisfied and the pair (C, A) is τ-exactly observable for some $\tau > 0$, i.e., there exists a constant $\gamma_\tau > 0$ such that

$$\int_0^\tau \|CT(t)x\|_Y^2 dt \geq \gamma_\tau^2 \|x\|^2, \quad \forall x \in D(A).$$

Then there exists a constant $\Theta_\tau > 0$ such that the stochastic semilinear system (2) is τ-exactly observable, whenever the Lipschtiz constant $L < \Theta_\tau$.

Contribution IV: Main Result

Sketch of proof

- Let $\xi_1, \xi_2 \in L^2_{\mathcal{F}_0}(\Omega, H)$, then there exists $\delta_\tau > 0$ such that

$$\mathbb{E}\int_0^\tau \|C_\Lambda T(t)(\xi_1 - \xi_2)\|^2_{\mathscr{Y}} dt \geq \delta_\tau^2 \mathbb{E}\|\xi_1 - \xi_2\|^2_H.$$

- If we denote by

$$\mathbb{K}_t(X(\cdot;\xi)) := \int_0^t T(t-s)G(X(s))ds + \int_0^t T(t-s)F(X(s))dW(s).$$

Then for any $\xi_1, \xi_2 \in L^2_{\mathcal{F}_0}(\Omega, H)$, we obtain for a constant $a_\tau^2 := 2\gamma^2 \upsilon_\tau(\tau^2 + \tau)$

$$\mathbb{E}\int_0^\tau \|C_\Lambda \left(\mathbb{K}_t(X(\cdot;\xi_1)) - \mathbb{K}_t(X(\cdot;\xi_2))\right)\|^2_Y dt \leq a_\tau^2 L^2 \mathbb{E}\|\xi_1 - \xi_2\|^2_H.$$

Contribution III: Main Result

– Moreover, we have

$$\begin{aligned}\mathbb{E}\int_0^\tau \|C_\wedge X(t,\xi_1) - C_\wedge X(t,\xi_2)\|_Y^2 dt &\geq \mathbb{E}\int_0^\tau \|C_\wedge T(t)(\xi_1-\xi_2)\|_Y^2 dt \\ &- \mathbb{E}\int_0^\tau \|C_\wedge\left(\mathbb{K}_t(X(\cdot;\xi_1)) - \mathbb{K}_t(X(\cdot;\xi_2))\right)\|_{\mathscr{Y}}^2 dt \\ &\geq \left(\delta_\tau^2 - a_\tau^2 L^2\right)\mathbb{E}\|\xi_1-\xi_2\|_H^2 \\ &:= h(L)\mathbb{E}\|\xi_1-\xi_2\|_H^2.\end{aligned}$$

– The function h is continuous from $\mathbb{R}^+$ to $(-\infty, \delta_\tau^2]$ and strictly decreasing, hence it is bijective. Then there exists a unique $\Theta_\tau > 0$ such that $h(\Theta_\tau) = 0$. Hence $h(L) > 0$ whenever $L < \Theta_\tau$. Thus the system (2) is τ-observable whenever $L < \Theta_\tau$.

Well-posedness of Stochastic systems with unbounded observation operators

Exact Observability for stochastic semilinear systems

Application to a semi-linear Dirichlet boundary Schrödinger equation

Other Contributions and Perspectives

Application to observed Schrödinger equations

☞ Let $\mathscr{O} = (0,\pi) \times (0,\pi)$, Γ be an open nonempty set of $\partial\mathscr{O}$. We consider the semi-linear boundary observation Schrödinger system

$$\begin{cases} dX(t,x) = (-i\Delta X(t,x) + f(X(t,x)))\,dt + g(X(t,x))dW(t), & x \in \mathscr{O},\ t \geq 0, \\ X(0,x) = \xi(x), & x \in \mathscr{O}, \\ \dfrac{\partial}{\partial \nu} X(t,x) = 0, & x \in \partial\mathscr{O},\ t \geq 0, \\ Y(t,x) = X(t,x) & x \in \Gamma,\ t \geq 0. \end{cases} \tag{4}$$

Here,

– $H := L^2(\mathscr{O})$, $Y := L^2(\Gamma)$.

The functions $g, f : H \to H$ satisfying for $l > 0$

$$\|g(x) - g(y)\| + \|f(x) - f(y)\| \leq l\|x - y\| \quad \forall x, y \in H.$$

Formulation of the Problem

☞ Define the following operators

$$A\psi = -i\Delta\psi, \qquad D(A) = \left\{\psi \in H^2(\mathscr{O}) : \frac{\partial}{\partial\nu}\psi = 0\right\},$$
$$C\psi = \psi_{|\Gamma}, \qquad \psi \in D(A).$$

With these notation (4) takes the following form

$$\begin{cases} dX(t) = (AX(t) + g(X(t)))dt + f(X(t))dW(t), \quad X(0) = \xi, & t \geq 0, \\ Y(t) = CX(t), & t \geq 0. \end{cases} \tag{5}$$

Application to observed Schrödinger equations

Verification of Assumptions

- The operator A is skew-adjoint and generates a strongly continuous semigroup $\mathbb{T} := (T(t))_{t\geq 0}$ on H.
- $g, f : H \to H$ satisfy the condition (3).
- There is exist $\tau > 0$ and $k_\tau > 0$ such that

$$\int_0^\tau \int_\Gamma \|X(t)\|^2 d\Gamma dt \geq k_\tau^2 \|\xi\|_H^2, \quad \forall \xi \in D(A).$$

☞ Consequence: The system (4) is τ-exactly observable whenever l is small enough.

[1] K. Ramdani, T. Takahashi, G. Tenenbaum and M. Tucsnak, A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator, J. Funct. Anal., 226 (2005) pp. 193–229.

Well-posedness of Stochastic systems with unbounded observation operators

Exact Observability for stochastic semilinear systems

Application to a semi-linear Dirichlet boundary Schrödinger equation

Other Contributions and Perspectives

Stochastic systems with unbounded control operators

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space.
- $(W(t))_{t\geq 0}$ a one dimensional Brownian motion.
- $A : D(A) \subset H \to H$ a generator of a $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ on H.
- $\mathscr{M} \in \mathcal{L}(H)$ a linear bounded operator.
- $B \in \mathcal{L}(U, H_{-1})$ control operator, where $(H_{-1} := \overline{H}^{\|\cdot\|_{-1}}, \quad \|x\|_{-1} = \|R(\lambda, A)x\|, \quad x \in H)$.

For $\xi \in L^2_{\mathcal{F}_0}(\Omega, H)$, we consider the stochastic control linear system

$$\begin{cases} dX(t) = (AX(t) + Bu(t))dt + \mathscr{M}(X(t))dW(t), \quad t > 0, \\ X(0) = \xi. \end{cases} \tag{6}$$

Stochastic systems with unbounded control operators

- For any $t \geq 0$, the mild solution of the stochastic linear system (6)

$$X(t) = T(t)\xi + \int_0^t T_{-1}(t-s)Bu(s)ds + \int_0^t T(t-s)\mathscr{M}(X(s))dW(s) \in H_{-1}, \quad \mathbb{P}-\text{a.s.}$$

- We define the following stochastic control maps

$$\tilde{\Phi}_t : L^2_{\mathbb{F}}(0,+\infty;U) \to L^2_{\mathcal{F}_t}(\Omega, H_{-1}), \quad \tilde{\Phi}_t u := \int_0^t T_{-1}(t-s)Bu(s)ds.$$

- If (A, B) is admissible, then

$$\tilde{\Phi}_t \in \mathcal{L}\left(L^2_{\mathbb{F}}(0,+\infty;U), L^2_{\mathcal{F}_t}(\Omega, H)\right), \quad \forall t \geq 0.$$

Theorem: Hadd, L 2021

Let (A, B) be admissible. Then there exists a unique mild solution $X(\cdot;\xi,u) \in \mathcal{C}_{\mathbb{F}}(0,+\infty; L^2(\Omega,H))$ of (6) such that

$$X(t;\xi,u) = T(t)\xi + \int_0^t T(t-s)\mathscr{M}(X(s,\xi,0))dW(s) + \Phi_t^W u, \qquad t \geq 0,$$

where

$$\Phi_t^W u = \tilde{\Phi}_t u + \int_0^t T(t-s)\mathscr{M}(\Phi_s^W u)dW(s), \quad \text{and}$$
$$\Phi_t^W \in \mathcal{L}\left(L^2_{\mathbb{F}}(0,+\infty;U), L^2_{\mathcal{F}_t}(\Omega;H)\right), \qquad t \geq 0.$$

Open Problem: Exact controllability of stochastic control problems

– (A, B) is exactly controllable on $[0, \tau]$ $\Rightarrow$ if $\exists \kappa_\tau > 0$ such that

$$\int_0^\tau \|B^* T^*(t) v\|^2 dt \geq \kappa_\tau^2 \|v\|^2, \quad v \in D(A^*).$$

☞ Open problem: If (A, B) is exactly controllable on $[0, \tau]$, is the stochastic control problem is exactly controllable on $[0, \tau]$?

Basic references and literature review

☞ We refer to:

[1] M. Baroun and B. Jacob, Admissibility and Observability of Observation Operators for Semilinear Problems, Integr. Equ. Oper. Theory, 64 (2009), pp. 1–20.

[2] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, Berlin, Heidelberg, 2000.

[3] Curtain, R. F., & Zwart, H. (2012). An introduction to infinite-dimensional linear systems theory (Vol. 21). Springer Science & Business Media.

[4] LAHBIRI, Fatima Zahra et HADD, Said. A functional analytic approach to infinite dimensional stochastic linear systems, SIAM Journal on Control and Optimization, 2021, vol. 59, no 5, p. 3762-3786.

[5] Q. Lü, Stochastic Well-posed Systems and Well-posedness of Some Stochastic Partial Differential Equations with Boundary Control and Observation, SIAM J. Control Optimization, 53 (2015), pp. 3457–3482.

[6] Q. Lü and X. Zhang, A Concise Introduction to Control Theory for Stochastic Partial Differential Equations, arXiv:2101.10678

[7] G. Weiss, Regular linear systems with feedback, Math. Control Signals Sys., 7 (1994), pp. 23–57.

Thank you for your attention