# Optimal transport of measures via autonomous vector fields


Nicola De Nitti

(joint work with X. Fernández-Real)

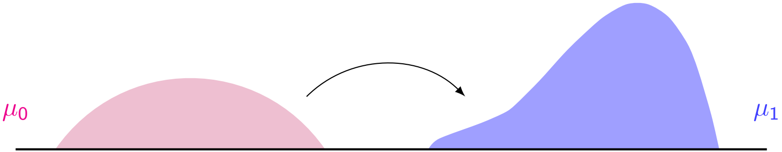
EPFL


August 28, 2024

# Presentation Outline

# A transport question by E. Zuazua

Given two probability measures, $\mu_0$ and $\mu_1$, find an autonomous (i.e., time-independent) vector field that transports $\mu_0$ to $\mu_1$.



# Problem I: exact controllability for ODEs (Lagrangian viewpoint)

Given two probability measures $\mu_0,\, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$, construct an *autonomous* vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ such that the corresponding flow, *i.e.*

(ODE)
$$\begin{cases} \partial_t \phi(t,x) = v(\phi(t,x)), & t > 0, \\ \phi(0,x) = x, & x \in \mathbb{R}^d, \end{cases}$$

is well-defined and satisfies

(AIM:I)
$$\phi(1,\cdot)_{\#}\mu_0 \equiv \mu_1,$$

We recall that the measure denoted by $\phi(1,\cdot)_{\#}\mu_0$ is defined by

$$(\phi(1,\cdot)_{\#}\mu_0)(A) \coloneqq \mu_0\left(\phi(1,\cdot)^{-1}(A)\right), \quad \text{for every measurable set } A \subset \mathbb{R}^d,$$

and is called *image measure* or *push-forward* of $\mu_0$ through $\phi(1,\cdot)$.

# Problem II: exact controllability for PDEs (Eulerian viewpoint)

Given two probability measures $\mu_0,\, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$, construct an *autonomous* vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ such that the solution $\mu : [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ to the Cauchy problem

$$\text{(CE)} \qquad \begin{cases} \partial_t \mu(t,x) + \mathrm{div}_x(v(x)\,\mu(t,x)) = 0, & t > 0,\ x \in \mathbb{R}^d, \\ \mu(0,x) = \mu_0(x), & x \in \mathbb{R}^d, \end{cases}$$

is well-defined and satisfies

$$\text{(AIM:II)} \qquad \mu(1,\cdot) \equiv \mu_1.$$

# Lagrangian vs Eulerian formulation

If $\mu_0$ and $v$ are smooth, then, by the *method of characteristics*, the (unique) solution $\mu$ of (CE) can be represented using the (unique) flow $\phi$ of (ODE), and viceversa.

That is, Problem I and Problem II are equivalent as a consequence of the *Lagrangian representation formula* for the solution of (CE):

$$\mu(t,\cdot) \equiv \phi(t,\cdot)_{\#}\mu_0, \qquad t \geq 0.$$

[Ambrosio–Bernard, *Rend. Lincei Mat. Appl.* 2008], [Bonicatto–Gusev, *Rend. Lincei Mat. Appl.* 2019], etc.: This is not necessarily true in more general situations.

# Presentation Outline

# Dacorogna–Moser's coupling

[Dacorogna–Moser, *AIHP* 1990]:

If $\mu_i$ (for $i \in \{0,1\}$) is absolutely continuous with smooth and positive density $\bar{\mu}_i$, then a *time-dependent* velocity field solving Problems I and II is given by

$$v(t,x) := \frac{\nabla f(x)}{(1-t)\,\bar{\mu}_0 + t\,\bar{\mu}_1},$$

where $f \in C^\infty(\mathbb{R}^d)$ is the unique solution of $-\Delta f = \bar{\mu}_1 - \bar{\mu}_0$ with zero mean.

# Localized and time-dependent vector fields

[Duprez–Morancey–Rossi, *SIAM J. Control Optim.* 2019]
[Duprez–Morancey–Rossi, *JDE* 2019]:

(Approximate) solution of Problems I and II using a *time-dependent* and localized perturbation of a given velocity field $v$:

$$v(x) + \chi_\omega(x)\, u(t,x).$$

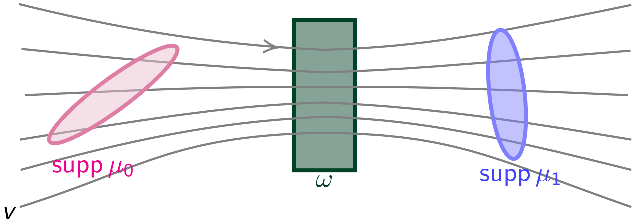



Figure: *Geometric condition:* the uncontrolled vector field $v$ needs to send the support of $\mu_0$ to $\omega$ forward in time and the support of $\mu_1$ to $\omega$ backward in time.

# Neural transport

[Ruiz-Balet–Zuazua, *SIAM Rev.* 2023]
[Alvarez-López–Slimane–Zuazua, *Neural Networks* 2024]:

(Approximate) solution to Problems I and II with "neural" velocity functions,

$$v(t,x) := w(t)\,\sigma\big(\langle a(t), x\rangle + b(t)\big),$$

with $\sigma(x) := \max\{x, 0\}$ (the so-called *activation function of the neural network*) and control parameters $a$, $w \in L^\infty((0,1);\mathbb{R}^d)$ and $b \in L^\infty((0,1);\mathbb{R})$.

The controls $a$, $w$, and $b$ were constructed *piecewise-constant in time* with an explicit (non-zero) *lower bound* on the number of jumps.

Cf. also [Li–Liu–Liverani–Zuazua, *arXiv:2407.17092*] for

$$v(t,x) := w\,\sigma(\langle a, x\rangle + b\,t + c).$$

# Presentation Outline

# When can we solve the problems?

*We can solve Problem I and Problem II under quite general assumptions.*

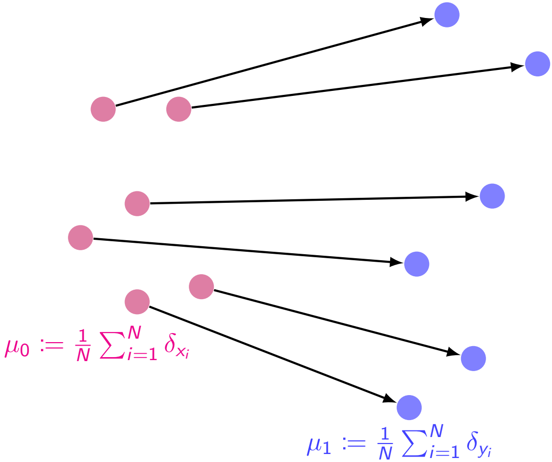
We consider $\mu_0, \mu_1 \in \mathcal{P}_{a.c.}(\mathbb{R}^d)$, with $d \geq 1$, and assume that the following conditions hold:

- $\operatorname{supp} \mu_0$ and $\operatorname{supp} \mu_1$ are convex;
- the densities $\bar{\mu}_0$ and $\bar{\mu}_1$ are continuous functions (in their respective supports);
- $\bar{\mu}_0 > 0$ in $\operatorname{supp} \mu_0$ and $\bar{\mu}_1 > 0$ in $\operatorname{supp} \mu_1$.

# Presentation Outline

# Dirac deltas in multi-d

If $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ are superpositions of Dirac deltas, for $d \geq 2$, it suffices to build non-intersecting paths (except, maybe, at the end-points) linking $x_i$ to $y_i$ for all $i \in \{1, \dots, N\}$.



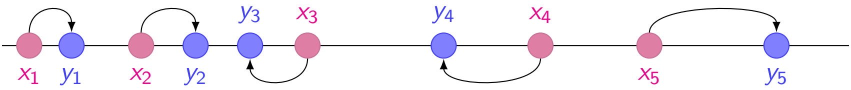
# Dirac deltas in 1-d

If

$$\mu_0 := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \qquad \mu_1 := \frac{1}{N}\sum_{i=1}^{N}\delta_{y_i},$$

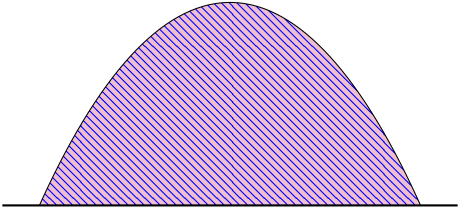
with $\{x_i\}_{i\in\{1,\dots,N\}}, \{y_i\}_{i\in\{1,\dots,N\}} \subset \mathbb{R}$ and $x_i \neq x_j,\ y_i \neq y_j,$ if $i \neq j,$

then there exists $v \in C^\infty(\mathbb{R})$ that solves Problem I.



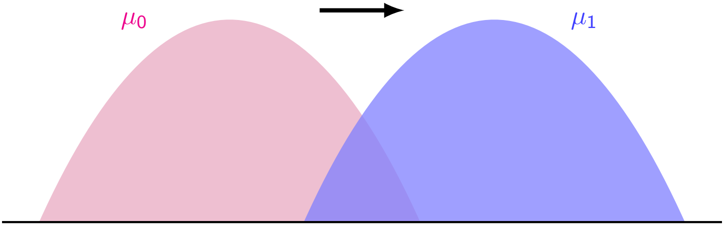
# Coinciding measures

$\mu_0 \equiv \mu_1$



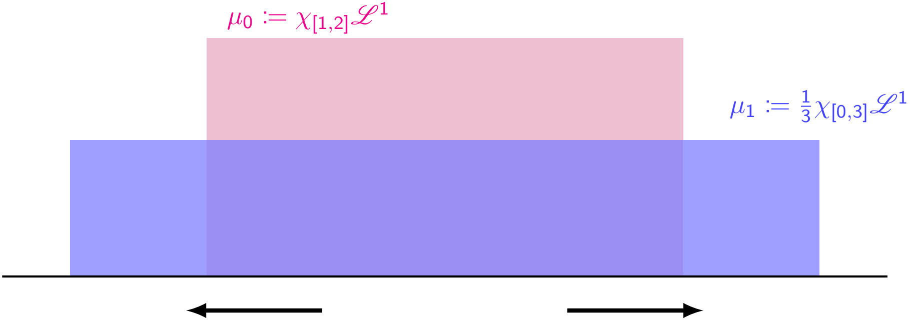
$v \equiv 0$

# Translated measures




$v \equiv \text{const}$

# $\mu_0 := \chi_{[1,2]}\mathscr{L}^1$ and $\mu_1 := \frac{1}{3}\chi_{[0,3]}\mathscr{L}^1$

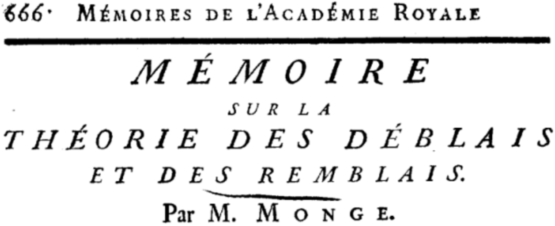



$$v(x) = (-3/2 + x)\log(3)$$

This yields $\phi(t,x) = -3/2(-1+3^t) + 3^t x$, so that $\phi(1,x) = 3x - 3$.

*How did we get this?*

# Monge's optimal transport problem (1781)



666· MÉMOIRES DE L'ACADÉMIE ROYALE

*MÉMOIRE*

*SUR LA*

*THÉORIE DES DÉBLAIS*

*ET DES REMBLAIS.*

Par M. MONGE.

*How to move dirt from one place (*déblais*) to another (*remblais*) while minimizing the "effort"?*

**Transport:** Find a mapping $\mathrm{T}$ between two measures such that $\mathrm{T}_{\#}\mu_0 = \mu_1$.

**Optimal:** Optimize with respect to a displacement cost $\mathrm{c}(x, y)$.

## Idea

*Realize $\mathrm{T}$ as time-1 map of the flow associated with an autonomous velocity.*

1. **1-d case:** realize the monotone transport map;
2. **multi-d case:** use Sudakov's disintegration approach.

## Related problems

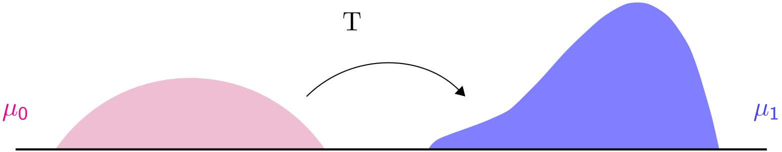
- Embedding homeomorphism into a flow: [Fort, *Proc. AMS* 1955].
- Inverse problem for ODEs of reconstructing the vector field from the time-$t_i$ map of the flow for some $\{t_i\}_{i\in\{1,\dots,N\}}$: [Alfaro Vigo–Álvarez–Chapiro–García–Moreira, *J. Comput. Dyn.* 2020], [Kuehn–Kuntz, *arXiv:2308.01213*].

# Presentation Outline

# One-dimensional Monge's problem

$$\mathrm{M_c}(\mu_0,\mu_1) := \min\left\{\int_{\mathbb{R}} \mathrm{c}(\mathrm{T}(x),x)\,d\mu_0(x) : \ \mathrm{T}:\mathbb{R}\to\mathbb{R} \text{ and } \mu_1 = \mathrm{T}_{\#}\mu_0\right\},$$

with cost $\mathrm{c}(x,y) := |x-y|^p$ for some $p \geq 1$.



## Theorem 1 (One-dimensional Monge's problem)

*Let* $\mu_0,\, \mu_1 \in \mathcal{P}(\mathbb{R})$ *and let us assume that* $\mu_0$ *is non-atomic (i.e., a diffuse measure:* $\mu_0(\{x\}) = 0$ *for any* $x \in \mathbb{R}$*).*

*Then there exists a unique (modulo countable sets) non-decreasing function* $\mathrm{T} : \operatorname{supp}\mu_0 \to \mathbb{R}$ *such that* $\mathrm{T}_\# \mu_0 \equiv \mu_1$*, given explicitly by*

$$\mathrm{T}(x) = \sup\Big\{z \in \mathbb{R} :\ \mu_1((-\infty, z]) \le \mu_0((-\infty, x])\Big\}, \quad \text{for} \quad x \in \operatorname{supp}\mu_0.$$

*Moreover, the function* $\mathrm{T}$ *is an optimal transport map (the unique optimal transport map if* $p > 1$*) and, provided that* $\operatorname{supp}\mu_1$ *is connected, it is continuous.*

*Finally, if* $\mu_0,\, \mu_1 \ll \mathscr{L}^1$ *and their densities* $\bar{\mu}_0$ *and* $\bar{\mu}_1$ *are continuous functions and satisfy* $0 < \lambda < \bar{\mu}_0, \bar{\mu}_1 < \Lambda < +\infty$ *(for some* $\lambda, \Lambda > 0$*) in their respective supports, then* $\mathrm{T}$ *is* $C^1$ *and its derivative is given by*

$$\mathrm{T}'(x) = \frac{\mu_0(x)}{\mu_1(\mathrm{T}(x))}.$$

# Problem I in 1-d

If $d = 1$, (ODE) reduces to

$$\text{(ODE-1d)} \qquad \begin{cases} \partial_t \phi(t,x) = v(\phi(t,x)), & t > 0,\ x \in \mathbb{R}, \\ \phi(0,x) = x, & x \in \mathbb{R}. \end{cases}$$

If the flow is unique (and defined up to time $t = 1$), then the map $\mathbb{R} \ni x \mapsto \phi(1,x)$ is *non-decreasing*.

Therefore, if a velocity $v : \mathbb{R} \to \mathbb{R}$ exists such that the corresponding flow $\phi$ exists, is unique, and satisfies $\phi(1,\cdot)_{\#}\mu_0 \equiv \mu_1$, then $\phi(1,\cdot)$ must coincide with the unique *monotone transport map* between $\mu_0$ and $\mu_1$

Let $\phi$ be the unique solution to

$$\begin{cases} \partial_t \phi = V(t, \phi(t,x)), & t > 0, \\ \phi(0,x) = x, & x \in \mathbb{R}, \end{cases}$$

where $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We claim that $x \mapsto \phi(1,x)$ is non-decreasing.

Let us suppose, by contradiction, that there exists $x_1 \leq x_2$ such that $\phi(1,x_2) < \phi(1,x_1)$.

Since $t \mapsto \phi(t,\cdot)$ is a continuous function, we can apply the intermediate-value theorem: $x_2 = \phi(0,x_2) > \phi(0,x_1) = x_1$ and $\phi(1,x_2) < \phi(1,x_1)$ imply that $\phi(\bar{t},x_2) = \phi(\bar{t},x_1) =: \bar{\phi}$ for some $\bar{t} \in (0,1)$.

This means that $\phi(t,x_1)$ and $\phi(t,x_2)$ solve the Cauchy problem

$$\begin{cases} \partial_t \psi(t) = V(t, \psi(t))), & t > \bar{t}, \\ \psi(\bar{t}) = \bar{\phi}, & x \in \mathbb{R}. \end{cases}$$

This yields a contradiction, because the solution of the Cauchy problem was assumed to be unique.

# Key idea

If $v \in C \cap L^\infty$ and $|v| > 0$, then there exists one and only one solution of (ODE-1d) in the following sense: $\phi(\cdot, x) \in C^1((0, +\infty))$ for every $x \in \mathbb{R}$ and

(SV) $$\int_x^{\phi(t,x)} \frac{1}{v(\xi)}\, d\xi = t, \qquad t > 0.$$

If $\phi(1, \cdot) \equiv \mathrm{T}$, then we have that a primitive of $1/v$ (*i.e.*, $F$ such that $F' = 1/v$) solves *Abel's functional equation*:

(A) $$F(\mathrm{T}(x)) = F(x) + 1, \quad x \in \operatorname{supp} \mu_0.$$

Differentiating with respect to $x$ yields *Aczél–Jabotinsky–Julia's equation*:

(AJJ) $$v(\mathrm{T}(x)) = \mathrm{T}'(x)\, v(x), \quad x \in \operatorname{supp} \mu_0.$$

Viceversa, a solution $v \in C \cap L^\infty$, with $|v| > 0$, of (AJJ) generates a unique flow $\phi$ that satisfies $\phi(1, \cdot) \equiv \mathrm{T}$ (up to a scaling constant to achieve $\mathrm{T}$ at $t = 1$).

# Solving linear homogeneous functional equations

To solve Problems I and II, we will build a suitable solution $v$ to Aczél–Jabotinsky–Julia's equation (AJJ), which belongs to the class of *linear homogeneous functional equations.*

[Kuczma, *Monogr. Mat.* 1968];
[Zdun, *Sci. Pub. Uni. Silesia* 1979];
[Kuczma–Choczewski–Ger, *Encycl. Math. Appl. Cambridge* 1990];
[Belitskii–Tkachenko, *Birkhäuser* 2003].

The velocity field is non-unique: we can construct it iteratively and it is obtained from an arbitrary prescription in an open set.

The construction is more or less delicate depending on the fixed points of $\mathrm{T}$.

## Theorem 2 (Exact controllability, $d = 1$)

*Let* $\mu_0, \mu_1 \in \mathcal{P}_{a.c.}(\mathbb{R})$ *be two probability measures with convex support, and continuous densities positive in their support.*

*Then there exists a velocity field* $v : \mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1) \to \mathbb{R}$ *such that*

$$\begin{aligned} |v| > 0 \quad &\text{in} \quad \mathrm{Conv}(\operatorname{supp}\mu_0 \cup \operatorname{supp}\mu_1) \setminus \mathcal{S}, \\ v \equiv 0 \quad &\text{in} \quad \mathcal{S}, \end{aligned}$$

*and*

$$\mathrm{T}(x) = \phi(1, x), \qquad x \in \operatorname{supp}\mu_0,$$

*where* $\mathrm{T}$ *is the* monotone optimal transport map, $\mathcal{S}$ *is the set of fixed points of the map* $\mathrm{T}$ *in* $\operatorname{supp}\mu_0$, *and* $\phi$ *is the unique solution of* (ODE-1d) *for* $x \in \operatorname{supp}\mu_0$.

*Moreover,* $v$ *is continuous except possibly at* $\partial\mathcal{S}$.

*If, additionally,* $|\bar{\mu}_0 - \bar{\mu}_1| > 0$ *in* $\partial\mathcal{S}$, *then* $v$ *can be taken to be continuous also at* $\partial\mathcal{S}$. *If, furthermore,* $\bar{\mu}_0$ *and* $\bar{\mu}_1$ *are Lipschitz continuous,* $v$ *can be taken Lipschitz continuous up to* $\partial\mathcal{S}$.

## Corollary 3 (Approximate controllability, $d = 1$)

*Let* $\mu_0,\, \mu_1 \in \mathcal{P}_{a.c.}(\mathbb{R})$ *be two probability measures with convex support, and continuous densities positive in their support.*

*For every* $\varepsilon > 0$, *there exists* $\mu_1^\varepsilon \in \mathcal{P}_{a.c.}(\mathbb{R})$ *such that* $\mathrm{dist}(\mu_1,\, \mu_1^\varepsilon) < \varepsilon$ *(in the sense of the* $L^1$ *or of the Wasserstein distance) and there exists a continuous velocity field* $v^\varepsilon : \mathrm{Conv}(\mathrm{supp}\,\mu_0 \cup \mathrm{supp}\,\mu_1^\varepsilon) \to \mathbb{R}$ *such that*

$$\begin{aligned} |v^\varepsilon| > 0 \quad & \textit{in} \quad \mathrm{Conv}(\mathrm{supp}\,\mu_0 \cup \mathrm{supp}\,\mu_1) \setminus \mathcal{S}, \\ v^\varepsilon \equiv 0 \quad & \textit{in} \quad \mathcal{S}, \end{aligned}$$

*and*

$$\mathrm{T}(x) = \phi(1, x), \qquad x \in \mathrm{supp}\,\mu_0,$$

*where* $\mathrm{T}$ *is the* monotone optimal transport map, $\mathcal{S}$ *of fixed points of the map* $\mathrm{T}$ *in* $\mathrm{supp}\,\mu_0$, *and* $\phi$ *is the unique solution of* (ODE-1d).

*If, furthermore,* $\bar{\mu}_0$ *and* $\bar{\mu}_1$ *are Lipschitz continuous,* $\mu_1^\varepsilon$ *and* $v^\varepsilon$ *can be taken Lipschitz continuous.*

## Remarks

- We claim that $\phi$ exists and is unique, even if $v$ is not globally continuous and can vanish on $\mathcal{S}$. In particular, we get uniqueness because $v$ satisfies an Osgood-type condition on $\mathcal{S}$:

  For any $\bar{x} \in \partial\mathcal{S}$ and $\varepsilon > 0$,

$$\int_{A_{\pm,\varepsilon}} \frac{dx}{|v(x)|} = \infty \quad \text{whenever} \quad A_{\pm,\varepsilon} \neq \emptyset,$$

  where

$$A_{+,\varepsilon} := (\bar{x}, \bar{x}+\varepsilon) \cap \operatorname{supp} \mu_0, \qquad A_{-,\varepsilon} := (\bar{x}-\varepsilon, \bar{x}) \cap \operatorname{supp} \mu_0.$$

- There exist measures $\mu_0$ and $\mu_1$, satisfying the hypotheses, such that $\bar{\mu}_0(\bar{x}) = \bar{\mu}_1(\bar{x})$ and either $v$ cannot be taken bounded, or there is no uniqueness of the flow (ODE-1d) (more precisely, it is not true $|v| > 0$ outside of $\mathcal{S}$). Moreover, if $v$ is continuous outside of $\bar{x}$, then it does not belong to $L^1_{loc}$ around $\bar{x}$.

- The $v$ constructed also gives a solution to Problem II in the appropriate sense (inspired by [Aizenman, *Duke Math. J.* 1978]).

  The velocity fields constructed do not have to be $L^1_{loc}$ in general at points $\partial\mathcal{S}$, and thus $\mu(t,\cdot) \equiv \phi(t,\cdot)_{\#}\mu_0$ need not be a distributional solution across $\partial\mathcal{S}$, but it satisfies (CE) as follows:

  *There exists a discrete set* $\partial\mathcal{S} = \partial\{x = \mathrm{T}(x)\}$ *where* $v \equiv 0$ *such that* $\mu(t,\cdot)$ *satisfies* (CE) *in the distributional sense in* $\mathrm{supp}(\mu(t,\cdot)) \setminus \partial\mathcal{S}$*, and it satisfies a no-flow condition through* $\partial\mathcal{S}$*; namely, trajectories starting outside of* $\partial\mathcal{S}$ *never reach* $\partial\mathcal{S}$ *in finite time.*

If $\mathrm{T}$ does not have fixed points, the construction of a solution is easy. If $\mathrm{T}$ has a fixed point, say at $x = a$, the situation is more difficult.

## Heuristics

As $x \to a^+$, we can approximate $v(\mathrm{T}(x)) \approx v(x)$ and $\mathrm{T}'(x) \approx \mathrm{T}'(a)$ and, heuristically, reduce (AJJ) to

$$\widetilde{v}(x) \approx \mathrm{T}'(a) v(x).$$

For two functions $v_1, v_2$ we obtain

$$\widetilde{v}_1(x) - \widetilde{v}_2(x) \approx \mathrm{T}'(a)(v_1(x) - v_2(x)).$$

# Presentation Outline

# Case without fixed-points

For the moment, let us assume

$$M_0 \coloneqq \operatorname{supp}\mu_0 = (a_0, b_0), \qquad M_1 \coloneqq \operatorname{supp}\mu_1 = (a_1, b_1).$$

**Case 1:** $\overline{M_0} \cap \overline{M_1} = \emptyset$**.** We can fix $v \equiv 1$ in $\overline{M_0}$, so that $v$ in $\overline{M_1}$ is given by

$$v(x) \coloneqq \mathrm{T}'(\mathrm{T}^{-1}(x))v(\mathrm{T}^{-1}(x)) = \mathrm{T}'(\mathrm{T}^{-1}(x)) \in \left[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}\right], \quad \text{for} \quad x \in \overline{M_1}.$$

In particular, $v$ can be chosen continuous and with $v(x) \in \left[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}\right]$ for $x \in \operatorname{Conv}(\overline{M_0 \cup M_1})$ to satisfy (AJJ).

**Case 2:** $\overline{M_0} \cap \overline{M_1} \neq \emptyset$**.** Without loss of generality, $a_0 < a_1$ (and therefore, $T(x) > x$ for $x \in \overline{M_0}$).

We define $\alpha_0 \coloneqq a_0$, $\alpha_1 \coloneqq a_1 = \mathrm{T}(\alpha_0)$, and $\alpha_i \coloneqq \mathrm{T}(\alpha_{i-1})$ for $i = 1, 2, \dots$. Then, there exists $N \in \mathbb{N}$ such that $\alpha_N \in (b_0, b_1]$. Indeed, $i \mapsto \alpha_i$ is increasing (owing to the monotonicity of $\mathrm{T}$) and, if $\alpha_i \le b_0$, $\alpha_{i+1} \le b_1$. If the sequence $\{\alpha_i\}_i$ had an accumulation point $\bar{\alpha} \le b_0$, then $\mathrm{T}(\bar{\alpha}) = \bar{\alpha}$ and $\bar{\alpha}$ is a fixed point for $\mathrm{T}$, which do not exist by assumption. Hence, the sequence must be finite.

Let us now fix $v(x) \in \left[\frac{\lambda}{\Lambda}, \frac{\Lambda}{\lambda}\right]$ to be any smooth function in $[a_0, a_1]$ with

$$\text{(R-1)} \qquad v(a_1) = \mathrm{T}'(a_0)v(a_0) = \frac{\mu_0(a_0)}{\mu_1(a_1)} v(a_0).$$

We then define, recursively, and denoting $\alpha_{N+1} \coloneqq b_1$,

$$\text{(R-2)} \qquad v(\mathrm{T}(x)) = \mathrm{T}'(x)v(x) \quad \text{for} \quad x \in [\alpha_i, \alpha_{i+1}],\ i = 1, 2, \dots, N.$$

This defines $v$ in the interval $[a_0, b_1]$ in a continuous way.

# Case with one fixed-point

Let $\bar{x}$ be the unique fixed point for $\mathrm{T}$, and let us assume, without loss of generality, that $\bar{x} = b_0 = b_1$

The sequence $\alpha_i$ is no longer finite, and $\alpha_i \to b_0 = b_1$ as $i \to +\infty$.

This allows us to recursively define a (continuous) vector field $v$ in $(a_0, b_0)$ by means of (R-2), after fixing it in $(a_0, \mathrm{T}(a_0))$ first. A priori, it could degenerate when approaching $\bar{x}$, though.

If $\bar{\mu}_0(\bar{x}) \neq \bar{\mu}_1(\bar{x})$, then we necessarily have $\mathrm{T}'(x) < 1$, which helps (studying (R-2)) to gain continuity up to $\bar{x}$.

# General case

Generalizing the arguments above, we can deal with the two remaining cases:

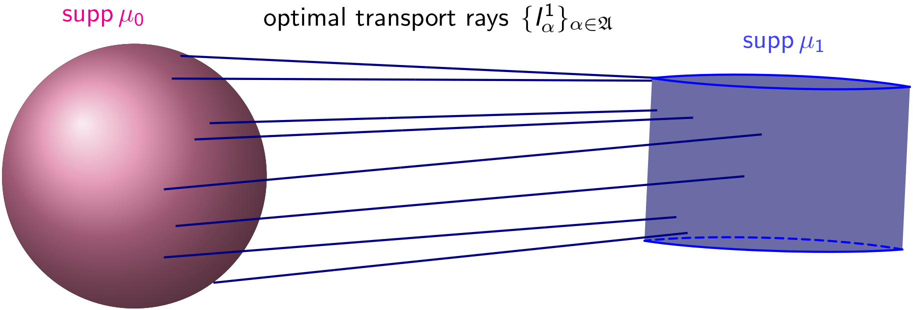
- transport map with exactly two fixed points;
- transport maps with arbitrarily many fixed points.

Finally, we remove the compact support assumption.

# Presentation Outline

# Multi-d problem via Sudakov's decomposition

Decompose multi-d optimal transport along 1-d segments (*optimal transport rays*).



## Theorem 4 (Exact controllability, $d \geq 1$)

*Let* $\mu_0, \mu_1 \in \mathcal{P}_{a.c.}(\mathbb{R}^d)$, *with* $d \geq 1$, *be two probability measures with convex support, and continuous densities positive in their support.*

*Then, there exists a vector field* $v : \mathbb{R}^d \to \mathbb{R}^d$ *such that*

$$\mathrm{T}(x) = \phi(1, x), \qquad x \in \mathbb{R}^d,$$

*where* $\mathrm{T}$ *is* Sudakov's transport map *(*$\mathrm{T}(x) := \mathrm{T}_\alpha(x)$ *if* $x \in I^1_\alpha$, *where* $\mathrm{T}_\alpha : I^1_\alpha \to I^1_\alpha$ *is the* monotone transport map on the ray $I^1_\alpha$*) and* $\phi$ *is the unique solution of* (ODE) *for* $x \in \operatorname{supp} \mu_0$.

# Presentation Outline

# Transport map with no fixed points

Let $\mu_0 := \chi_{[0,1]}\mathscr{L}^1$ and $\mu_1 := \frac{1}{2}\chi_{[2,4]}\mathscr{L}^1$. The monotone transport map between $\mu_0$ and $\mu_1$ is

$$\mathrm{T}(x) = 2x + 2.$$

A solution to Abel's and Julia's equations can be given explicitly as follows:

$$F(x) = c + \frac{\log|2+x|}{\log(2)}, \qquad c \in \mathbb{R},\ x \in \mathbb{R},$$

$$v(x) = (\log(4) + x\log(2)), \qquad x \in \mathbb{R}.$$

This yields $\phi(t,x) = -2 + 2^t(2+x)$, so that $\phi(1,x) = 2x + 1$.

The map $\mathrm{T}$ has a fixed point at $x = -2$, but it does not belong to the intervals where $\mu_0$ and $\mu_1$ are supported (and $F$ is not defined there).

# Transport map with one "good" fixed point

Let $\mu_0 := \chi_{[1,2]}\mathscr{L}^1$ and $\mu_1 := \frac{1}{3}\chi_{[0,3]}\mathscr{L}^1$. The monotone transport map between $\mu_0$ and $\mu_1$ is

$$\mathrm{T}(x) = 3x - 3.$$

A solution to Abel's and Julia's equations can be given explicitly as follows:

$$F(x) = c + \frac{\log(-3/2 + x)}{\log(3)}, \quad c \in \mathbb{R},\ x \in \mathbb{R},$$
$$v(x) = (-3/2 + x)\log(3), \quad x \in \mathbb{R}.$$

This yields $\phi(t, x) = -3/2(-1 + 3^t) + 3^t x$, so that $\phi(1, x) = 3x - 3$.

We observe that the map $\mathrm{T}$ has a fixed point, $\bar{x} = 3/2$ and $v(3/2) = 0$, while $F$ is not defined there.

# Gaussian measures

Let $\mu_0 \coloneqq \mathcal{N}\left(m_0, \sigma_0^2\right)$ and $\mu_1 \coloneqq \mathcal{N}\left(m_1, \sigma_1^2\right)$ be two Gaussian measures in $\mathbb{R}$: the densities are

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

The monotone transport map between $\mu_0$ to $\mu_1$ is given by

$$\mathrm{T}(x) = \frac{\sigma_1}{\sigma_0} x - \frac{\sigma_1}{\sigma_0} m_0 + m_1$$

(here, we take $\sigma_0$, $\sigma_1 > 0$). $\mathrm{T}$ coincides with the identity map if $m_0 = m_1$ and $\sigma_0 = \sigma_1$; has no fixed points if $\sigma_0 = \sigma_1$ and $m_0 \neq m_1$; and has one fixed point at $\bar{x} = \sigma_0 \frac{m_0 - m_1}{\sigma_0 - \sigma_1}$ if $\sigma_0 \neq \sigma_1$. At $\bar{x}$, the densities of the two measures measures do not coincide.

A solution to Abel's and Julia's equations can be given explicitly as follows:

$$F(x) = c + \frac{\log\left(\left|x - \frac{\sigma_1 m_0 - \sigma_0 m_1}{\sigma_1 - \sigma_0}\right|\right)}{\log\left(\frac{\sigma_1}{\sigma_0}\right)}, \qquad x \in \mathbb{R}, \text{ for any } c \in \mathbb{R},$$

$$v(x) = x \log\left(\frac{\sigma_1}{\sigma_0}\right) - \log\left(\frac{\sigma_1}{\sigma_0}\right) \frac{\sigma_1 m_0 - \sigma_0 m_1}{\sigma_1 - \sigma_0}, \qquad x \in \mathbb{R}.$$

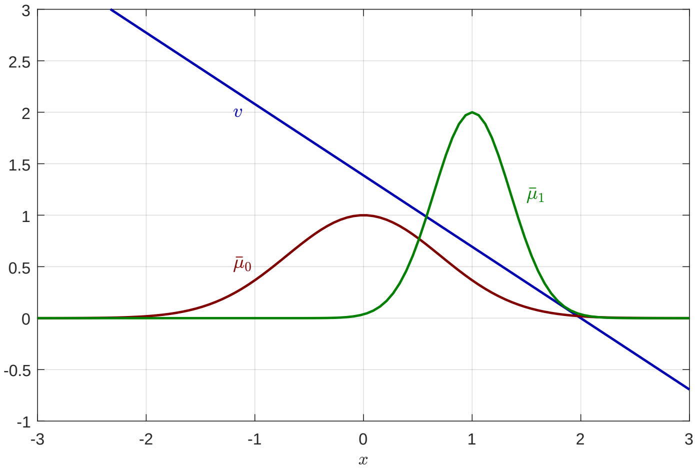



Figure: The vector field transporting a Gaussian $\bar{\mu}_0(x) = e^{-x^2}$ into a translated and rescaled Gaussian $\bar{\mu}_1(x) = 2e^{-4(x-1)^2}$ is given by the linear function $v$ here depicted. In particular, since the supports are unbounded, even if we are in a setting where the velocity field is smooth, it does not need to be globally bounded.

# Affine transport maps

If, in general, $\mu_0(dx) := f(x)\,\mathscr{L}^1(dx)$ and $\mu_1(dx) := \alpha f(\alpha(x-\beta))\,\mathscr{L}^1(dx)$ for some $\alpha > 0$, $\beta \in \mathbb{R}$, where the density $f$ is positive and continuous in its (convex) support, then the monotone map transporting $\mu_0$ into $\mu_1$ is

$$\mathrm{T}(x) = \frac{x}{\alpha} + \beta$$

which has a single fixed point at

$$x_{\alpha\beta} := \frac{\alpha\beta}{\alpha - 1}.$$

If $\alpha = 1$, this was just a translation and we can fix $v \equiv c$ constant in the whole space. Otherwise, we can take

$$v(x) = \begin{cases} x - x_{\alpha\beta} & \text{if} \quad \alpha \in (0,1), \\ x_{\alpha\beta} - x & \text{if} \quad \alpha > 1, \end{cases}$$

and then adjust a multiplicative constant on $v$ so that

$$\left| \int_0^\beta \frac{dx}{x - x_{\alpha\beta}} \right| = 1.$$

# Transport map with one "bad" fixed point

Let $\mu_0 \coloneqq \frac{1}{2}\chi_{[0,2]}\mathscr{L}^1$ and $\mu_1 \coloneqq \left(\frac{1}{2} - \frac{1}{9}x\right)\chi_{[0,3]}\mathscr{L}^1$. The monotone transport map that brings $\mu_1$ to $\mu_0$ is

$$\mathrm{T}^{-1}(x) = x - \frac{1}{9}x^2$$

It has a single fixed point at $\bar{x} = 0$, where the densities of both measures coincide.

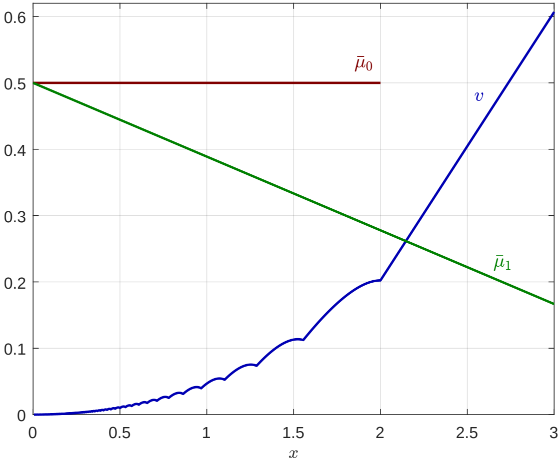



Figure: The velocity field $v$ (in blue) can be constructed arbitrarily in the interval $[2,3]$, and this fixes the values uniquely in $[0,2]$ as well. In this case, we have chosen a linear construction that matches the end-points in $[2,3]$. This extends to a continuous map, but since we are not trying to match higher derivatives, such a $v$ is not $C^1$.

# Transport map with two "good" fixed points

Let $\mu_0 := (1 - x)\chi_{[-1/2,1/2]}\mathscr{L}^1$ and $\mu_1 := (1 + x)\chi_{[-1/2,1/2]}\mathscr{L}^1$. The monotone transport map between $\mu_0$ and $\mu_1$ is

$$\mathrm{T} = \frac{1}{2}(-2 + \sqrt{2(3 + 4x - 2x^2)}),$$

which has two fixed points, $\mathcal{S} = \{-1/2, 1/2\}$. Moreover, $\bar{\mu}_0 \neq \bar{\mu}_1$ on $\mathcal{S}$.

Using Theorem 2, we can construct a Lipschitz continuous velocity field in $[-1/2, 1/2]$ solving Problems I and II.

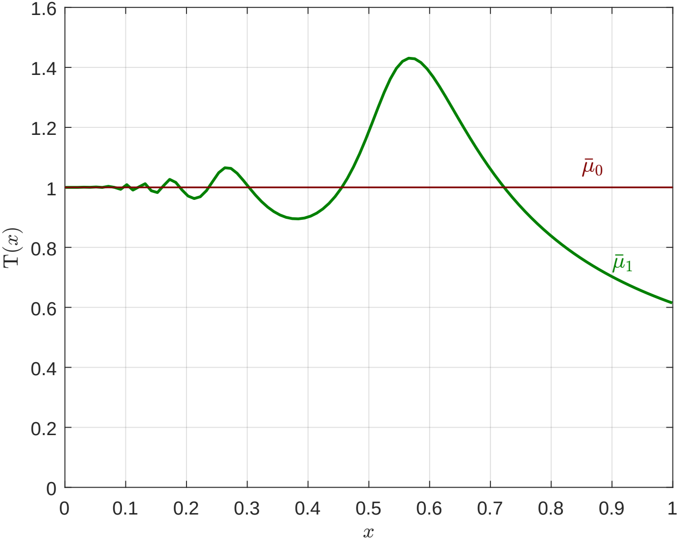
# Transport map with a sequence of “good” fixed points

Let $\mu_0 \coloneqq \chi_{[0,1]}\mathscr{L}^1$ and $\mathrm{T}(x) = x + \frac{1}{5}x^3\sin(\pi/x) \in C^1([0,+\infty))$, which has fixed points

$$\mathcal{S} = \{0\} \cup \left\{ \frac{1}{n} : \ n \in \mathbb{Z} \setminus \{0\} \right\}.$$

In $\mathcal{S}$, 0 is an accumulation point. We define $\mu_1 \coloneqq \mathrm{T}_\# \mu_0$ (so we have $\mu_1 = \bar{\mu}_1 \mathscr{L}^1$, with $\bar{\mu}_1 = (\mathrm{T}^{-1})' \chi_{[0,1]} \in C([0,1]) \cap C^\infty((0,1))$). Moreover, $\bar{\mu}_0 \neq \bar{\mu}_1$ in $\mathcal{S} \setminus \{0\}$.

Using Theorem 2, we can construct a Lipschitz continuous velocity field solving Problem I in $(0,1]$.


1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
T($x$)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$x$
$\bar{\mu}_0$
$\bar{\mu}_1$

Thank you for your attention!

Nicola De Nitti and Xavier Fernández-Real. *Optimal transport of measures via autonomous vector fields*. 2024. arXiv:2405.06503 [math.OC].